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BORSUK–ULAM PROPERTY FOR GRAPHS

Daciberg Lima GONÇALVES and Jesús GONZÁLEZ

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Abstract. For finite connected graphs Γ and G, with Γ admitting a free involution τ, we characterize the based homotopy classes $\alpha \in [\Gamma, G]$ for which the Borsuk–Ulam property holds in the sense of Gonçalves, Guaschi, and Casteluber Laass, i.e., the homotopy classes α such that each of the representatives $f \in \alpha$ satisfies $f(x) = f(\tau \cdot x)$ for some $x \in \Gamma$. This is attained through a graph-braid-group perspective aided by the use of discrete Morse theory.

1. Introduction and main result

In its classical formulation, the Borsuk–Ulam theorem asserts that, for any continuous map

$$f\colon S^n \to \mathbb{R}^n, \tag{1}$$

there is a point $x \in S^n$ such that both x and its antipodal $-x$ have the same image under f. Such a phenomenon has been intensively studied in the past 15 years within generalized contexts, namely, for maps $f\colon M \to N$ between spaces M and N, where M admits a free involution. For instance, the case where M ranges over surfaces or suitable families of 3-manifolds is now reasonably well understood [**3–8, 12, 13**]. The case where N has non-trivial homotopy information leads to a more refined problem, as the Borsuk–Ulam question can then have different answers for different homotopy classes[†] in $[M, N]$.

Definition 1.1. [**17**] Assume M admits a free involution τ. We say that the Borsuk–Ulam property holds for a homotopy class $\alpha \in [M, N]$ if for every representative $f \in \alpha$ there is a point $x \in M$ such that $f(x) = f(\tau \cdot x)$. If the above condition holds for all homotopy classes in $[M, N]$, we say that the triple (M, τ, N) satisfies the Borsuk–Ulam property.

We give a complete answer to the Borsuk–Ulam problem in the case where both M and N are one-dimensional compact connected objects. We will thus focus on maps $f\colon \Gamma \to G$ between finite connected graphs Γ and G, addressing the Borsuk–Ulam property with respect to some fixed free involution τ on Γ.

Remark 1.2. In the classical situation (1) with $n = 1$, the circle plays no essential role. Indeed, by considering the differences $f(x) - f(\tau \cdot x)$, it can be seen that any map $f\colon \Gamma \to \mathbb{R}$ satisfies the Borsuk–Ulam property. Such a pleasant situation changes drastically when $\mathbb{R}$ (or an interval, for that matter) is replaced by a more general graph G which, in what follows, will be assumed

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[†]Unless otherwise noted, spaces are assumed to come equipped with base points which must be preserved by maps between spaces. Likewise, homotopy classes are meant in the based sense.

not to be homeomorphic to an interval. In particular the configuration spaces $\mathrm{Conf}_2(G)$ and $\mathrm{UConf}_2(G)$ consisting, respectively, of pairs (x_1, x_2) and of subsets $\{x_1, x_2\}$ with $x_1 \neq x_2$ are both connected.

The Borsuk–Ulam property for (Γ, τ, G) as above is described next.

THEOREM 1.3. *If G is not homeomorphic to a circle or to an interval, then the Borsuk–Ulam property fails for all homotopy classes in* $[\Gamma, G]$*, i.e., for every* $\alpha \in [\Gamma, G]$ *there is a representative* $f \in \alpha$ *satisfying* $f(x) \neq f(\tau \cdot x)$ *for all* $x \in \Gamma$.

When G is a circle, the behavior of the Borsuk–Ulam property sits in between Remark 1.2 and Theorem 1.3. The explicit answer, given in Theorem 1.4 below, generalizes [**17**, Proposition 6] and depends on the Euler characteristic $\chi(\Gamma)$. The latter number is even, in view of the free involution τ, and at most 0, since Γ is connected, say $\chi(\Gamma) = -2m$ with $m \geq 0$. In particular Γ is homotopy equivalent to a wedge of $2m + 1$ circles, and a homotopy class $\Gamma \to S^1$ is determined by a $(2m+1)$-tuple of integer numbers (once an orientation is fixed for each circle).

THEOREM 1.4. *If G is homeomorphic to a circle* S^1*, then the Borsuk–Ulam property holds for most of the homotopy classes in* $[\Gamma, S^1]$*. Explicitly, under the identification* $[\Gamma, S^1] = \mathbb{Z}^{2m+1}$ *discussed above, the homotopy classes of maps* $\Gamma \to S^1$ *for which the Borsuk–Ulam property fails are precisely the* $(2m+1)$*-tuples* $(p, p_1, p_1, p_2, p_2, \ldots, p_m, p_m)$ *with p odd (and* $p_1, \ldots, p_m$ *arbitrary).*

Observe that Theorems 1.3 and 1.4 can be stated by replacing based homotopy classes by free homotopy classes. This is clear in the case of Theorem 1.3, while the case of Theorem 1.4 follows from the fact that $\pi_1(S^1)$ is abelian, so that based homotopy classes and free homotopy classes coincide.

As in [**14–18**], we study the Borsuk–Ulam property for graphs through a sharp algebraic model in terms of braid groups. In our case (graphs), the critical information comes from a detailed control of the topological combinatorics associated to graph configuration spaces (in both the ordered and unordered contexts) provided by Farley and Sabalka's discrete gradient field on Abrams' homotopy model. All the details needed are reviewed in Section 2. Section 3 is devoted to the proof of Theorems 1.3 and 1.4.

