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A NONASSOCIATIVE QUATERNION SCALAR FIELD THEORY

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A nonassociative Groenewold–Moyal (GM) plane is constructed using quaternion-valued function algebras. The symmetrized multiparticle states, the scalar product, the annihilation/creation algebra and the formulation in terms of a Hopf algebra are also developed. Nonassociative quantum algebras in terms of position and momentum operators are given as the simplest examples of a framework whose applications may involve string theory and nonlinear quantum field theory.

Keywords: Field theory; quaternionic quantum mechanics; nonassociative algebras; Hopf algebras.

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1. Introduction

Noncommutative geometry[1] has a wide range of applications in quantum field theory,[2,3] in the construction of noncommutative physical models. These noncommutative theories are associative. A more general framework could be conceived whereby, in addition to noncommutativity, the algebra is also nonassociative. Our aim is to find an example where noncommutative and nonassociative algebra appears naturally in the context of field theory. Since most field theories are based on associative algebra, our aim is to obtain a deformation parameter θ such that associativity is recovered when θ goes to zero.

In the following pages, this goal is achieved by means of construction: we start with a field theory where the base space is composed of $\mathbb{R}^D$ and target space is composed of quaternions $\mathbb{H}$. The second step is to deform $\mathbb{R}^D$ into noncommutative algebra such that $[x^\mu, x^\nu] = i\theta^{\nu\mu}$. It turns out that the resulting algebra of fields is nonassociative. As expected, when $\theta^{\mu\nu}$ goes to zero, associativity is recovered.

Let us consider a quaternion-valued field theory, and write the field $\mathcal{F} : \mathbb{R}^D \to \mathbb{H}$ in a symplectic notation as $\mathcal{F} = f_0 + jf_1$, so that $f_{i=0,1} : \mathbb{R}^D \to \mathbb{C}$. In this theory, the sources of noncommutativity are the quaternion complex units i, j and $k = ij = -ji$. By deforming the commutative multiplication of the complex-valued functions $f_{i=0,1}$ to a noncommutative, we obtain a theory with nonassociativity as a by-product of the superposition of the two different noncommutativities.

Nonassociative phenomena appear in many places, and further information can be found in reviews on the subject.[4,5] However, while nonassociativity is common place in algebra,[6] examples of nonassociativity in physics are a collection of disconnected problems. The most obvious proposals for finding a physical phenomenon that may be described by nonassociativity involve the octonion field.[7–12] Although octonion algebra is a standard example of nonassociativity, it does not have an associative limit. Recently, nonassociative structures have appeared in general relativity,[13–17] string theory[18–20] and brane theory.[21–26] The model proposed in this paper is an attempt to obtain a very simple example of nonassociativity where associativity can be recovered at a suitable limit.

The field theory described in this paper has a natural interpretation since its target space may be understood as the tangent space of a hyper-complex manifold. In the same way a complex manifold is locally complex, a hyper-complex manifold is locally quaternionic. In the context of supersymmetric models, there can be various types of complex and hyper-complex manifolds as found in supersymmetric extensions of nonlinear sigma models,[27–31] string compactification on $K3$ surfaces,[32] generalized hyper-Kähler applied to string theory,[33,34] and even speculations on the nature of time.[35] Therefore, the model presented here can be understood as a linearized version of such non-linear sigma models.

Our results are related to quaternion quantum mechanics and quantum field theory.[35–41] However, these latter models do not consider multiparticle states, and consequently in these theories it is impossible to build states with particle statistics, a problem that has been solved here by defining annihilation and creation operators of symmetrized states.

This paper is organized as follows: in Sec. 2, we present the nondeformed quaternion scalar field theory and its multiparticle states and statistics. In Sec. 3, a deformed algebra of functions is formulated according to the Groenewold–Moyal (GM) procedure. We then show that the resulting algebra is nonassociative. Examples of nonassociative quantum algebra obtained from introducing quaternion unity are presented as well. The last section contains our conclusions and future perspectives.

