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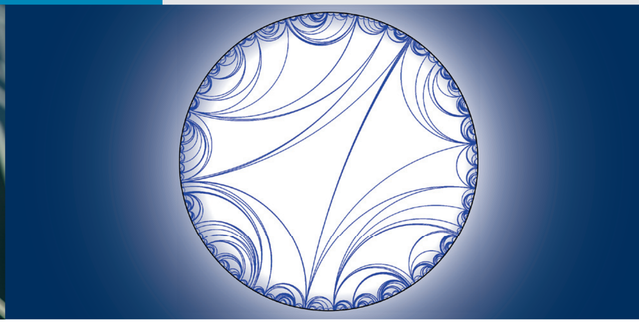

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Large emissions. Hawking-Penrose black hole model

Eugeny Pechersky, Sergei Pirogov† and Anatoly Yambartsev‡*

Abstract. *We propose a formalism about the large deviations of emissions. As an example we study the large deviations asymptotics for an introduced stochastic version of the Hawking-Penrose black hole model with special attention to the large emission regime. One of our goals is to find the most probable trajectory corresponding to a certain amount of the emission during the time interval.*

1 Introduction

This paper is devoted to applications of the large deviations theory. The large deviations theory is an area of probability theory studying rare events with vanishing positive probability. It means that such an event may occur but very rarely. It can happen as a catastrophic event like an overload of the queueing system or a crisis phenomenon in the

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economy. Rare events also naturally appear in nature. Perhaps there are processes with a stochastic component in their dynamics such that some rare event drastically changes the behaviour of the process forever.

In this paper, we briefly review our previous results and continue our research on rare large emissions [8]–[7]. In general, the emission can be represented as a function of a base (original) process: the emission counts the occurrence of some set of specific transitions of an original process. Conditioning on a large emission event changes the behaviour of the original process. This change is characterised by a rate function. We study here the rate function for the proposed stochastic version of the Hawking-Penrose black hole model.

2 General settings

Markov processes. Markov processes ξ are the basic object of our studies in this work. The processes are pure jumps on the finite time interval $[0,T]$. We consider the processes having a finite state space $\mathcal{N} = \{0,1,\dots,N\}$. The process paths are right-continuous step functions

$$x \in \mathcal{X}; \quad x : [0,T] \to \mathcal{N}.$$

The jumps of any $x \in \mathcal{X}$ are equal to -1 or $+1$ only. We suppose that the corresponding rate transitions can be represented in the following form: for any state $k \in \mathcal{N}$

$$r_+(k,N) = \lambda_N k^{\gamma_+} u_+(k,N)\big(1-\delta(N-k)\big), \tag{7.1}$$

$$r_-(k,N) = \mu_N k^{\gamma_-} u_-(k)\big(1-\delta(k-1)\big), \tag{7.2}$$

where $\delta(m) = 1$ for $m = 0$ and $\delta(m) = 0$ otherwise. The power functions k^{γ_+} and k^{γ_-} describe the main functional part of the intensities of the jumps. Real numbers $\lambda_N > 0$ and $\mu_N > 0$ depend on N only. The functions $u_+(k,N)$ and $u_-(k)$ present some linear dependence on N and k of the intensities. We will restrict ourselves here only to two cases for the functions $u_\pm$: $u_+(k,N) = N-k$ or $u_+(k,N) \equiv 1$, and $u_-(k) = k$ or $u_-(k) \equiv 1$. The multipliers $\big(1-\delta(N-k)\big)$ and $\big(1-\delta(k-1)\big)$ do not allow the process to go out of $\mathcal{N}$.

The rate $r_+(k,N)$ corresponds to the jump $k \to k+1$, and the rate $r_-(k,N)$ corresponds to the jump $k \to k-1$. The infinitesimal operator on the function set $\mathbf{F}$; $f \in \mathbf{F} : \mathcal{N} \to \mathbb{R}$ is

$$\mathbf{L}f(k) = r_+(k,N)\big[f(k+1)-f(k)\big] + r_-(k,N)\big[f(k-1)-f(k)\big]. \tag{7.3}$$