2. Graph braid groups via discrete Morse theory

We start by collecting the ingredients we need about Forman's discrete Morse theory and Farley and Sabalka's gradient field on Abrams' discrete model for (ordered and unordered) graph configuration spaces. For details, the reader is referred to [**1, 2, 9–11, 19**].

2.1. Discrete Morse theory

Let X be a connected finite regular CW complex with cell poset $\mathcal{F}$ partially ordered by inclusion.† For a cell $a \in \mathcal{F}$, we use $a^{(p)}$ as a shorthand of $\dim(a) = p$. The Hasse diagram of $\mathcal{F}$,

†Cells are meant in the closed sense.

$H_{\mathcal{F}}$, is thought of as a directed graph with arrows $a^{(p+1)} \searrow b^{(p)}$ oriented from the higher-dimensional cell to the lower-dimensional cell. Let W be a partial matching on $H_{\mathcal{F}}$, i.e., a directed subgraph of $H_{\mathcal{F}}$ all the vertices of which have degree one. The modified Hasse diagram $H_{\mathcal{F}(W)}$ is obtained from $H_{\mathcal{F}}$ by reversing all arrows of W. A reversed edge is denoted as $b^{(p)} \nearrow a^{(p+1)}$, in which case a is said to be collapsible and b is said to be redundant. A path λ in $H_{\mathcal{F}(W)}$ is a chain of up-going and down-going arrows:

$$a_0 \nearrow b_1 \searrow a_1 \nearrow \cdots \nearrow b_k \searrow a_k.$$

The path λ is said to be a cycle when $a_0 = a_k$. If there are no cycles, W is called a gradient field and, in such a case, cells of X that are neither redundant nor collapsible are said to be critical. In what follows we assume that W is a gradient field on X.

The subgraph of the 1-skeleton $X^{(1)}$ consisting of all vertices and of all collapsible edges forms a maximal forest F_X with as many components as there are critical 0-cells in X. Add critical edges as needed in order to form a maximal tree T_X (see [**9**, Proposition 2.3] together with the two-line paragraph following the proof of that proposition). Fix a vertex $v_0 \in X^{(0)}$ as base point. Collapsing T_X to v_0 yields a generating set $\{\beta_e\}_e$ for the fundamental group $\pi_1(X; v_0)$, where e runs over the set of (arbitrarily oriented) critical 1-cells of X that are not part of T_X. Explicitly, for each vertex $u \in X^{(0)}$, let β_u be the unique path in T_X determined by the ordered sequence of non-repeating edges connecting v_0 to u. Then, for a critical 1-cell e from u_1 to u_2 that is not part of T_X, the loop

$$\beta_{u_1} \star e \star \beta_{u_2}^{-1} \tag{2}$$

represents the homotopy class $\beta_e \in \pi_1(X; v_0)$, which will simply be denoted as $e \in \pi_1(X; v_0)$ – the context clarifies whether we refer to the actual cell or to the corresponding homotopy class. Farley and Sabalka go further, describing a set of relations among the homotopy generators $\{\beta_e\}_e$, thus obtaining a presentation for $\pi_1(X, v_0)$. The relations depend on the critical 2-cells and the redundant 1-cells. We omit details as we will have no need to use the relations.

2.2. *Farley–Sabalka gradient field on Abrams' model*

Let G be a finite connected graph. By inserting a few non-essential vertices, we can assume G is simplicial, i.e., that G contains neither loops nor multiple edges. Let $\mathrm{Conf}_2(G)$ denote the ordered configuration space of pairs $(x, y) \in G^2$ with $x \neq y$, and let $\mathrm{UConf}_2(G)$ denote the orbit space by the involution $(x, y) \mapsto (y, x)$. Abrams' homotopy model $D_2(G)$ for $\mathrm{Conf}_2(G)$ is the subcomplex of $G \times G$ whose cells are the ordered pairs $c = (c_1, c_2)$ of cells† of G with $c_1 \cap c_2 = \varnothing$. The orbit complex $UD_2(G)$ resulting from the involution $(c_1, c_2) \mapsto (c_2, c_1)$ is the corresponding homotopy model for $\mathrm{UConf}_2(G)$. Thus, cells of $UD_2(G)$ are sets $c = \{c_1, c_2\}$ of disjoint cells c_i of G. In both the ordered and unordered settings, we have the following:

- The cells c_1 and c_2 are called the ingredients of c.
- The dimension of c is the sum of the dimensions of c_1 and c_2.
- The orientation of a one-dimensional cell with vertex ingredient v and edge ingredient e will be inherited from that of e. For instance, if e is oriented from v_1 to v_2 (so $u \notin \{v_1, v_2\}$), then the ordered 1-cell (e, u) is oriented from (v_1, u) to (v_2, u).

†A cell of G is either a vertex or a (closed) edge.