2. The Quaternion Scalar Field Theory

The aforementioned quaternion field theories have only one-particle state. This means that multiparticle states cannot be built according to boson–fermion statistics. In this section, this void is filled in the mathematical structure of quaternion field theory following the Hopf algebra formalism of Ref. 42, where the Poincaré group acts on the GM plane with a deformed co-product. In this section the deformation is the hyper-complex quaternion structure. A second deformation, in the usual multiplication, is introduced in Sec. 3.

2.1. *Poincaré invariance*

If g is an element belonging to the Poincaré group, the action of the symmetry group ($\triangleright$) on spacetime functions $\mathcal{F}, \mathcal{G} \in \mathbb{H}$ must obey

$$g \triangleright (\mathcal{F} \cdot \mathcal{G}) = (g \triangleright \mathcal{F}) \cdot (g \triangleright \mathcal{G}) \,, \tag{1}$$

where the dot represents ordinary multiplication. The symplectic notation is adopted for quaternionic functions, so that $\mathcal{F} = f_0 + f_1 j$, with $f_{i=1,2}$ $\mathbb{C}$-functions, and j is the complex element of quaternion algebra, and thus $ij = -ji$. In terms of Hopf algebras, the action of the elements of an algebra over a product of complex functions is determined by the co-product. By way of example, the translation generator $\hat{p} = i\partial_x$ of the Poincaré group acts on complex function algebra according to the co-product

$$\Delta(\hat{p}) = \mathbb{1} \otimes \hat{p} + \hat{p} \otimes \mathbb{1} \,, \tag{2}$$

which, acting on $f, g \in \mathbb{C}$ with multiplication m, is subject to the consistency constraint $m(\Delta(\hat{p})(f \otimes g)) = \hat{p}(f \cdot g)$, where m takes the elements of the tensor product and multiplies them. On the other hand, taking the quaternion functions fj and gj, again with f, g $\mathbb{C}$-functions, we obtain $m(\Delta(p)(fj \otimes gj)) \neq p(fj \cdot gj)$. This difficulty is solved by defining a quaternion tensor product, namely

$$(f \otimes g) \cdot (m \otimes n) = (f \cdot m) \otimes (g \cdot n) \,, \tag{3}$$

$$(f \otimes gj) \cdot (m \otimes n) = (f \cdot \bar{m}) \otimes (gj \cdot n) \,, \tag{4}$$

$$(f \otimes g) \cdot (mj \otimes n) = (f \cdot mj) \otimes (\bar{g} \cdot n) \,, \tag{5}$$

$$(f \otimes gj) \cdot (mj \otimes n) = (f \cdot \bar{m}j) \otimes (\bar{g}j \cdot n) \,. \tag{6}$$

f, g, m and n are complex-valued functions and barred functions are the complex conjugates. This result follows for $f, n \in \mathbb{H}$ as well. This kind of structure is similar to the $\mathbb{Z}_2$ tensor product found in Lie super-algebras.[43] Adopting this tensor product, the co-product satisfies the identity $\Delta(\hat{p}\hat{q}) = \Delta(\hat{p})\Delta(\hat{q})$, and the first element of a multiparticle quaternionic state is given: a well-defined co-product. As the co-product of the translation operator of the Poincaré algebra has the expression (2), either in the quaternion case or in the complex case, it will have the same behavior when the multiplication operation is deformed according to the Moyal procedure in both cases.[44] Thus, the deformed co-product of the rotation operator of the Poincaré group in the noncommutative complex function algebra is valid for the deformed quaternion spaces discussed in the next section as well.

