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"Finite Groups for the Genetic
Code I: Codon Representations"

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Abstract

We investigate the possibility of interpreting the degeneracy of the genetic code, i.e., the feature that different codons (base triplets) of DNA are transcribed into the same amino acid, as the result of a symmetry breaking process, in the context of finite groups. In the present paper, we give the complete list of all codon representations (64-dimensional irreducible representations) of simple finite groups and their satellites (central extensions and extensions by outer automorphisms), whereas in the second part, we shall present the corresponding branching rules and discuss which of them – if any – reproduce the multiplet structure of the genetic code.

Keywords: Finite groups, representation theory, genetic code.

1 Introduction

The discovery of the molecular structure of DNA in 1953 by Watson and Crick has been a landmark of science, laying the foundation for understanding the physico-chemical basis for the storage and transfer of genetic information. DNA is a macromolecule in the form of a double helix, encoding all genetic information in a language with 64 three-letter words built from an alphabet with a set of four different letters (A, C, G and T – the four nucleic bases attached to the backbone of a DNA molecule). These words are called codons and form sentences called genes. Each codon can be translated into one of twenty amino acids or a termination signal. This leads to a degeneracy of the code in the sense that different codons represent the same amino acid, that is, different words have the same meaning. In fact, the codons which code for the same amino acids form multiplets as follows:

- 3 sextets   Arg, Leu, Ser
- 5 quadruplets Ala, Gly, Pro, Thr, Val
- 2 triplets   Ile, Term
- 9 doublets   Asn, Asp, Cys, Gln, Glu, His, Lys, Phe, Tyr
- 2 singlets   Met, Trp

When a protein is synthesized, an appropriate segment of one of the two strings in the DNA molecule (or more precisely, the mRNA molecule built from it) is read and the corresponding amino acids are assembled sequentially. The linear chain thus obtained will then fold to the final configuration of the protein.

These well-known facts, however, provide no explanation as to why just this special language has been chosen by nature. Since its discovery, the genetic code has essentially remained a table connecting codons (base triplets) with the amino acids they represent, but a complete understanding of its structure is still missing.

A new approach to the question was suggested in 1993 by Hornos and Hornos [1] who proposed to explain the degeneracy of the genetic code as the result of a symmetry breaking process. The demand of this approach can be compared to explaining the arrangement of the chemical elements in the periodic table as the result of an underlying dynamical symmetry which is reflected in the electronic shell structure of atoms. Another comparable example is the explanation of the multiplet structure of hadrons as a result

of a "flavor" $SU(3)$ symmetry, which led to the quark model and to the prediction of new particles. An interesting and important feature of this "flavor" symmetry is its internal or dynamical nature, that is, it is an internal property of the dynamical equations of the system, rather than being related to the structure of space-time.

In the same spirit, the idea of the above mentioned authors was to explain the multiplet structure of the genetic code through the multiplets found in the codon representation (= irreducible 64-dimensional representation) of an appropriate simple Lie algebra and its branching rules into irreducible representations of its semisimple subalgebras. They checked the tables of branching rules of McKay and Patera [2] for semisimple subalgebras of simple Lie algebras of rank ≤ 8. The most suitable multiplet structure found is derived from the codon representation of the symplectic algebra $\mathfrak{sp}(6)$ by the following sequence of symmetry breakings:

$$
\begin{aligned}
\mathfrak{sp}(6) \quad \supset \quad & \mathfrak{sp}(4) \oplus \mathfrak{su}(2) & I \\
\supset \quad & \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) & II \\
\supset \quad & \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) & III/IV/V
\end{aligned}
$$

The sequence of steps $I - V$ is associated with the evolution of the genetic code in the beginning of life. For a recent detailed exposition, see Ref. [3].

This work, which had a strong resonance in the scientific community [4, 5], has raised a lot of new interesting problems. One of them is that the last step in the symmetry breaking remains incomplete: the lifting of degeneracy by breaking the last two $\mathfrak{su}(2)$ subalgebras to $\mathfrak{u}(1)$ is not followed by all codon multiplets. Only if some of them continue to represent a single amino acid can the actual multiplet structure of the genetic code be obtained. This "freezing" had already been proposed by biologists [6] who claimed that a completely accomplished evolution of the genetic code should have resulted in 28 amino acids [7] – in perfect agreement with the mathematical model. However, the phenomenon that some of the multiplets preserve a symmetry while it is broken in others, even though it does not contradict any biological principle or observation (see Ref. [8] or Ref. [9] for extensive reviews), is somewhat awkward from a mathematical point of view.

The proposal of the present project, already stated in Ref. [1], is to study the same problem within its most traditional and natural mathematical context, namely that of finite groups, rather than of compact Lie groups. In fact, there is no particular reason for using continuous symmetries, rather than discrete ones, in this kind of investigation. Another direction, also contemplated in Ref. [1], is to replace ordinary Lie algebras by other algebras in

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their "vicinity", such as quantum groups and Lie superalgebras; a systematic analysis for the case of Lie superalgebras can be found in Refs [10, 11]. Of course, the reader may wonder why the whole program has been carried out first for Lie algebras and even for Lie superalgebras and not for finite groups, but the reason is quite simple: the case of finite groups is technically the most difficult one. Indeed, until recently, we did not really believe it would be possible to overcome the formidable problems associated with the question of obtaining a complete classification and arrive at a definite conclusion. Fortunately, we were wrong!

2 Simple finite groups and their satellites

Our first task in what follows is to specify the class of finite groups within which the search for codon representations has been conducted. This class is formed by the *simple* finite groups, i.e., those that have no non-trivial normal subgroups, together with their so-called satellites, obtained as central extensions or extensions by outer automorphisms or a combination thereof. Our basic reference will be the atlas of finite groups [12], henceforth simply referred to as the ATLAS.

As a preliminary remark, we recall that any group G has (at least) two natural normal subgroups, namely its *center* $Z(G)$, consisting of those elements that commute with all elements of G, and its commutator subgroup or *derived subgroup* G', formed by products of elements $g_1 g_2 g_1^{-1} g_2^{-1}$ with g_1 and g_2 running through the elements of G; moreover, a group G is called *perfect* if it is equal to its derived subgroup G'. Of course, when G is simple, there are only two alternatives: either G is abelian ($Z(G) = G$, $G' = \{1\}$) or G is perfect ($Z(G) = \{1\}$, $G' = G$).

Among the finite groups, the simple ones constitute the basic building blocks from which all others can be constructed. Indeed, the simple constituents of any finite group can be determined from its so-called *composition series* – a finite sequence of subgroups such that each of them is a maximal normal subgroup of the previous one, implying that the quotient formed by any two consecutive subgroups of the series is simple. This series is unique in the sense that any two such sequences have the same length and provide (up to isomorphism) the same simple quotient groups, though possibly in a different order. As a result, any finite group can be constructed from simple finite groups by repeated application of an appropriate extension procedure. Various such procedures that allow to build new groups from given ones are

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known, the ones of interest here being central extensions and extensions by outer automorphisms, which we proceed to explain briefly.

The general idea of a group extension allows two different interpretations which, in a certain sense, are dual to each other; both of them are most easily formulated in terms of short exact sequences. Namely, given three groups A, B, C forming a short exact sequence

$$\{1\} \longrightarrow C \longrightarrow B \longrightarrow A \longrightarrow \{1\}, \tag{1}$$

one says that B is an *extension of C by A* or an *extension of A by C*, depending on the circumstances. These two options correspond to the idea that an extension of a group G by a group H is either a larger group $\hat{G}$ containing G as a normal subgroup with H appearing as the quotient group $\hat{G}/G$ or a larger group $\tilde{G}$ containing H as a normal subgroup with G appearing as the quotient group $\tilde{G}/H$. Both options play an extensive role in group theory, with a wide range of applications; in particular, extensions by outer automorphisms are of the first type and central extensions are of the second type.

