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"TEORIA DOS ANÉIS"

Encontro IME.USP - IMECC.UNICAMP

Realizado no IME.USP em 18
de junho de 1993

Outubro 1993

“TEORIA DE ANÉIS”

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O encontro IME-IMECC em teoria de anéis é o segundo da série e o primeiro que contou com a participação de professores visitantes de outros centros e que teve o apoio de outras instituções, no caso, da FAPESP e da FAPEAL. Foi organizado pelo grupo de Representações de Álgebras do IME, o qual indicou a nós para cuidar de sua coordenação. Participaram professores e alunos de doutoramento do IMECC-UNICAMP e do IME-USP.

O presente relatório, além da lista de expositores e do programa, contem os textos das conferências e das comunicações na forma que estes foram entregues por seus autores. Alguns deles se constituem desta forma em preprints de artigos submetidos a revistas internacionais.

Agradecemos às instituições citadas e também em particular ao IME-USP pela publicação deste.

H. Merklen
São Paulo, agosto de 1993

EXPOSITORES[1].

Severino Collier Coutinho, UFRJ, conferencista convidado
Aron Simis, UFB e UNICAMP, conferencista convidado
Héctor Merklen, IME-USP
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Roberto Costa, IME-USO
Henrique Guzzo Jr., IME-USP
Nikolaus Vonessen, UFAl, conferencista convidado
Zinovy Reichstein, Berkeley e UFAl, conferencista convidado.

PROGRAMA

9:30 – 10:10 hs.	S. Collier Coutinho
10:25 – 11:05	A. Simis
11:10 – 11:25	H. Merklen
11:30 – 11.45	F. Coelho
11:45 – 14:30	Almoço e descanso
14:30 – 15:10	A. Skowroński
15:15 – 15:30	R. Costa
15:35 – 15:50	H. Guzzo
15:50 – 16:20	Café
16:20 – 17:00	N. Vonessen
17:10 – 17:50	Z. Reichstein

[1] Na ordem do programa.

ÍNDICE

GEOMETRY AND THE REPRESENTATION THEORY OF WEYL ALGEBRAS

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1. Introduction

By representation theroy we understand the study of modules of a Weyl algebra, especially modules of finite length. Typical problems include the construction of irreducible and indecomposable modules and the calculation of their extension groups.

The geometry of the title refers to both algebraic and symplectic geometry. To a module M over the n-th complex Weyl algebra, one associates a variety $Ch(M)$, its *characteristic variety*. This is an algebraic subvariety of affine complex space of dimension $2n$.

The characteristic variety is involutive with respect to the symplectic structure of $\mathbf{C}^{2n}$. This imposes certain restrictions on $Ch(M)$. For example, its dimension cannot be less than n. A module over the Weyl algebra whose characteristic variety has dimension exactly n is called a *holonomic* module. Holonomic modules always have finite length.

The theory of modules over the Weyl algebras – and over more general rings of differential operators – has been mostly concerned with holonomic modules. There are several reasons for this. The so-called *regular* holonomic modules are genralisations to higher dimension of the differential equations with regular singularities. These were studied by L. Fuchs, after Riemann's memoirs on the hypergeometric function. Hilbert asked for a generalisation of Fuch's results in the *21st Problem*. The theory of D-modules provides a vast generalisation of Hilbert's 21st Problem in the form of the *Riemann-Hilbert Correspondence* proved by Kashiwara and Mebkhout.

One of the remarkable facts about holonomic modules is their ubiquity. They come up in mathematical physics (*special functions and Feynmann amplitudes*), in the representation theory of algebraic groups, in the study of singularities of algebraic and analytic varieties (*intersection homology*) and in number theory (*exponential sums*), to mention but a few.

The Riemann-Hilbert Correspondence, mentioned above, provides a link between (regular) holonomic modules and geometry. The correspondence associates to every such module a complex of locally constant sheaves that lives in a derived category. This is useful in the construction and classification of these modules. For example, through the correspondence one may prove an equivalence of categories between regular holonomic modules with a gien support and a category of quivers. The quiver is constructed from a stratification of the support.

Although holonomic modules have been intensely studied, modules of higher dimension have been almost completely ignored. Actually, until 1983 it was widely believed – for no good reason – that all irreducible modules over the Weyl algebra were holonomic. In that year, J. T. Stafford constructed irreducible modules over the n-th Weyl algebra, for $n > 2$, whose characteristic varieties were hypersurfaces. His proof is highly computational.

In 1985 J. Bernstein and V. Lunts [1] and [7] used symplectic geometry to show that most hypersurfaces of $\mathbf{C}^{2n}$ are characteristic varieties of irreducible modules over the Weyl algebra. Their papers suggest that a deep knowledge of the geometry of the characteristic variety may be a key to many interesting results in the representation theory of non-holonomic modules. In this paper we shall discuss some of these results and problems that they suggest.

The second section is a crash course on filtered and graded modules for the Weyl algebra and associated characteristic varieties, followed by a summary of the results proved by Bernstein and Lunts. Section 3 contains a review of the bounds obtained from geometrical invariants for the Krull and Gelfand-Kirillov dimensions of modules over Weyl algebras followed by a summary of some of the results obtained by Bernstein and Lunts in [1, 7]. In section 4 we discuss the construction of irreducible objects and the calculation of extension groups for the category of modules whose characteristic variety is a generic hypersurface. Section 5 is dedicated to some of the problems suggested by this way of approaching the representation theory of D-modules.

2. Characteristic Varieties

We begin by establishing some notation. Let A_n denote the *n-th complex Weyl algebra.* This is the subalgebra os $\mathrm{End}_{\mathbf{C}}(\mathbf{C}[x_1, ..., x_n])$ generated by $x_1, ..., x_n$ and $\partial_1, ..., \partial_n$. The Weyl algebra may be filtered in many different ways. We consider one of these, here: the *Bernstein filtration.*

The *Bernstein filtration*, which we will denote as $B = \{B_i\}$ is obtained by giving both the x's and the ∂'s degree 1. Thus B_0 is a complex vector space of dimension $2n$ with basis $\{x_1, ..., x_n, \partial_1, ..., \partial_n\}$. Note that each B_k is a complex vector space of finite dimension. This is one of the reasons why it is best to work with this filtration.

The *associated graded ring* of A_n for this filtration is isomorphic to the polynomial ring in n variables. The isomorphism is established using the *symbol map.* Let us do this in more detail. The *symbol map of degree k* is the linear map of complex vector spaces defined by the canonical projection

$$\sigma_k : B_k \to B_k/B_{k-1}$$

Let $\sigma_1(x_i) = y_i$ and $\sigma_1(\partial_i) = y_{i+n}$. Then the graded ring S_n of A_n associated to the Bernstein filtration is generated by $y_1, ..., y_{2n}$. One easily checks that these symbols are algebraically independent. Thus S_n is a polynomial ring of $2n$ variables.

Before we proceed, let us make two useful definitions. First, if $d \in B_k/B_{k-1}$ then we say that d has *degree k*. If d has degree k then its *principal symbol sigma(d)* is defined to be $\sigma_k(d)$.

We now extend the filtered and graded methos to A_n-modules. Let T be a filtration of M, the graded module *associated* to T is defined by

$$gr^T(M) = \oplus_{k \geq 0} T_k$$

If $gr^T(M)$ is a finitely generated S_n-module, then we say that T is a *good filtration*. A finitely generated left A_n-module always admits a good filtration, which may be constructed as follows. If M is generated by $u_1, ..., u_m$, then we may construc a filtration $F = \{F_k\}$ of M by setting

$$F_k = \sum_{i=1}^{m} B_k \cdot u_i$$

The graded module $gr^F(M)$ is generated by the images of the u's if gr^F, hence is finitely generated over S_n.

Keeping the notation of the above paragraph, let $I(M)$ denote the radical of the annihilator of $gr^F(M)$ in S_n. To calculate this ideal we have to choose a good filtration for M, but it turns out that $I(M)$ is independent of the choice of good filtration. Thus $I(M)$ is an invariant of M – it is called the *characteristic ideal* of M. The set $Z(I(M))$ of zeros of $I(M)$ in $\mathbf{C}^{en}$ is an algebraic variety – it is called the *characteristic variety* of M and it is denoted by $Ch(M)$. Note that $Ch(M)$ is a homogeneous subvariety of $\mathbf{C}^{2n}$. These varieties have very special properties; to discuss these we must recall the *symplectic structure* of $\mathbf{C}^{2n}$.

The space $\mathbf{C}^{2n}$ has a symplectic structure defined by the *standard skew-symmetric form* $\omega = \sum_0^n dy_k \wedge dy_{n+k}$. Let I be the isomorphism $I : (\mathbf{C}^{2n})^* - \mathbf{C}^{2n}$ defined by the form ω. A variety X in $\mathbf{C}^{2n}$ is said to be *involutive* if its tangent spaces are co-isotropic for ω. This admits a more algebraic definition in terms of the ideal $I(X)$ of the variety X. Consider first two polynomials $f, g \in S_n$. The *Poisson bracket* of f and g is defined at $p \in \mathbf{C}^{2n}$ by the formula $\{f,g\}(p) = \omega(Idf(p), Idg(p))$. In this notation X is involutive if $\{f,g\} \in I(X)$ whenever $f, g \in I(X)$. We also say that $I(X)$ is involutive, when this happens.

If M is a finitely generated left A_n-module, then its characteristic variety $Ch(M)$ is an involutive subvariety of $\mathbf{C}^{2n}$. This very important result admits a purely algebraic proof. as shown by Gabber in [5]. A well-known consequence of the involutivity of $Ch(M)$ is that $\dim Ch(M) \geq n$. Thus, curves in $\mathbf{C}^{2n}$ are not characteristic varieties unless $n = 1$. Since the *Gelfand-Kirillov* dimension GKdim(M) of M is equal to the dimension of $Ch(M)$, see [8, 8.6.18], it follows that GKdim(M) $\geq n$.

It is time for an example. Let J be a left ideal of A_n. Put $M = A_n/J$. Thus M is a cyclic A_n-module, hence finitely generated. Let F be the filtration of M defined by the generator $1 + J$. Define the ***symbol ideal*** of J by $\sigma(J) = \sum_{k\geq 0} \sigma_k(J \cap B_k)$. Thus $\mathrm{Ann}_{S_n} gr^F(M) = \sigma(J)$. The characteristic ideal $I(M)$ is the radical of $\sigma(J)$. An important special case occurs when $d \in A_n$ and $J = A_n \cdot d$. Then $I(A_n/J)$ is the ideal of S_n generated by $\sigma(d)$, and $Ch(M)$ is the hypersurface of equation $\sigma(d) = 0$. Thus GKdim(A_n/J) $= 2n - 1$.

The most important class of A_n-modules is that of holonomic modules. A finitely generated non-zero A_n-module is ***holonomic*** if its Gewlfand-Kirillov dimension is exactly n - that is, it has minimal Gelfand-Kirillov dimension. For example, if J is the left ideal of A_n generated by $\partial_1, \ldots, \partial_n$, then A_n/J is holonomic. In fact, using the results of the previous paragraph, we have that $Ch(A_n/J)$ is the linear variety of equations $y_{n+1} = \cdots = Y_{2n} = 0$, which has dimension n.

Holonomic modules have many nice properties. For example, submodules, quotients and extensions of holonomic modules are holonomic. Also, holonomic modules always have finite length. In the theory of D-modules they are shown to be preserved by inverse and direct image functors. Finally, the category of holonomic A_n-modules has a duality functor defined from $\mathrm{Ext}^2(\cdot, A_n)$.

3. Involutive Dimension

In the previous section we showed that the Gelfand-Kirillov dimension of an A_n-module may be calculated using its characteristic variety. In this section we discuss a bound for the *Krull dimension* in terms of invariants of invariants of the characteristic variety.