The construction of the Farley–Sabalka gradient field on $D_2(G)$ and of its quotient on $UD_2(G)$ requires some preliminary notation. Start by choosing a maximal tree T of G. Edges of G outside T are called *deleted edges*. Fix a planar embedding of T and a root of T (i.e., a vertex of degree one in T), which is denoted by 0. The rest of the vertices of G are consecutively numbered $1, 2, \ldots$ as we first find them in the walk along T that starts at 0 and that takes the leftmost branch at any given intersection, turning around when a vertex of degree one is reached. An edge e bounded by vertices u and v with $u < v$ is denoted by $e = (u, v)$, and is oriented from u to v. Under such conditions we also write $v = \sigma(e)$, the *source* of e, and $u = \tau(e)$, the *target* of e. The source–target notation is compatible with the fact that, by collapsing T down to its root 0, we can think of $\pi_1(G, 0)$ as the free group generated by the deleted edges (each with the vertex-ordering orientation). Note that if (u, v) is non-deleted, u is determined as the vertex adjacent to v in T that is located in the T-path leading from v back to 0, so we can safely write $e_v = (u, v)$. In particular, non-deleted edges e_v will be ordered according to the order of the corresponding v.

Remark 2.1. The relation $u + 1 \leq v$ holds for any deleted edge (u, v). A strict inequality

$$u + 1 < v \tag{3}$$

can actually be assured through a slight adjustment of the tree T. Explicitly, assume

$$u + 1 = v. \tag{4}$$

By hypothesis, G is simplicial, so u must have degree one in T. By the construction of the T-order, there is some non-deleted edge $(w, u + 1)$ with the vertex w lying on the segment joining 0 and u. See the left-hand side of Figure 1, where the dashed line represents the deleted edge $(u, u + 1)$, and where the horizontal portion together with the figure Y on top represent the part of T generated by all the vertices smaller than or equal to u. Now replace in T the edge $(w, u + 1)$ by the edge $(u, u + 1)$ to get the adjusted tree T' depicted on the right-hand side of Figure 1. Note that the ordering of vertices determined by T agrees with the one determined by T'. Furthermore, the deleted edges of T are those of T', except for the deleted edge $(u, u+1)$ of T, which has been replaced by the deleted edge $(w, u+1)$ of T', for which we have $w+1 \leq u < u+1$, thus correcting (4). Repeating the process as needed, we end up with a fully adjusted tree all the deleted edges of which satisfy (3). In what follows, we assume the adjusting process has been made.

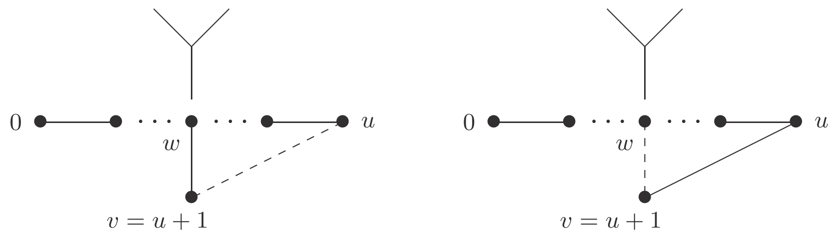


FIGURE 1. The original tree T (left) and the adjusted tree T' (right).

Let $c = (c_1, c_2)$ (or $c = \{c_1, c_2\}$) be a cell of $D_2(G)$ (or of $UD_2(G)$). A vertex ingredient $v = c_i$ of c is said to be critical in c if either $v = 0$ or, else, if replacement of v by e_v in c fails to yield a cell of $D_2(G)$ (or of $UD_2(G)$). Likewise, an edge ingredient $e = c_j$ of c is said to be critical in c if either e is deleted or, else, if $e = e_v$ and there is a vertex ingredient u of c adjacent to $\tau(e)$ with $\tau(e) < u < v$. With such notation, c is critical in the Farley–Sabalka gradient field provided c_1 and c_2 are both critical in c. Otherwise, if the smallest† of the non-critical ingredients c_1 and c_2 is

(i) a vertex v, then c is redundant and $c \nearrow d$, where d is obtained from c by replacing v by e_v, or
(ii) an edge e_v, then c is collapsible and $d \nearrow c$, where d is obtained from c by replacing e_v by v.

In other words, the only critical 0-cells in $D_2(G)$ are $(0, 1)$ and $(1, 0)$, while $\{0, 1\}$ is the only critical 0-cell in $UD_2(G)$. All other 0-cells are redundant. Likewise, for a 1-cell c with vertex ingredient u and edge ingredient e we have the following:

(iii) If e is deleted, then c is critical if u is critical in c, otherwise c is redundant.
(iv) If e is non-deleted, say $e = e_v$, and
  - either $u = 0$ or $v < u$, then c is collapsible;
  - $0 < u < v$ with u critical in c (in which case $\tau(e_u) = \tau(e_v)$ is forced), then c is critical;
  - $0 < u < v$ with u non-critical in c, then c is redundant.

Lastly, a 2-cell c is critical if both of its ingredients are deleted, otherwise c is collapsible.

2.3. *Graph braid groups*

We now recover the assumption in Remark 1.2 about G not being homeomorphic to an interval, so we can use the facts reviewed above in order to describe generators for

$$P_2(G) = \pi_1(D_2(G)) = \pi_1(\mathrm{Conf}_2(G))$$

and

$$B_2(G) = \pi_1(UD_2(G)) = \pi_1(\mathrm{UConf}_2(G)),$$

the (pure and full, respectively) braid groups of two (ordered and unordered, respectively) non-colliding particles in G. In particular, the notation set up in Sections 2.1 and 2.2 will be in effect throughout the rest of the paper.