2.2. *State statistics*

States endowed with well-defined statistics have a permutation operator which interchanges the positions of the functions describing the particles in a state. As the particles are represented by quaternion functions, for a generic quaternion state

$$\mathcal{F}\otimes\mathcal{G}=f_0\otimes g_0+f_0\otimes g_1 j+f_1 j\otimes g_0+f_1 j\otimes g_1 j\,, \tag{7}$$

the following operators are defined:

$$\hat{\sigma}\triangleright(\mathcal{F}\otimes\mathcal{G})=(\mathcal{G}\otimes\mathcal{F})\,, \tag{8}$$

$$\hat{\tau}\triangleright(\mathcal{F}\otimes\mathcal{G})=g_0\otimes f_0+g_1 j\otimes\bar{f}_0+\bar{g}_0\otimes f_1 j+\bar{g}_1 j\otimes\bar{f}_1 j\,, \tag{9}$$

$$\hat{\kappa}\triangleright(\mathcal{F}\otimes\mathcal{G})=f_0\otimes g_0+\bar{f}_0\otimes g_1 j+f_1 j\otimes\bar{g}_0+\bar{f}_1 j\otimes\bar{g}_1 j\,, \tag{10}$$

so that $\hat{\sigma}^2=\hat{\tau}^2=\hat{\kappa}^2=\mathbb{1}$ and $\hat{\kappa}=\hat{\sigma}\hat{\tau}$. Symmetrized states and anti-symmetrized states are defined as follows

$$|\mathcal{F}_1,\mathcal{F}_2\rangle_\pm=\frac{1}{2}(\mathbb{1}\pm\hat{\tau})\triangleright|\mathcal{F}_1,\mathcal{F}_2\rangle\,, \tag{11}$$

with $|\mathcal{F}_1,\mathcal{F}_2\rangle=\mathcal{F}_1\otimes\mathcal{F}_2$, the defined states are eigen-states of the permutation operator $\hat{\tau}$ according to

$$\hat{\tau}|\mathcal{F}_1,\mathcal{F}_2\rangle_\pm=\pm|\mathcal{F}_1,\mathcal{F}_2\rangle_\pm\,. \tag{12}$$

After defining symmetrized states and anti-symmetrized states, the particle statistics is guaranteed, and a scalar product is needed, which is presented below.

2.3. *The scalar product*

By expressing a one-particle state as $|\mathcal{F}\rangle=|f_0\rangle+|f_1 j\rangle$, so that the orthogonality condition $\langle f|gj\rangle=\langle fj|g\rangle=0$ holds, a complex-valued scalar product is obtained as a sum of usual scalar products of complex functions

$$\begin{aligned}\langle\mathcal{F}|\mathcal{G}\rangle&=\langle f_0|g_0\rangle+\langle f_1 j|g_1 j\rangle\\&=\langle f_0|g_0\rangle+\langle g_1|f_1\rangle\,.\end{aligned} \tag{13}$$

In the above scalar product, $\langle q\mathcal{F}|\mathcal{G}\rangle\neq\langle\mathcal{F}|\bar{q}\mathcal{G}\rangle$, where $q\in\mathbb{H}$ and $\mathcal{F},\mathcal{G}:\mathbb{R}^4\to\mathbb{H}$. As a consequence, when q is a quaternionic operator, this fact will result in the splitting of the creation/annihilation operator algebra in pure complex and pure quaternionic operators which separately obey the condition, as shown in the next item. On the other hand, when defining the scalar product of two-particle states as

$$\langle\mathcal{F},\mathcal{G}|\mathcal{M},\mathcal{N}\rangle=(|\mathcal{F},\mathcal{G}\rangle,\hat{\kappa}|\mathcal{M},\mathcal{N}\rangle) \tag{14}$$

$$=\langle f_0|m_0\rangle\langle\mathcal{G}|\mathcal{N}\rangle+\langle f_1 j|m_1 j\rangle\overline{\langle\mathcal{G}|\mathcal{N}\rangle}\,, \tag{15}$$

it is observed that if $\mathcal{F}=\mathcal{M}$ and $\mathcal{G}=\mathcal{N}$, then

$$|\mathcal{F}\otimes\mathcal{G}|^2=|\mathcal{F}|^2|\mathcal{G}|^2\,. \tag{16}$$

The scalar product also obeys the necessary self-adjointness condition

$$(\hat{\tau}|\mathcal{F},\mathcal{G}\rangle,|\mathcal{M},\mathcal{N}\rangle)=(|\mathcal{F},\mathcal{G}\rangle,\hat{\tau}|\mathcal{M},\mathcal{N}\rangle)\,. \tag{17}$$

Thus, the scalar product constructed above is valid for multiparticles, something which has not been observed in previous quaternion quantum theories.