First of all recall that the (non-commutative) Krull dimension measures how much a module deviates from being artinian. To be more exact, let M be a finitely generated left A_n-module. Denote its Krull dimension by Kdim(M). If M is artinian, then Kdim(M) $= 0$. The definition proceeds inductively. Let α be a cardinal, and suppose that Kdim(M) $\not< \alpha$. Then Kdim(M) $= \alpha$ if any descending chain $M_0 \supset M_1 \supset \ldots$ of submodules of M satisfies $\mathrm{Kdim}(M_i/M_{i-1}) \geq \alpha$ for only finitely many $i \geq 0$.

One may show that Kdim(A_n) $= n$. On the other hand, holonomic modules, being artinian, have Krull dimension zero. In [2] Björk suggested that for a finitely generated left A_n-module M, one ought to have that GKdim(M) $= n+$ dim(M).

Note that this agrees with both of the above examples. However J. T. Stafford showed in [9] that there exist irreducible A_n-modules of Gelfand-Kirillov dimension $2n-1$, and these do not satisfy the above formula. However it is known that $\mathrm{GKdim}(M) \geq \mathrm{Kdim}(M) + n$, see [8, Corollary 8.5.6], for example.

This inequality may be sharpened if we use an invariant of the characteristic variety, the *involutive dimension.* Let X be an involutive homogeneous subvariety of $\mathbf{C}^{2n}$. Its irreducible components must also be homogeneous and involutive. The *involutive dimension* $\beta(X)$ of X is the longest chain of irreducible involutive subvarieties contained in X. If M is a finitely generated left A_n-module, then the *involutive dimension* of M is the involutive dimension of $Ch(M)$. We shall denote if by $\beta(M)$. The next result comes from [3, Theorem 2.6].

TTHEOREM 3.1. *Let M be a finitely generated left A_n-module. Then $Kdim(M) \leq \beta(M)$.*

This allows us to conclude that if $\beta(M) = 0$ then M must have finite length. Actually, a little more may be said in this case.

PROPOSITION 3.2. *Let J be a left ideal of A_n. Suppose that $\sigma(J)$ is a prime ideal of S_n and that $\beta(\sigma(J)) = 0$. Then J is a maximal left ideal of A_n. Therefore, A_n/J is an irreducible left A_n-module.*

The inequality of Theorem 3.1 may be strict. In [3, Theorem 3.2] we construct an irreducible A_2-module, whose characteristic variety is irreducible, and has involutive dimension 1. The inequality $\mathrm{GKdim(M)} \geq \beta(M) + n$ also holds, and it too may be strict. This directs us to the work of Bernstein and Lunts. First of all, note that a hypersurface is always an involutive subvariety. Bernstein and Lunts showed that, contrary to what one might expect, most homogeneous hypersurfaces have involutive dimension 0. To describe their results in greater detail it is necessary to introduce some notation.

Let $S_n(k)$ be the vector space of homogeneous polynomials of degree k over S_n. The set $S_n(k)$ has a natural structure of affine complex space, induced by its vector space strcture. We say that a property $\mathbf{P}$ is *generic* in $S_n(k)$ if the set $\{P \in S_n(k) : P does not satisfy \mathbf{P}\}$ is contained in a countable union of hypersurfaces of $S_n(k)$. This is commonly shortened to "if $P \in S_n(k)$ is *generic* then $\mathbf{P}$ holds".

Let $P \in S_n(k)$. The *hamiltonian vector field* h_P is defined on a polynomial $f \in S_n(k)$ by the formula $h_P(f) = \{P, f\}$. We shall say that a subvariety X of $\mathbf{C}^{2n}$ is *preserved* by h_P if it is tangent to h_P at every one of its smooth points. We may now state the main result of Bernstein and Lunts, Cf [1, Theorem 1] and [7, Theorem 1].

THEOREM 3.3. *Let $k \geq 4$ ne an integer and let P be a generic polynomial of $S_n(k)$. If X is a homogeneous subvariety of $Z(P)$ which is preserved by h_P, then $dim(X) \leq 1$.*

An immediate consequence of this theorem is the following.

COROLLARY 3.4. *Let $k \geq 4$ be an integer and let P be a generic polynomial of $S_n(k)$.*

The variety Z(P) is homogeneous and has involutive dimension 0.

This provides us with a geometrical method to construct families of irreducible A_n-modules of Gelfand-Kirillov dimension $2n-1$. Let d be an operator of A_n of degree $k \geq 4$. Suppose that the symbol $\sigma(d)$ is generic in $S_n(k)$. It follows from Corollary 3.4 and Proposition 3.2 that the module $A_n/A_n \cdot d$ is irreducible.

The irreducible modules constructed in this way have another importante characteristic, they have finite *multiplicity*. Let M be a finitely generated left A_n-module, and let F a good filtration of M. Lep P be a prime ideal of S_n. The *multiplicity* $m_P(M)$ of M *with respect to* P if the length of the $(S_n)_P$-module $gr^F(M)_P$. The multiplicity thus defined is independent of the good filtration F. Let Ibe the set of all involutive homogeneous prime ideals P of S_n such that $\beta(Z(P)) = 0$. The *multiplicity* of M is the sum of the $m_P(M)$ with M running over all primes in I. The multiplicity is always ≥ 0, and it is additive over shor exact sequences.

Suppose, as above, that $d \in A_n$ has degree $k \geq 4$ and that $\sigma(d)$ is generic in $S_n(k)$. Then $Z(\sigma(d))$ is a homogeneous irreducible involutive subvariety of $\mathbf{C}^{2n}$ and $\beta(Z(\sigma(d))) = 0$. Let Q be the prime ideal corresponding to $Z(\sigma(d))$. Put $M = A_n/A_n \cdot d$. Then the characteristic ideal $I(M) = Q$, and $m(M) = m_Q(M) = 1$. One should note at this point that the irreducible A_n-module constructed by Stafford in [9] has *infinite* multiplicity.

Let $\mathbf{M}_d$ be the full subcategory of finitely generated left A_n-modules of finite multiplicity and Gelfand-Kirillov dimension d. If $d = n$, then $\mathbf{M}_d$ is the category of holonomic A_n-modules. In [1, §3.1] Bernstein and Lunts show that for any D, $n \leq d \leq 2n$, the category $\mathbf{M}_d$ shares many of the nice properties that make holonomic modules so attractive. For example, submodules, quotients and extensions of modules in $\mathbf{M}_d$ belong to $\mathbf{M}_d$. There is a duality functor if $\mathbf{M}_d$ which is defined on an object M by turning the right module $\mathrm{Ext}^d(M, A_n)$ into a left module in the standard way.

It is possible to use Theorem 3.3 to construct homogeneous involutive subvarieties of $\mathbf{C}^{2n}$ of involutive dimension 0 which are not hypersurfaces. Since the purpose of these constructions is to produce examples of irreducible A_n-modules, we run into a problem. Notice first that if X is a homogeneous hypersurface of $\mathbf{C}^{2n}$, then $Z = Z(f)$, for some homogeneous polynomial $f \in S_n$. Thus, if $d \in A_n$ has symbol f, we have that the characteristic variety $Ch(A_n/A_n \cdot d) = X$. The situation is far more complicated if X is not a hypersurface. For example, suppose that the ideal of X is generated by f_i for $1 \leq i \leq m$, and that $d_i \in A_n$ satisfy $\sigma(d_i) = f_i$. Let J be the left ideal of A_n generated by $d_1, \ldots, d_M$. One might expect that the $Ch(A_n/J = X$, but this need not be true. This suggest the following problem.

PROBLEM 3.5. *Let X be a homogeneous involutive variety of $\mathbf{C}^{2n}$. Is there a finitely generated left A_n-module M such that $Ch(M) = X$?*

There are not many cases to which the answer to this problem seems to be known.

4. The Codimension one case

In this section we shall study the hypersurface case in more detail. To make the problem tractable, we shall not work with the category $\mathbf{M}_{2n-1}$ but we a subcategory thereof. Let X be a homogeneous hypersurface in $\mathbf{C}^{2n}$. Let $\mathbf{M}X$ be the full subcategory of $\mathbf{M}_{2n-1}$ of all objects M with $Ch(M) = X$. This subcategcory is closed for submodules, quotients and extensions.

In this section we shall assume that $k \geq 4$ is an integer and that P is a generic polynomial in $S_n(k)$. Set $X = Z(P)$. In [4, 4, §3] we prove the following result about the irreducible objects of $\mathbf{M}(X)$.

Theorem 4.1. *$\mathbf{M}(X)$ constins infinitely many irreducible objects of multiplicity m, for every integer $m \geq 1$.*

Let d_1, d_2 be elements of A_n whose principal symbol is P. We have seen in §3 that $A_n/A_n \cdot d_i$ is irreducible of multiplicity 1 for $i = 1, 2$. It is shown in [4, Theorem 3.1] that $A_n/A_n \cdot d_1 \not\cong A_n/A_n \cdot d_2$. This suggests the following problem.

Problem 4.2. *Suppose that M is an irreducible object in $\mathbf{M}(X)$ of multiplicity 1. Is there a $d \in A_n$ with symbol P such that $M \cong A_n/A_n \cdot d$?*

We may also calculate the extension groups of irreducible objects of $\mathbf{M}(X)$. That way we can construct indecomposable objects of length 2 in this category.

Theorem 4.3. *Suppose that $d_1, d_2 \in A_n$ have principal symbol P,, Then:*

=0 (1) $Ext^1(A_n/A_n \cdot d_1, A_n/A_n \cdot d_2) \neq 0)$

(2) If $d = d_1 = d_2$, then $Ext^1(A_n/A_n \cdot d, A_n/A_n \cdot d)$ is a vector space of infinite dimension over $\mathbf{C}$.

Compare with this the holonomic case: if M, N are holonomic modules over A_n then $Ext^k(M, N)$ is a finite dimensional vector space for every $k \geq 0$.

If $d \in A_n$ has symbol P, then the extensions of $A_n/A_n \cdot d$ by itself admit a very simple description. For an element $a \in A_n$ put $I(a) = \{x \in A_n : x \cdot a \in A_n \cdot d\}$. Thus these extensions are isomorphic to modules of the form $A_n/I(a) \cdot d$, for some $a \in A_n$. In fact since $A_n/A_n \cdot d$ is irreducible, it follows that $A_n/I(a) \cong A_n/A_n \cdot d$. Thus $A_n/A_n \cdot d$ has a projective resolution of length 1. In particular $I(a)$ is a projective ideal of A_n. We may prove even more, as shown in [4, Theorem 5.2].

Theorem 4.4. *If $a \in A_n \setminus (\mathbf{C} + A_n \cdot d)$, then the ideal $I(a)$ is a non-cyclic projective left ideal of A_n.*

The proof follows the method established by Stafford in [10], but it is less computational, and it produces a whole family of examples. There are many questions related to the results stated in this section that one would like to know the answer of. We mention three.

Problem 4.5. *If P is a generic polynomial in $S_n(3)$, is it true that $\beta(Z(\sigma(P))) = 0$?*

The hypothesis $k \geq 4$ is necessary for technical reasons in Theorem 3.3, so one ought to be able to remove it. On the other hand, if P is homogeneous of degree 2, then $\beta(Z(\sigma(P))) \geq 1$.

PROBLEM 4.6. *Construct an infinite family of hypersurfaces Y such that $\beta(Y) = 1$.*

It follows from *Queen's Lemma* [8, Theorem 9.5.5], that the endomorphism ring of any object in $\mathrm{M}(X)$ is a finite dimensional algebra over $\mathbf{C}$.

PROBLEM 4.7. *Which finite dimensional complex algebras are endomorphism rings of objects in $\mathrm{M}(X)$?*

5. FINAL REMARKS

The results of the previous sections cannot be generalized to other rings of differential operators. They depend on the fact that the components of the Bernstein filtration have finite dimension over $\mathbf{C}$, and this filtration has no natural generalization. To handle general rings of differential operators (*over affine, smooth, irreducible varieties*) one must use the *filtration by order*. This means giving the x's degree 0 and the ∂'s degree 1.

Bernstein and Lunts have also shown in [1, §4] how this filtration can be used to construc irreducible A_n-modules by geometrical methods.

