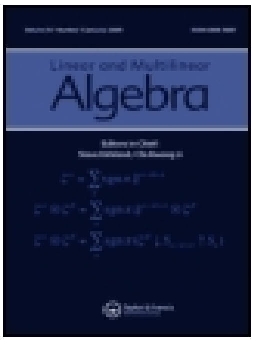

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Cocharacters of group graded algebras and multiplicities bounded by one

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ABSTRACT

Let G be a finite group and A a G-graded algebra over a field F of characteristic zero. We characterize the T_G-ideals $Id^G(A)$ of graded identities of A such that the multiplicities $m_{\langle\lambda\rangle}$ in the graded cocharacter of A are bounded by one. We do so by exhibiting a set of identities of the T_G-ideal. As a consequence we characterize the varieties of G-graded algebras whose lattice of subvarieties is distributive.

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1. Introduction

Let G be a finite group and A a G-graded algebra over a field F of characteristic zero. This paper is concerned with the graded polynomial identities satisfied by A. More precisely in the free G-graded algebra of countable rank we consider the set $Id^G(A)$ of graded polynomials vanishing when evaluated in A. $Id^G(A)$ has a natural structure of T_G-ideal, i.e. an ideal invariant under the graded endomorphisms of the free G-graded algebra and, since the base field is of characteristic zero, $Id^G(A)$ is determined by its multilinear polynomials.

An efficient way of studying such T_G-ideals is through the representation theory of symmetric groups [1,2]. If $G = \{g_1, \dots, g_q\}$, we consider the space of multilinear polynomials of degree n in n_1 fixed variables of homogeneous degree g_1, n_2 fixed variables of homogeneous degree g_2, and so on, where $n = n_1 + \cdots + n_q$. Then we act on such space (modulo $Id^G(A)$) with the direct product of symmetric groups $S_{n_1} \times \cdots \times S_{n_q}$. Given any q-tuple $\langle n\rangle = (n_1, \dots, n_q)$, the corresponding $S_{n_1} \times \cdots \times S_{n_q}$-character, called $\chi_{\langle n\rangle}(A)$, decomposes into irreducibles

$$\chi_{\langle n\rangle}(A) = \sum_{\langle\lambda\rangle \vdash n} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(q)}, \tag{1}$$

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where $\langle\lambda\rangle = (\lambda(1), \ldots, \lambda(q))$, $\chi_{\lambda(i)}$ is the irreducible S_{n_i}-character corresponding to the partition $\lambda(i) \vdash n_i$, $1 \le i \le q$, and $m_{\langle\lambda\rangle} \ge 0$ are the multiplicities.

Since the representation theory of the symmetric group in characteristic zero is well-known, the main problem in this setting is to determine the multiplicities $m_{\langle\lambda\rangle}$. This is in general a difficult problem and some results have been obtained in recent years (see for instance [3–6,12]).

The purpose of this note is to characterize the T_G-ideals $Id^G(A)$ such that the multiplicities $m_{\langle\lambda\rangle}$ in (1) are bounded by one. We shall obtain such characterization by exhibiting a set of polynomials lying in the T_G-ideal.

We should mention that in case G is an abelian group, there is a duality between the representation theory of a product of symmetric groups and that of the wreath products $GwrS_n$ (see [2,7]). In this case one can define an action of the group $GwrS_n$ on the space of multilinear G-graded polynomials of degree n (modulo $Id^G(A)$) (see [8]). The corresponding character denoted $\chi^{GwrS_n}(A)$ decomposes into irreducibles

$$\chi^{GwrS_n}(A) = \sum_{\langle\lambda\rangle\vdash n} m'_{\langle\lambda\rangle}\chi_{\langle\lambda\rangle}$$

where $\chi_{\langle\lambda\rangle}$ is the irreducible $GwrS_n$ character associated to the multipartition $\langle\lambda\rangle$. Then the main result proved here gives also a characterization of the T_G-ideals such that the multiplicities $m'_{\langle\lambda\rangle}$ are bounded by one.

The problem studied here originated from a paper of Ananin and Kemer [9] where they characterized the T-ideals of the free associative algebra such that the multiplicities in the ordinary cocharacter are bounded by one. This result was later extended in [10] to superalgebras and algebras with involution using the representation theory of Z_2wrS_n.