More explicitly, a group $\hat{G}$ containing G as a normal subgroup is said to be an *extension of G by outer automorphisms* if the centralizer of G in $\hat{G}$ reduces to the center of G, that is, there is no element of $\hat{G}\backslash G$ commuting with all elements of G. The corresponding short exact sequence of groups has the form

$$\{1\} \longrightarrow G \longrightarrow \hat{G} \longrightarrow A \longrightarrow \{1\}. \tag{2}$$

To motivate this definition, note that as $\hat{G}$ is supposed to contain G as a normal subgroup, $\hat{G}$ acts on G by conjugation and this action provides a homomorphism of $\hat{G}$ into the group $Aut(G)$ of automorphisms of G which restricts to the standard homomorphism of G into the normal subgroup $Inn(G)$ of inner automorphisms of G. The extra hypothesis stated above asserts that these two homomorphisms have the same kernel and therefore induce an injective homomorphism of quotient groups, so that $A = \hat{G}/G$ can be considered as a subgroup of the group $Out(G) = Aut(G)/Inn(G)$ of outer automorphisms of G:

$$A \subset Out(G). \tag{3}$$

Similarly, a group $\tilde{G}$ is said to be a *central extension* or *covering group of G* if G can be written as the quotient of $\tilde{G}$ by an appropriate subgroup M of its center (note that M is automatically abelian and a normal subgroup of $\tilde{G}$). The corresponding short exact sequence of groups has the form

$$\{1\} \longrightarrow M \longrightarrow \tilde{G} \longrightarrow G \longrightarrow \{1\}. \tag{4}$$

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When the hypothesis that M is contained in the center of $\tilde{G}$,

$$M \subset Z(\tilde{G}) , \tag{5}$$

is replaced by the stronger hypothesis that M is contained in the intersection between the center and the derived subgroup of $\tilde{G}$,

$$M \subset Z(\tilde{G}) \cap \tilde{G}' , \tag{6}$$

one says that $\tilde{G}$ is a *proper central extension* or *proper covering group* of G. The appropriate concept for describing such extensions is the *Schur multiplier* of G, which we shall denote by $M(G)$: it can be defined abstractly as the second Hochschild cohomology group $H^2(G, \mathbb{C}^*)$ of G with coefficients in the multiplicative group $\mathbb{C}^*$ of nonzero complex numbers:

$$M(G) = H^2(G, \mathbb{C}^*) . \tag{7}$$

Following the notation of the ATLAS, we shall use the generic symbol $G.A$ to denote any group obtained as an extension of the group G by some group of outer automorphisms A and the generic symbol $M.G$ to denote any group obtained as a proper central extension of the group G by some abelian group M.

In what follows, we shall consider only finite groups – even though the concepts introduced so far make sense even for infinite groups. In particular, if G is a finite group, so are its automorphism group $Aut(G)$ and – as it turns out – its Schur multiplier $M(G)$, as well as their various subgroups and quotient groups. In this case, just as all groups A which arise for extensions $G.A$ of a given group G by outer automorphisms are subgroups of a largest one, namely $Out(G)$, one finds that all abelian groups M which arise for proper central extensions $M.G$ of a given group G are quotient groups of a largest one, namely $M(G)$. For the extensions themselves, however, similar statements can be obtained only under additional assumptions on G:

- If G is a finite group with trivial center, so $G \cong Inn(G)$, there is, up to isomorphism, a unique largest extension of G by outer automorphisms, namely the group $Aut(G)$ itself: it is maximal in the sense that any other extension $G.A$ of G by outer automorphisms can be obtained as a subgroup of $Aut(G)$, namely the inverse image of A under the natural projection of $Aut(G)$ onto $Out(G)$.

  When G has a non-trivial center $Z(G)$, the desired extension $G.A$ may not exist at all or there may exist several non-isomorphic versions of

it; this will in general depend on the specific nature of the action of A on $Z(G)$ (induced from the obvious action of $Aut(G)$ on G which leaves $Z(G)$ invariant and factors to an action of $Out(G)$ on $Z(G)$ since $Inn(G)$ acts trivially on $Z(G)$).

- If G is a perfect finite group, so $G' = G$, there is, up to isomorphism, a unique largest proper covering group $\tilde{G}$ of G, called the *universal covering group* of G: its center is just the Schur multiplier of G,

$$Z(\tilde{G}) = M(G) , \qquad (8)$$

and it is maximal in the sense that any other proper covering group $M.G$ of G can be obtained as a quotient group of $\tilde{G}$, namely the one by that subgroup $\tilde{M}$ of $M(G)$ for which $M = M(G)/\tilde{M}$. (We note in passing that for perfect groups G, proper covering groups and perfect covering groups are the same thing: if $\tilde{G}$ is a covering group of a perfect group G, the condition that $\tilde{G}$ be a proper covering is easily shown to be equivalent to the requirement that $\tilde{G}$ be a perfect group.)

When G has a non-trivial derived subgroup, the desired extension $M.G$ may not exist at all or there may exist several non-isomorphic versions of it. In the special case where M is the entire Schur multiplier $M(G)$, existence of a covering group of G with center isomorphic to $M(G)$ can still be guaranteed but uniqueness is lost, so it seems appropriate to replace the term "universal" by the term "maximal", that is, G will in general admit several non-isomorphic *maximal proper covering groups*. To handle this more general case, it is useful to introduce the notion of *isoclinism* − a more general equivalence relation between groups than that of isomorphism − since any two maximal proper covering groups of G turn out to be isoclinic. For more details, the reader is referred to the introduction to the ATLAS and to Ref. [14].

In particular, when starting out from a perfect simple finite group G or, more generally, from a perfect finite group with trivial center, it is clear how to construct the extensions $G.A$ and $M.G$, for any subgroup A of $Out(G)$ and any quotient group M of $M(G)$. Not nearly as clear is what is to be meant by the double extension $M.G.A$, since the two interpretations that come to mind are both plagued by ambiguities. In view of the relation $(G.A)' = G.A'$, $G.A$ is no longer perfect (except when A is perfect, which is rarely the case), while $M.G$ has center M, so the two candidate groups $M.(G.A)$ and $(M.G).A$ may not exist or may require additional data in order to be well-defined. At any rate, these groups $G.A$, $M.G$ and $M.G.A$ (inasfar as they exist) are collectively known as the *satellites* of G.

Fortunately, the problem of determining all irreducible representations with a given dimension of a finite simple group G and of its satellites does not require considering all satellite groups of G but only a certain subset of them. To explain why this is so and what is the subset that must be analyzed, we have to make a digression into the representation theory of finite groups.

Covering groups play a fundamental role in group theory because they allow to lift projective representations to linear representations – a procedure that, according to well-known theorems going back to Wigner and to Bargmann, is essential for applications to quantum theory. In fact, our desire to determine not only the linear codon representations of the simple finite groups but also the projective ones is motivated by the speculation that the origin of symmetry in the genetic code might in some way be related to quantum theory. Moreover, including the representations of the proper covering groups of simple groups corresponds to the strategy adopted in earlier investigations of the same subject in different contexts, primarily that of compact Lie groups and Lie algebras.

In standard textbooks on finite groups, the maximal proper covering groups mentioned above are also known as *representation groups,* or more precisely as representation groups over $\mathbb{C}$ since the notion can be extended to contemplate other ground fields than that of complex numbers. This term indicates the fact that passing from G to a representation group $\tilde{G}$ of G allows to include projective representations: not only every linear representation but also every projective representation of G is induced from a linear representation of $\tilde{G}$. As mentioned before, any two such representation groups of G are (at least) isoclinic. Moreover, isoclinic groups have essentially the same representations, since any irreducible representation of one of them acting in a complex vector space can, through multiplication of the representing operators by appropriate scalars in $\mathbb{C}^*$, be converted to an irreducible representation of the other acting in the same vector space; the same conversion rule applies to the irreducible characters. In particular, the answer to the question whether there exist irreducible projective representations of G of a given dimension does not depend on which representation group $\tilde{G}$ is chosen to lift them to irreducible linear representations. A further simplification arises due to the fact that every irreducible linear representation of $\tilde{G}$ maps the center of $\tilde{G}$ to a finite subgroup of $\mathbb{C}^*$ and that this is necessarily a cyclic group, so after dividing out the kernel of the original representation of $\tilde{G}$, it provides an irreducible linear representation of an appropriate proper covering group $M.G$ of G for which M is cyclic. It is in this way, namely through the characters for irreducible linear representations of G and of all proper covering groups of G of the form $\mathbf{Z}_n.G$, that the tables of the ATLAS provide

a complete classification of all irreducible representations – linear as well as projective – of G, for a large number of simple finite groups, as well as their extensions by cyclic groups of outer automorphisms.

There is also a close relationship between representations – linear as well as projective – of a finite group G and of any one of its extensions $G.A$ by outer automorphisms. The basic result here is a theorem due to Clifford (for linear representations) and to Mackey (for projective representations), based exclusively on the fact that G is a normal subgroup of $G.A$: it states that an irreducible representation of $G.A$ will under restriction to G decompose into the direct sum of a certain number, say r, of copies of a representation of G which in turn is the direct sum of a certain number, say s, of irreducible representations of G that are mutually inequivalent but conjugate (under an outer automorphism belonging to A) [13, p. 268]; in particular, all of these have the same dimension d, implying that the dimension of the original representation of $G.A$ is rsd. Conversely, this means that the irreducible representations of $G.A$ are obtained by fusing a certain number s of mutually inequivalent but conjugate irreducible representations of G, all of the same dimension d, into a single representation of G of dimension sd which, when repeated with a certain multiplicity r, can finally be extended to an irreducible representation of $G.A$ of dimension rsd. Of particular interest is the case $r = 1$, which can be divided into two subcases:

- $r = 1$ and $s = 1$.

  This means that when restricted to G, the given irreducible representation of $G.A$ remains irreducible, or conversely, that the given irreducible representation of G can be extended to an irreducible representation of $G.A$. This extension is not unique, but the various inequivalent extensions can be classified, namely by the group $Hom(A, \mathbb{C}^*)$ of homomorphisms of A into $\mathbb{C}^*$ [13, p. 295]. In the ATLAS, this situation is referred to as the "split case", in the sense that the extension splits the given representation of G into several inequivalent representations of $G.A$.