PROBLEM 5.2. *Can a generic hypersurface of* $\mathbf{T}^*(\mathbf{P}^n(\mathbf{C}))$, *contain a proper involutive subvariety?*

It may be appropiate to end in a more speculative note. First, one may observe that the Weyl algebra is the algebra obtained by "quantization" of the algebra of functions (*observables*) in the symplectic variety $\mathbf{C}^{2n}$. Given a general symplective variety X it is often possible to quantize its algebra of functions by deformation theoretic methods, cf. [6].

PROBLEM 5.3. *Is it possible to use methods described in this paper to construct irreducible representations of the quantization of the algebra of functions in a symplectic algebraic manifold ?*

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The module of derivations of a Stanley-Reisner ring

Aron Simis, UFBa e UNICAMP[2]

Abstract

An explicit description is given of the module $\mathrm{Der}((k[\underline{X}]/I, k[\underline{X}]/I)$ of the derivations of the residue ring $k[\underline{X}]/I$ where I is an ideal generated by monomials whose exponents are prime to the characteristic of the field k. (This includes the case of square free monomials in any characteristic and the case of arbitrary monomials in characteristic zero.) In the case where I is generated by square free monomials, this description is interpreted in terms of the corresponding abstract simplicial complex Δ. Sharp bounds for the depth of this module are obtained in terms of the depths of the face rings of certain subcomplexes Δ_i related to the stars of the vertices v_i of Δ. The case of a Cohen–Macaulay simplicial complex Δ is discussed in some detail: it is shown that $\mathrm{Der}((k[\Delta], k[\Delta])$ is a Cohen–Macaulay module if and only if depth $\Delta_i \geq \dim \Delta - 1$ for every vertex v_i. A measure of triviality of the complexes Δ_i is introduced in terms of certain *star corners* of v_i. Thus, the absence of these corners implies that $\mathrm{Der}(k[\Delta], k[\Delta])$ is a Cohen–Macaulay module. Examples of such complexes are the doubly Cohen–Macaulay complexes of Baclawski and complexes whose faces are the independent sets of vertices of a suspension graph. A curious corollary of the main structural result is an affirmative answer in the present context of the conjecture of Herzog-Vasconcelos on finite projective dimension of the $k[\underline{X}]/I$-module $\mathrm{Der}(k[\underline{X}]/I, k[\underline{X}]/I)$. differentials of $k[\Delta]$.

[2] Joint work with P. Brumatti.

Module categories with infinite radical square zero are of finite type

Héctor Merklen[1]

We have recently study artin algebras A whose categories of finitely generated modules have a nilpotent infinite radical (see [2, 3]). Here we present an alternate proof to the following, the main result of [2]. This proof depends heavely on recent results relating the infinite radical to tame concealed algebras.

Theorem 1 *Let A be an artin algebra and let J be the Jabobson radical of the category* mod$A$ *of finitely generated A-modules. If $(J^\infty)^2 = 0$, then A is of finite representation type.*

REMARK. The converse is obviously true because if A is representation finite then $J^\infty = 0$.
PROOF. Let us consider only finitely generated A-modules. Also, let us assume, as we can, that A is *minimal representation- infinite*, *i.e.*, every proper factor algebra is of finite representation type.

By a well known result of Auslander, there exists an infinite sequence of proper epimorphisms between indecomposable modules as the following.

$$(*) \qquad \cdots \to M_r \xrightarrow{f_r} M_{r-1} \to \cdots M_1 \xrightarrow{f_1} M_0.$$

In connection to this, we note first that $\mathrm{Hom}_A(P, M_i) = J^\infty(P, M_i)$ for all $i \geq 0$ and all projective modules P. In fact, if f is a non-zero A-morphism from a projective P to some M_i, then, for each $t > i$ there exists a lifting $g_t : P \to M_t$ such that $f = f_i \cdots f_t g_t$, meaning that f belongs to all powers of J.

Next, we claim that $(*)$ contains a faithful module, implying that we can assume that all the M_i's are faithful. Indeed, the annihilators of the M_i's form a descending chain. If non of them is zero, we would get an infinite sequence as $(*)$ where all modules would have one and the same annihilator, say, I. Then, if $B = A/I$, B would have infinite non-isomorphic indecomposable modules, contradicting our assumption that A is minimal representation-infinite.

We show now that $J^\infty(--, A) = 0$. For, if for some X h is a non-zero map in $J^\infty(X, A)$, and if ι is a monomorphism $A \to (M_i)^m$, for some i and

[1]Joint work with E. N. Marcos, F. U. Coelho and A. Skowroński.

some m, then ιh would be a non-zero element of $(J^\infty)^2$, contradiction. A dual argument shows that we have also $J^\infty\,(DA, --) = 0$ (where D denotes the usual duality of mod$A$).

Finally, from [1, (3.4)] or [4, (3.3)], it is easy to deduce that a minimal representation-infinite artin algebra satisfying those properties is either tame concealed or wild concealed with two points. However, it is known that such algebras do not have $(J^\infty)^2 = 0$.

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ON THE MODULE CATEGORIES WITH INFINITE RADICAL CUBE ZERO

FLÁVIO ULHOA COELHO

1. Introduction

This is a report of a joint work with E. Marcos, H. Merklen and A. Skowroński.

Let A be an artin algebra over a commutative ring R, that is, A is an R-algebra which is finitely generated as an R-module. All algebras in this note are basic, connected and indecomposable. By an A-module it is meant a finitely generated left A-module. Let rad(mod$A$) denote the Jacobson radical of mod$A$, that is, the ideal of mod$A$ generated by all non-invertible morphisms and by $\text{rad}^\infty(\text{mod}A)$ the intersection of all powers $\text{rad}^i(\text{mod}A)$ of rad(mod$A$). The study of $\text{rad}^\infty(\text{mod}A)$ gives important informations on the category mod$A$, in particular, in the components of the Auslander-Reiten quiver Γ_A of A (see definition below). We are particularly interested in the case when $\text{rad}^\infty(\text{mod}A)$ is nilpotent. We say that an algebra A is *representation-finite* if mod$A$ has only finitely many non-isomorphic indecomposable modules. Otherwise, A is called *representation-infinite*. The following result has been proven in [4].

Theorem 1.1. *If* $(\text{rad}^\infty(\text{mod}A))^2 = 0$, *then A is representation-finite.*

We now consider algebras A such that $(\text{rad}^\infty(\text{mod}A))^3 = 0$. We first observe that there are representation-infinite algebras with this property. In order to introduce such examples, we will first recall some notions.

2. Auslander-Reiten quivers

For a given artin algebra A, its Auslander-Reiten quiver Γ_A is defined as follows. The vertices of Γ_A is in a one-to-one correspondence with the isomorphism class of the indecomposable modules in mod$A$. For the definition of the arrows in Γ_A we recall the notion of irreducible morphisms: if X and Y are modules in mod$A$ then a morphism $f: X \longrightarrow Y$ is *irreducible* if (i) f is not a split morphism; and (ii) whenever $f = gh$, then either g is a split epimorphism or h is a split monomorphism. Suppose $[X]$ and $[Y]$ are two vertices in Γ_A corresponding, respectively, to indecomposable

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modules X and Y. Now, by definition, there is an arrow from $[X]$ to $[Y]$ if and only if there is an irreducible morphism between X and Y.

We would like to stress some facts on the quiver defined above. First, Γ_A is locally finite, that is, for each vertex $[X]$, there are at most finitely many arrows with $[X]$ as a start or an end point. Moreover, there are no arrows from a vertex to itself. Finally, Γ_A is a so-called translation quiver, that is, there exists a bijection $\tau\colon \Gamma' \longrightarrow \Gamma''$, where Γ' (respectively, Γ'') is the set of vertices not corresponding to projective (respectively, injective) modules, such that for each $x \in \Gamma'$, there exists an arrow $y \longrightarrow x$ if and only if there exists an arrow $\tau x \longrightarrow y$. The quiver Γ_A is, in general, not connected. For details on the above construction we refer the reader to, for instance, [1, 6]. We also recall the following result due to Auslander-Reiten [2](1.7).

Proposition 2.1. *If $f \in rad(X,Y)$, then $f = \Sigma g_i + h$, where $h \in rad^{\infty}(modA)$ and, for each i, g_i is a composite of irreducible morphisms.*

Corollary 2.2. *Any morphism between modules belonging to distinct components of Γ_A belongs to $rad^{\infty}(modA)$.*

3. Tame concealed algebras

Let H be a hereditary algebra. It is known that in this case R is a field and H is in fact a finite dimensional algebra over R. Moreover, there exists a bilinear form on the Grothendieck group $K_0(H)$ of H given by

$$< M, N > = \dim_R \mathrm{Hom}_H(M,N) - \dim_R \mathrm{Ext}^1_H(M,N)$$

which induces a quadratic form q_H on $K_0(H) \otimes_{\mathbb{Z}} \mathbb{Q}$. It is well-known that H is representation-finite if and only if q_H is positive definite. The algebra H is said to be of *tame* type if it is not representation-finite and q_H is positive semidefinite.

Let now H be a representation-infinite hereditary algebra and let n denote the rank of $K_0(H)$. Let T be a multiplicity-free preprojective tilting H-module, that is, $\mathrm{Ext}^1_H(T,T) = 0$, $\mathrm{rad}^{\infty}(-,T) = 0$ and T is a direct sum of n pairwise non-isomorphic indecomposable H-modules. The algebra $B = \mathrm{End}_H(T)$ is called a *concealed algebra* and if H is tame hereditary then B is called *tame concealed.*

Let now A denote a tame concealed algebra (which can be, in particular a hereditary algebra). The Auslander-Reiten quiver of A defined as above has

-a component consisting of modules X such that $\mathrm{rad}^{\infty}(-,X) = 0$ and containing all indecomposable projectives, called *preprojective component*;

-a component consisting of modules X such that $\mathrm{rad}^{\infty}(X,-) = 0$ and containing all indecomposable injectives, called *preinjective component*; and

-an infinite family of generalized standard pairwise orthogonal stable tubes $(\mathcal{T}_\rho)_{\rho\in\Omega}$, that is, for each $\lambda \in \Omega$, $\mathcal{T}_\rho$ is a quiver of the form $\mathbb{Z}A_\infty/(\tau^m)$, for some m, and $\mathrm{rad}^{\infty}(X,Y) = 0$ for all X and Y belonging to components in this family.

With this description of the components of the Auslander-Reiten quiver of A, it is not difficult to see that $(\text{rad}^\infty(\text{mod}A))^3 = 0$ and since A is not representation-finite, we infer from (1.1) that $(\text{rad}^\infty(\text{mod}A))^2 \neq 0$. For details on the results discussed above we refer to [6].

4. Main result

Let A be an artin algebra and C be a component of Γ_A. The component C is called *regular* if it does not contain neither projective nor injective modules and it is called *faithful* if it contains all indecomposable summands of a faithful module. Recall that a module Z is *faithful* if $\text{ann}Z = 0$.

¿From now on, we assume that $(\text{rad}^\infty(\text{mod}A))^3 = 0$. The main result in this note is the following.

Theorem 4.1. *Let A be an artin algebra such that $(\text{rad}^\infty(\text{mod}A))^3 = 0$. If Γ_A contains a faithful regular component, then A is tame concealed.*

For a proof of this result, we refer the reader to [5]. However, we shall discuss quickly some intermediate steps in order to show the techniques used. By hypothesis, Γ_A contains a regular component. Let then $(\mathcal{T}_\rho)_{\rho\in\Omega}$ be the family of all regular components. We shall first show that $(\mathcal{T}_\rho)_{\rho\in\Omega}$ is a family of generalized standard pairwise orthogonal components. Indeed, suppose there exists a non-zero morphism $f \in \text{rad}^\infty(X,Y)$ with X and Y belonging to the family $(\mathcal{T}_\rho)_{\rho\in\Omega}$ and consider the projective cover $\pi\colon P_A(X) \longrightarrow X$ of X and the injective envelope $\iota\colon Y \longrightarrow I_A(Y)$ of Y. By Corollary 2.2 both π and ι belong to $\text{rad}^\infty(\text{mod}A)$. Therefore, $\iota f \pi$ is a non-zero morphim in $(\text{rad}^\infty(\text{mod}A))^3$, a contradiction. Using results from *tilting theory* we can also conclude that $\mathcal{T}_\lambda$ is a stable tube

Let now Γ be a faithful regular component of Γ_A. Then there exists a module $Z \in \text{add}\Gamma$ and a monomorphism $q\colon A \longrightarrow Z$. Consider now the injective envelope $\iota\colon Z \longrightarrow I_A(Z)$ of Z. Again by Corollary 2.2 both q and ι belong to $\text{rad}^\infty(\text{mod}A)$. Since ιq is a monomorphism, we conclude that $\text{rad}^\infty(-,A) = 0$. Similarly, we have also that $\text{rad}^\infty(DA,-) = 0$. By [3](3.4) or [7](3.3) we infer that A is concealed. Moreover, A is in fact tame concealed because Γ_A contains stable tubes (see [6]).