The case of $B_2(G)$ is slightly easier, as $UD_2(G)$ has a single critical 0-cell, so that the 0-cells and the collapsible 1-cells of $UD_2(G)$ span a maximal tree UDT of the 1-skeleton of $UD_2(G)$. Thus, after collapsing UDT to its base point, which is taken to be the critical 0-cell $\{0, 1\}$, we get that the critical 1-cells $\{u, e\}$ of $UD_2(G)$ yield corresponding generators $\{u, e\} \in B_2(G)$.

Since $D_2(G)$ has two critical 0-cells, we need to identify the two components of the maximal forest DF determined by the 0-cells and the collapsible 1-cells of $D_2(G)$. Recall that any collapsible 1-cell c of $D_2(G)$ with vertex ingredient u and edge ingredient e_v satisfies

†With respect to the vertex-edge ordering discussed in the previous paragraph.

$u = 0$ or $v < u$. In particular, c joins 0-cells (w_1, w_2) and (w'_1, w'_2) satisfying $w_1 < w_2$ if and only if $w'_1 < w'_2$. Consequently, DF consists of two trees DT_u and DT_d, where the vertices (w_1, w_2) in the former (latter) tree satisfy $w_1 < w_2$ $(w_1 > w_2)$. Now, as reviewed above, we need to add a single critical 1-cell of $D_2(G)$ to DF in order to get a maximal tree DT. For our purposes, the required critical 1-cell will have the form $(a,(b,c))$ with $b < a < c$. (Note that such an edge goes from $(a,b) \in DT_d$ to $(a,c) \in DT_u$.) The explicit values of a, b, and c will be spelled out later, depending of the actual graph G. All other critical 1-cells $(r,(s,t))$ and $((s,t),r)$ of $D_2(G)$ yield corresponding generators $(r,(s,t)), ((s,t),r) \in P_2(G)$ after collapsing DT to its base point, which is now taken to be $(0,1)$.

Proposition 2.2. *The inclusion* $\iota\colon P_2(G) \hookrightarrow B_2(G)$ *induced by the* 2*-fold covering* $D_2(G) \to UD_2(G)$ *is determined on generators by*

$$\iota(r,(s,t)) = \begin{cases} \{r,(s,t)\} & \text{if } r < s < t, \\ \{a,(b,c)\}^{-1} \cdot \{r,(s,t)\} & \text{if } s < r < t, \\ \{a,(b,c)\}^{-1} \cdot \{r,(s,t)\} \cdot \{a,(b,c)\} & \text{if } s < t < r; \end{cases}$$

$$\iota((s,t),r) = \begin{cases} \{a,(b,c)\}^{-1} \cdot \{r,(s,t)\} \cdot \{a,(b,c)\} & \text{if } r < s < t, \\ \{r,(s,t)\} \cdot \{a,(b,c)\} & \text{if } s < r < t, \\ \{r,(s,t)\} & \text{if } s < t < r. \end{cases}$$

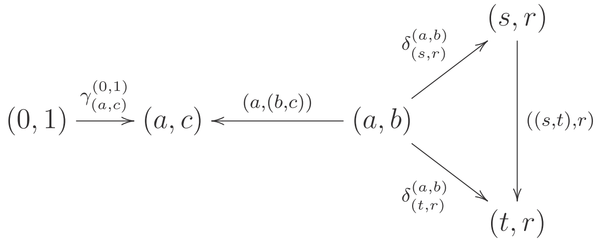
In particular, $\iota((b,c),a) = \{a,(b,c)\}^2$.

Proof. Since the 2-fold covering projection $D_2(G) \to UD_2(G)$ is cellular and preserves the critical/collapsible/redundant nature of cells, the effect of the induced monomorphism ι is easily readable in terms of the generators described above. Here we consider only the case of $\iota((s,t),r)$ with $r < s < t$, an assumption that will be in force throughout the rest of the proof, leaving to the reader the fully parallel details in the other five cases.

As a path in $D_2(G)$, the edge $((s,t),r)$ goes from $(s,r) \in DT_d$ to $(t,r) \in DT_d$. Then the paths $\beta_{(s,r)}$ and $\beta_{(t,r)}$ in (2) are given by

$$\beta_{(s,r)} = \gamma^{(0,1)}_{(a,c)} \star (a,(b,c))^{-1} \star \delta^{(a,b)}_{(s,r)} \quad \text{and} \quad \beta_{(t,r)} = \gamma^{(0,1)}_{(a,c)} \star (a,(b,c))^{-1} \star \delta^{(a,b)}_{(t,r)},$$

where γ-paths and δ-paths consist of collapsible cells. Explicitly, $\gamma^{(0,1)}_{(a,c)}$ is the unique simple path in DT_u connecting $(0,1)$ to (a,c), while $\delta^{(a,b)}_{(s,r)}$ and $\delta^{(a,b)}_{(t,r)}$ are the unique simple paths in DT_d connecting (a,b) to (s,r) and (t,r), respectively, as shown below:



The loop (2) representing $((s,t),r) \in P_2(G)$ is then

$$\gamma^{(0,1)}_{(a,c)} \star (a,(b,c))^{-1} \star \delta^{(a,b)}_{(s,r)} \star ((s,t),r) \star (\gamma^{(0,1)}_{(a,c)} \star (a,(b,c))^{-1} \star \delta^{(a,b)}_{(t,r)})^{-1}, \qquad (5)$$

and the asserted expression for $\iota((s,t),r)$ now follows by noticing that the portions corresponding to γ-paths and δ-paths are sent by ι into UDT and, so, get squeezed to the base point $\{0,1\}$. □

From this point on we will think of $P_2(G)$ as a honest subgroup of $B_2(G)$, omitting to write the symbol ι when thinking of an element in $P_2(G)$ as an element of $B_2(G)$.