As an outcome of our result we are able to classify the varieties of G-graded algebras whose lattice of subvarieties are precisely those whose T_G-ideal is generated by the polynomials determined in the main theorem. For the ordinary case (trivial G-grading) or the supercase or the case of algebras with involution we refer the reader to [9,11].

2. The general setting

Throughout $G = \{g_1 = 1, g_2, \ldots, g_q\}$ is a finite group, F is a field of characteristic zero and $F\langle X|G\rangle$ is the free G-graded algebra on a countable set $X = \{x_1, x_2, \ldots\}$. The set X decomposes into a disjoint union of countables sets

$$X = \cup_{i=1}^{q} X_{g_i},$$

where $X_{g_i} = \{x_j^{g_i} | j \ge 1\}$ is a set of variables of homogeneous degree g_i, $1 \le i \le q$. A monomial $x_{j_1}^{g_{i_1}} \cdots x_{j_t}^{g_{i_t}}$ has homogeneous degree $g_{i_1} \cdots g_{i_t}$ and $F\langle X|G\rangle = \bigoplus_i \mathcal{F}_{g_i}$ is a G-graded algebra where $\mathcal{F}_{g_i}$ is the subspace generated by all monomials of homogeneous degree g_i.

Let $A = \bigoplus_{g\in G} A_g$ be a G-graded algebra. Recall that a polynomial $f = f(x_{j_1}^{g_{i_1}}, \ldots, x_{j_t}^{g_{i_t}})$ of $F\langle X|G\rangle$ is a G-graded polynomial identity of A, if f vanishes under all evaluations of the graded variables $x_{j_k}^{g_{i_k}}$ into elements of $A_{g_{i_k}}$. Then $Id^G(A) = \{f \in F\langle X|G\rangle | f \equiv 0 \text{ on } A\}$, the set of graded polynomial identities of A, is a T_G-ideal of $F\langle X|G\rangle$, i.e. an ideal invariant under all G-graded endomorphisms of $F\langle X|G\rangle$. As in the ordinary case (trivial G-grading), since $\mathrm{char} F = 0$, one can study only the multilinear G-graded identities of A. To this end, for every $n \geq 1$, we set

$$P_n^G = \mathrm{span}_F\{x_{\sigma(1)}^{h_1} \cdots x_{\sigma(n)}^{h_n} \mid \sigma \in S_n, h_1, \ldots, h_n \in G\}.$$

We act on P_n^G with the wreath product

$$GwrS_n = \{(h_1, \ldots, h_n; \sigma) \mid h_1, \ldots, h_n \in G, \sigma \in S_n\}$$

via

$$(h_1, \ldots, h_n; \sigma)x_{\tau(1)}^{k_1} \cdots x_{\tau(n)}^{k_n} = x_{\sigma\tau(1)}^{h_1 k_{\sigma(1)}^{-1}}, \ldots, x_{\sigma\tau(n)}^{h_n k_{\sigma(n)}^{-1}}.$$

(see [8]).

Now $P_n^G \cap Id^G(A)$ is invariant under this action and we consider the character $\chi^{GwrS_n}(A)$ of the $GwrS_n$-module $\frac{P_n^G}{P_n^G \cap Id^G(A)}$.

In case G is an abelian group, the irreducible $GwrS_n$-characters are easily described: if $n = n_1 + \cdots + n_q, n_i \geq 0, 1 \leq i \leq q$, we let $\langle\lambda\rangle = (\lambda(1), \ldots, \lambda(q))$ be a sequence of partitions $\lambda(i) \vdash n_i, 1 \leq i \leq q$, and we write $\langle\lambda\rangle \vdash n$. Then we have

$$\chi^{GwrS_n}(A) = \sum_{\langle\lambda\rangle \vdash n} k_{\langle\lambda\rangle} \chi_{\langle\lambda\rangle} \tag{2}$$

where $\chi_{\langle\lambda\rangle}$ is the irreducible $GwrS_n$-character corresponding to $\langle\lambda\rangle$ and $k_{\langle\lambda\rangle} \geq 0$ is the multiplicity.