For finite dimensional algebras over algebraically closed fields we have the following consequence. Recall that an algebra A is said to be *minimal representation-infinite* if it is representation-infinite but for each ideal I, the algebra A/I is representation-finite.

Corollary 4.2. *Let A be a connected finite dimensional algebra over an algebraically closed field. Then A is tame concealed if and only if A is minimal representation-infinite and $(\text{rad}^\infty(\text{mod}A))^3 = 0$.*

Proof. The necessity is clear. To prove the sufficiency, it is enough to observe that if A is a representation-infinite algebra as in the statement, then there exists a regular component Γ. Since A is minimal representation-infinite, it follows that Γ is faithful and then, by Theorem 4.1. A is tame concealed. □

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Composition factors of indecomposable modules

Andrzej Skowroński

Let A be an artin algebra over a commutative artin ring R. We denote by $\mathrm{mod}A$ the category of finitely generated right A-modules, by $\mathrm{ind}A$ the full subcategory of $\mathrm{mod}A$ formed by the indecomposable modules, and by $\mathrm{rad}(\mathrm{mod}A)$ the *Jacobson radical* of $\mathrm{mod}A$, that is, the ideal in $\mathrm{mod}A$ generated by all non-invertible morphisms in $\mathrm{ind}A$. Moreover, we denote by $D : \mathrm{mod}A \to \mathrm{mod}A^{op}$ the *standard duality* $D = \mathrm{Hom}_A(--, I)$, where I is the injective envelope of $R/\mathrm{rad}R$ in $\mathrm{mod}R$.

Let $\mathrm{K}_0(A)$ be the Grothendieck group of A. For a module M in $\mathrm{mod}A$ we denote by $[M]$ its image in $\mathrm{K}_0(A)$. Clearly, if $S_1, ..., S_n$ is a complete set of pairwise non-isomorphic simple A-modules, then $[S_1], ..., [S_n]$ is a $\mathbb{Z}$-basis of $\mathrm{K}_0(A)$. Thus, $[M] = [N]$ if and only if M and N have the same (simple) composition factors including the multiplicities. We may ask

When $[M] = [N]$ for M and N in $\mathrm{ind}A$?

When M in $\mathrm{ind}A$ is uniquely determined up to isomorphism by $[M]$?

We shall show that the above questions are related with the properties of cycles in $\mathrm{mod}A$. Recall that a *cycle* of *length* t in $\mathrm{mod}A$ is a sequence $M_0 \to M_1 \to \cdots \to M_t = M_0$ of non-zero non-isomorphisms between modules in $\mathrm{ind}A$. If $t \leq 2$ such a cycle is said to be *short*. A *loop* in $\mathrm{mod}A$ is a cycle of length 1. Finally, a cycle in $\mathrm{mod}A$ is said to be *finite* if the morphisms forming this cycle do not belong to $\mathrm{rad}^\infty(\mathrm{mod}A)$.

THEOREM 1 [1] *Let X and Y be modules in* $\mathrm{mod}A$. *Suppose that $P_1 \to P_0 \to X \to 0$ is a minimal projective presentation of X and $0 \to Y \to I_0 \to I_1$ is a minimal injective copresentation of Y. Then*

$$(i) \quad |\,\mathrm{Hom}_A(X,Y)\,| - |\,\mathrm{Hom}_A(Y,\tau_A X)\,| = |\,\mathrm{Hom}_A(P_0,Y)\,| - |\,\mathrm{Hom}_A(P_1,Y)\,|$$

$$(ii) \quad |\,\mathrm{Hom}_A(Y,X)\,| - |\,\mathrm{Hom}_A(\tau_A^- X,Y)\,| = |\,\mathrm{Hom}_A(Y,I_0)\,| - |\,\mathrm{Hom}_A(Y,I_1)\,|\,.$$

Then we get the following fact on indecomposable modules having the same composition factors.

THEOREM 2 [3] *Let M and N be two non-isomorphic modules in* $\mathrm{ind}A$ *such that $[M] = [N]$. Then M and N lie on short cycles.*

Proof. Suppose that M does not lie on a short cycle. Then $\mathrm{Hom}_A(M, \tau_A M) = 0$. Indeed, otherwise ther is a short cycle $M \to V \to M$ where V is an indecomposable direct summand of the middle term of an Auslander-Reiten sequence $0 \to \tau_A M \to E \to M \to 0$. Let $P_1 \to P_0 \to M \to 0$ be a minimal projective presentation of M. Then we get the equalities

$$| \mathrm{Hom}_A(M,M) | - | \mathrm{Hom}_A(M,\tau_A M) | = | \mathrm{Hom} P_0, M) | - | \mathrm{Hom}_A(P_1,M) |$$

$$= | \mathrm{Hom}_A(P_0,N) | - | \mathrm{Hom} P_1,N) | = | \mathrm{Hom}_A(M,N) | - | \mathrm{Hom}_A(N,\tau_A M) |,$$

because $[M] = [N]$.

But $| \mathrm{Hom}_A(M,M) | - | \mathrm{Hom}_A(M,\tau_A M) | = | \mathrm{Hom}_A(M,M) | > 0$, hence $\mathrm{Hom}_A(M,N) \neq 0$. Similarly, we show that also $\mathrm{Hom}_A(N,M) \neq 0$. Consequently, we have a short cycle $M \to N \to M$ a contradiction. This shows that M lies on a short cycle in mod$A$.

Corollary 1 [3] *Let M be an indecomposable A-module which does not lie on a short cycle in mod$A$. Then M is uniquely determined by its composition factors.*

In the representation theory of algebras an important role is played by periodic modules. Recall that a module X in ind$A$ is *periodic* if $\tau_A^m X \cong X$ for some $m \geq 1$. If $\tau_A X \cong X$, the module X is said to be τ_A-invariant. It has been shown in [2] that if A is a tame algebra over an algebraically closed field then, in each dimension d, almost all indecomposable A-modules are τ_A-invariant, and hence periodic. We are interested in the composition factors of periodic modules lying in stable tubes of the Auslander-Reiten quiver Γ_A of A. Recall that a *stable tube of rank r* in Γ_A is a connected component $\mathcal{T}$ of the form $\mathbb{Z}\mathbf{A}_\infty/(\tau_A^r)$, where $\mathbf{A}_\infty$ is the quiver $0 \to 1 \to 2 \to \cdots$. Then the τ_A-orbit in $\mathcal{T}$ formed by the modules having exactly one predecessor is called the *mouth* of $\mathcal{T}$. For any module M om a stable tube $\mathcal{T}$ there exists a unique sectional path $X_1 \to X_2 \to \cdots \to X_s = M$ in $\mathcal{T}$ with X_1 lying on the mouth of $\mathcal{T}$, and s is called the *quasi-length* of M which we shall denote by $\mathrm{ql}(M)$. Following [4], a stable tube $\mathcal{T}$ is called *generalized standard* if $\mathrm{rad}^\infty(X,Y) = 0$ for any modules X and Y in $\mathcal{T}$.

We have the following facts on generalized standard stable tubes proved in [5].

Proposition 1 *Let $\mathcal{T}$ be a stable tube in Γ_A. The following conditions are equivalent.*

(i) $\mathcal{T}$ *is generalized standard*

(ii) $\mathrm{rad}^\infty(M,M) = 0$ *for any indecomposable module M in $\mathcal{T}$*

(iii) $\mathrm{rad}^\infty(X,Y) = 0$ *for any indecomposable modules X, Y lying on the mouth of $\mathcal{T}$.*

Lemma 1 *Let $\mathcal{T}$ be a generalized standard stable tube in Γ_A, and X and Y be two modules in $\mathcal{T}$. Then*

$$|\operatorname{Hom}_A(Y, \tau_A X)| = |\operatorname{Ext}^1_A(X, Y)| = |\operatorname{Hom}_A(\tau_A^- Y, X)|.$$

Proposition 2 *Let $\mathcal{T}$ be a generalized standard stable tube of rank r in Γ_A and M be a module in $\mathcal{T}$. Then*

(i) $|\operatorname{Hom}_A(M, M)| \geq |\operatorname{Ext}^1_A(M, M)|$

(ii) $|\operatorname{Hom}_A(M, M)| = |\operatorname{Ext}^1_A(M, M)|$ *if and only if r divides* $\operatorname{ql}(M)$

Applying the above results we proved in [5] the following facts on the composition factors of modules lying in generalized standard stable tubes.

Theorem 3 *Let $\mathcal{T}$ be a generalized standard stable tube of rank $r > 1$ in Γ_A. Assume that M and N are non-isomorphic indecomposable modules in $\mathcal{T}$. Then $[M] = [N]$ if and only if $\operatorname{ql}(M) = \operatorname{ql}(N) = cr$ for some $c \geq 1$.*

It follows from Prop. 1 that if a stable tube $\mathcal{T}$ consists of modules which don't lie on infinite short cycles, then $\mathcal{T}$ is generalized standard. Then we get the following results.

Corollary 2 *Let $\mathcal{T}$ be a stable tube of rank $r > 1$ in Γ_A consisting of modules which don't lie on infinite short cycles, and M be an indecomposable module in $\mathcal{T}$. Then M is uniquely determined by the composition factors if and only if r does not divide $\operatorname{ql}(M)$.*

Corollary 3 *Let $\mathcal{T}, \mathcal{T}'$ be different stable tubes in Γ_A consisting of modules which don't lie on infinite shor cycles. Let r be the rank of $\mathcal{T}$, r' be the rank of $\mathcal{T}'$. Assume that $[M] = [M']$ for some indecomposable modules M in $\mathcal{T}$ and M' in $\mathcal{T}'$. Then r divides $\operatorname{gl}(M)$, r' divides $\operatorname{gl}(M')$, and the tubes $\mathcal{T}$ and $\mathcal{T}'$ are orthogonal.*

Theorem 4 *Let $\mathcal{T}_i$, $i \in I$, be a family of pairwise different stable tubes in Γ_A consisting of modules which don't lie on infinite short cycles. Assume that each $\mathcal{T}_i$ contains an indecomposable module M_i such that $[M_i] = [M_j]$ for all $i, j \in I$. Denote by r_i the rank of $\mathcal{T}_i$. Then*

$$\sum_{i \in I} (r_i - 1) \leq n - 2$$

where n is the rank of $\mathrm{K}_0(A)$.

As an application of the above results we obtain the following theorem on module categories without infinite short cycles.

THEOREM 5 *Let A be an artin algebra such that every short cycle in* mod$A$ *is finite. Then, for each positive integer d, all but finitely many isomorphism classes of indecomposable modules of length d are τ_A-invariant.*

COROLLARY 4 *Let A be an artin algebra such that every short cycle in* mod$A$ *is finite. Then, all but countable many components of Γ_A are stable tubes of rank 1.*

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Shape identities in genetic algebras[1]

R. Costa

§ 1. Preliminaries

There exist several classes of nonassociative algebras (Bernstein, train, stochastic, etc) whose investigation has provided a number of significant contributions to theoretical Population Genetics. Such classes have been defined at different times by several authors and all algebras belonging to these classes are generally called "genetic". A good account of the known results up to 1980 can be found in [9]. In recent years many authors have tried to investigate the difficult problem of classification of these algebras. As a major part of this problem, we should consider the question of nonassociative identities satisfied by these algebras. The relation with Jordan algebras has been treated by several authors, as for instance, in [7]. Holgate has studied the entropic law in genetic algebras in [4]. More recently, the problem of finding identities of minimal degree satisfied by all algebras in a given class has been studied, with the assistence of computers. In this paper we investigate those genetic algebras which satisfy some multilinear identities. As a by product, a numerical invariant, the level, is defined.