Corollary 2.3. *For a critical 1-cell e of $D_2(G)$ other than $(a,(b,c))$, the conjugate*

$$\{a,(b,c)\} \cdot e \cdot \{a,(b,c)\}^{-1} \in P_2(G) \lhd B_2(G)$$

is described as follows:

For $r<s<t$,

$$\begin{aligned}\{a,(b,c)\}\cdot(r,(s,t))\cdot\{a,(b,c)\}^{-1} &= ((b,c),a)\cdot((s,t),r)\cdot((b,c),a)^{-1},\\ \{a,(b,c)\}\cdot((s,t),r)\cdot\{a,(b,c)\}^{-1} &= (r,(s,t)).\end{aligned}$$

For $s<r<t$,

$$\begin{aligned}\{a,(b,c)\}\cdot(r,(s,t))\cdot\{a,(b,c)\}^{-1} &= ((s,t),r)\cdot((b,c),a)^{-1},\\ \{a,(b,c)\}\cdot((s,t),r)\cdot\{a,(b,c)\}^{-1} &= \begin{cases}((b,c),a) & \text{if } (r,(s,t))=(a,(b,c)),\\ ((b,c),a)\cdot(r,(s,t)) & \text{otherwise.}\end{cases}\end{aligned}$$

For $s<t<r$,

$$\begin{aligned}\{a,(b,c)\}\cdot(r,(s,t))\cdot\{a,(b,c)\}^{-1} &= ((s,t),r),\\ \{a,(b,c)\}\cdot((s,t),r)\cdot\{a,(b,c)\}^{-1} &= ((b,c),a)\cdot(r,(s,t))\cdot((b,c),a)^{-1}.\end{aligned}$$

Proposition 2.4. *If G is not homeomorphic to an interval, then the morphism $\theta\colon B_2(G) \to \mathbb{Z}_2$ induced in fundamental groups by the classifying map of the double covering $D_2(G) \to UD_2(G)$ is given on generators by*

$$\theta(\{r,(s,t)\}) = \begin{cases}1 & \text{if } s<r<t,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Since θ vanishes on $P_2(G)$, the relations

$$\begin{aligned}(r,(s,t)) &= \{r,(s,t)\} \quad \text{for } r<s<t,\\ ((s,t),r) &= \{r,(s,t)\} \quad \text{for } s<t<r\end{aligned}$$

force $\theta(\{r,(s,t)\}) = 0$ provided $r<s$ or $t<r$. On the other hand, for $s<r<t$, the relation

$$((s,t),r) = \{r,(s,t)\} \cdot \{a,(b,c)\}$$

gives $\theta(\{r,(s,t)\}) = \theta(\{a,(b,c)\})$, which is forced to be 1, since θ is surjective (recall that G is not an interval, so that $\mathrm{Conf}_2(G)$ is connected). □

PROPOSITION 2.5. *The morphism $(p_1)_\# : P_2(G) \to \pi_1(G)$ induced in fundamental groups by the projection $p_1 : D_2(G) \to G$ onto the first coordinate is trivial on generators $(r, (s, t))$, while*

$$(p_1)_\#((s,t), r) = \begin{cases} (s,t) & \text{if } (s,t) \text{ is a deleted edge,} \\ 1 & \text{otherwise.} \end{cases}$$

Proof. The conclusion for $(p_1)_\#((s, t), r)$ when $r < s < t$ follows from the fact that (5) is a representing loop for $((s, t), r) \in P_2(G)$, and noticing that the p_1-image of a collapsible 1-edge lands in the tree T and, so, it plays no role in $\pi_1(G)$. The other five cases are treated similarly and are left as an exercise for the reader. □

3. Borsuk–Ulam property

Let (Γ, τ, G) be as in Section 1. We assume in effect all hypotheses, ingredients, and constructions set up in Section 2.3 around G and its braid groups. Theorem 3.1 below is a 'graph' version of [**17**, Theorem 7], proven with the same argument, using that graph configuration spaces are $K(\pi, 1)$.

THEOREM 3.1. *The classes $\alpha \in [\Gamma, G]$ for which the Borsuk–Ulam property fails are precisely those fitting on the commutative diagram of groups*

$$\begin{array}{ccc} & & \pi_1(G) \\ & \overset{\alpha_\#}{\nearrow} & \big\uparrow{\scriptstyle (p_1)_\#} \\ \pi_1(\Gamma) & \xrightarrow{\ \varphi\ } & P_2(G) \\ \big\downarrow & & \big\downarrow \\ \pi_1(\Gamma/\tau) & \xrightarrow{\ \psi\ } & B_2(G) \\ {\scriptstyle \theta_1}\big\downarrow & & \big\downarrow{\scriptstyle \theta_2} \\ \mathbb{Z}_2 & = & \mathbb{Z}_2 \end{array} \tag{6}$$

for suitable morphisms φ and ψ. Here the two central group inclusions are induced by the obvious 2-fold covering projections, while morphisms θ_i are induced by the corresponding classifying maps. In particular, both downward vertical sequences are short exact.