There is another useful action of a product of symmetric groups on a subspace of multilinear polynomials as follows. For $n = n_1 + \cdots + n_q$ as above, set $\langle n\rangle = (n_1, \ldots, n_q)$ and let $X_{\langle n\rangle} = \left(x_1^{g_1}, \ldots, x_{n_1}^{g_1}, x_1^{g_2}, \ldots, x_{n_2}^{g_2}, \ldots, x_1^{g_q}, \ldots, x_{n_q}^{g_q}\right)$. Then set

$$P_{\langle n\rangle} = \mathrm{span}\{w_{\sigma(1)} \cdots w_{\sigma(n)} | (w_1, \ldots, w_n) = X_{\langle n\rangle}, \sigma \in S_n\}.$$

We act on $P_{\langle n\rangle}$ with the group $S_{n_1} \times \cdots \times S_{n_q}$ by letting S_{n_i} act on the set of variables $\{x_1^{g_i}, \ldots, x_{n_i}^{g_i}\}$, $1 \leq i \leq q$. Then $P_{\langle n\rangle} \cap Id^G(A)$ is invariant under this action and we let $\chi_{\langle n\rangle}(A)$ be the character of the quotient space $\frac{P_{\langle n\rangle}}{P_{\langle n\rangle} \cap Id^G(A)}$.

By decomposing the character $\chi_{\langle n\rangle}(A)$ into irreducibles we get

$$\chi_{\langle n\rangle}(A) = \sum_{\langle\lambda\rangle \vdash n} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(q)} \tag{3}$$

where $\langle\lambda\rangle = (\lambda(1), \ldots, \lambda(q)), \lambda(i) \vdash n_i, \chi_{\lambda(i)}$ is the corresponding irreducible S_{n_i}-character, $1 \leq i \leq q$ and $m_{\langle\lambda\rangle} \geq 0$ is the multiplicity. $\chi_{\langle n\rangle}(A)$ will be called the $\langle n\rangle$-th cocharacter of A.

Now if G is an abelian group it is known that the multiplicities $k_{\langle\lambda\rangle}$ in (2) and $m_{\langle\lambda\rangle}$ in (3) coincide for the same multipartition $\langle\lambda\rangle \vdash n$. (see [2,10]).

In the above discussion when the grading group is trivial one has the natural notions of free associative algebra $F\langle X\rangle$, T-ideal $Id(A)$ of the polynomial identities satisfied by an algebra A, etc. The symmetric group S_n acts on P_n, the space of multilinear polynomials in

$x_1, \dots, x_n$ and the S_n-character of the quotient space $P_n/(P_n \cap Id(A))$ is $\chi_n(A)$, the n-th cocharacter of A.

3. Multiplicities bounded by one

In this section we shall prove our main result.

Theorem 1: *Let G be a finite group and A a G-graded algebra over a field F of characteristic zero. Let*

$$\chi_{\langle n\rangle}(A) = \sum_{\langle\lambda\rangle\vdash n} m_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(q)}$$

be the $\langle n\rangle$-th cocharacter of A. Then $m_{\langle\lambda\rangle} \leq 1$, for all $\langle\lambda\rangle \vdash n$ and for all $n \geq 1$, if and only if A satisfies the following graded identities

$$\alpha x_1^{g_1}[x_1^{g_1}, x_2^{g_1}] + \beta[x_1^{g_1}, x_2^{g_1}]x_1^{g_1} \equiv 0 \tag{4}$$

$$\gamma x_1^g x_2^h + \delta x_2^h x_1^g \equiv 0, \tag{5}$$

where $g_1 = 1$, for all $g, h \in G, g \neq h$ and for some $\alpha, \beta, \gamma, \delta \in F$ such that $(\alpha, \beta) \neq (0,0)$ and $(\gamma, \delta) \neq (0,0)$.

***Proof*:** Suppose first that $m_{\langle\lambda\rangle} \leq 1$, for all $\langle\lambda\rangle \vdash n$ and for all $n \geq 1$. We consider the sequence $\langle n\rangle = (n, 0, \dots, 0)$ and the $\langle n\rangle$-th cocharacter of A becomes

$$\chi_{\langle n\rangle}(A) = \sum_{\lambda\vdash n} m_\lambda \chi_\lambda \tag{6}$$

where $\lambda = \lambda(1) \vdash n$. Notice that $\chi_{\langle n\rangle}(A)$ coincides with the ordinary cocharacter $\chi_n(A_{g_1})$ of $A_{g_1} = A_1$. Since A_{g_1} is a subalgebra of A with cocharacter $\chi_n(A_{g_1}) = \sum_{\lambda\vdash n} m_\lambda\chi_\lambda$ whose decomposition is given in (6) and the multiplicities m_λ are bounded by one, by the result of Ananin and Kemer ([9]) we get that A_1 and so A satisfies the identity given in (4).