§ 2. Shape identities

For the sake of motivation, we assume that A is a commutative stochastic algebra over the real field $\mathbb{R}$. This means A has a basis $\{a_0, a_1, \ldots, a_n\}$ such that the constants of multiplication $\gamma_{ijk} \in \mathbb{R}$, defined by $a_i a_j = \sum_{k=0}^{n} \gamma_{ijk} a_k$ satisfy the following relations:

$$0 \leq \gamma_{ijk} \leq 1 \quad (i,j,k = 0,1,\ldots,n) \text{ and } \sum_{k=0}^{n} \gamma_{ijk} = 1 \quad (i,j = 0,1,\ldots,n) \tag{1}$$

These algebras are close to the genetic models. We may think of the a_i as representatives of the genotypes of the population and the element $x = \sum_{i=0}^{n} \alpha_i a_i$, with $0 \leq \alpha_i \leq 1$ and $\sum_{i=0}^{n} \alpha_i = 1$, as a representative of a distribution of genotypes. Moreover the product xy of two such elements, when expressed in the basis $\{a_0, a_1, \ldots, a_n\}$, represents the distribution of frequencies of genotypes in the filial generation, obtained by random mating among the individuals in x and y. See [9] for more details. With this in mind, suppose x, y, z and u are four distributions of frequencies and consider the following five filial generations, where the order

x, y, z, u is preserved:

$$\mu_1 = ((xy)z)u, \quad \mu_2 = (x(yz))u, \quad \mu_3 = (xy)(zu), \quad \mu_4 = x((yz)u), \quad \mu_5 = x(y(zu)) \tag{2}$$

If there are real numbers α, β, γ, δ, ϵ, not all zero, such that

$$\alpha\mu_1 + \beta\mu_2 + \gamma\mu_3 + \delta\mu_4 + \epsilon\mu_5 = 0 \tag{3}$$

[1]The complete version of this paper will appear in Linear Algebra and its Application

this will mean that the five distributions of frequencies are linearly dependent: one of them can be obtained algebraically from the remaining four. We can think also of a similar problem involving k distributions $x_1, \dots, x_k$ and algebraic relations involving all the possibilities for the filial generations. We introduce more precisely this problem, in the general setting of nonassociative algebras. Suppose $x_1, \dots, x_n, \dots$ are nonassociative variables and consider the free nonassociative algebra generated by this set of variables over some field F of characteristic 0. For a fixed k, consider all monomials involving $x_1, \dots, x_k$, obtained by different arrangements of parentheses in the word $x_1 \dots x_k$. The number k' of these multilinear monomials is given by $k' = \frac{1}{k}\binom{2k-2}{k-1}$, which is also the dimension of the vector space $S_k(F)$ generated by these monomials in the above algebra. For instance, $S_4(F)$ is generated by the 5 monomials in (2). In general, every element Every monomial $\mu = \mu(x_1, \dots, x_k) \in S_k(F)$ can be decomposed uniquely in the form

$$\mu(x_1, \dots, x_k) = \mu_1(x_1, \dots, x_s)\mu_2(x_{s+1}, \dots, x_k) \tag{4}$$

for some $1 \le s < k$, where μ_1 and μ_2 are also multilinear. The k' monomials which form a basis for $S_k(F)$ can be totally ordered inductively in the following way: If

$$\mu = \mu_1(x_1, \dots, x_r)\mu_2(x_{r+1}, \dots, x_k)$$
$$\mu' = \mu_1'(x_1, \dots, x_s)\mu_2'(x_{s+1}, \dots, x_k)$$

then $\mu < \mu'$ if $s < r$. If $r = s$ then $\mu < \mu'$ when $\mu_1 < \mu_1'$. In the case $r = s$ and $\mu_1 = \mu_1'$, then $\mu < \mu'$ when $\mu_2 < \mu_2'$.

Given a nonassociative algebra A over F and $S = S(x_1, \dots, x_k) \in S_k(F)$, $S \neq 0$, if $S(a_1, \dots, a_k) = 0$ for all $a_i \in A$, then S will be called a shape identity in A. The set of all shape identities in k variables on A (plus the zero polynomial) is a vector subspace of $S_k(F)$. The level of A is the number of variables in a shape identity satisfied by A, with the smallest number of variables.

¿From the general theory of nonassociative algebras, as it appears for instance in [5], it is well known that, given $S = S(x_1, \dots, x_k) \in S_k(F)$, $S \neq 0$, the class of all algebras A over F satisfying the identity $S = S(a_1, \dots, a_k) = 0$, for all $a_i \in A$, is a variety. So, by Birkhoff's theorem, this class is closed under the operations of forming direct products, homomorphic images and subalgebras. In particular, the level of subalgebras and that of homomorphic images is not greater than the level of a given algebra.

Proposition 1 *Let A be a nonassociative algebra over F and B a subalgebra of A such that, for some subspace X of A, $A = B \bigoplus X$ and $AX = 0$. Then A and B satisfy the same homogeneous polynomial identities of degree ≥ 2. In particular, A and B satisfy the same shape identities and hence they have the same level.*

Proof: It is enough to observe that if $a = b + x$ and $a' = b' + x'$, where $a, a' \in A$, $b, b' \in B$ and $x, x' \in X$, then $aa' = bb'$. ∎

When a nonassociative algebra A has a nonzero idempotent e and satisfies a shape identity $S = S(x_1, \dots, x_k) = \sum_\mu \alpha_\mu \mu \in S_k(F)$ then $\sum_\mu \alpha_\mu = 0$. It is enough to replace $(x_1, \dots, x_k)$ by

$(e, \ldots, e)$ and observe that $\mu(e, \ldots, e) = e$. We assume, from now on, that $S = S(x_1, \ldots, x_k) = \sum_\mu \alpha_\mu \mu \in S_k(F)$ belongs, in fact, to the hyperplane of $S_k(F)$ defined by $\sum_\mu \alpha_\mu = 0$. This assumption is due to the importance of idempotents in genetic algebra theory.

Proposition 2 *For a baric algebra (A, ω) to satisfy a shape identity $S = S(x_1, \ldots, x_k)$ it is enough that $S(a_1, \ldots, a_k) = 0$ for all a_i such that $\omega(a_i) = 1$.*

Proof: Is is a consequence of the multilinearity of S. Take any elements $b_1, \ldots, b_k \in A$. If $\omega(b_i) \neq 0$, then $a_i = \frac{1}{\omega(b_i)} b_i$ has weight 1. If $\omega(b_i) = 0$, write b_i as a difference of two elements of weight 1. ∎

Suppose we have a family (A_i, ω_i) of baric algebras over the field F, all of them satisfying the same shape identity $S = S(x_1, \ldots, x_k)$. Then their direct product $\prod_i A_i$ also satisfies this identity. The subset of $\prod_i A_i$, consisting of all families (a_i) such that $\omega_i(a_i)$ is constant relative to i is clearly a subalgebra of $\prod_i A_i$ (so also satisfies S) and moreover, it has a weight function ω given by $\omega((a_i)) = \omega_i(a_i)$. This baric algebra is called the direct product of the family (A_i, ω_i). This is summarized in the

Proposition 3 *The direct product of any family of baric algebras satisfying a shape identity also satisfies this identity.*

Theorem 1 *Every baric algebra (A, ω) such that $\ker\omega$ is nilpotent of index k satisfies at least $k' = \frac{1}{k}\binom{2k-2}{k-1}$ linearly independent shape identities in $3k$ variables.*

Proof: For a nonassociative algebra A, denote $a = a(x, y, z)$ the associator of x, y and z, that is, $a = a(x, y, z) = (xy)z - x(yz)$, so $\omega(a(x, y, z)) = 0$ when A is baric. Consider the formal product of the k associators in the free nonassociative algebra $F\{x_1, \ldots, x_n, \ldots\}$:

$$a_1(x_1, x_2, x_3)a_2(x_4, x_5, x_6) \ldots a_k(x_{3k-2}, x_{3k-1}, x_{3k})$$

There are k' different arrangements of parentheses in this product, thus resulting in k' different multilinear polynomials $S_1, \ldots, S_{k'}$ in $x_1, \ldots, x_{3k}$, such that, when each S_i is expressed as a linear combination of the monomials in $x_1, \ldots, x_{3k}$, these variables appear always in natural order. So each S_i ($i = 1, \ldots, k'$) is a shape polynomial in $3k$ variables, that is, $S_i \in S_{3k}(F)$. As $ker\omega$ is nilpotent of index k, we have $S_i(a_1, \ldots, a_{3k}) = 0$ for all $a_1, \ldots, a_{3k}$ in A and $i = 1, \ldots, k'$. The linear independence of the shape polynomials $S_1, \ldots, S_{k'}$ can be deduced from the fact that each S_i is expressed as a linear combination of a set of 2^k monomials of degree $3k$ in the variables $x_1, \ldots, x_{3k}$ (with coefficients 1 or -1) and the fact that two different shape polynomials, say S_i and S_j, are expressed as linear combinations of two disjoint sets of these monomials. This property implies that $S_1, \ldots, S_{k'}$ are

This theorem ensures the existence of shape identities (unfortunately, with a large number of variables) for several classes of genetic algebras. Every special train algebra satisfies shape identities, as the kernel of the weight function is nilpotent, by definition. Nuclear Bernstein algebras have also a nilpotent kernel, a remarkable result of Grishkov [3]. Train algebras which are also Jordan algebras have a nilpotent kernel, due to a classical theorem of A.A.Albert. Hence they satisfy shape identities. All genetic algebras in Gonshor's sense also satisfy shape identities. Recently, S.Walcher [8] studied baric algebras which satisfy the equation $(x^2)^2 = \omega(x)^3 x$. They have nilpotent kernels and so the above theorem applies. Finally we observe that train algebras of rank 3 are special train so they satisfy shape identities. Some other examples can be found in the literature.

The question of existence of shape identities for wider classes of genetic algebras is left open.

§ 3. Shape identities for the gametic algebras

It is possible to calculate all shape identities satisfied by these algebras, denoted here by $G(n+1,2)$, due to their extremely simple algebraic structure. The product of x and y is given by

$$2xy = \omega(x)y + \omega(y)x \tag{5}$$

In particular, if $\omega(x) = \omega(y) = 1$ then

$$2xy = y + x \tag{6}$$

There do not exist shape identities in 3 variables, as this would mean associativity for $G(n+1,2)$. Let $\mu = \mu(x_1, \dots, x_k)$ be a multilinear monomial in $S_k(F)$ and take elements $a_1, \dots, a_k \in G(n+1,2)$, with $\omega(a_i) = 1$. The value of μ on $(a_1, \dots, a_k)$ is given by the following generalization of (6):

$$\mu(a_1, \dots, a_k) = \sum_{i=1}^{k} \beta_i(\mu) a_i \tag{7}$$

where $\beta_i(\mu)$ are suitable elements in the field F, which depend essentially on the arrangement of parentheses in μ. The precise description of these elements is the following. Given the monomial μ, we construct a $(3n-4)$-uple of integers as follows: replace in μ each right turned parenthesis by 1, each of the variables by 0 and each left turned parenthesis by -1. By summing up the coordinates of the above $(3n-4)$-uple, from left to right, until we reach the i-th zero, we get a non negative integer s_i. Then $\beta_i(\mu)$ is the multiplicative inverse in the field F of 2^{s_i+1}. In particular, $\beta_i(\mu)$ is never 1. It is also clear that $\sum_{i=1}^{k} \beta_i(\mu) = 1$, for every monomial $\mu \in S_k(F)$. For this, it is enough to apply ω to (7).