Example 3.2. Assume $G = T$, a tree (not homeomorphic to an interval). Then θ_2 is surjective. Since $\pi_1(\Gamma/\tau)$ is free (see paragraph below), it is possible to choose a lifting $\psi : \pi_1(\Gamma/\tau) \to B_2(G)$ of θ_1 along θ_2. The restricted map $\varphi : \pi_1(\Gamma) \to P_2(G)$ then completes diagram (6) with $\alpha_0 \in [\Gamma, G]$ necessarily the unique (trivial) homotopy class. This proves Theorem 1.3 when G is contractible. Therefore, throughout the rest of this section we assume that $G \neq T$. In particular $\pi_1(G) = F(z_1, \ldots, z_k)$, the free group on generators $z_i = (x_i, y_i)$, where $\{(x_i, y_i)\}_{i=1,\ldots,k}$ is the set of deleted edges of G (recall $x_i < y_i$ for all i, which gives the orientation of the representing loop for z_i – after collapsing T to a point). For convenience, we will assume that the deleted edges have been arranged so that $y_1 < y_2 < \cdots < y_k$.

Ignoring vertices of degree 2, τ is forced to act at the level of vertices and edges. Thus Γ/τ has a natural graph structure. A simple Euler characteristic argument then shows that the free groups $\pi_1(\Gamma)$ and $\pi_1(\Gamma/\tau)$ have respective ranks $2m+1$ and $m+1$, where $m = -\chi(\Gamma)/2 \geq 0$. In particular,

$$[\Gamma, G] = F(z_1, \dots, z_k)^{2m+1}. \tag{7}$$

More explicitly, we have the next lemma.

LEMMA 3.3. *It is possible to choose generators*

$$a, a_1, a_1', a_2, a_2', \dots, a_m, a_m'$$

of $\pi_1(\Gamma)$ *as well as generators* $c, c_1, c_2, \dots, c_m$ *of* $\pi_1(\Gamma/\tau)$ *satisfying*

$$a = c^2, \quad a_i = c_i, \quad a_i' = c c_i c^{-1}, \quad \theta_1(c) = 1 \in \mathbb{Z}_2 \quad \textit{and} \quad \theta_1(c_i) = 0 \in \mathbb{Z}_2$$

for $i = 1, 2, \dots, m$. *In this setting,* $\alpha \in [\Gamma, G]$ *is identified under (7) with the tuple*

$$(\alpha_\#(a), \alpha_\#(a_1), \alpha_\#(a_1'), \dots, \alpha_\#(a_m), \alpha_\#(a_m')).$$

Proof. This lemma is part of the folklore. Since we cannot find an explicit reference, we sketch a proof. Suppose we have an epimorphism $\theta : \pi_1(\Gamma/\tau) \to \mathbb{Z}_2$ and let $\{e_0, e_1, \dots, e_m\}$ be an arbitrary base. Assume without loss of generality that $\theta(e_0) = 1 \in \mathbb{Z}_2$ and set $c := e_0$. If $\theta(e_i) = 0$ for $i > 0$, then set $c_i := e_i$. Otherwise, let $c_i := e_0 \cdot e_i$. Then we have constructed a base $\{c, c_1, \dots, c_m\}$ with the desired θ-properties. Now we apply the Reidemeister–Schreier process to find a presentation of $\ker(\theta)$, using the set of generators $c, c_1, \dots, c_m$ and, as Schreier system, $\{1, c\}$. It follows that the kernel has a presentation given by elements as in the statement of the lemma subject to no relation. The result follows. □

Proof of Theorem 1.4. It is a standard fact that the right-hand side column in (6) becomes

$$\pi_1(S^1) = \mathbb{Z} \xleftarrow{=} \mathbb{Z} = P_2(S^1) \xrightarrow{\cdot 2} B_2(S^1) = \mathbb{Z} \xrightarrow{\theta_2 = \mathrm{proj}} \mathbb{Z}_2. \tag{8}$$

Consequently, morphisms $\psi : \pi_1(\Gamma/\tau) \to B_2(G)$ satisfying $\theta_2 \circ \psi = \theta_1$ are in one-to-one correspondence with tuples of integer numbers $(p, q_1, \dots, q_m)$, with p odd and each q_i even, where the correspondence is such that $\psi(c) = p$ and $\psi(c_i) = q_i$ for $1 \leq i \leq m$. Say $q_i = 2p_i$. Lemma 3.3 and (8) then imply that the restriction to $\pi_1(\Gamma)$ of such a ψ is given by $\varphi(c) = p$ and $\varphi(a_i) = p_i = \varphi(a_i')$ for $i = 1, 2, \dots, m$. The result follows. □

The proof of Theorem 1.3 follows the strategy in the previous proof, except that the needed algebraic manipulations are far more subtle, and depend on the results in Section 2.3. With this in mind, we assume from this point on that G is not homeomorphic to a circle (or to a tree, in view of Example 3.2), and pick key elements

$$\rho, \lambda_1, \lambda_2, \dots, \lambda_k \in P_2(G) \quad \text{and} \quad \sigma \in B_2(G)$$

as described below. We then set $\lambda_i' := \sigma \lambda_i \sigma^{-1} \in P_2(G)$.