- $r = 1$ and $s > 1$.

  This means that when restricted to G, the given irreducible representation of $G.A$ splits into s mutually inequivalent but conjugate irreducible representations, or conversely, that s mutually inequivalent but conjugate irreducible representations of G fuse into a single irreducible representation of $G.A$. In the ATLAS, this situation is referred to as the "fusion case".

Moreover, there are various theorems that impose restrictions on the possible values of r, s and d, depending on the structure of A. One of these is the theorem of Conlon [13, p. 276] which states that if A is cyclic and $s = 1$, then $r = 1$ as well, so we are back to the split case. For simplicity, we shall in what follows refer to the case $r > 1$ and $s > 1$ as the "generalized fusion case".

For our investigation, the split case is of less interest than the fusion case since irreducible representations of $G.A$ that stay irreducible under restriction to G may already be detected among the irreducible representations of G of the same dimension, from which they are obtained by extension; moreover, the classification of all possible extensions is a simple exercise: given one of them, any other one is obtained by twisting with the corresponding homomorphism of A into $\mathbb{C}^*$. Considering the fusion case and generalized fusion case, we can affirm first of all that if several irreducible representations of G (equivalent or not) fuse in an extension $G.A$ of G by some group A of outer automorphisms, then they must already fuse, at least partially, in at least one extension $G.\mathbb{Z}_n$ of G by some cyclic subgroup $\mathbb{Z}_n$ of A, and this is a kind of information that can be read from the tables in the ATLAS. Finally, a strong restriction comes from the fact that we are interested in determining irreducible representations of dimension 64, so that all three numbers r, s and d must be powers of 2, and in the fusion case or generalized fusion case, $s \geqslant 2$, so the group G must admit at least two inequivalent but conjugate representations of dimension d that fuse in at least one extension $G.\mathbb{Z}_n$ of G by some outer automorphism of G of even order n, with d assuming one of the values 2, 4, 8, 16 or 32.

It seems worth noting that the two extension procedures described above have natural counterparts in the theory of compact Lie groups and Lie algebras. Roughly speaking, extension by outer automorphisms corresponds to passing from a connected Lie group to a Lie group with several connected components, whereas central extensions and covering groups play entirely analogous roles for finite groups and for compact Lie groups; in particular, the Schur multiplier and the universal covering group of finite group theory correspond to the fundamental group and to the universal covering group of Lie group theory, respectively. Indeed, it is well known that the representation theory (over $\mathbb{C}$) of a connected Lie group G is in one-to-one correspondence with that of its Lie algebra $\mathfrak{g}$ if and only if G is simply connected; if not, G must be replaced by its universal covering group $\tilde{G}$ in order for this correspondence to hold. Of course, what is lost in the transition from continuous groups to discrete groups are the topological aspects of notions such as "connected" or "simply connected". On the other hand, new light is

theory when one realizes that, even for compact Lie groups, such an object may not exist, or more precisely, may exist in several different variants (which are isoclinic but not isomorphic): this does indeed happen for compact Lie groups that are not connected, even when their Lie algebra is simple.

As an instructive example, consider the simple finite group A_n of all even permutations of n objects and the simple compact connected Lie group $SO(2n)$ of all rotations in $2n$-dimensional euclidean space, assuming that $n \geqslant 5$. Apart from the exceptional case of A_6, all of them have outer automorphism group isomorphic to $\mathbb{Z}_2$, and the corresponding extensions are well known: they are the full permutation group S_n and the full orthogonal group $O(2n)$, respectively. Similarly, apart from the exceptional cases of A_6 and A_7, the Schur multiplier of A_n and the fundamental group of $SO(2n)$ are also isomorphic to $\mathbb{Z}_2$: the corresponding universal covering groups are usually denoted by $\mathbb{Z}_2.A_n$ and by $Spin(2n)$, respectively. Finally, the groups S_n and $O(n)$ admit two non-isomorphic double coverings, usually denoted by $\mathbb{Z}_2^+.S_n$ and $\mathbb{Z}_2^-.S_n$ and by $Pin^+(2n)$ and $Pin^-(2n)$, respectively, providing an explicit example for the non-universality of the double covering in the sense described above. As mentioned before, however, this ambiguity is harmless for representation theory.

3   Determination of codon representations

The task of determining all codon representations of the simple finite groups and their satellites is feasible due to the existence of a classification of the simple finite groups – one of the great achievements of Mathematics in the 20th century. These groups can be divided into four types: the cyclic groups $\mathbb{Z}_p$ (of prime order p), the alternating groups A_n (for $n \geqslant 5$), the simple groups of Lie type and the sporadic groups. Among these, the cyclic groups can be discarded immediately because they are abelian and hence all their irreducible representations are one-dimensional. For the remaining cases, we use a series of general results on dimensions of irreducible representations that can be found in the literature, together with the character tables of the ATLAS or the GAP library.

The easiest case is that of the sporadic groups, whose character tables are completely listed in the ATLAS. The result is that only one sporadic group qualifies, namely the second Janko group J_2: it has two pseudo-real projective codon representations which under extension by the full outer automorphism group $\mathbb{Z}_2$ of J_2 fuse into one irreducible pseudo-real projective representation of $J_2.\mathbb{Z}_2$ of dimension 128.

The other types of simple finite groups are characterized by the fact that they form infinite families, parametrized either by one natural number n, as in the case of the alternating groups, or by one natural number q or two natural numbers n and q, with the restriction that q has to be a prime power, as in the 16 families of simple finite groups of Lie type, also known as the (untwisted or twisted) finite Chevalley groups. The basic strategy here is the same as for the classical series of simple Lie algebras or of basic classical Lie superalgebras: one recognizes that the "lowest" possible dimension d_1 for an irreducible representation grows with n and with q; a similar statement holds for the "second lowest" dimension d_2, the "third lowest" dimension d_3, etc..[1] More concretely, various authors have given exact formulas or at least lower bounds for d_1, d_2, d_3 ... as functions of n and of q (where applicable) which allow to impose upper bounds on n and on q (where applicable) in order for the relevant simple finite group or one of its satellites to have any (non-trivial) irreducible representation of dimension ≤ 64 at all. With these cutoffs, the remaining cases can be analyzed explicitly with the help of the character tables of the ATLAS or the GAP library, leading to the list of codon representations shown in Tables 1 and 2 as the final result, where the star indicates pairs of complex conjugate representations.

G	$\lvert G \rvert$	N_l	N_p
A_8	20.160	1	1
A_{10}	1.814.400	0	2
A_{14}	43.589.145.600	0	1
A_{15}	653.837.184.000	0	2*
A_{65}	65!/2	1	0
S_8	40.320	2	2
S_{13}	6.227.020.800	0	1
S_{14}	87.178.291.200	0	2*
S_{65}	65!	2	0

Table 1: Number N_l of linear and N_p of projective codon representations of simple finite groups and their satellites: alternating and symmetric groups

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[1] Note that there may very well exist several inequivalent representations of dimension d_1, d_2, d_3, ...; when this is the case, their number will be denoted by N_1, N_2, N_3, ...

G	$\lvert G\rvert$	$M(G)$	$Out(G)$	N_l	N_p
$A_2(4)=PSL_3(4)$	20.160	$\mathbf{Z}_3\times\mathbf{Z}_4\times\mathbf{Z}_4$	D_{12}	1	$1+2^*+2^*$
$B_2(3)=PSp_4(3)$	25.920	$\mathbf{Z}_2$	$\mathbf{Z}_2$	1	1
$^2B_2(8)=Sz(8)$	29.120	$\mathbf{Z}_2\times\mathbf{Z}_2$	$\mathbf{Z}_3$	1	1
$^2A_2(4)=PSU_3(4)$	62.400	$\{1\}$	$\mathbf{Z}_4$	1	
$A_1(64)=PSL_2(64)$	262.080	$\{1\}$	$\mathbf{Z}_6$	1	
J_2	604.800	$\mathbf{Z}_2$	$\mathbf{Z}_2$	0	2
$A_1(127)=PSL_2(127)$	1.024.128	$\mathbf{Z}_2$	$\mathbf{Z}_2$	0	2^*
$B_3(2)=PSp_6(2)$	1.451.520	$\mathbf{Z}_2$	$\{1\}$	0	2^*
$G_2(3)$	4.245.696	$\mathbf{Z}_3$	$\mathbf{Z}_2$	2^*	0

$G.A$	$\lvert G\rvert$			N_l	N_p
$G_2(2)=\,^2A_2(3).\mathbf{Z}_2$	6.048			1	
$A_2(4).(\mathbf{Z}_2)_1$	20.160	$(\mathbf{Z}_2)_1=Z(D_{12})$		2	2
$A_2(4).(\mathbf{Z}_2)_2$	20.160	$(\mathbf{Z}_2)_2\neq Z(D_{12})$		2	$2+2^*+2^*$
$A_2(4).(\mathbf{Z}_2)_3$	20.160	$(\mathbf{Z}_2)_3\neq Z(D_{12})$		2	$2+2^*+2^*$
$A_2(4).\mathbf{Z}_3$	20.160			$1+2^*$	0
$A_2(4).\mathbf{Z}_6$	20.160			$2+2^*+2^*$	0
$B_2(3).\mathbf{Z}_2$	25.920			2	2^*
$^2B_2(8).\mathbf{Z}_3$	29.120			$1+2^*$	0
$^2A_2(4).\mathbf{Z}_2$	62.400			2	
$^2A_2(4).\mathbf{Z}_4$	62.400			$2+2^*$	
$A_1(64).\mathbf{Z}_2$	262.080			2	
$A_1(64).\mathbf{Z}_3$	262.080			$1+2^*$	
$A_1(64).\mathbf{Z}_6$	262.080			$2+2^*+2^*$	
$G_2(3).\mathbf{Z}_2$	4.245.696			2^*+2^*	0

Table 2: Number N_l of linear and N_p of projective codon representations of
simple finite groups and their satellites:
Chevalley groups and sporadic groups

The remainder of this section will be devoted to presenting the relevant arguments in more detail.

