Linear (Multi) Algebra I: Linear Systems, Matrices and Vector Spaces over Superfields

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Motivated by recent advances in abstract theories of quadratic forms, this work embarks on a discussion of matrices, linear systems, and vector spaces over superfields. The aim is to produce an expansion of Linear Algebra into the realm of multivalued structures. Specifically, we introduce and analyze matrices and determinants within the framework of commutative superrings. Additionally, we investigate linear systems and vector spaces over superfields, providing definitions that align with the contextual requirements of abstract theories of quadratic forms and broader semantic studies over multialgebras, as in [3]. We finish with an application of our theoretical developments, establishing an isotropy interpolation principle applicable to both algebraic and abstract theories of quadratic forms.

A k-ary multioperation on a set A is just a function $A^k \to P(A) \setminus \{\emptyset\}$. The data of k-ary multioperation on A is equivalent to a $k+1$-ary relation on A, satisfying a convenient $\forall\exists$ axiom. Therefore, a multialgebraic structure is just a certain kind of first-order structure.

The concept of a multialgebraic structure – an algebraic-like entity endowed with multiple-valued operations – has been under investigation since the 1930s. Notably, in the 1950s, Krasner introduced the notion of hyperrings, which are essentially rings with a multivalued addition. Since the middle of the 2000s decade, the notion of multiring, as discussed in Marshall's work [6], has obtained more attention: a multiring is a lax hyperring, satisfying a weak distributive law, but hyperfields and multifields coincide. Additionally, superrings, as recentlty considered by Ameri et al. [1], are characterized by both their multivalued operations of sum and product. Extensive algebraic inquiries into multialgebras have been conducted, as evidenced by studies such as those by Golzio [4] and Pelea [7].

Multirings have been studied for applications in many areas: in abstract quadratic forms theory, tropical geometry, algebraic geometry , valuation theory, Hopf algebras, etc. A more detailed account of variants of the concept of polynomials over hyperrings is even more recent: see [5], [1], [2], and [8].

There are numerous significant distinctions among rings, hyper/multirings, and superrings. However, the analogical extensions of concepts from the algebraic realm to the multi-algebraic

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domain yield surprisingly profound implications in other theoretical frameworks. For instance, the utilization of polynomials over superfields and their evaluation, coupled with an appropriate semantics, yields a quantifier elimination procedure as demonstrated in [8]. Additionally, Marshall's quotient over a suitable analogue of quadratic extensions leads to the derivation of the Arason-Pfister Hauptsatz for special groups, as elucidated in [9].

The present work embarks on a discussion of matrices, linear systems, and vector spaces over superfields. The aim is to produce an expansion of Linear Algebra into the realm of multivalued structures. We have achieved our aim with a considerable degree of success: the development of the theory proceeds relatively smoothly. Specifically, we have established that matrices $M_{mn}(R)$ over a superring R form a non-commutative superring if R is full and scalation of linear systems exhibits relatively well-behaved properties. Moreover, natural examples such as F^n, polynomials, matrices, extensions, etc., are readily available for (multi) vector spaces over superfields. Additionally, for hyperfields, we have obtained the:

Theorem. *Let F be a hyperfield and V be a finitely generated (multi) F-vector. If V is full then V has a basis.*

With two conditions over the ground superfield F, linearly closeness (every homogeneous linear system with more variable than equations has a non-trivial solution), and the full vector space with rigid basis (essentially, $\lambda_1 v_1 + ... + \lambda_n v_n$ is a singleton), we have obtained the following important Theorems:

Theorem. *Let F be a linearly closed superfield and V be a full and finitely generated (multi) F-vector space with $V = \langle v_1, ..., v_n \rangle$ $(v_1, ..., v_n \in V)$. If V is rigidly generated by $\{v_1, ..., v_n\}$ then every linear independent subset of V has at most n elements.*

Theorem. *Let F be a linearly closed superfield and V be a (multi) F-vector space. If B_1 and B_2 are rigid basis of V then $|B_1| = |B_2|$.*

The two preceding theorems suggest that the generalization of linear algebra methods for superfields will proceed smoothly if we confine ourselves to full (multi) vector spaces over a linearly closed superfield F. Indeed, this abstract framework are effective when applied to multi-structures arising from abstract theories of quadratic forms. We have established the following two general structural theorems. The first one offers many examples of linearly closed superfields, beginning with a hyperbolic hyperfield (i.e., a hyperfield F where $1 - 1 = F$):

Theorem. *Every hyperbolic hyperfield is a linearly closed superfield.*

And the second providing an Example where we can calculate the dimension of a (multi)-vector space in this general setting:

Theorem. *Let F be a linearly closed superfield and $p \in F[X]$ be an irreducible polynomial with* $\deg p = n + 1$. *Then $F(p)$ is also linearly closed full (multi) F-vector space and* $\dim(F(p)) = n + 1$.

Surprisingly, we have obtained a nice application in the context of classical algebraic theory of quadratic forms over fields:

Theorem (Isotropy Interpolation)**.** *Let $K = M(F) := F/_m(F^2 \setminus \{0\})$ for a field F (of characteristic not 2) or $K = G \sqcup \{0\}$, for a formally real special group G. Consider a matrix $A \in M_{n\times m}(K)$, saying $A = (a_{ij})$. If $m > n$, there exists $d_1, ..., d_n \in F$, not all zero, such that all the forms $\{\varphi_1, ..., \varphi_n\}$ with*

$$\varphi_i := \langle a_{i1}d_1, a_{i2}d_2, ..., a_{im}d_m \rangle$$

are isotropic.