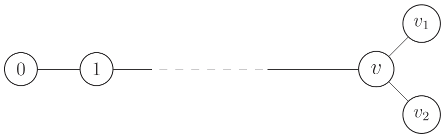


FIGURE 2. Part of the nonlinear tree T showing the essential vertex v.

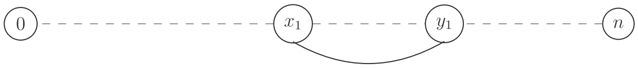


FIGURE 3. The linear tree T together with the deleted edge (x_1, y_1).

When T has an essential vertex v. Choose vertices v_1 and v_2 with $v < v_1 < v_2$ and $\tau(e_{v_1}) = v = \tau(e_{v_2})$. See Figure 2. In this situation we choose the critical 1-cell connecting the trees DT_d and DT_u to be $(a, (b, c)) := (v_1, (v, v_2))$ (see Section 2.3), and set

$$\sigma := \{v_1, (v, v_2)\}, \quad \rho := ((v, v_2), v_1) \quad \text{and} \quad \lambda_i := (x_i + 1, (x_i, y_i)),$$

for $1 \leq i \leq k$ (recall, from Remark 2.1, that $x_i + 1 < y_i$). Corollary 2.3 and Propositions 2.2 and 2.5 then give

$$\rho = \sigma^2, \quad p_1(\rho) = 1, \quad p_1(\lambda_i) = 1 \quad \text{and} \quad p_1(\lambda'_i) = z_i, \tag{9}$$

for $i = 1, 2, \ldots, k$. Here and below, we write p_1 instead of $(p_1)_\#$. The context clarifies the abuse of notation.

When T is linear. In this situation we choose the critical 1-cell connecting the trees DT_d and DT_u to be $(a, (b, c)) := (x_1 + 1, (x_1, y_1))$. See Figure 3. Recall that we have chosen $y_1 < y_i$ for $i > 1$. Furthermore, the condition $x_1 + 1 < y_1$ is forced on us because G is simplicial. We then set $\sigma := \{x_1 + 1, (x_1, y_1)\}$, $\rho := ((x_1, y_1), x_1 + 1)$, $\lambda_1 := (x'_1, (x_1, y_1))$, and $\lambda_i := (x_i + 1, (x_i, y_i))$, for $2 \leq i \leq k$, where $x'_1 = 0$ when $x_1 > 0$, whereas $x'_1 = y_1 + 1$ when $x_1 = 0$ (so $y_1 < n$, even if $k = 1$, since G is not homeomorphic to a circle). Corollary 2.3 and Propositions 2.2 and 2.5 then give

$$\rho = \sigma^2, \quad p_1(\rho) = z_1, \quad p_1(\lambda_i) = 1, \quad p_1(\lambda'_1) = z_1 \quad \text{and} \quad p_1(\lambda'_i) = z_i z_1^{-1}, \tag{10}$$

for $i = 2, 3, \ldots, k$.

Proof of Theorem 1.3. Let λ, λ', and z denote, respectively, the sequences of symbols $(\lambda_1, \ldots, \lambda_k)$, $(\lambda'_1, \ldots, \lambda'_k)$, and $(z_1, \ldots, z_k)$, and consider an arbitrary sequence of words

$$(w(z), w_1(z), w'_1(z), \ldots, w_m(z), w'_m(z)) \in F(z_1, \ldots, z_k)^{2m+1}. \tag{11}$$

We have to prove that the homotopy class $\alpha \in [\Gamma, G]$ corresponding to (11) under (7) and Lemma 3.3 fits in the commutative diagram (6) for suitable morphisms ψ and φ.

Assume T has an essential vertex v and consider the setup in (9). The map $\psi\colon \pi_1(\Gamma/\tau) \to B_2(G)$ defined by

$$\psi(c) = w(\lambda)\sigma \quad \text{and} \quad \psi(c_i) = \sigma w_i(\lambda)\sigma^{-1} w'_i(\lambda), \quad i = 1, \ldots, m,$$

satisfies $\theta_1 = \theta_2 \circ \psi$, in view of Proposition 2.4 and Lemma 3.3. Furthermore, the restricted map $\varphi\colon \pi_1(\Gamma) \to P_2(G)$ satisfies

$$\begin{aligned}\varphi(a) &= \psi(c)^2 = w(\lambda)\sigma w(\lambda)\sigma = w(\lambda)\cdot \sigma w(\lambda)\sigma^{-1}\cdot\rho,\\ \varphi(a_i) &= \psi(c_i) = \sigma w_i(\lambda)\sigma^{-1}\cdot w_i'(\lambda),\\ \varphi(a_i') &= \psi(c)\psi(c_i)\psi(c)^{-1} = w(\lambda)\sigma\sigma w_i(\lambda)\sigma^{-1}w_i'(\lambda)\sigma^{-1}w(\lambda)^{-1}\\ &= w(\lambda)\rho w_i(\lambda)\rho^{-1}\cdot \sigma w_i'(\lambda)\sigma^{-1}\cdot w(\lambda)^{-1}.\end{aligned}$$

So

$$\begin{aligned}p_1(\varphi(a)) &= p_1(w(\lambda))\cdot p_1(\sigma w(\lambda)\sigma^{-1})\cdot p_1(\rho) = p_1(\sigma w(\lambda)\sigma^{-1})\\ &= p_1(w(\lambda')) = w(p_1(\lambda')) = w(z),\\ p_1(\varphi(a_i)) &= p_1(\sigma w_i(\lambda)\sigma^{-1})\cdot p_1(w_i'(\lambda)) = p_1(w_i(\lambda'))\\ &= w_i(p_1(\lambda')) = w_i(z),\\ p_1(\varphi(a_i')) &= p_1(w(\lambda)\rho w_i(\lambda)\rho^{-1})\cdot p_1(\sigma w_i'(\lambda)\sigma^{-1})\cdot p_1(w(\lambda))^{-1}\\ &= p_1(\sigma w_i'(\lambda)\sigma^{-1}) = p_1(w_i'(\lambda')) = w_i'(z),\end{aligned}$$

which yields the result in the case under consideration.