3.1 Alternating groups

The representation theory of the alternating groups A_n and the symmetric groups S_n is presented in many textbooks, so we shall restrict ourselves to briefly commenting a few aspects that are relevant for our purposes. First of all, we observe that, according to the character tables of the ATLAS, the first three simple alternating groups A_5, A_6 and A_7 do not admit any codon representations, and the same holds for their extensions by outer automorphisms. Therefore, we may without loss of generality assume that $n \geqslant 8$; this guarantees that both the Schur multiplier and the outer automorphism group of A_n are equal to $\mathbf{Z}_2$:

$$M(A_n) \ = \ \mathbf{Z}_2 \quad , \quad \mathrm{Out}(A_n) \ = \ \mathbf{Z}_2 \qquad \text{for } n \geqslant 8 \ .$$

In particular, $S_n = A_n.\mathbf{Z}_2$ is the maximal extension of A_n by outer automorphisms. As we have seen before, the irreducible representations of A_n and of S_n are then related in one of two possible ways:

- The *split case* is that of an irreducible representation of A_n which extends to an irreducible representation of S_n (it then does so in precisely two inequivalent ways), or conversely, of an irreducible representation of S_n which under restriction to A_n remains irreducible. The relation is 1 : 2 (one irreducible representation of A_n splitting into two of S_n under extension).

- The *fusion case* is that of two irreducible representations of A_n which fuse to give a single irreducible representation of S_n, or conversely, of an irreducible representation of S_n which under restriction to A_n splits into two irreducible representations of A_n. Obviously, the relation here is 2 : 1 (two irreducible representations of A_n fusing into one of S_n under extension).

Exactly the same situation holds not only for linear representations but also for projective ones, which can be lifted to linear representations of the double covering groups $\mathbf{Z}_2.A_n$ and $\mathbf{Z}_2^{\pm}.S_n$ (recall that the latter comes in two isoclinic variants); this happens because $\mathbf{Z}_2.A_n$ turns out to be isomorphic to the derived subgroup of $\mathbf{Z}_2^{\pm}.S_n$, just as A_n is the derived subgroup of S_n [15, 16].

In order to exclude the existence of codon representations of A_n or S_n from a certain value of n onwards, it is convenient to distinguish between linear and proper projective representations.

Starting with the linear ones, we use a theorem that can be found in Ref. [17], according to which the three lowest dimensions of irreducible linear representations of S_n are given by

$$d_1(S_n) = n - 1 ,$$
$$d_2(S_n) = \tfrac{1}{2} n(n - 3) ,$$
$$d_3(S_n) = \tfrac{1}{2} (n - 1)(n - 2) ,$$

provided that $n \geqslant 14$. The only number among these that can take the value 64 or 128 is $d_1(S_n)$, and since the irreducible representation of S_{129} of dimension 128 remains irreducible when restricted to A_{129}, we conclude that there is no linear codon representation of A_n or S_n when $n \geqslant 14$ except for $n = 65$: this is the case where the irreducible linear representation of lowest possible dimension provides a real codon representation of A_{65} which upon extension by its unique outer involution splits into two real codon representations of S_{65}. Of course, this is an enormous group, as can be seen by comparing its order

$$2^{62} \cdot 3^{30} \cdot 5^{15} \cdot 7^{10} \cdot 11^5 \cdot 13^5 \cdot 17^3 \cdot 19^3 \cdot 23^2 \cdot 29^2 \cdot 31^2 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot 59 \cdot 61 ,$$

which is a number of order $\sim 10^{93}$, to the order

$$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 ,$$

of the monster group, the largest of the sporadic groups, which is "only" a number of order $\sim 10^{55}$.

Turning to the projective representations, we use a theorem that can be found in Ref. [18], according to which the dimension of any proper projective representation of S_n and of A_n is divisible by a certain power of 2, namely by $2^{[(n-s)/2]}$ and by $2^{[(n-s-1)/2]}$, respectively, where the square brackets denote taking the integral part and s is the number of terms in the decomposition of n into powers of 2 ($n = 2^{w_1} + \ldots + 2^{w_s}$), or in other words, the number of digits that, in the binary representation of n, are equal to 1. Now it is easily seen that $n - s$ is a monotonically non-decreasing function of n, and this implies that there is no projective codon representation of A_n or of S_n when $n \geqslant 16$.

To show that $n - s$ is indeed a monotonically non-decreasing function of n, suppose that n lies in the interval between, say, the

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k^{th} and the $(k+1)^{\text{st}}$ power of 2 $(2^k \leqslant n < 2^{k+1})$, so that its binary representation consists of a digit 1 followed by k digits that are equal to 0 or 1. Assuming first that n is not equal to $2^{k+1} - 1$. and hence $n+1$ lies in the same interval, we conclude that when passing from n to $n+1$, there are two possibilities: if n is even, its last digit will be converted from 0 to 1, so s increases by 1 and $n - s$ remains constant, while if n is odd, its last digit will be converted from 1 to 0 and some previous digit (the last one among those that are equal to 0) will be converted from 0 to 1, so s remains constant and $n - s$ increases by 1. Finally, when $n = 2^{k+1} - 1$, passing from n to $n + 1$ makes s drop from $k + 1$ to 1 and hence $n - s$ undergoes a sudden increase from $2^{k+1} - k - 2$ to $2^{k+1} - 1$.

With these cutoffs, it is possible to read off the remaining information from the character tables of the ATLAS or, in the cases $n = 14$ and $n = 15$, from general theorems on the basic representations of the alternating groups [16, Theorems 6.8 and 6.9, pp. 70-73], to conclude the following.

- A_8 has one real linear and one pseudo-real projective codon representation, both of which are split: the former extends to two real linear and the latter to two pseudo-real projective codon representations of S_8.

- A_{10} has two real projective codon representations which fuse into a single irreducible real projective representation of S_{10} of dimension 128.

- A_{13} has two irreducible pseudo-real projective representations of dimension 32 which fuse into a single pseudo-real projective codon representation of S_{13}.

- A_{14} has one pseudo-real projective codon representation which is split: it extends to a pair of complex conjugate codon representations of S_{14}.

- A_{15} has a pair of complex conjugate projective codon representations which fuse into a single irreducible real projective representation of S_{15} of dimension 128.

This result is schematically represented in Table 1. It coincides essentially with the list presented by the authors of Ref. [19], except for the projective representations of A_8, S_8 and A_{10} which are neither ordinary nor spin representations and, probably for this reason, do not appear in Ref. [19].

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3.2 Simple finite groups of Lie type

The simple finite groups of Lie type bear this name because they are constructed as groups of automorphisms of simple Lie algebras over finite fields; they are also widely known as *finite Chevalley groups*. Their definition is based on the fact that every simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ has a *Chevalley basis* in which all structure constants are integers, so that any such Lie algebra admits a so-called $\mathbb{Z}$-form $\mathfrak{g}_{\mathbb{Z}}$ and hence, for any field $\mathbb{F}$, a sibling $\mathfrak{g}_{\mathbb{F}}$ – a simple Lie algebra over $\mathbb{F}$ that, with respect to the Chevalley basis, has the same integer structure constants as the original simple Lie algebra over $\mathbb{C}$. This fact explains why the simple finite groups of Lie type are classified in terms of the Cartan labels A_n $(n \geqslant 1)$, B_n $(n \geqslant 2)$, C_n $(n \geqslant 3)$, D_n $(n \geqslant 4)$ and E_6, E_7, E_8, F_4, G_2 for the classical and exceptional simple Lie algebras over $\mathbb{C}$, respectively, together with a prime power $q = p^f$ to characterize the finite Galois field $\mathbb{F}_q$ used in their definition. (The restrictions on the values of n mentioned above are the standard ones, imposed to exclude Lie algebras that are not simple and to avoid repetitions, since D_1 is abelian, D_2 is not simple, $D_3 \cong A_3$, $C_2 \cong B_2$ and $C_1 \cong B_1 \cong A_1$.) Among them are six families of *classical finite groups* that can be realized as matrix groups with matrix entries from the field $\mathbb{F}_q$:

$$A_n(q) \cong PSL_{n+1}(q) \ (n \geqslant 1) ,$$
$$B_n(q) \cong PSO_{2n+1}(q)' \ (n \geqslant 2) ,$$
$$C_n(q) \cong PSp_{2n}(q) \ (n \geqslant 3) ,$$
$$D_n(q) \cong PSO_{2n}^+(q)' \ (n \geqslant 4) ,$$
$${}^2A_n(q) \cong PSU_{n+1}(q) \ (n \geqslant 2) ,$$
$${}^2D_n(q) \cong PSO_{2n}^-(q)' \ (n \geqslant 4) .$$