We construct a matrix B_k over the field F, of type $k' \times k$, where the i-th row is the sequence $(\beta_1(\mu), \dots, \beta_k(\mu))$ and where μ is the i-th monomial, in the ordering described in § 2, of the set of all monomials in k variables. Then B_1, B_2, B_3 and B_4 are given by:

$B_1 = (1)$ (by convention)

$$B_2 = \begin{pmatrix} 1/2 & 1/2 \end{pmatrix}$$

$$B_3 = \begin{pmatrix} 1/4 & 1/4 & 1/2 \\ 1/2 & 1/4 & 1/4 \end{pmatrix}$$

$$B_4 = \begin{pmatrix} 1/8 & 1/8 & 1/4 & \vdots & 1/2 \\ 1/4 & 1/8 & 1/8 & \vdots & 1/2 \\ \cdots & \cdots & \cdots & & \cdots \\ 1/4 & 1/4 & \vdots\ 1/4 & & 1/4 \\ \cdots & \cdots & \cdots & & \cdots \\ 1/2 & \vdots\ 1/8 & 1/8 & & 1/4 \\ 1/2 & \vdots\ 1/4 & 1/8 & & 1/8 \end{pmatrix}$$

In order to describe B_k in block form, as suggested by B_4, we introduce some ad hoc notation. If X is any r $\times$ s matrix over F, and p is a non negative integer, then X^P is the following matrix of type pr $\times$ s :

$$X^P = \begin{pmatrix} X \\ X \\ \vdots \\ X \end{pmatrix}$$

where we have p copies of X. Then X_t^P will denote the matrix obtained from X^P by the following permutation of rows : collect in X^P all the

p copies of the first row of X, which appear in rows 1, r+1, 2r+1, ... of

X^P and consider them as the first p rows of X_t^P. Repeat the same procedure with the p copies of the second row of X, which appear in rows 2, r+2, 2r+2, ... of X^P, considering them as rows p+1, ..., 2p of X_t^P and so on, with the remaining rows of X. The fact that each monomial μ in the variables $x_1, \ldots, x_k$ can be uniquely expressed as a product of two monomials $\mu_1(x_1, \ldots, x_r)\mu_2(x_{r+1}, \ldots, x_k)$, for some r, will produce the following block form for $2B_k$:

$$2B_k = \begin{bmatrix} B_{k-1}^{1'} & (B_1)_t^{(k-1)'} \\ B_{k-2}^{2'} & (B_2)_t^{(k-2)'} \\ B_{k-3}^{3'} & (B_3)_t^{(k-3)'} \\ \vdots & \vdots \\ B_2^{(k-2)'} & (B_{k-2})_t^{2'} \\ B_1^{(k-1)'} & (B_{k-1})_t^{1'} \end{bmatrix}$$

The proof of the following lemma (omitted here) uses induction on k and the block form of B_k.

Lemma 1 : *If $k \geq 4$ the rank of $2B_k$ (and hence that of B_k) is k.*

Theorem 2 *The subspace of all elements $S = \sum_\mu \alpha_\mu \mu \in S_k(F)$ which vanish on $G(n+1,2)$ has dimension $k' - k$, when $k \geq 4$. In particular, there is, up to a scalar factor, just one shape identity in four variables in $G(n+1,2)$, namely*

$$S = -2\mu_1 + \mu_2 + 2\mu_3 + \mu_4 - 2\mu_5 \tag{8}$$

where $\mu_1, \dots, \mu_5$ are given by (2). As a consequence, the level of $G(n+1,2)$ is 4, for every $n \geq 1$.

Proof: Suppose $S = \sum_{\mu} \alpha_\mu \mu \in S_k(F)$ is a shape identity for $G(n+1,2)$. Then for arbitrary elements $a_1, \dots, a_k$ of weight 1 in $G(n+1,2)$, we have

$0 = S(a_1, \dots, a_k) = \sum_\mu \alpha_\mu \mu(a_1, \dots, a_k) = \sum_\mu \alpha_\mu \sum_{i=1}^k \beta_i(\mu) a_i =$
$= \sum_{i=1}^k \sum_\mu \alpha_\mu \beta_i(\mu) a_i$. This implies that $\sum_\mu \alpha_\mu \beta_i(\mu) = 0$, i = 1, ..., k. By our lemma, this system of k linear equations in k'

unknowns α_μ has rank k, so $k' - k$ of the total number of unknowns will be free, proving the statement. When $k = 4$, $k' = 5$ so we are left with just one identity, up to a scalar factor; a direct calculation, omitted here, shows that $S = -2\mu_1 + \mu_2 + 2\mu_3 + \mu_4 - 2\mu_5$ vanishes in $G(n+1,2)$ for every n. ∎

The following corollary covers the case of independent loci.

Corollary *The direct product of any family of gametic algebras for simple Mendelian inheritance at one locus satisfies the shape identity* (9).

Proof: Apply Proposition 3. ∎

§ 4. Duplication

The concept of duplicate algebra was introduced by I.M.H. Etherington in order to obtain an algebraic model for zygotes once the corresponding model for gametes is known. See [2] and [9].

Let (A, ω) be a commutative baric algebra and A^D its commutative duplicate. As in [6], we consider the exact sequence

$$0 \longrightarrow N(A) \longrightarrow A^D \xrightarrow{\mu} A^2 \longrightarrow 0$$

where μ is the unique homomorphism sending a generator $x.y$ of A^D to $xy \in A^2$. If $\eta: A^2 \longrightarrow A^D$ is a linear mapping such that $\mu \circ \eta = \mathrm{id}_{A^2}$, let $\varphi: A^2 \times A^2 \longrightarrow A^D$ be defined by $\varphi(x,y) = \eta(x)\eta(y) - \eta(xy)$. Then every element of A^D can be represented by an ordered pair (x,y) where $x \in A^2$ and $y \in N(A)$ and the product in A^D can be represented by

$$(x,y)(x',y') = (xx', \varphi(x,x')). \tag{9}$$

This means A^D is a semi-direct product of A^2 and $N(A)$, denoted $A^2 \times_{s.d} N(A)$.

Theorem 3 *Let (A,ω) be a commutative baric algebra and suppose A^2 satisfies a polynomial identity $f = f(x_1, \dots, x_k) = 0$. Then A^D satisfies the polynomial identity $f' = f(x_1, \dots, x_k)x_{k+1} = 0$. In particular, if A^2 satisfies a shape identity $S = S(x_1, \dots, x_k)$, then A^D satisfies the shape identity $S' = S(x_1, \dots, x_k)x_{k+1}$, and so the level of A^D is 1 plus the level of A^2.*

Proof: We give the proof for the particular case of shape identities but obviously the same argument works for arbitrary polynomial identities. If $\mu = \mu(x_1, \ldots, x_k)$ is a monomial of $S_k(F)$, and $\mu = \mu_1(x_1, \ldots, x_s)\mu_2(x_{s+1}, \ldots, x_k)$, then for $z_i = (x_i, y_i) \in A^2 \times_{s.d} N(A)$, we have the following equality, which generalizes (10):

$$\mu(z_1, \ldots, z_k) = (\mu(x_1, \ldots, x_k), \varphi(\mu_1(x_1, \ldots, x_s), \mu_2(x_{s+1}, \ldots, x_k))).$$

If $S = \sum_\mu \alpha_\mu \mu \in S_k(F)$, then

$$S(z_1, \ldots, z_k) = (S(x_1, \ldots, x_k), z)$$

where z is a rather complicated element in $N(A)$ depending on the coefficients α_μ and of the factorizations of μ. But if S is an identity in A^2, we have

$$S(z_1, \ldots, z_k) = (0, z)$$

and so $S(z_1, \ldots, z_k)z_{k+1} = 0$, for every $z_{k+1} \in A^D$ because $A^D N(A) = 0$. ■

Corollary 1 *The zygotic and copular algebras for simple Mendelian inheritance at one locus with an arbitrary number of alleles satisfy respectively the shape identities in 5 and 6 variables*

$$(-2\mu_1 + \mu_2 + 2\mu_3 + \mu_4 - 2\mu_5)x_5 = 0 \text{ and } ((-2\mu_1 + \mu_2 + 2\mu_3 + \mu_4 - 2\mu_5)x_5)x_6 = 0.$$

Proof: It is enough to apply twice the Theorem 3. ■

Corollary 2 *The direct product of a family of zygotic (rep. copular) algebras for simple Mendelian inheritance at one locus with an arbitrary number of alleles satisfies the shape identity*

$$(-2\mu_1 + \mu_2 + 2\mu_3 + \mu_4 - 2\mu_5)x_5 = 0 \quad (\text{resp. } ((-2\mu_1 + \mu_2 + 2\mu_3 + \mu_4 - 2\mu_5)x_5)x_6 = 0).$$

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Embedding nil algebras in train algebras[1]

Henrique Guzzo Jr.

Dedicated to the memory of Philip Holgate

§ 1. Preliminaries

In [1], Abraham gave the first example of a commutative train algebra which is not special triangular. A second example was given later by Holgate [5]. Basically Abraham used an example (due to Suttles [7]) of a commutative nil algebra A of nil index 4 which is not nilpotent, to which he added a unity (resp. an idempotent e such that $ea = \frac{1}{2}a$) thus obtaining a train algebra of rank 5 (resp. of rank 4) which is not special triangular. We intend in this note to explore further this idea, in the more general context of non commutative algebras of arbitrary right nil index n and possibly infinite dimensional. We prove, in particular, that every right nil algebra of nil index n can be embedded as the kernel of a right train algebra of rank n or $n+1$. We should observe that every commutative nilpotent algebra A of dimension n can be embedded trivially in many ways in a special triangular, hence train, algebra. In fact, from the sequence $A \supseteq A^2 \supseteq \ldots \supseteq A^{k-1} \supseteq A^k = 0$ we obtain by the usual lifting process a basis $c_1, \ldots, c_n$ such that for $i < j$, $c_i c_j$ is a linear combination of $c_{j+1}, \ldots, c_n$. Then we add an idempotent c_0 such that $c_0 A^i \subseteq A^i$ for all $i = 1, \ldots, k$. This contruction can be done also for non commutative nil algebras, thus yelding right or left train algebras.

We will omit henceforth the word "right". The case of left nil algebras is treated similarly. For basic facts about genetic algebra theory, the reader can consult [8] or [4].

Let $\mathbb{N}$ be the set of natural numbers and F a field. We define recursively two functions from $\mathbb{N} \times \mathbb{N} \times F$ to F, denoted by $(n, j; r)$ and $[n, j; r]$, by:

i) For $n < j$, $(n, j; r) = 0$; for $n \geq j$, $(n, 0; r) = 1$ and

$$(n, j; r) = (-1)^j \left(\binom{n-1}{j} r^j + \binom{n-1}{j-1} r^{j-1} \right) \quad \text{if } j \geq 1.$$

ii) For $n < j$, $[n, j; r] = 0$; for $n \geq j$, $[n, 0; r] = r^{1-\delta_{0n}}$ and

$$[n, j; r] = \sum_{i=0}^{n-j} [n-1-i, j-1; r] r^i \quad \text{if } j \geq 1,$$

where δ_{ij} is Kronecker's symbol. Observe that for $r = 1$, $(n, j; 1) = (-1)^j \binom{n}{j}$.