2.4. *Creation and annihilation operators*

In principle, the creation $\mathbb{a}^\dagger_{\mathcal{F}}$ operator and annihilation operator $\mathbb{a}_{\mathcal{F}}$ of a quaternionic state are

$$\mathbb{a}^\dagger_{\mathcal{F}}=a^\dagger_{f_0}+a^\dagger_{f_1j}\qquad\text{and}\qquad\mathbb{a}_{\mathcal{F}}=a_{f_0}+a_{f_1j}\,. \tag{18}$$

However, as the scalar product constructed above is such that $\langle q\mathcal{F}|\mathcal{G}\rangle\neq\langle\mathcal{F}|\bar{q}\mathcal{G}\rangle$, where $q\in\mathbb{H}$ and $\mathcal{F},\mathcal{G}:\mathbb{R}^4\to\mathbb{H}$, the creation/annihilation algebra will be built in terms of $a^\dagger_{f_0}$, $a^\dagger_{f_1j}$, a_{f_0} and a_{f_1j}. These operators create complex fields, and thus satisfy commutation rules with the quaternionic unity j, namely

$$za_{f_0}=a_{f_0}z\,,\qquad ja_{f_0}=a_{\bar{f}_0}j\,,\qquad za_{f_1j}=a_{f_1j}\bar{z}\qquad\text{and}\qquad ja_{f_1j}=a_{\bar{f}_1j}j\,. \tag{19}$$

The operator creates/annihilates either a bosonic or a fermionic state, thus the wave function must be either symmetrized or anti-symmetrized. In order to construct the algebra, the scalar product must have the same result as that obtained by the creation annihilation operators. The necessary scalar products are

$${}_\pm\langle f\otimes g,m\otimes n\rangle_\pm=\langle f,m\rangle\langle g,n\rangle\pm\langle f,n\rangle\langle g,m\rangle\,, \tag{20}$$

$${}_\pm\langle f\otimes gj,m\otimes nj\rangle_\pm=\langle f,m\rangle\langle gj,nj\rangle\,, \tag{21}$$

$${}_\pm\langle fj\otimes gj,mj\otimes nj\rangle_\pm=\langle fj,mj\rangle\langle\bar{g}j,\bar{n}j\rangle\pm\langle fj,\bar{n}j\rangle\langle\bar{g}j,mj\rangle\,, \tag{22}$$

so that the plus sign corresponds to the symmetric bosonic states and the minus sign corresponds to the fermionic anti-symmetric states. For the bosonic case, the operator algebra reproduces the above results as

$$[a_f,a_g]=[a^\dagger_f,a^\dagger_g]=0\,, \tag{23}$$

$$a_fa_{gj}-a_{gj}a_{\bar{f}}=a^\dagger_fa^\dagger_{gj}-a^\dagger_{gj}a^\dagger_{\bar{f}}=0\,, \tag{24}$$

$$a_{fj}a_{gj}-a_{\bar{g}j}a_{\bar{f}j}=a^\dagger_{fj}a^\dagger_{gj}-a^\dagger_{\bar{g}j}a^\dagger_{\bar{f}j}=0\,, \tag{25}$$

$$a_{fj}a^\dagger_g-a^\dagger_{\bar{g}}a_{fj}=0\,, \tag{26}$$

$$[a_f,a^\dagger_g]=\langle f|g\rangle\,, \tag{27}$$

$$a_{fj}a^\dagger_{gj}-a^\dagger_{\bar{g}j}a_{\bar{f}j}=\langle fj|gj\rangle\,, \tag{28}$$

remembering that f and g are complex-valued functions, and that $\langle a, zb\rangle = z\langle a, b\rangle$ and $\langle a, jzb\rangle = \bar{z}\langle a, jb\rangle$ are adopted. On the other hand, for an anti-symmetric fermionic state, the operator algebra is