(Here, as usual, the prime denotes the derived subgroup.) The remaining ones can be arranged into ten series:

$$E_6(q) ,$$
$$E_7(q) ,$$
$$E_8(q) ,$$
$$F_4(q) ,$$
$$G_2(q) ,$$
$${}^3D_4(q) ,$$
$${}^2E_6(q) ,$$

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$$^2B_2(q) \quad (q = 2^{2l+1}) \, ,$$
$$^2F_4(q) \quad (q = 2^{2l+1}) \, ,$$
$$^2G_2(q) \quad (q = 3^{2l+1}) \, .$$

Further restrictions must be imposed on the range of q in order to exclude groups that are not simple and to avoid repetitions; these are the following:

- $A_1(q)$: $q \geqslant 7$ and $q \neq 9$

  Indeed, $A_1(2) \cong S_3$ and $A_1(3) \cong A_4$ are solvable, while $A_1(4) \cong A_5$, $A_1(5) \cong A_5$ and $A_1(9) \cong A_6$ already occur among the alternating groups.

- $A_n(q)$ with $n = 2$ or $n = 3$: $q \geqslant 3$

  $A_2(2) \cong A_1(7)$ already appears elsewhere in the classification, while $A_3(2) \cong A_8$ already occurs among the alternating groups.

- $B_2(q)$: $q \geqslant 3$

  $B_2(2) \cong S_6$ is not simple and its derived subgroup $B_2(2)' \cong A_6$ (of index 2) already occurs among the alternating groups.

- $C_n(q)$: q odd

  If q is even, hence a power of 2, the group $C_n(q) \cong B_n(q)$ already appears elsewhere in the classification.

- $G_2(q)$: $q \geqslant 3$

  $G_2(2)$ is not simple and its derived subgroup $G_2(2)' \cong {}^2A_2(3)$ (of index 2) already appears elsewhere in the classification.

- $^2A_n(q)$ with $n = 2$ or $n = 3$: $q \geqslant 3$

  $^2A_2(2)$ is not simple while $^2A_3(2) \cong B_2(3)$ already appears elsewhere in the classification.

- $^2B_2(q)$: $q \geqslant 8$

  $^2B_2(2)$ is not simple.

- $^2F_4(q)$: $q \geqslant 8$

  $^2F_4(2)$ is not simple, but its derived subgroup (of index 2) is a simple group $^2F_4(2)' \cong T$ known as the *Tits group*, which does not appear anywhere else in the classification.

- $^2G_2(q)$: $q \geqslant 27$

  $^2G_2(3)$ is not simple and its derived subgroup $^2G_2(3)' \cong A_1(8)$ (of index 3) already appears elsewhere in the classification.

For the six families of classical finite groups G, exact formulas for the lowest dimension $d_1(G)$ and for the number $N_1(G)$ of irreducible representations (linear as well as projective) of dimension $d_1(G)$ have been given in Ref. [20]. For the other ten series of finite Chevalley groups G, the literature contains lower bounds $b(G)$ for $d_1(G)$ which were originally derived in Ref. [21] and later improved in Ref. [22]. For the convenience of the reader, these results are reproduced in Tables 3-5.

In what follows, we shall first of all apply the formulas in Tables 3-5 to determine the groups G for which $d_1(G) \leqslant 64$, since it is clear that when $d_1(G) > 64$, neither G itself nor any of its extensions by outer automorphisms has a codon representation. Given the fact that $d_1(G)$ is a polynomial in q whose exponents are affine functions of n, an inequality such as $d_1(G) \leqslant N$, where N is any given number, imposes upper bounds on q and n, so the set of candidate groups is finite. An even sharper requirement is that G should have irreducible representations of dimension 2^k where k takes one of the values $1, 2, 3, 4, 5, 6$, since as shown in the previous section, this is a necessary condition for G to admit some extension by outer automorphisms that has a codon representation. As we shall see, this greatly reduces the number of cases that must be analyzed in detail. In fact, it turns out that the question can to a large extent be settled by consulting the character tables of the ATLAS, while in the few cases not covered by the ATLAS, the necessary information can be extracted from Table 6 which, for the groups of interest, gives the lowest three dimensions for irreducible representations (rather than just the lowest one). These numbers are derived from the results of Ref. [20], either directly (cf. Table IV and Table V) or indirectly from the statement (cf. the beginning of Sect. 5 and Theorem 5.2) that for $n \geqslant 2$ and odd q, the first five irreducible representations of $C_n(q) = PSp_{2n}(q)$ have dimension $(q^n - 1)/2$ (two representations), $(q^n + 1)/2$ (two representations) and $q(q^{n-1} - 1)(q^n - 1)/2(q + 1)$ (one representation) while all others must have dimension $\geqslant (q^{2n} - 1)/2(q + 1)$.

With these general remarks out of the way, we proceed to the analysis of the individual cases. We start by investigating which of the Chevalley groups admit codon representations or, more generally, irreducible representations of dimension 2^k where k takes one of the values $1, 2, 3, 4, 5, 6$. In a second step, we shall analyse the fate of these representations under extension by outer automorphisms.

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G	$d_1(G)$	$N_1(G)$	Conditions
$A_1(q)$	$q-1$	$q/2$	q even, $q \geqslant 8$
	$\frac{1}{2}(q-1)$	2	q odd, $q \neq 9$
	3	4	for $q = 9$
$A_n(q)$ $(n \geqslant 2)$	$\dfrac{q(q^n-1)}{q-1}$	1	$q \neq 2,3$ if $n=3$ $q \neq 2,4$ if $n=2$
	26	2	for $n=3$, $q=3$
	6	6	for $n=2$, $q=4$
$B_2(q)$	$\frac{1}{2}q(q-1)^2$	1	q even, $q \geqslant 4$
	$\frac{1}{2}(q^2-1)$	2	q odd
$B_n(q)$ $(n \geqslant 3)$	$\dfrac{q(q^{n-1}-1)(q^n-1)}{2(q+1)}$	1	q even, with $q \geqslant 4$ if $n=3$
	8	1	for $n=3$, $q=2$
$B_n(q)$ $(n \geqslant 3)$	$\dfrac{q^{2n}-1}{q^2-1}$	1	q odd, $q \geqslant 5$
	$\dfrac{q(q^{n-1}-1)(q^n-1)}{q^2-1}$	1	for $n \geqslant 4$, $q=3$
	27	2	for $n=3$, $q=3$
$C_n(q)$ $(n \geqslant 3)$	$\frac{1}{2}(q^n-1)$	2	q odd
$D_n(q)$ $(n \geqslant 4)$	$\dfrac{q(q^{n-2}+1)(q^n-1)}{q^2-1}$	1	$q \geqslant 4$
	$\dfrac{(q^{n-1}-1)(q^n-1)}{q^2-1}$	2	for $n \geqslant 5$, $q=3$
		1	for $n \geqslant 5$, $q=2$
	260	6	for $n=4$, $q=3$
	8	3	for $n=4$, $q=2$

Table 3: Lowest dimensions and numbers for irreducible representations of the untwisted classical finite Chevalley groups (cf. Table II of Ref. [20])

G	$d_1(G)$	$N_1(G)$	Conditions
$^2A_n(q)$ $(n \geqslant 2)$	$\dfrac{q(q^n - 1)}{q+1}$	1	n even, with $q \geqslant 3$ if $n = 2$
	$\dfrac{q^{n+1} - 1}{q+1}$	q	n odd, with $q \geqslant 4$ if $n = 3$
	6	4	for $n = 3,\, q = 3$
$^2D_n(q)$ $(n \geqslant 4)$	$\dfrac{q(q^{n-2} - 1)(q^n + 1)}{q^2 - 1}$	1	

Table 4: Lowest dimensions and numbers for irreducible representations of the twisted classical finite Chevalley groups (cf. Table II of Ref. [20])

G	$b(G)$	Conditions
$E_6(q)$	$q^9(q^2 - 1)$	
$E_7(q)$	$q^{15}(q^2 - 1)$	
$E_8(q)$	$q^{27}(q^2 - 1)$	
$F_4(q)$	$\frac{1}{2}q^7(q^3 - 1)(q - 1)$ $q^6(q^2 - 1)$	q even, $q \geqslant 4$ q odd
$G_2(q)$	$q(q^2 - 1)$	$q \geqslant 5$
$^2E_6(q)$	$q^9(q^2 - 1)$	
$^3D_4(q)$	$q^3(q^2 - 1)$	
$^2B_2(q)$	$\sqrt{q/2}\,(q - 1)$	$q = 2^{2l+1}, q \geqslant 32$
$^2F_4(q)$	$\sqrt{q/2}\,q^4(q - 1)$	$q = 2^{2l+1}, q \geqslant 8$
$^2G_2(q)$	$q(q - 1)$	$q = 3^{2l+1}, q \geqslant 27$