In order to prove the existence of the identities in (5) we consider two distinct homogeneous components of A, say A_g and A_h with $g \neq h$.

For $n = 2$ we consider $\langle 2\rangle = (0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots)$ where the two $1'$s correspond to the g and h components. Then we have $P_{\langle 2\rangle} \cong \text{span}\{x_1^g x_2^h, x_2^h x_1^g\}$ and the decomposition of this space into irreducible $S_1 \times S_1$-modules is $P_{\langle 2\rangle} \cong M_1 \oplus M_2$, where $M_1 = \text{span}\{x_1^g x_2^h\}$ and $M_2 = \text{span}\,\{x_2^h x_1^g\}$.

Since M_1 and M_2 are isomorphic as $S_1 \times S_1$-modules and the multiplicities in $\chi_{\langle 2\rangle}(A)$ are bounded by one by hypothesis, we get that $x_1^g x_2^h$ and $x_2^h x_1^g$ are linearly dependent modulo $Id^G(A)$. This says that A satisfies the graded identity in (5) and we are done.

Suppose now that A satisfies the identities in (4) and in (5). Notice that the identities in (5) allow to reorder the graded variables in a multilinear monomial of $P_{\langle n\rangle}$ (modulo $Id^G(A)$). In such reordering all variables of the same homogeneous degree form a submonomial. In other words, there is a permutation $\sigma \in S_q$ and a reordering of the group elements $(g_{\sigma(1)}, \dots, g_{\sigma(q)})$ such that

$$P_{\langle n\rangle} = \text{span}\{x_{i_1}^{g_{\sigma(1)}} \cdots x_{i_{n_{\sigma(1)}}}^{g_{\sigma(1)}} \cdots x_{j_1}^{g_{\sigma(n)}} \cdots x_{j_{n_{\sigma(1)}}}^{g_{\sigma(n)}}\} \pmod{Id^G(A)}.$$

This says that if $\langle\lambda\rangle = (\lambda(1),\ldots,\lambda(q)) \vdash n$ where $\lambda(i) \vdash n_i$, $1 \leq i \leq q$, then

$$m_{\langle\lambda\rangle} \leq \max\{m_{\lambda(1),\emptyset,\ldots,\emptyset},\ldots,m_{\emptyset,\ldots,\emptyset,\lambda(q)}\}.$$

Hence in order to prove that $m_{\langle\lambda\rangle} \leq 1$, for all $\langle\lambda\rangle \vdash n$, it is enough to prove that $m_{\emptyset,\ldots,\lambda(i),\ldots,\emptyset} \leq 1$ for all i, $1 \leq i \leq q$. In other words, we may restrict ourselves to consider the spaces

$$P_{n_i}^{g_i} = span\{x_{\sigma(1)}^{g_i}\cdots x_{\sigma(n_i)}^{g_i} \mid \sigma \in S_{n_i}\} \pmod{Id^G(A)},$$

for all $g_i \in G$.

Let us write $m_{\emptyset,\ldots,\lambda(i),\ldots,\emptyset} = m_{\lambda(i)}$, $\langle\lambda\rangle = \lambda(i)$. Now, if $i = 1$, $g_1 = 1$, and by the result of Ananin and Kemer ([9]) we get that $m_{\lambda(1),\emptyset,\ldots,\emptyset} \leq 1$.

Hence we may assume that $g_i \neq g_1$. Let us write $g_i = g$. Since $g^2 \neq g$, by (5) we have that $\gamma x_1^g x_2^{g^2} + \delta x_2^{g^2} x_1^g \equiv 0$ is a graded identity of A for some $(\gamma,\delta) \neq (0,0)$. As a consequence, since $x_2^g x_3^g$ has homogeneous degree g^2 in the free algebra, we get that

$$\gamma x_1^g x_2^g x_3^g + \delta x_2^g x_3^g x_1^g \equiv 0$$

is a graded identity of A. If $\gamma = 0$ or $\delta = 0$, then $m_{\lambda(i)} = 0$, for all $\lambda(i) \vdash n$ such that $|\lambda(i)| \geq 3$. On the other hand for all $|\lambda(i)| \leq 2$, the irreducible representations have multiplicity one in the regular representation, so still the conclusion holds.