These functions satisfy the following identities, which are proved either by induction or either directly from the definition:

For all $n, j \in \mathbb{N}$ and $r \in F$, we have the equalities:

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[1] This paper will appear in Proceedings of the Edinburgh Mathematical Society

a) $[n,n;r]=1;$

b) $[n,n-1;r]=nr$ $\quad (n\geq 1);$

c) $(n,j;r)=(n-1,j;r)-(n-1,j-1;r)r$ $\quad (n\geq 2,\ j\geq 1);$

d) $[n,j;r]=[n-1,j;r]r+[n-1,j-1;r]$ $\quad (n\geq 1,\ j\geq 1).$

Is a similar way we prove that:

e) $\sum_{j=0}^{k}(n,j;r)=(-1)^k\binom{n-1}{k}r^k$ $\quad (0\leq k\leq n-1);$

f) $\sum_{j=0}^{n}(n,j;r)=0$ $\quad (n\geq 1);$

g) $\sum_{i=0}^{n-j}[n-i,j;r](n-1,i;r)=0$ $\quad (0\leq j\leq n-2);$

h) $\sum_{i=1}^{n}(-1)^{i-1}(2r-1)^{i-1}[n,i;r]=1$ $\quad (n\geq 1).$

Recall that a baric algebra over the field F is an ordered pair (A,ω) where A is any algebra over F and $\omega\colon A\longrightarrow F$ is a nonzero homomorfism. The set $N=\{x\in A:\omega(x)=0\}$ is a two sided ideal of A, of codimension 1. If c is any element of A such that $\omega(c)=1$, we have a direct sum decomposition $A=Fc\bigoplus N$, due to the equality $a=\omega(a)c+(a-\omega(a)c)$, for all $a\in A$. If A has an idempotent e such that $\omega(e)=1$ (this happens for most of the relevant examples in the theory of genetic algebras) then $A=Fe\bigoplus N$ so for all $b\in A$, $b=\omega(b)e+a$, where $a\in N$. For $b\in A$ let b^k, the (right) principal power of b, be defined by $b^1=b$, $b^{k+1}=b^kb$ for $k\geq 1$. When there are elements $\gamma_1,\ldots,\gamma_{n-1}\in F$ such that

$$b^n+\gamma_1\omega(b)b^{n-1}+\ldots+\gamma_{n-1}\omega(b)^{n-1}b=0,\quad \text{for all } b\in A, \tag{1}$$

A will be called a principal train algebra. The rank of (A,ω) is the degree of the equation (1) of minimal degree satisfied by (A,ω). Our next proposition will enable us to treat the problem stated in the preliminaries. We remark that when (A,ω) satisfies $b^n+\gamma_1\omega(b)b^{n-1}+\ldots+\gamma_{n-1}\omega(b)^{n-1}b=0$ then necessarily $1+\gamma_1+\ldots+\gamma_{n-1}=0$. For this, apply ω to this identity, with $\omega(b)=1$.

Proposition 1 *Suppose we are given a baric algebra (A,ω) with a central idempotent e of weight 1 such that $ea=ra$ $(=ae)$, for some fixed $r\in F$, for all $a\in N=\ker\omega$. Then for a given $b=\omega(b)e+a$ and for given elements $\gamma_0=1,\gamma_1,\ldots,\gamma_{n-1}\in F$, we have*

$$b^n+\gamma_1\omega(b)b^{n-1}+\ldots+\gamma_{n-1}\omega(b)^{n-1}b=\left(\sum_{i=0}^{n-1}\gamma_i\right)\omega(b)^ne+\sum_{j=1}^{n}\left(\sum_{i=0}^{n-j}[n-i,j;r]\gamma_i\right)\omega(b)^{n-j}a^j.$$

Proof (sketch): We proceed by induction, the case $n=2$ being trivial. Then

$$b^{n+1}+\gamma_1\omega(b)b^n+\ldots+\gamma_n\omega(b)^nb=[b^n+\gamma_1\omega(b)b^{n-1}+\ldots+\gamma_{n-1}\omega(b)^{n-1}b]\,b+\gamma_n\omega(b)^nb=$$

$$=\left[\left(\sum_{i=0}^{n-1}\gamma_i\right)\omega(b)^ne+\sum_{j=1}^{n}\left(\sum_{i=0}^{n-j}[n-i,j;r]\gamma_i\right)\omega(b)^{n-j}a^j\right](\omega(b)e+a)+\gamma_n\omega(b)^n(\omega(b)e+a).$$

The rest of the proof is obtained by combining the properties of the function $[n,\,j;\,r]$ stated above. We omit the details, which are not interesting in their own. ∎

The following lemma is well known:

Lemma *Let A be a right nil algebra of nil index $s\geq 2$. If $a\in A$ and $a^{s-1}\neq 0$, then $a,a^2,\ldots,a^{s-1}$ are linearly independent.*

§ 2. The case $2r\neq 1$

Theorem 1 *Let (A,ω) be a baric algebra, $e\in A$ a central idempotent of weight 1 such that $ea=ra$ $(=ae)$ for all $a\in\ker\omega$. Suppose that $\ker\omega$ is a right nil algebra of nil index $s\geq 2$. Then for every $b\in A$, we have $b^{s+1}+\gamma_1\omega(b)b^s+\ldots+\gamma_s\omega(b)^sb=0$ where $\gamma_i=(s,\,i;\,r)$ for $i=0,1,\ldots,s$. Moreover, no other similar relation involving the powers $b,b^2,\ldots,b^s$ will hold and hence the rank of A is $s+1$.*

Proof: From Proposition 1, for all $b\in A$, $b=\omega(b)e+a$, $a\in\ker\omega$, we have

$$b^{s+1}+\gamma_1\omega(b)b^s+\ldots+\gamma_s\omega(b)^sb=\sum_{j=1}^{s+1}\left(\sum_{i=0}^{s+1-j}[s+1-i,j;r]\gamma_i\right)\omega(b)^{s+1-j}a^j,$$

because $1+\gamma_1+\ldots+\gamma_s=\sum_{i=0}^{s}(s,\,i;\,r)=0$ by equation f) above. But in the above sum, $a^s=a^{s+1}=0$ so that

$$b^{s+1}+\gamma_1\omega(b)b^s+\ldots+\gamma_s\omega(b)^sb=\sum_{j=1}^{s-1}\left(\sum_{i=0}^{s+1-j}[s+1-i,j;r]\gamma_i\right)\omega(b)^{s+1-j}a^j.$$

¿From equation g), we have

$$\sum_{i=0}^{s+1-j}[s+1-i,j;r]\gamma_i=\sum_{i=0}^{s+1-j}[s+1-i,j;r](s,i;r)=0\quad\text{for}\quad 1\leq j\leq s-1.$$

This shows that (A,ω) is a train algebra of rank $\leq s+1$. We show that $s+1$ is a minimal degree. Suppose, to the contrary, that for some $t\leq s+1$ and constants $\alpha_1,\ldots,\alpha_{t-1}$ in F we have $b^t+\alpha_1\omega(b)b^{t-1}+\ldots+\alpha_{t-1}\omega(b)^{t-1}b=0$ for all $b\in A$. In particular, when $\omega(b)=0$, we have $b^t=0$ and so $s\leq t$. We are left with two possibilities: $t=s$ or $t=s+1$. Suppose we have $t=s$. Choose $a\in\ker\omega$ with $a^{s-1}\neq 0$ so that $a,a^2,\ldots,a^{s-1}$ are linearly independent. Apply to the element $b=e+a$ the equality in Proposition 1 to get $0=b^s+\alpha_1b^{s-1}+\ldots+\alpha_{s-1}b=$
$\sum_{j=1}^{s}\left(\sum_{i=0}^{s-j}[s-i,j;r]\alpha_i\right)a^j=$

$$= \sum_{j=1}^{s-1}\left(\sum_{i=0}^{s-j}[s-i,j;r]\alpha_i\right)a^j \quad \text{as} \quad a^s = 0.$$

We have now a triangular system of linear equations in the unknowns α_i:

$$\sum_{i=0}^{s-j}[s-i,j;r]\alpha_i = 0, \quad 1 \le j \le s-1.$$

Multiply the j-th equation by $(-1)^{j-1}(2r-1)^{j-1}$ and sum up all the resulting equations to get $\sum_{j=1}^{s-1}\left(\sum_{i=0}^{s-j}(-1)^{j-1}(2r-1)^{j-1}[s-i,j;r]\alpha_i\right) = 0$. Reordering the summands, we get $\sum_{i=1}^{s-1}\left(\sum_{j=1}^{s-i}(-1)^{j-1}(2r-1)^{j-1}[s-i,j;r]\right)\alpha_i + \sum_{j=1}^{s-1}(-1)^{j-1}(2r-1)^{j-1}[s,j;r] = 0.$

By equation h), we have

$$0 = -(-1)^{s-1}(2r-1)^{s-1}[s,s;r] + 1 + \alpha_1 + \ldots + \alpha_{s-1} = (-1)^s(2r-1)^{s-1}[s,s;r],$$

and by equation a), we have $[s, s; r] = 1$ and so $2r = 1$, a contradiction. We must have then $t = s+1$, which proves that the rank of A is $s+1$. ■

Corollary 1 (Abraham [1]) *Suppose (A,ω) is a baric algebra with a unity element 1_A and $\ker\omega$ is nil of nil index $s \ge 2$. Then (A,ω) is a train algebra of rank $s+1$, satisfying the equation $(x-\omega(x)1_A)^s x = 0$.*

Proof: Our theorem, with $e = 1_A$ and $r = 1$, says that (A,ω) is a train algebra of rank $s+1$ and the coefficients γ_i of its train equation are given by:

$$\gamma_i = (s,i;1) = (-1)^i\left(\binom{s-1}{i} + \binom{s-1}{i-1}\right) = (-1)^i\binom{s}{i}, \quad \text{for } 0 \le i \le s.$$

Then for every $b \in A$, $0 = b^{s+1} + \gamma_1\omega(b)b^s + \ldots + \gamma_s\omega(b)^s b =$

$$= \left(b^s + \binom{s}{1}(-\omega(b))b^{s-1} + \ldots + \binom{s}{s-1}(-\omega(b))^{s-1} + (-\omega(b))^s 1_A\right)b =$$

$$= (b - \omega(b)1_A)^s\, b.$$ ■

Given any algebra N over the field F and a linear mapping $\tau: N \longrightarrow N$, we can obtain a baric algebra with a central idempotent of weight 1 in the following way (see [2] for details): take $A = Fe \oplus N$ and the product and a weight function ω given by

$$(\alpha,a)(\beta,b) = (\alpha\beta, ab + \tau(\alpha b + \beta a)); \quad \omega(\alpha,a) = \alpha,$$

where $\alpha,\beta \in F$; $a,b \in N$. Then $(1,0) = e$ satisfies the above conditions. If N is nil of nil index s and τ is the homothety $h_r: a \mapsto ra$, where $2r \ne 1$, $r \in F$, then the resulting baric algebra satisfies the equation of Theorem 1. We denote this algebra by $[N, h_r]$.

Corollary 2 *Suppose M and N are two nil algebras of nil indices m and n respectively, $m, n \geq 2$. Suppose $r, s \in F$ satisfy $2r \neq 1$ and $2s \neq 1$. Then $[M, h_r]$ and $[N, h_s]$ are isomorphic (as baric algebras) if and only if M and N are isomorphic and $r = s$.*

Proof: Suppose $[M, h_r]$ and $[N, h_s]$ are isomorphic, let φ be an isomorphism. By [2. Prop.1 and Corollary] there exist an isomorphism $\theta\colon M \longrightarrow N$ and $c \in N$ such that $\varphi(\alpha, a) = (\alpha, \alpha c + \theta(a))$, $c^2 + 2sc = c$ and the diagram

$$\begin{array}{ccc} M & \xrightarrow{\theta} & N \\ {\scriptstyle h_r}\downarrow & & \downarrow{\scriptstyle h_s + L_c} \\ M & \xrightarrow{\theta} & N \end{array}$$

is commutative. But $c^2 + 2sc = c$ implies $c^n = (1 - 2s)^{n-1}c$ and so $c = 0$. From the diagram, for all $a \in M$, $r\theta(a) = s\theta(a)$ with means necessarily $r = s$.

The proof of the converse is trivial. ∎

§ 3. The case $\mathbf{r} = \frac{1}{2}$

We assume now the characteristic of F is not 2. The following theorem is a generalization of Abraham's example in [1].