Table 5: Lower bounds for the dimensions of irreducible representations of the non-classical finite Chevalley groups (cf. Table I of Ref. [20])

G	$|G|$	$d_1(G)$	$d_2(G)$	$d_3(G)$
$A_5(2) = PSL_6(2)$	20.158.709.760	62	217	588
$B_2(7) = PSp_4(7)$	138.297.600	24	25	126
$B_2(9) = PSp_4(9)$	1.721.606.400	40	41	288
$B_2(11) = PSp_4(11)$	12.860.654.400	60	61	550
$C_3(5) = PSp_6(5)$	228.501.000.000.000	62	63	1240
$C_4(3) = PSp_8(3)$	65.784.756.654.489.600	40	41	780
$^2A_3(4) = PSU_4(4)$	1.018.368.000	51	52	221
$^2A_4(3) = PSU_5(3)$	258.190.571.520	60	61	549
$^2A_6(2) = PSU_7(2)$	227.787.103.272.960	42	43	860

Table 6: Lowest three dimensions for irreducible representations of some classical finite Chevalley groups

- $A_1(q)$ $(q \geqslant 7,\ q \neq 9)$:

  According to Table 3, we have $d_1(A_1(q)) \leqslant 64$ if and only if $q \leqslant 64$ when q is even or $q \leqslant 129$ when q is odd. Further information can be extracted from the generic character tables for $A_1(q)$ found, e.g., in Refs [23, 24].

  - If q is even, hence $q = 2^f$, $A_1(q)$ has trivial Schur multiplier and outer automorphism group $\mathbb{Z}_f$ and has precisely

    * $q/2$ irreducible representations of dimension $q - 1$,
    * one irreducible representation of dimension q,
    * $q/2 - 1$ irreducible representations of dimension $q + 1$.

    This shows, first of all, that there is only one even value of q, namely $q = 64$, for which $A_1(q)$ admits a codon representation. For other values of q, we still have to consider the possibility of obtaining a codon representation of some extension of $A_1(q)$ by outer automorphisms by fusing a certain number of irreducible representations of dimension r where r is either $q-1$ or q or $q+1$. Of course, this forces r to be one of the numbers 2, 4, 8, 16 or 32, so the possibilities $r = q - 1$ and $r = q + 1$ are ruled out,

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and given the fact that the outer automorphism group as well as all of its subgroups are cyclic, we conclude from Conlon's theorem quoted above that the desired result can only be obtained by fusing several inequivalent representations of dimension q with q taking one of the values 8, 16 or 32. But this is impossible since there is just one such representation.

- If q is odd, with $q = p^f$ where p is an odd prime, $A_1(q)$ has Schur multiplier $\mathbf{Z}_2$ and outer automorphism group $\mathbf{Z}_2 \times \mathbf{Z}_f$ and has precisely

  * two irreducible representations of dimension $(q - 1)/2$,
  * two irreducible representations of dimension $(q + 1)/2$,
  * $(q - 1)/2$ irreducible representations of dimension $q - 1$,
  * one irreducible representation of dimension q,
  * $(q - 3)/2$ irreducible representations of dimension $q + 1$.

This shows, first of all, that there is only one odd value of q, namely $q = 127$, for which $A_1(q)$ admits codon representations. (Indeed, other values for q such as 129, 65 or 63 are forbidden by the condition that q should be a prime power.) For other values of q, we still have to consider the possibility of obtaining a codon representation of some extension of $A_1(q)$ by outer automorphisms by fusing a certain number of irreducible representations of dimension r where r is either $(q - 1)/2$ or $(q + 1)/2$ or $q - 1$ or q or $q + 1$. Of course, this forces r to be one of the numbers 2, 4, 8, 16 or 32, so the possibility $r = q$ is ruled out, and given the fact that q should be a prime power such that $q \geqslant 7$ and $q \neq 9$, there are precisely three solutions, namely $q = 7$, $q = 17$ and $q = 31$. All these values of q are prime numbers and therefore the corresponding outer automorphism group is the cyclic group $\mathbf{Z}_2$, which according to Conlon's theorem quoted above implies that the desired result can only be obtained by fusing two inequivalent irreducible representations of dimension 32. Now the character tables of the ATLAS show that $A_1(7)$ and $A_1(17)$ have no irreducible representations of this dimension, whereas $A_1(31)$ has a lot of them (7 linear ones and another 7 projective ones), but all of them are split under the extension to $A_1(31) . \mathbf{Z}_2$.

- $A_n(q)$ $(n \geq 2)$:

  - $n = 2$, $q \geqslant 3$:

    According to Table 3, we have $d_1(A_2(q)) \leqslant 64$ if and only if $q \leqslant 7$. According to the ATLAS, $A_2(7)$ and $A_2(5)$ have no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$, $A_2(4)$ has one linear and five projective codon representations, apart from two irreducible projective representations of dimension 8 which under extension by any cyclic subgroup of its full outer automorphism group D_{12} either split or fuse into one irreducible projective representation of dimension 16, and finally $A_2(3)$ (whose Schur multiplier is trivial) has four irreducible representations of dimension 16 which under extension by its full outer automorphism group $\mathbf{Z}_2$ fuse into two irreducible representations of dimension 32.

  - $n = 3$, $q \geqslant 3$:

    According to Table 3, we have $d_1(A_3(q)) \leqslant 64$ if and only if $q = 3$, and according to the ATLAS, $A_3(3)$ has no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$.

  - $n = 4$:

    According to Table 3, we have $d_1(A_4(q)) \leqslant 64$ if and only if $q = 2$, and according to the ATLAS, $A_4(2)$ has no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$.

  - $n = 5$:

    According to Table 3, we have $d_1(A_5(q)) \leqslant 64$ if and only if $q = 2$, and according to Table 6, $A_5(2)$ has no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$.

  - $n \geqslant 6$:

    According to Table 3, we have $d_1(A_n(q)) > 64$ when $n \geqslant 6$, for all possible values of q.

- $B_2(q)$ $(q \geqslant 3)$:

  According to Table 3, we have $d_1(B_2(q)) \leqslant 64$ if and only if $q = 4$ when q is even or $q \leqslant 11$ when q is odd. Moreover, according to Table 6, $B_2(11)$, $B_2(9)$ and $B_2(7)$ have no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$, and according to the ATLAS, the same is true for $B_2(5)$ and $B_2(4)$, while $B_2(3)$ has one linear and one projective codon representation, apart from two irreducible projective representations of dimension 4 which under extension by its full outer

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automorphism group $\mathbf{Z}_2$ fuse into a single irreducible projective representation of dimension 8.

- $B_n(q)$ $(n \geqslant 3, q$ even$)$:

  - $n = 3$:

    According to Table 3, we have $d_1(B_3(q)) \leqslant 64$ if and only if $q = 2$, and according to the ATLAS, $B_3(2)$ has two projective codon representations, apart from a single irreducible projective representation of dimension 8.

  - $n = 4$:

    According to Table 3, we have $d_1(B_4(q)) \leqslant 64$ if and only if $q = 2$, and according to the ATLAS, $B_4(2)$ has no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$.

  - $n \geqslant 5$:

    According to Table 3, we have $d_1(B_n(q)) > 64$ when $n \geqslant 5$, for all possible even values of q.

- $B_n(q)$ $(n \geqslant 3, q$ odd$)$:

  - $n = 3$:

    According to Table 3, we have $d_1(B_3(q)) \leqslant 64$ if and only if $q = 3$, and according to the ATLAS, $B_3(3)$ has no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$.

  - $n \geqslant 4$:

    According to Table 3, we have $d_1(B_n(q)) > 64$ when $n \geqslant 4$, for all possible odd values of q.

- $C_n(q)$ $(n \geqslant 3, q$ odd$)$:

  - $n = 3$:

    According to Table 3, we have $d_1(C_3(q)) \leqslant 64$ if and only if $q \leqslant 5$. Moreover, according to Table 6, $C_3(5)$ has no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$, and according to the ATLAS, the same is true for $C_3(3)$.

  - $n = 4$:

    According to Table 3, we have $d_1(C_4(q)) \leqslant 64$ if and only if $q = 3$, and according to Table 6, $C_4(3)$ has no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$.

- $n \geqslant 5$:

    According to Table 3, we have $d_1(C_n(q)) > 64$ when $n \geqslant 5$, for all possible odd values of q.

- $D_n(q)$ $(n \geqslant 4)$:

    - $n = 4$:

        According to Table 3, we have $d_1(D_4(q)) \leqslant 64$ if and only if $q = 2$, and according to the ATLAS, $D_4(2)$ has no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$, except for a single irreducible projective representation of dimension 8.