Now suppose that $\gamma \neq 0$ and $\delta \neq 0$. Then $x_1^g x_2^g x_3^g \equiv \alpha x_2^g x_3^g x_1^g \pmod{Id^G(A)}$, for some $\alpha \neq 0$, and this says that

$$P_{n_i}^g = \mathrm{span}\{x_1^g x_2^g \cdots x_n^g, x_1^g x_2^g \cdots x_n^g x_{n-1}^g\} \pmod{Id^G(A)}.$$

Then $\chi_{(n_i)}$ and $\chi_{(1^{n_i})}$ are the only irreducible characters appearing in the decomposition of the S_{n_i}-character of $\frac{P_{n_i}^g}{P_{n_i}^g \cap Id^G(A)}$ both with multiplicity one. This completes the proof of the theorem. $\square$

As we mention in Section 2, when G is an abelian group one can study the space of multilinear G-graded polynomials in n variables through the representation theory of the wreath product $GwrS_n$. By the discussion in that section, since the multiplicities in the $GwrS_n$-character and in the $S_{n_1} \times \cdots \times S_{n_q}$-character are the same, we get the following

Theorem 2: *Let G be a finite abelian group and A a G-graded algebra over a field F of characteristic zero. Let $\chi^{GwrS_n}(A) = \sum_{\langle\lambda\rangle\vdash n} m_{\langle\lambda\rangle}\chi_{\langle\lambda\rangle}$. Then $m_{\langle\lambda\rangle} \leq 1$ for all $\langle\lambda\rangle \vdash n$ and for all $n \geq 1$, if and only if A satisfies the graded identities in (4) and (5).*

4. Varieties whose lattice of subvarieties is distributive

Let $\mathcal{V}$ be a variety of G-graded algebras. In this section we shall characterize the varieties $\mathcal{V}$ such that the lattice of subvarieties is distributive, i.e. for any subvarieties $\mathcal{U},\mathcal{W},\mathcal{S}$ of $\mathcal{V}$ we have that $(\mathcal{U}\cap\mathcal{W})\cup\mathcal{S} = (\mathcal{U}\cup\mathcal{S})\cap(\mathcal{W}\cup\mathcal{S})$. In terms of ideals of identities this means that if Q,R,S are T_G-ideals containing $Id^G(\mathcal{V})$, then $(Q+R)\cap S = (Q\cap S)+(R\cap S)$.

If $\mathcal{V}$ is the variety generated by the G-graded algebra A we write $\mathcal{V} = \mathrm{var}^G(A)$ we also write $Id^G(\mathcal{V}) = Id^G(A)$. Next we prove the following.

Theorem 3: *Let $\mathcal{V}$ be a variety of G-graded algebras over a field F of characteristic zero. Then $\mathcal{V}$ is distributive if and only if $\mathcal{V}$ satisfies the graded identities*

$$\alpha x_1^{g_1}[x_1^{g_1}, x_2^{g_1}] + \beta[x_1^{g_1}, x_2^{g_1}]x_1^{g_1} \equiv 0$$
$$\gamma x_1^g x_2^h + \delta x_2^h x_1^g \equiv 0,$$

where $g_1 = 1$, for all $g, h \in G, g \neq h$ and for some $\alpha, \beta, \gamma, \delta \in F$ such that $(\alpha, \beta) \neq (0,0)$ and $(\gamma, \delta) \neq (0,0)$.

***Proof*:** Suppose that $\mathcal{V}$ satisfies the given graded identities. If $\mathcal{V} = \mathrm{var}^G(A)$, then by Theorem 1 all multiplicities $m_{\langle\lambda\rangle}$ in the $\langle n\rangle$-cocharacter $\chi_{\langle n\rangle}(A)$ are bounded by one, for all $\langle\lambda\rangle \vdash n$, and for all $n \geq 1$. Now let Q, R, S be the T_G-ideals containing $Id^G(A)$. Since char $F = 0$, they are determined by their multilinear graded polynomials, for instance, Q is determined by $\{Q \cap P_{\langle n\rangle}\}_{\langle n\rangle, n\geq 1}$. Since $P_{\langle n\rangle} \cap Id^G(A)$ is an $S_{n_1} \times \cdots \times S_{n_q}$-module (here $\langle n\rangle = (n_1, \ldots, n_q)$), by complete reducibility we have

$$P_{\langle n\rangle} = (P_{\langle n\rangle} \cap Id^G(A)) \oplus M_1 \oplus \cdots \oplus M_r,$$

where $M_1, \ldots, M_r$ are irreducible $S_{n_1} \times \cdots \times S_{n_q}$-modules. Also, since $m_{\langle\lambda\rangle} \leq 1, M_i \not\cong M_j$, for all $i \neq j, 1 \leq i, j \leq r$.