$$\{a_f, a_g\} = \{a_f^\dagger, a_g^\dagger\} = 0\,, \tag{29}$$

$$a_f a_{gj} + a_{gj} a_{\bar{f}} = a_f^\dagger a_{gj}^\dagger + a_{gj}^\dagger a_{\bar{f}}^\dagger = 0\,, \tag{30}$$

$$a_{fj} a_{gj} + a_{\bar{g}j} a_{\bar{f}j} = a_{fj}^\dagger a_{gj}^\dagger + a_{\bar{g}j}^\dagger a_{\bar{f}j}^\dagger = 0\,, \tag{31}$$

$$a_{fj} a_g^\dagger + a_{\bar{g}}^\dagger a_{fj} = 0\,, \tag{32}$$

$$\{a_f, a_g^\dagger\} = \langle f|g\rangle\,, \tag{33}$$

$$a_{fj} a_{gj}^\dagger + a_{\bar{g}j}^\dagger a_{\bar{f}j} = \langle fj|gj\rangle\,. \tag{34}$$

Thus, the constructed quaternionic scalar field theory has all the necessary structures: Poincaré invariant one-particle and multiparticle states; symmetrized and anti-symmetrized states with well-defined statistics; a scalar product and a creation/annihilation operator algebra. This theory can be deformed according to the GM procedure generalizing the well-known noncommutative complex field theories, and this is carried out in Sec. 3.

3. The Deformed Product

Noncommutative geometry is obtained by changing the ordinary commutative product of complex-valued functions f and g into the GM deformed product

$$f(x)\star g(x) = f(x)g(x) + \sum_{n=1}^{\infty}\left(\frac{i}{2}\right)\frac{1}{n}\theta^{i_1 j_1}\cdots\theta^{j_n j_n}\partial_{i_1}\cdots\partial_{i_n} f(x)\partial_{j_1}\cdots\partial_{j_n} g(x)\,, \tag{35}$$

so that θ^{ij} is anti-symmetric in its indices. Linear functions generate the commutator between coordinates

$$x^i \star x^j - x^j \star x^i = i\theta^{ij} \tag{36}$$

and in the limit where $\theta^{ij} \to 0$ the commutative geometry is recovered. Both the commutative product and the noncommutative product of functions are associative.

A more general picture may be obtained by deforming the usual product of quaternion-valued functions according to the GM prescription. The quaternionic-valued functions $\mathcal{F} : \mathbb{R}^D \to \mathbb{H}$ over a D-dimensional Euclidean space with coordinates x^i are represented by

$$\mathcal{F} = f_1 + f_2 j\,, \qquad \text{so that} \quad f_{a=1,2} : \mathbb{R}^D \to \mathbb{C}\,, \tag{37}$$

f_a are defined on a Schwarz space, thus allowing a Fourier transform $\tilde{\mathcal{F}}$, where

$$\tilde{\mathcal{F}} = \tilde{f}_1 + \tilde{f}_2 j \qquad \text{and} \qquad \tilde{f}_a(k) = \int d^D x\, e^{-ik_\mu x^\mu} f_a(x)\,. \tag{38}$$

Accordingly, the Weyl symbol of a quaternion function may be introduced as well, so that

$$\hat{W}[\mathcal{F}] = \hat{W}[f_1] + \hat{W}[f_2]j \qquad \text{and} \qquad \hat{W}[f_a] = \int \frac{d^D k}{(2\pi)^D} e^{-ik_\mu x^\mu} \tilde{f}_a \,. \tag{39}$$