Particle systems. Particle systems are the main interpretation of the studied Markov processes. The particle system is a set $\mathfrak{P}$ of $N = |\mathfrak{P}|$ particles. Each particle $\mathfrak{p} \in \mathfrak{P}$ can be in one of two states from $S = \{0,1\}$. Let $s_\mathfrak{p} \in S$ be the state of the particle $\mathfrak{p}$. The state $s_\mathfrak{p} = 0$ is a *ground state* of the particle $\mathfrak{p}$. The state $s_\mathfrak{p} = 1$ is an *excited* state of the particle. The state $s_\mathfrak{p}$ of any particle is random variable changing over time, $s_\mathfrak{p} \equiv s_\mathfrak{p}(t)$. For this settings, the values of the Markov process ξ are the number of the excited particles

$$\xi(t) = \sum_{\mathfrak{p}\in\mathfrak{P}} s_\mathfrak{p}(t).$$

The physical terminology is commonly used in order to characterise the particle state. This terminology is related to the energy of the system. However, in our example of the stochastic version of the black hole, the state of a particle will be interpreted in other terms: the excited state 1 means that the particle is located inside of the black hole, and the ground state 0 means that the particle is out of the black hole.

Large deviations. Our aim is to study an emission of the particle system. More exactly, we would like to understand behaviour of the probability of the large emission on the interval $[0,T]$. The emissions in terms of the process $\xi(t)$ are negative jumps of the process: the emission occurs at the moment $\tau \in [0,T]$ means that $\xi(\tau) - \xi(\tau - 0) = -1$.

We introduce a process $\eta(t), t \in [0,T]$, of the emissions in the following way. Let

$$\Theta_-(t) = \{t_i : \ t_i \le t \text{ and } \xi(t_i) - \xi(t_i - 0) = -1\}$$

be the set of the time instances of the emissions during $[0,t]$, $t \le T$. Then

$$\eta(t) = |\Theta_-(t)| \tag{7.4}$$

is the number of the emissions occurred during $[0,t]$. The process $\eta(t)$ takes its values in $\mathbb{Z}_+$ and it is the monotone increasing process. Further, we consider the pair $\big(\xi(t), \eta(t)\big)$ of dependent processes. The infinitesimal generator of the joint process is

$$\mathbf{L}^r f(k,m) = r_+(k,N)\big[f(k+1,m) - f(k,m)\big] + r_-(k,N)\big[f(k-1,m+1) - f(k,m)\big], \tag{7.5}$$

where $k \in \mathcal{N}, m \in \mathbb{Z}_+$.

The large emission, which we study, is the event

$$\big\{\eta(T) \ge \widetilde{B}T\big\}, \tag{7.6}$$

where $\widetilde{B} > 0$ is large. The event (7.6) is a rare event arising during stochastic dynamics of the processes $\big(\xi(t), \eta(t)\big)$. The study of rare events is the subject of the large deviations theory. The answer which can be obtained by this theory has an asymptotic form. Therefore instead of one process, a sequence of scaled processes is considered, where the sequence is defined by a scaling parameter connected to the original process. In the large deviation theory, the probabilities of the rare events are being found asymptotically under the scaling parameter. We will take as the scaling parameter for our case the number N of the particles in the system.

Therefore we will consider the scaled version of our problem where the scaling is taken by growing N, $N \to \infty$. The scaled processes are

$$\big(\xi_N(t), \eta_N(t)\big) = \left(\frac{\xi(t)}{N}, \frac{\eta(t)}{N}\right). \tag{7.7}$$

The jumps of $\xi_N(t)$ are $\pm\frac{1}{N}$ and the jumps of $\eta_N(t)$ are equal to $\frac{1}{N}$.

The intensities of the processes $\big(\xi_N(t), \eta_N(t)\big)$ are as follows

$$\begin{aligned} R_+\left(\frac{k}{N}, N\right) &= \lambda_N N^{\gamma_+ + \mathrm{e}_+} u_+\left(\frac{k}{N}, 1\right)\left(\frac{k}{N}\right)^{\gamma_+}\left(1 - \delta\left(1 - \frac{k}{N}\right)\right), \\ R_-\left(\frac{k}{N}, N\right) &= \mu_N N^{\gamma_- + \mathrm{e}_-} u_-\left(\frac{k}{N}\right)\left(\frac{k}{N}\right)^{\gamma_-}\left(1 - \delta\left(\frac{k-1}{N}\right)\right), \end{aligned}$$

where

$$\mathrm{e}_+ = \begin{cases} 1, \text{ if } u_+(k,N) = N - k, \\ 0, \text{ if } u_+(k,N) \equiv 1, \end{cases} \qquad \mathrm{e}_- = \begin{cases} 1, \text{ if } u_+(k) = k, \\ 0, \text{ if } u_+(k) \equiv 1. \end{cases} \tag{7.8}$$

It is convenient to put $\widetilde{B} = NB$ in the scaled version of the system, where $B > 0$ is large enough. The large emission $\widetilde{B}$ is large if NB is large which is the same as large N. In terms of the scaled processes the event (7.6) is

$$\big\{\eta_N(T) \geq BT\big\}, \tag{7.9}$$

where B is large enough.