    - $n \geqslant 5$:

        According to Table 3, we have $d_1(D_n(q)) > 64$ when $n \geqslant 5$, for all possible values of q.

- ${}^2A_n(q)$ $(n \geqslant 2)$:

    - $n = 2, q \geqslant 3$:

        According to Table 4, we have $d_1({}^2A_2(q)) \leqslant 64$ if and only if $q \leqslant 8$. According to the ATLAS, ${}^2A_2(8)$, ${}^2A_2(7)$ and ${}^2A_2(5)$ have no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$, ${}^2A_2(4)$ (whose Schur multiplier is trivial) has one codon representation, and finally ${}^2A_2(3)$ (whose Schur multiplier is trivial) has two irreducible representations of dimension 32 which under the extension by its full outer automorphism group $\mathbf{Z}_2$ fuse into a single codon representation of ${}^2A_2(3).\mathbf{Z}_2$.

    - $n = 3, q \geqslant 3$:

        According to Table 4, we have $d_1({}^2A_3(q)) \leqslant 64$ if and only if $q \leqslant 4$. Moreover, according to Table 6, ${}^2A_3(4)$ has no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$, and according to the ATLAS, the same is true for ${}^2A_3(3)$.

    - $n = 4$:

        According to Table 4, we have $d_1({}^2A_4(q)) \leqslant 64$ if and only if $q \leqslant 3$. Moreover, according to Table 6, ${}^2A_4(3)$ has no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$, and according to the ATLAS, the same is true for ${}^2A_4(2)$.

    - $n = 5$:

        According to Table 4, we have $d_1({}^2A_5(q)) \leqslant 64$ if and only if $q = 2$, and according to the ATLAS, ${}^2A_5(2)$ has no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$.

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- $n = 6$:

  According to Table 4, we have $d_1(^2A_6(q)) \leqslant 64$ if and only if $q = 2$, and according to Table 6, $^2A_6(2)$ has no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$.

- $n \geqslant 7$:

  According to Table 4, we have $d_1(^2A_n(q)) > 64$ when $n \geqslant 7$, for all possible values of q.

- $^2D_n(q)$ $(n \geqslant 4)$:

  - $n = 4$:

    According to Table 4, we have $d_1(^2D_4(q)) \leqslant 64$ if and only if $q = 2$, and according to the ATLAS, $^2D_4(2)$ has no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$.

  - $n \geqslant 5$:

    According to Table 4, we have $d_1(^2D_n(q)) > 64$ when $n \geqslant 5$, for all possible values of q.

- $E_n(q)$ $(n = 6, 7, 8)$, $^2E_6(q)$:

  According to Table 5, these groups have no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$.

- $F_4(q)$:

  According to Table 5, we have $d_1(F_4(q)) > 64$ when $q \geqslant 3$, and according to the ATLAS, $F_4(2)$ has no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$.

- $G_2(q)$ $(q \geqslant 3)$:

  According to Table 5, we have $d_1(G_2(q)) > 64$ when $q \geqslant 5$, and according to the ATLAS, $G_2(4)$ has no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$, while $G_2(3)$ has two linear codon representations.

- $^3D_4(q)$:

  According to Table 5, we have $d_1(^3D_4(q)) > 64$ when $q \geqslant 3$, and according to the ATLAS, $^3D_4(2)$ has no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$.

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- $^2B_2(q)$ $(q = 2^{2l+1}$ with $l \geqslant 1)$:

  According to Table 5, we have $d_1(^2B_2(q)) > 64$ when $q \geqslant 32$, and according to the ATLAS, $^2B_2(8)$ has one linear and one projective codon representation.

- $^2F_4(q)$ $(q = 2^{2l+1}$ with $l \geqslant 1)$:

  According to Table 5, we have $d_1(^2F_4(q)) > 64$ when $q \geqslant 8$. Moreover, according to the ATLAS, $F_4(2)$ the Tits group $^2F_4(2)'$ has no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$.

- $^2G_2(q)$ $(q = 3^{2l+1}$ with $l \geqslant 1)$:

  According to Table 5, these groups have no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$.

Passing to the second step, we argue as follows.[2]

- $G = A_1(64) = PSL_2(64)$:

  In classical notation, G is the projective special linear group $PSL_2(64)$ in 2 dimensions over the field $\mathbb{F}_{64}$; it has trivial Schur multiplier and outer automorphism group $\mathbb{Z}_6$. (The first of these two statements implies that G is in fact identical with the classical matrix group $SL_2(64)$.) Its codon representation identified above turns out to be real and, as a consequence of Conlon's theorem quoted above, must be split under extension by any cyclic subgroup of $Out(G)$: more precisely, it extends to two real codon representations of $G.\mathbb{Z}_2$, to one real plus a pair of complex conjugate codon representations of $G.\mathbb{Z}_3$ and to two real plus two pairs of complex conjugate codon representations of $G.\mathbb{Z}_6$.

- $G = A_1(127) = PSL_2(127)$:

  In classical notation, G is the projective special linear group $PSL_2(127)$ in 2 dimensions over the field $\mathbb{F}_{127}$; it has Schur multiplier $\mathbb{Z}_2$ and outer automorphism group $\mathbb{Z}_2$. (The first of these statements means that G is the quotient of the classical matrix group $SL_2(127)$, which is also its universal covering group, by its center $\mathbb{Z}_2$.) Its codon representations identified above turn out to form a pair of complex conjugate projective representations which under extension by the full outer automorphism group $\mathbb{Z}_2$ fuse into a single irreducible real projective representation of $G.\mathbb{Z}_2$ of dimension 128: this follows from the fact that the outer

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[2]It should be noted that for groups with trivial Schur multiplier, there is no distinction between linear and projective representations.

automorphism of G, which can be represented as conjugation by the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 126 \end{pmatrix},$$

permutes the characters of these two representations, as can be inferred from the generic character tables of Ref. [25].

- $G = A_2(4)$:

  In classical notation, G is the projective special linear group $PSL_3(4)$ in 3 dimensions over the field $\mathbb{F}_4$. This group is by far the most complicated one for our analysis because it has a large Schur multiplier $\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ as well as a large outer automorphism group D_{12}. The necessary information can be read from the character tables of the ATLAS, but this requires, as a prerequisite, understanding what are the cyclic quotient groups M of $M(G)$, what are the cyclic subgroups A of $Out(G)$ and finally which of the bicyclic extensions $M.G.A$ are well-defined and which are not.

  - Cyclic quotient groups of $M(G)$:

    To find the cyclic quotient groups $M = M(G)/\tilde{M}$ of $M(G)$, we need some information about the order of the elements of $M(G)$ and about the subgroups $\tilde{M}$ of $M(G)$ they generate. Among the 48 elements of $M(G)$, there are exactly three of order 2, two of order 3, twelve of order 4, six of order 6 and twenty-four of order 12: they generate cyclic subgroups of $M(G)$ as follows: three of order 2, one of order 3, six of order 4, three of order 6 and six of order 12. Thus if $\tilde{M}$ has order 2 or 3 or 6, then M will be isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_4 \times \mathbb{Z}_4$ or $\mathbb{Z}_4 \times \mathbb{Z}_2$, respectively, and hence will not be cyclic. Similarly, if $\tilde{M}$ has order 4 (or 12) and is generated by two elements of order 2 (or one element of order 3 together with two elements of order 2), then $\tilde{M} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ (or $\tilde{M} \cong \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$), so M will be isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ (or $\mathbb{Z}_2 \times \mathbb{Z}_2$) and hence will not be cyclic. On the other hand, if $\tilde{M}$ has order 4 (or 12) and is generated by one element of order 4 (or one element of order 3 together with one element of order 4), then $\tilde{M} \cong \mathbb{Z}_4$ (or $\tilde{M} \cong \mathbb{Z}_3 \times \mathbb{Z}_4$), so M will be isomorphic to the cyclic group $\mathbb{Z}_3 \times \mathbb{Z}_4 \cong \mathbb{Z}_{12}$ (or $\mathbb{Z}_4$): there are precisely six different subgroups $\tilde{M}$ and corresponding quotient groups M of $M(G)$ of this kind, which can be divided into three classes of two subgroups or quotient groups each, enumerated by the element of order 2 in $M(G)$ that belongs to $\tilde{M}$. Similarly,

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if $\tilde{M}$ has order 8 (or 24), it must be generated by one element of order 4 together with one independent element of order 2 (or one element of order 3 together with one element of order 4 and one independent element of order 2), so M will be isomorphic to the cyclic group $\mathbf{Z}_3 \times \mathbf{Z}_2 \cong \mathbf{Z}_6$ (or $\mathbf{Z}_2$): there are precisely three different subgroups $\tilde{M}$ and corresponding quotient groups M of $M(G)$ of this kind, which can enumerated by the element of order 2 in $M(G)$ that lies outside $\tilde{M}$. Finally, if $\tilde{M}$ has order 16, then $\tilde{M} \cong \mathbf{Z}_4 \times \mathbf{Z}_4$ and $M = \mathbf{Z}_3$. In the notation employed in the ATLAS, the resulting cyclic quotient groups are denoted by 2, $2'$, $2''$, 3, 4_1, $4_1'$, $4_1''$, 4_2, $4_2'$, $4_2''$, 6, $6'$, $6''$ and 12_1, $12_1'$, $12_1''$, 12_2, $12_2'$, $12_2''$, respectively.