Assume now that T is linear and consider the setup in (10). Write the elements $w(z)z_1^{-1}$, $w_i(z)z_1^{-1}$, and $w_i'(z)z_1^{-1}$ of $F(z_1, z_2, \dots, z_k)$ as words on the generators $t_1, t_2, \dots, t_k$ of $F(z_1, z_2, \dots, z_k)$ given by $t_1 := z_1$ and $t_i := z_i z_1^{-1}$ for $i \geq 2$. Say $w(z)z_1^{-1} = \ell(t)$, $w_i(z)z_1^{-1} = \ell_i(t)$, and $w_i'(z)z_1^{-1} = \ell_i'(t)$, where t stands for the tuple $(t_1, t_2, \dots, t_k)$. The map $\psi\colon \pi_1(\Gamma/\tau) \to B_2(G)$ defined by

$$\psi(c) = \ell(\lambda)\sigma \quad\text{and}\quad \psi(c_i) = \sigma\ell_i(\lambda)\sigma\lambda_1^{-1}\ell_i'(\lambda)\lambda_1, \quad i = 1, \dots, m,$$

satisfies $\theta_1 = \theta_2 \circ \psi$, in view of Proposition 2.4 and Lemma 3.3. Furthermore, the restricted map $\varphi\colon \pi_1(\Gamma) \to P_2(G)$ satisfies

$$\begin{aligned}\varphi(a) &= \psi(c)^2 = \ell(\lambda)\sigma\ell(\lambda)\sigma = \ell(\lambda)\cdot\sigma\ell(\lambda)\sigma^{-1}\cdot\rho,\\ \varphi(a_i) &= \psi(c_i) = \sigma\ell_i(\lambda)\sigma\lambda_1^{-1}\ell_i'(\lambda)\lambda_1 = \sigma\ell_i(\lambda)\sigma^{-1}\cdot\rho\lambda_1^{-1}\ell_i'(\lambda)\lambda_1,\\ \varphi(a_i') &= \psi(c)\psi(c_i)\psi(c)^{-1} = \ell(\lambda)\sigma\cdot\sigma\ell_i(\lambda)\sigma\lambda_1^{-1}\ell_i'(\lambda)\lambda_1\cdot\sigma^{-1}\ell(\lambda)^{-1}\\ &= \ell(\lambda)\rho\ell_i(\lambda)\cdot\sigma\lambda_1^{-1}\ell_i'(\lambda)\lambda_1\sigma^{-1}\cdot\ell(\lambda)^{-1}.\end{aligned}$$

So

$$\begin{aligned}p_1(\varphi(a)) &= p_1(\ell(\lambda))\cdot p_1(\sigma\ell(\lambda)\sigma^{-1})\cdot p_1(\rho) = p_1(\ell(\lambda'))z_1\\ &= \ell(p_1(\lambda'))z_1 = \ell(t)z_1 = w(z),\\ p_1(\varphi(a_i)) &= p_1(\sigma\ell_i(\lambda)\sigma^{-1})\cdot p_1(\rho\lambda_1^{-1}\ell_i'(\lambda)\lambda_1) = p_1(\sigma\ell_i(\lambda)\sigma^{-1})z_1\\ &= p_1(\ell_i(\lambda'))z_1 = \ell_i(t)z_1 = w_i(z),\\ p_1(\varphi(a_i')) &= p_1(\ell(\lambda)\rho\ell_i(\lambda))\cdot p_1(\sigma\lambda_1^{-1}\ell_i'(\lambda)\lambda_1\sigma^{-1})\cdot p_1(\ell(\lambda))^{-1}\\ &= z_1 p_1(\sigma\lambda_1^{-1}\ell_i'(\lambda)\lambda_1\sigma^{-1}) = z_1 p_1((\lambda_1')^{-1}\ell_i'(\lambda')\lambda_1')\\ &= z_1(p_1(\lambda_1'))^{-1}\cdot p_1(\ell_i'(\lambda'))\cdot p_1(\lambda_1') = z_1 z_1^{-1}\ell_i'(t)z_1 = w_i'(z),\end{aligned}$$

which completes the proof. □

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Daciberg Lima Gonçalves
Departamento de Matemática, IME
Universidade de São Paulo
Rua do Matão 1010
CEP 05508-090
São Paulo-SP
Brazil
(E-mail: dlgoncal@ime.usp.br)

Jesús González
Departamento de Matemáticas
Centro de Investigación y de Estudios Avanzados del IPN
Av. Instituto Politécnico Nacional 2508
San Pedro Zacatenco
México City 07000
México
(E-mail: jesus@math.cinvestav.mx)