The theory of the large deviations also allows to extract a large deviation path of the process which produces the given deviation of the large emissions during the interval $[0,T]$. We apply the large deviations theory in a topological space of paths $\mathbf{F}_2 : \mathbb{D} \to \mathbb{R}$ on $[0,T]$, where $\mathbb{D} = [0,1] \times \mathbb{R}_+$. The paths $\big(\mathbf{x}(\cdot), \mathbf{y}(\cdot)\big) \in \mathbf{F}_2$ satisfy the following conditions

1) the paths $\big(\mathbf{x}(\cdot), \mathbf{y}(\cdot)\big)$ are real-valued right-continuous paths defined on $[0,T]$ with left-hand limits;

2) the path $\mathbf{y}(\cdot)$ is non-negative and non-decreasing;

3) a topology in $\mathbf{F}_2$ is defined by Lindvall metric ([1]).

The theory of large deviations allows one to solve both mentioned tasks: to find the asymptotics of the large emission probability as $N \to \infty$ as well as the path of the dynamics that realises the large deviation on the interval $[0,T]$. Solving these problems by the method of large deviations we follow constructions and results in [3].

Further, we assume that

$$\lim_{N\to\infty} \lambda_N N^{\gamma_+ + \mathrm{e}_+ - 1} = \lambda^R, \qquad \lim_{N\to\infty} \mu_N N^{\gamma_- + \mathrm{e}_- - 1} = \mu^R,$$

where $\lambda^R > 0$ and $\mu^R > 0$ are parameters of the model.

For any $x \in (0,1)$, sequences of integers k such that $\frac{k}{N} \to x$ as $N \to \infty$ then

$$\begin{aligned} R_+(x) &:= \lim_{N\to\infty} R_+\left(\frac{k}{N}, N\right) = \lambda^R u_+(x) x^{\gamma_+}, \\ R_-(x) &:= \lim_{N\to\infty} R_-\left(\frac{k}{N}, N\right) = \mu^R u_-(x) x^{\gamma_-}. \end{aligned} \tag{7.10}$$

The *principle of the large deviations* introduced by Varadhan is a basic construction of the large deviations theory (see [10]). To have the large deviations principle means to know a *rate function I*. In our case it is the function $I : \mathbb{D} \to \mathbb{R}_+$, which has the following integral functional form

$$I(\mathbf{x},\mathbf{y}) = \int_0^T \sup_{\varkappa_1,\varkappa_2} \left(\varkappa_1 \dot{\mathbf{x}} + \varkappa_2 \dot{\mathbf{y}} - R_+(\mathbf{x})[e^{\varkappa_1} - 1] - R_-(\mathbf{x})[e^{-\varkappa_1+\varkappa_2} - 1]\right) \mathrm{d}t. \tag{7.11}$$

The function $\mathbf{x} : [0,T] \to [0,1]$ is a density of excited particles, the non-decreasing function $\mathbf{y} : [0,T] \to \mathbb{R}_+$ is the dynamics of the emissions. The functions $\varkappa_1$ and $\varkappa_2 : [0,T] \to \mathbb{R}$ are dual variables to $\mathbf{x}$ and $\mathbf{y}$ accordingly.

Generally speaking, the functions $\mathbf{x}$ and $\mathbf{y}$ from $\mathbf{F}_2$ can be discontinuous. However, the rate function I in (7.11) has finite values on the absolutely continuous $\mathbf{x}$ and $\mathbf{y}$ only.