The Weyl symbol allows the GM product to be introduced, thus replacing the usual multiplication, so that the complex-valued functions obey $\hat{W}[f_a \star f_b] = \hat{W}[f_a] \star \hat{W}[f_b]$, which results in

$$\mathcal{F} \star \mathcal{G} = f_1 \star g_1 + (f_1 \star g_2)j + j(\bar{f}_2 \star g_1) + j((\bar{f}_2 \star g_2)j) \,, \tag{40}$$

where the bar means the conjugate of the complex function. This star product of quaternion functions is, of course, noncommutative; nevertheless, it is also nonassociative, so that

$$(\mathcal{F} \star \mathcal{G}) \star \mathcal{H} - \mathcal{F} \star (\mathcal{G} \star \mathcal{H}) \neq 0 \,. \tag{41}$$

Equation (41) does not has a simple and illuminating form, and so we do not express it and emphasize the nonassociativity of the quaternionic fields only. This is an interesting by-product for introducing a noncommutative local structure on a former noncommutative complex structure. This simple theory has a number of possible applications, as cited in Sec. 1.

3.1. *Nonassociative quaternion quantum algebras*

The simplest example of a nonassociative deformed theory comes from quantum mechanics and its celebrated commutation relation

$$[\hat{x}, \hat{p}] = i\hbar, \tag{42}$$

whose $\hbar \to 0$ limit, or classical limit, turns the operators into a commutative algebra. Introducing the quaternion complex unity j naturally generates a nonassociative structure. As j does not commute with $[\hat{x}, \hat{p}]$, it does not associate with the products of the commutator anymore. The associator $(\hat{x}, j, \hat{p}) = (\hat{x}j)\hat{p} - \hat{x}(j\hat{p})$ may be calculated in the specific case where the quantum quaternion algebra is an alternative algebra. Using the Moufang identities,[6] the resulting associator is

$$(\hat{x}, j, \hat{p}) = k\hbar \,, \tag{43}$$

so that $k = ij$. This example in which quantum mechanics turns to a nonassociative theory is somewhat surprising, but it shows very simply how combining noncommutative structures generates a nonassociative one. In this case, the associative limit goes to a commutative complex theory, but this is a classical one. In this sense, the commutativity and associativity are coupled. A non-coupled case comes in the more general framework discussed previously.

On the other hand, it is possible to further extend the quantum algebra. Defining the operators

$$\hat{z}^\dagger = \frac{1}{\sqrt{2}}(p + ix) \qquad \text{and} \qquad \hat{z} = \frac{1}{\sqrt{2}}(p - ix) \,, \tag{44}$$

so that $[\hat{z}, \hat{z}^\dagger] = \hbar$, and with the use of the associator (43),

$$(\hat{z}, j, \hat{z}^\dagger) = (\hat{z}^\dagger, j, \hat{z}) = (\hat{z}, \hat{z}, j) = (j, \hat{z}, \hat{z}) = (\hat{z}^\dagger, \hat{z}^\dagger, j) = (j, \hat{z}^\dagger, \hat{z}^\dagger) = 0\,, \tag{45}$$

$$(\hat{z}^\dagger, \hat{z}, j) = -(j, \hat{z}^\dagger, \hat{z}) = -(\hat{z}, \hat{z}^\dagger, j) = (j, \hat{z}, \hat{z}^\dagger) = -(\hat{z}, j, \hat{z}) = (\hat{z}^\dagger, j, \hat{z}^\dagger) = j\,. \tag{46}$$

This is also a nonassociative and noncommutative algebra, although it is not alternative as that formed by $\hat{x}$, $\hat{p}$ and j, but its classical limit is also a classical quaternion theory as expected. The above examples are the simplest examples of the deformed algebras, whose geometry is to be analyzed in forthcoming studies.

4. Conclusion

In this paper, two novel quaternion quantum scalar field theories have been presented. Both of them are noncommutative because of the quaternion nature of their fields. In one of them ordinary commutative multiplication is defined, and in this case a multiparticle quaternion scalar theory has been constructed. The second theory is a deformation of the former one according to the GM procedure. This second theory is a noncommutative and nonassociative one. These theories are well-defined, and may be used in a number of physical applications, as the models are quite general. Developments in quaternion scalar fields and nonassociative geometry are the most immediate applications. We expect that results derived from this linear model will be useful when applied to hyper-Kähler structures in string theory and nonlinear sigma models.

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