Now, since $Q \supseteq Id^G(A)$,

$$P_{\langle n\rangle} \cap Q = M_{i_1} \oplus \cdots \oplus M_{i_s},$$

for some proper subset $\{i_1, \ldots, i_s\}$ of $\{1, \ldots, r\}$. The same remark applies to $P_{\langle n\rangle} \cap R$ and to $P_{\langle n\rangle} \cap S$. But then it is easily verified that

$$(P_{\langle n\rangle} \cap Q + P_{\langle n\rangle} \cap R) \cap (P_{\langle n\rangle} \cap S) = P_{\langle n\rangle} \cap Q \cap S + P_{\langle n\rangle} \cap R \cap S,$$

and $\mathcal{V}$ is distributive.

Suppose now that $\mathcal{V} = \mathrm{var}^G(A)$ is distributive. For every $n \geq 1$, consider $\langle n\rangle = (n, 0, \ldots, 0)$ and the corresponding $\langle n\rangle$-cocharacter $\chi_{\langle n\rangle}(A) = \sum_{\lambda\vdash n} m_\lambda \chi_\lambda$.

As we remarked before $\chi_{\langle n\rangle}(A)$ coincides with the ordinary cocharacter $\chi_n(A_1)$ of $A_1 = A_{g_1}$. Let $F\langle X\rangle$ be the free algebra on the set X; clearly $F\langle X\rangle$ can be embedded naturally in $F\langle X|G\rangle$ and if we let $\mathcal{U}$ be the variety of algebras (with trivial grading) generated by A_1, then $Id(\mathcal{U}) = Id(A_1) = Id^G(\mathcal{V}) \cap F\langle X\rangle$. Since the multiplicities in the cocharacter of A_1 are bounded by one, by [9] A_1 and, so, A satisfies the graded identity in (4).

Now suppose by contradiction that there exist $g, h \in G, g \neq h$ such that $\alpha x_1^g x_2^h + \beta x_2^h x_1^g \not\equiv 0$ is not a graded identity of $\mathcal{V}$, for all $(\alpha, \beta) \neq (0,0)$. We shall construct three algebras having special properties.

Let B be the Grassmann algebra on a two dimensional vector space; hence $B = \mathrm{span}\{e_1, e_2, e_1e_2 | e_1^2 = e_2^2 = 0, e_1e_2 = -e_2e_1\}$. We G-grade B by setting $B_g = Fe_1, B_h = Fe_2, B_{gh} = Fe_1e_2$ and $B_a = 0$, for all other $a \in G$. Then B is G-graded and satisfies the graded identity $x_1^g x_2^h + x_2^h x_1^g \equiv 0$.

Let C be the algebra of 3×3 strictly upper triangular matrices over F. We G-grade C as follows. We set $C_g = Fe_{12}, C_h = Fe_{23}, C_{gh} = Fe_{13}$ and $C_a = 0$, for all other $a \in G$. Then C is G-graded and satisfies $x_1^g x_2^h \equiv 0$. Notice that $x_2^h x_1^g \not\equiv 0$ is not a graded identity of C.

Let D be still the algebra of 3×3 strictly upper triangular matrices over F with the following grading. Set $D_g = Fe_{23}, D_h = Fe_{12}, D_{hg} = Fe_{13}$ and $D_a = 0$, for all other $a \in G$. Then D satisfies the graded identity $x_2^h x_1^g \equiv 0$. We also have that $x_1^g x_2^h \not\equiv 0$ is not a graded identity of D.

Now consider the following three T_G-ideals containing $Id^G(\mathcal{V})$

$$Q = Id^G(\mathcal{V} \cap \operatorname{var}^G(B)),\ R = Id^G(\mathcal{V} \cap \operatorname{var}^G(C)),\ S = Id^G(\mathcal{V} \cap \operatorname{var}^G(D)).$$

We have that $x_1^g x_2^h \in (Q + R) \cap S$ but $x_1^g x_2^h \notin (Q + S) \cap (R + S)$. This says that $\mathcal{V}$ is not distributive, a contradiction. □

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