Main information on the rare events is contained in the rate function I, (7.11). According to the large deviations theory, we can estimate the probability of the event (7.9) by the rate function as the following

$$\lim_{N\to\infty} \frac{1}{N} \ln \Pr(\eta_N \geq BT) = -\inf_{(x,y)\in\mathscr{B}} I(\mathbf{x},\mathbf{y}), \tag{7.12}$$

where $\mathscr{B} = \{(\mathbf{x},\mathbf{y}) : \mathbf{y}(0) = 0, \mathbf{y}(T) = BT\}$ is a set of the paths describing the event (7.9). If we have found a path $(\mathbf{x}_B,\mathbf{y}_B)$ such that

$$I(\mathbf{x}_B,\mathbf{y}_B) = \inf_{(\mathbf{x},\mathbf{y})\in\mathscr{B}} I(\mathbf{x},\mathbf{y}) \tag{7.13}$$

then this path is a mean path of a conditioned process under the event (7.9).

Remark that expression (7.12) holds only if

$$\inf_{(\mathbf{x},\mathbf{y})\in\mathscr{B}^o} I(\mathbf{x},\mathbf{y}) = \inf_{(\mathbf{x},\mathbf{y})\in\overline{\mathscr{B}}} I(\mathbf{x},\mathbf{y}), \tag{7.14}$$

where $\mathscr{B}^o$ is interior and $\overline{\mathscr{B}}$ is closure of $\mathscr{B}$. If (7.14) does not hold we only obtain bounds for $\lim_{N\to\infty}\frac{1}{N}\ln\Pr(\eta_N \geq BT)$.

The rate function (7.11) is Legendre transform of the Hamiltonian

$$H(\mathbf{x},\mathbf{y},\varkappa_1,\varkappa_2) = R_+(\mathbf{x})[e^{\varkappa_1}-1] + R_-(\mathbf{x})[e^{-\varkappa_1+\varkappa_2}-1]. \tag{7.15}$$

If the pair $(\mathbf{x}_B,\mathbf{y}_B)$ satisfies (7.13), then it is a solution of the Hamiltonian system

$$\begin{cases} \dot{\mathbf{x}} = \dfrac{\partial H}{\partial\varkappa_1} = R_+(\mathbf{x})\exp\{\varkappa_1\} - R_-(\mathbf{x})\exp\{-\varkappa_1+\varkappa_2\}, \\ \dot{\mathbf{y}} = \dfrac{\partial H}{\partial\varkappa_2} = R_-(\mathbf{x})\exp\{-\varkappa_1+\varkappa_2\}, \\ \dot{\varkappa}_1 = -\dfrac{\partial H}{\partial x} = -R'_+(\mathbf{x})[e^{\varkappa_1}-1] - R'_-(\mathbf{x})[e^{-\varkappa_1+\varkappa_2}-1], \\ \dot{\varkappa}_2 = -\dfrac{\partial H}{\partial y} = 0, \end{cases} \tag{7.16}$$

where $R'_\omega(x), \omega\in\{+,-\}$ is the derivative over x.

To find $(\mathbf{x}_B,\mathbf{y}_B)$, (see (7.13)) we have to solve this system under suitable boundary conditions. For the considered cases, the boundary conditions are $\mathbf{y}(0) = 0,\ \mathbf{y}(T) = BT$ and arbitrary $\mathbf{x}(0) = x_0$.

In many cases when $R_+(x)$ and $R_-(x)$ depend on x, finding the solution is a rather difficult problem. Peculiar properties of the system can facilitate the search of the solutions. These facilitating properties are that the right sides of every equation do not depend on y and that $\varkappa_2$ is a constant.

A general property of the solutions is in the following equation followed from (7.16)

Lemma 7.1

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln\dot{\mathbf{y}} = (\dot{\mathbf{x}}+\dot{\mathbf{y}})\frac{\mathrm{d}}{\mathrm{d}x}\ln\big(R_+(\mathbf{x})R_-(\mathbf{x})\big) - R'_+(\mathbf{x}) - R'_-(\mathbf{x}). \tag{7.17}$$

Proposition 7.2 If $\mathbf{x}$ is constant, then $\mathbf{y}$ is linear on $[0,T]$.

For the next proposition denote $A(\mathbf{x}) = \frac{\mathrm{d}}{\mathrm{d}x} \ln\big(R_+(\mathbf{x})R_-(\mathbf{x})\big)$ and $C(\mathbf{x}) = B \cdot \frac{\mathrm{d}}{\mathrm{d}x} \ln\big(R_+(\mathbf{x})R_-(\mathbf{x})\big) - R'_+(\mathbf{x}) - R'_-(\mathbf{x})$.

Proposition 7.3 If $\dot{\mathbf{y}} = B$, then the solution of (7.17) in the form $\mathbf{x}(t) \equiv x_0$, $\mathbf{y}(t) = Bt$ exists, where x_0 is a root of the equation $C(x) = 0$.