– Cyclic subgroups of $Out(G)$:

The geometrically most intuitive among the various descriptions of $Out(G)$ is as the symmetry group D_{12} of the regular hexagon, formed by six rotations by angles that are multiples of $60°$, three reflections in axes joining the centers of opposite edges and three reflections in axes joining opposite corners. The cyclic subgroups of this group are easily seen to be the three subgroups $\mathbf{Z}_6$, $\mathbf{Z}_3$ and $Z(D_{12})$ consisting of the rotations by angles that are multiples of $60°$, $120°$ and $180°$, respectively, together with the six subgroups generated by the reflections in the six axes mentioned above; in the notation employed in the ATLAS, they are denoted by 6, 3, 2_1 and 2_2, $2_2'$, $2_2''$, 2_3, $2_3'$, $2_3''$, respectively. This notation indicates the invariant differences that exist between the seven subgroups of order 2: 2_1 is the center $Z(D_{12})$ of D_{12}, while the other six non-central subgroups of order 2 fall into two conjugacy classes, each consisting of three subgroups.

– Bicyclic extensions $M.G.A$ of G:

According to the ATLAS, the bicyclic extension $M.G.A$ of G exists

   ∗ either when M is trivial or isomorphic to $\mathbf{Z}_3$: in this case, A is allowed to be an arbitrary cyclic subgroup of $Out(G)$,

   ∗ or when A is trivial or isomorphic to $\mathbf{Z}_2$; in this case, M is allowed to be an arbitrary cyclic quotient group of $M(G)$.

Now we can use the character tables of the ATLAS to determine the behavior of the various codon representations of G under extension by one of the cyclic subgroups $A \cong \mathbf{Z}_n$ ($n = 2, 3, 6$) of $Out(G)$ described above. First of all, we note that the linear codon representation of G is real and is split under extension by any cyclic subgroup A of $Out(G)$:

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more precisely, it extends to two real linear codon representations of $G.\mathbf{Z}_2$ (where $\mathbf{Z}_2$ stands for any one of the seven subgroups of $Out(G)$ of order 2), to one real plus a pair of complex conjugate linear codon representations of $G.\mathbf{Z}_3$ and to two real plus two pairs of complex conjugate linear codon representations of $G.\mathbf{Z}_6$. Next, consider the five projective codon representations of G: one of them is real while the remaining four form two complex conjugate pairs. The real one can be lifted to a real linear codon representation of any twofold covering group $\mathbf{Z}_2.G$ of G (here $\mathbf{Z}_2$ stands for any one of the three quotient groups of $M(G)$ of order 2), and this representation is split under extension by any cyclic subgroup A of $Out(G)$ of order 2, extending to two real linear codon representations of $M.G.A$. Similarly, each of the two pairs of complex conjugate projective codon representations of G can be lifted to a pair of complex conjugate linear codon representations of an appropriate fourfold covering group $\mathbf{Z}_4.G$ of G (here $\mathbf{Z}_4$ stands for any one of three among the six quotient groups of $M(G)$ of order 4), and these representations fuse into a single irreducible real linear representation of $M.G.A$ of dimension 128 if A is the center of $Out(G)$ or if A belongs to one of the two conjugacy classes of non-central subgroups of order 2 of $Out(G)$, but they are split under extension by A if A belongs to the other conjugacy class of non-central subgroups of order 2 of $Out(G)$: in that case, they extend to two pairs of complex conjugate linear codon representations of $M.G.A$. Finally, none of the projective codon representations of G admits an extension to $G.A$ when A has order 3 or 6 – in accordance with the fact that the corresponding bicyclic extension $M.G.A$ does not exist when M has order 2 or 4.

Summing up the number of linear and projective codon representations of each extension $G.A$, we arrive at the numbers given in Table 2.

- $G = B_2(3) = C_2(3) = {}^2A_3(2)$:

  In classical notation, G can be represented in various different forms: as the derived subgroup $PSO_5(3)'$ of the projective orthogonal group $PSO_5(3)$ in 5 dimensions over the field $\mathbf{F}_3$ or the projective symplectic group $PSp_4(3)$ in 4 dimensions over the field $\mathbf{F}_3$ or the projective special unitary group $PSU_4(2)$ in 4 dimensions over the field $\mathbf{Z}_2$; it has Schur multiplier $\mathbf{Z}_2$ and outer automorphism group $\mathbf{Z}_2$. (The various matrix group constructions are useful to represent various extensions: for example, G itself is identical with the classical matrix group $SU_4(2)$, $\mathbf{Z}_2.G$ is identical with the classical matrix group $Sp_4(3)$ and $G.\mathbf{Z}_2$ is identical with the classical matrix group $SO_5(3)$.) Its codon representations

identified above turn out to be real in the case of the linear one and pseudo-real in the case of the projective one; moreover, both are split under extension by the full outer automorphism group $\mathbf{Z}_2$: the former extends to two real linear and the latter to a pair of complex conjugate projective codon representations of $G.\mathbf{Z}_2$.

- $G = {}^2B_2(8)$:

This group, also known as the Suzuki group, has Schur multiplier $\mathbf{Z}_2 \times \mathbf{Z}_2$ and outer automorphism group $\mathbf{Z}_3$. Both of its codon representations turn out to be real. The linear codon representation is split under extension by the full outer automorphism group $\mathbf{Z}_3$: more precisely, it extends to one real plus a pair of complex conjugate linear codon representations of $G.\mathbf{Z}_3$. The projective codon representation admits no extension at all – in accordance with the fact that the bicyclic extension $M.G.A$ with A equal to $Out(G) = \mathbf{Z}_3$ and M any quotient group of $M(G) = \mathbf{Z}_2 \times \mathbf{Z}_2$ isomorphic to $\mathbf{Z}_2$, does not exist.

- $G = {}^2A_2(4)$:

In classical notation, G is the projective special unitary group $PSU_3(4)$ in 3 dimensions over the field $\mathbf{F}_4$; it has trivial Schur multiplier and outer automorphism group $\mathbf{Z}_4$. (The first of these two statements implies that G is in fact identical with the classical matrix group $SU_3(4)$.) Its codon representation turns out to be real and, as a consequence of Conlon's theorem quoted above, must be split under extension by any cyclic subgroup A of $Out(G)$: more precisely, it extends to two real codon representations of $G.\mathbf{Z}_2$ or to two real plus a pair of complex conjugate codon representations of $G.\mathbf{Z}_4$.

- $G = B_3(2) = C_3(2)$:

In classical notation, G can be represented in various different forms: as the derived subgroup $PSO_7(2)'$ of the projective orthogonal group $PSO_7(2)$ in 7 dimensions over the field $\mathbf{F}_2$ or the projective symplectic group $PSp_6(2)$ in 6 dimensions over the field $\mathbf{F}_2$; it has Schur multiplier $\mathbf{Z}_2$ and trivial outer automorphism group. Thus it suffices to note that its codon representations identified above turn out to form a pair of complex conjugate projective representations, since the extension problem does not pose itself.

- $G = G_2(3)$: This group, based on the exceptional Lie algebra G_2, has Schur multiplier $\mathbf{Z}_3$ and outer automorphism group $\mathbf{Z}_2$. Its codon representations identified above turn out to form a pair of complex conjugate linear representations, both of which are split under extension

by the full outer automorphism group $\mathbf{Z}_2$: they extend to two pairs of complex conjugate linear codon representations of $G.\mathbf{Z}_2$.

These results are schematically represented in Table 2.

4 Conclusions and Outlook

The main result of the present paper, the first in a sequence of two, is the complete list of all codon representations (64-dimensional irreducible representations) of the simple finite groups and their satellites, or to state it differently, of all linear and projective codon representations of the simple finite groups and their extensions by cyclic groups of outer automorphisms: they are enumerated in Table 1 and Table 2. In summary, we find 14 simple finite groups with a total of 22 codon representations (8 linear and 14 proper projective), plus 18 groups obtained from simple finite groups through extension by a cyclic group of outer automorphisms with a total of 51 codon representations (36 linear and 15 proper projective), where extensions by conjugate subgroups of $Out(G)$ and complex conjugate representations have been counted only once because they are not regarded as essentially different. These numbers are comparable to the ones found for compact connected Lie groups/algebras (10 groups/algebras with 10 codon representations) or for basic classical Lie superalgebras (12 superalgebras with 18 codon representations), given the fact that the analysis in these two cases includes the analogue of proper covering groups but not that of extensions by outer automorphisms.

In the second paper of this series, we shall investigate the branching schemes for the codon representations enumerated above, the main goal being to identify candidates for schemes capable of reproducing the standard genetic code.

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