3 Hawking-Penrose black hole

In this section, we apply the general settings described above to the Hawking-Penrose black hole model. The goal here is to investigate of the large emissions of the black hole. This model is one of the earliest and simplest descriptions of Schwarzschild black hole (see [4, 9]). The black hole both emits and absorbs the matter. The absorption is the result of the gravitation, and the emission is the result of the Hawking radiation.

The model we propose is as follows. There is the *Universe* composed by a finite piece of a space and a matter in the space. The matter is a finite set $\mathfrak{P}$ of particles. Some part of Universe space is a specific area which is called the *black hole*. Some portion $\mathfrak{P}_1 \subseteq \mathfrak{P}$ of the particles of the matter is located in the black hole. The remaining portion $\mathfrak{P}_2 = \mathfrak{P} \setminus \mathfrak{P}_1$ is located in the Universe outside the black hole. The Schwarzschild black hole has the shape of a ball. The radius of the ball is proportional to the number of the particles in $\mathfrak{P}_1$.

There is a dynamic of the particles between the parts $\mathfrak{P}_1$ and $\mathfrak{P}_2$.

We construct a stochastic dynamic of the particle jumps between $\mathfrak{P}_1$ and $\mathfrak{P}_2$. The particle jumps are described by a Markov process. The change of the particle number in the black hole that is of the set $\mathfrak{P}_1$, changes the radius of the black hole. One of the features of this stochastic dynamic is that the surface value of the black hole affects the laws of the jumps thereby determining the properties of the Markov processes. In physics, this feature is explained by the so-called *holographic principle* which means that the black hole surface (*horizon*) contains all the information about the black hole state. Our main interest is the big emission of the black hole, that is the large number of jumps $\mathfrak{P}_1 \to \mathfrak{P}_2$.

Formal descriptions: Markov process. The dynamics of the particles between the sets $\mathfrak{P}_1$ and $\mathfrak{P}_2$ is defined by a Markov process $\xi(t)$, where the value of $\xi(t)$ is number of the particles in the set $\mathfrak{P}_1$, $\xi(t) = |\mathfrak{P}_1|$. We consider the jump dynamics on the finite interval $[0,T]$ driven by the Markov process $\xi(t)$ taking its values in $\mathcal{N} \setminus \{0\} = \{1,\ldots,N\}$. It means that the number of the particle in Universe is N. Here we exclude the point 0 from the state space $\mathcal{N}$. It means that the black hole contains at least one particle, and does not disappear.

As in the general setting described above, the jumps of the process are $+1$ or -1. The jump $+1$ means the increment of the set $\mathfrak{P}_1$ (the matter in the black hole) by one particle, and -1 is the decrement of the same set by one particle.

The intensities of the jumps of ξ depend on the size of the horizon, which in turn depends on the amount of matter in the black hole. Let $\xi(t) = k, t \in [0,T]$, that is k particles are located in the black hole at the moment t. Then the jump intensities are

$$r_+(k,N) = \lambda_N k^2 (N-k)\big(1-\delta(N-k)\big),$$
$$r_-(k,N) = \mu_N k^{-2}\big(1-\delta(k-1)\big),$$

(cf. (7.1) and (7.2)).

For studying the large emission from the black hole, we introduce the process $\eta(t)$ (see (7.4)) as described in the section *Large deviations*. Next, we have to study the scaled version of the processes

$$\big(\xi_N(t), \eta_N(t)\big) = \left(\frac{\xi(t)}{N}, \frac{\eta(t)}{N}\right)$$

(see (7.7)). The intensities of the scaled version of the processes are

$$R_+(x) = \lim_{N\to\infty} R_+\left(\frac{k}{N}, N\right) = \lambda^R x^2(1-x),$$
$$R_-(x) = \lim_{N\to\infty} R_-\left(\frac{k}{N}, N\right) = \mu^R x^{-2},$$

assuming that $k/N \to x \in (0,1)$.

According the general settings of Section 2, we obtain the rate function

$$\begin{aligned} I(\mathbf{x},\mathbf{y}) = \int_0^{\mathscr{T}} \sup_{\varkappa_1(t),\varkappa_2(t)} \Big\{ &\varkappa_1(t)\dot{\mathbf{x}}(t) + \varkappa_2(t)\dot{\mathbf{y}}(t) \\ &- \lambda^R \mathbf{x}^2(t)(1-\mathbf{x}(t))[e^{\varkappa_1(t)}-1] - \mu^R \tfrac{1}{\mathbf{x}^2(t)}[e^{-\varkappa_1(t)+\varkappa_2(t)}-1]\Big\} \mathrm{d}t, \end{aligned} \tag{7.18}$$

where $\mathbf{x}(t)$ is a density of the particles in the black hole at the moment $t \in [0,T]$, and $\mathbf{y}(t)$ is a path of the particle emission on the interval $[0,T]$ which means the number of the particles emitted on the interval $[0,t]$.

As in the general case, the rate function is Legendre transform of the Hamiltonian

$$H(\mathbf{x},\mathbf{y},\varkappa_1,\varkappa_2) = \lambda^R \mathbf{x}^2(1-\mathbf{x})[e^{\varkappa_1}-1] + \mu^R \mathbf{x}^{-2}[e^{-\varkappa_1+\varkappa_2}-1].$$

In order to find the probability $\Pr\big(\eta_N(T) \geq BT\big)$, and an optimal path $\big(\mathbf{x}_B(\cdot),\mathbf{y}_B(\cdot)\big)$ on $[0,T]$, we have to find a solution of the equation system with suitable boundary conditions.

$$\begin{cases} \dot{\mathbf{x}} = \dfrac{\partial H}{\partial \varkappa_1} = \lambda(1-\mathbf{x})\mathbf{x}^2 \exp\{\varkappa_1\} - \mu \frac{1}{\mathbf{x}^2} \exp\{-\varkappa_1+\varkappa_2\}, \\ \dot{\mathbf{y}} = \dfrac{\partial H}{\partial \varkappa_2} = \mu \frac{1}{\mathbf{x}^2} \exp\{-\varkappa_1+\varkappa_2\}, \\ \dot{\varkappa}_1 = -\dfrac{\partial H}{\partial x} = -\lambda\big(2(1-\mathbf{x})\mathbf{x}-\mathbf{x}^2\big)[e^{\varkappa_1}-1] + \mu \frac{2}{\mathbf{x}^3}[e^{-\varkappa_1+\varkappa_2}-1], \\ \dot{\varkappa}_2 = -\dfrac{\partial H}{\partial y} = 0, \end{cases} \tag{7.19}$$

These solutions are extremals of the integral functional (7.18). We need the solution which finds the extremal hitting the infimum of (7.12). The event $\mathscr{B}$ defines the corresponding boundary conditions. That is $\mathbf{y}(0) = 0$, $\mathbf{y}(T) = BT$ and $\mathbf{x}(0) = x_0$ is chosen such that it gives the minimum of (7.18).

The solution of (7.19) under prescribed boundary conditions is a rather difficult problem because of the high non-linearity of the system.

We find the extremal determining a solution $\big(\mathbf{x}_B,\mathbf{y}_B\big)$, where $\mathbf{x}_B$ is a constant and $\mathbf{y}_B$ is a linear function, (see Propositions 7.2 and 7.3). It means that the corresponding conditional processes (ξ_N,η_N) considered at large N has its average values $\mathbf{x}_B$ and $\mathbf{y}_B$.

Definition 7.4 For a constant $B > 0$, the path $(\mathbf{x}_B(t),\mathbf{y}_B(t))$ is called a *stationary emission regime* if

1) there is a constant x_B such that $\mathbf{x}_B(t) \equiv x_B$, for all $t \in [0,T]$,

2) $\mathbf{y}_B(t) = Bt$, for all $t \in [0,T]$,

3) the path $\big(\mathbf{x}_B(t),\mathbf{y}_B(t)\big)$ are extremal of I with the boundary conditions $\mathbf{x}_B(0) = \mathbf{x}_B(T) = x_B$ and $\mathbf{y}_B(0) = 0,\ \mathbf{y}_B(T) = BT$.

According the above definition of stationary emission regime, using Propositions 7.2 and 7.3 we obtain the following theorem [8].

Theorem 7.5 For any $B > 0$, there exists a constant x_B such that the paths $\mathbf{x}(t) \equiv x_B$, $\mathbf{y}(t) = Bt$ realise the stationary emission regime. We have $x_B \to 0$ as $B \to \infty$ with the asymptotics

$$x_B \sim \left(\frac{2\mu}{B}\right)^{\frac{1}{3}}.$$

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