Theorem 1′ *Suppose (A, ω) is a baric algebra, $e \in A$ is a central idempotent of weight 1 such that $ea = \frac{1}{2}a$ $(= ae)$. Suppose that $N = \ker\omega$ is nil of nil index $s \geq 2$. Then for every $b \in A$, we have $b^s + \gamma_1\omega(b)b^{s-1} + \dots + \gamma_{s-1}\omega(b)^{s-1}b = 0$ where $\gamma_i = \left(s-1, i; \tfrac{1}{2}\right)$ for $i = 0, 1, \dots, s-1$. Moreover, no other similar relation involving the powers $b, \dots, b^{s-1}$ will hold and so the rank of A is s.*

Proof: For $b = \omega(b)e + a$, $b \in A$ and $a \in \ker\omega$, we have, as in the proof of Theorem 1,

$$b^s + \gamma_1\omega(b)b^{s-1} + \dots + \gamma_{s-1}\omega(b)^{s-1}b = \sum_{j=1}^{s-1}\left(\sum_{i=0}^{s-j}[s-i, j; \tfrac{1}{2}]\gamma_i\right)\omega(b)^{s-j}a^j \quad \text{as } a^s = 0.$$

¿From equation f),

$$\sum_{i=0}^{s-j}[s-i, j; \tfrac{1}{2}]\gamma_i = \sum_{i=0}^{s-j}[s-i, j; \tfrac{1}{2}](s-1, i; \tfrac{1}{2}) = 0, \quad 1 \leq j \leq s-2.$$

When $j = s-1$, we have,

$$\sum_{i=0}^{1}[s-i, s-1; \tfrac{1}{2}](s-1, i; \tfrac{1}{2}) = [s, s-1; \tfrac{1}{2}](s-1, 0; \tfrac{1}{2}) + [s-1, s-1; \tfrac{1}{2}](s-1, 1; \tfrac{1}{2}) =$$

$$= \frac{s}{2} + (-1)\left(\binom{s-2}{1}\frac{1}{2} + \binom{s-2}{0}\right) = 0.$$

This proves that $b^s + \gamma_1\omega(b)b^{s-1} + \ldots + \gamma_{s-1}\omega(b)^{s-1}b = 0$ for all $b \in A$. No other similar relation, involving only $b^{s-1}, \ldots, b$ can hold because this would imply that all elements b of weight 0 would satisfy $b^{s-1} = 0$, a contradiction to the nil index of $\ker\omega$. ■

It is obvious that if N is nil algebra of nil index s, then $\left[N, h_{\frac{1}{2}}\right]$ satisfies the equation of Theorem1′.

We can easily prove the following:

Corollary *Suppose M and N are F-algebras. Then $\left[M, h_{\frac{1}{2}}\right]$ and $\left[N, h_{\frac{1}{2}}\right]$ are isomorphic (as baric algebras) if and only if M and N are isomorphic.*

§ 4. Complements

In this paragraph, we collect some properties of the algebras $[M, h_r]$. For any baric algebra (A, ω), the ideal generated by all the elements $x^2 - \omega(x)x$, $x \in A$ is called the Etherington's ideal of A. For several classes of baric algebras, we can describe more explicitly this ideal, see, for instance, [7, Lemma 9.19] for Bernstein algebras.

Proposition 2 *For a given algebra M over F, the Etherington's ideal of $[M, h_r]$ is M when $r \neq \frac{1}{2}$ and is the ideal generated by the squares of elements of M, when $r = \frac{1}{2}$.*

Proof: For $b = \omega(b)e + a$ (notations as above), we have $b^2 - \omega(b)b = a^2 + \omega(b)(2r-1)a$. When $r = \frac{1}{2}$, we have the desired result. When $r \neq \frac{1}{2}$, we have $b^2 - \omega(b)b - a^2 = \omega(b)(2r-1)a$ so a belongs to the above ideal, that is, M is contained in the Etherington's ideal. The converse inclusion is obvious. ■

In [2], the authors have introduced the concept of decomposable baric algebra. It means that $N = \ker\omega$ can be decomposed as a direct sum of two nonzero two sided ideals of the baric algebra, contained in N. In our case, we have obviously the following result.

Proposition 3 *$[M, h_r]$ is decomposable if and only if M is decomposable as a direct sum $M = M_1 \oplus M_2$ where M_i are nonzero two sided ideals of M.*

Remark: Suppose (A, ω) is a baric algebra having a central idempotent e of weight 1 such that $ea = ra$ $(= ae)$ for all $a \in \ker\omega$, for some $2r \neq 1$, $r \in F$. Suppose moreover that $\ker\omega$ is nil of nil index $s \geq 2$. Let e' be another idempotent of weight 1, say $e' = e + c$, $c \in \ker\omega$. Then $e + c = e' = e'^2 = e + 2ce + c^2 = e + 2rc + c^2$, so $c^2 = (1-2r)c$. From this we have $c^k = (1-2r)^{k-1}c$ for $k \geq 2$. By the nil property, $0 = c^s = (1-2r)^{s-1}c$ so $c = 0$, and A has a unique idempotent of weight 1.

In the case $r = \frac{1}{2}$, we may have many idempotents; moreover, for another idempotent e', it may happen that $e'a \neq \frac{1}{2}a$. For an example, go back to Abraham's example in [1] and [5]. It is easy to see that $e_0 = c_0 + c_1$ is an idempotent of weight 1 and $e_0c_3 = \frac{1}{2}c_3 + c_4$.

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Torus Actions on Rings[1]

Nikolaus Vonessen

Let R be an algebra over a fixed algebraically closed base field, and denote by $\mathrm{Aut}(R)$ its group of algebra automorphisms. The rank of $\mathrm{Aut}(R)$ is the dimension of the largest torus it contains, i.e., the dimension of the largest torus which acts faithfully on R. In analogy to results from algebraic and differential geometry, we prove bounds on the rank of $\mathrm{Aut}(R)$ for certain 'free' algebras R, e.g., (non-)commutative polynomial algebras and algebras of generic matrices. In these cases, if R is generated by m 'indeterminates', then the rank of $\mathrm{Aut}(R)$ is $\leq m$. If the actions of the tori are additionally assumed to be rational, we obtain bounds for some larger classes of algebras, e.g., prime PI-algebras.

[1]Joint work with Zinovy Reichstein.

On the Noether Problem and Universal Division Algebras

Zinovy Reichstein[1]

Abstract We discuss a connection between two longstanding open problems in algebra: the Noether Conjecture and the Crossed Product Problem. We introduce both problems and explain in what way they are related. The results presented in Section 4 are based on joint work with Nikolaus Vonessen [RV2].

1 Quotients

In this section we discuss some preliminary notions and results used in the formulation of the Noether Problem. For a more detailed discussion of this material we refer the reader to [DC] or [Sp].

A complex algebraic group G is called *reductive* if every representation of G (finite or infinite-dimensional) decomposes as a direct sum of irreducible representations. Every finite group is reductive by Maschke's theorem. On the other hand, the group of all non-singular upper-triangular $n \times n$-matrices is not reductive because its natural representation on $\mathbf{C}^n$ is not isomorphic to a direct sum of irreducible representations.

Examples of infinite reductive groups are $\mathbf{C}^*$, $\mathrm{GL}_n(\mathbf{C})$, $\mathrm{SL}_n(\mathbf{C})$, $O_n(\mathbf{C})$, $\mathrm{PGL}_n(\mathbf{C})$; see [W, Ch. VIII, sect. 11]. The group PGL_n will be of special interest to us in the sequel. Recall that it is defined as the quotient of GL_n by the subgroup of scalar matrices; it can also be viewed as the automorphism group of the $n-1$-dimensional projective space $\mathbf{P}^{n-1}(\mathbf{C})$.

¿From now on we will only consider reductive complex algebraic groups G. Suppose we are given a finite-dimensional complex representation $r : G \longrightarrow \mathrm{GL}(V)$. We can view elements of G as coordinate changes in V. If we want to describe elements of V modulo these coordinate changes, we have to construct some sort of a quotient of V by G.

The space of orbits with the natural topology is likely to be highly irregular. For example, consider the natural action of $\mathbf{C}^*$ on $V = \mathbf{C}$. This action has two orbits, and the resulting two point orbit space is not Hausdorff. For this reason one usually considers the categorical quotient $V//G$ instead of the orbit space. The categorical quotient is an algebraic variety which parameterises the closed orbits for the action. It is constructed as follows.

Recall that a function f on V is called *G-invariant* (or simply *invariant* if the reference to the representation is clear from the context) if $f(x) = f(gx)$ for every $g \in G$. Here gx denotes the result of applying the linear transformation $r(g)$ to the vector $x \in V$. The ring of all polynomial G-invariant functions on V is denoted by $\mathbf{C}[V]^G$.

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[1] Partially supported by an NSF postdoctoral fellowship.

Theorem 1.1 (Hilbert, [DC, Ch. 3], [Sp, 2.4]) *$\mathbf{C}[V]^G$ is finitely generated as a $\mathbf{C}$-algebra.*

This result is sometimes called "Hilbert's Main Theorem" partly because it answered a longstanding conjecture, partly because of Hilbert's pioneering use of non-constructive arguments in its proof, and partly because both the Hilbert Basis Theorem and the Nullstellensatz first appeared as its byproducts. Hilbert's original proof was given only for $G = \mathrm{SL}_n$. His arguments were extended to all reductive groups by Weyl [W, Ch. VIII, sect. 14] and, in a more general setting, by Nagata [Na].

The categorical quotient $V//G$ is defined as the spectrum of $\mathbf{C}[V]^G$. Hilbert's theorem asserts that it is an affine variety. It comes equipped with the quotient map $\pi : V \longrightarrow V//G$ which is dual to the natural inclusion of rings $\mathbf{C}[V]^G \hookrightarrow \mathbf{C}[V]$. Arguing as in the proof of Hilbert's theorem one can show that π is surjective and that it separates the closed G-orbits in V. This is the most one can expect of a quotient map in the category of algebraic varieties.

In more concrete terms, let $f_1, \ldots, f_N$ be a set of generators for $\mathbf{C}[V]^G$ or, in classical terminology, a set of of "basic invariants". The quotient map $\pi : V \longrightarrow \mathbf{C}^N$ is then given by sending $x \in V$ to $(f_1(x), \ldots, f_N(x))$. The categorical quotient $V//G$ is the image of this map. Up to isomorphism it is independent of the choice of basic invariants. Of course, the embedding of $V//G$ in $\mathbf{C}^N$ (and, indeed, N itself) does depend of the choice of basic invariants. In fact, the complexity of this embedding depends on the algebraic relations among $f_1, \ldots, f_N$. The simplest case is the one where $f_1, \ldots, f_N$ are algebraically independent. In other words, the ring $\mathbf{C}[V]^G$ is isomorphic to a polynomial ring over $\mathbf{C}$ or, equivalently, $V//G$ is isomorphic to an affine space. Here is a prototypical example for this situation:

Example 1.2 The symmetric group $G = S_n$ acts on $V = \mathbf{C}^n$ by permuting the coordinates. In this case $\mathbf{C}[V]^G$ is called the ring of symmetric polynomials. The elementary symmetric polynomials $s_1, \ldots, s_n$ form an algebraically independent set of basic invariants. The quotient map in this case is given by

$$\begin{aligned} \pi : \quad V = \mathbf{C}^n &\longrightarrow \mathbf{C}^n = V//G \\ x &\longrightarrow (s_1(x), \ldots, s_n(x)) \,. \end{aligned}$$

The cases where $V//G$ is isomorphic to an affine space has been thoroughly investigated; see [KPV], [Sch]. Unfortunately, such actions are rather rare.

Example 1.3 Consider the representation of $G = \mathbb{Z}/2\mathbb{Z}$ on $V = \mathbf{C}^2$ given by sending the non-trivial element of G to $-I_{2\times 2}$. The polynomials $u = x^2, v = y^2$, and $w = xy$ form a system of basic invariants for this action. The categorical quotient is then isomorphic to the quadric surface $uv = w^2$ in $\mathbf{C}^3$. Since this surface is singular at the origin, it cannot be isomorphic to an affine space.

2 The Noether Problem

A closer look at Example 1.3 shows that the situation can be remedied if we allow rational invariants rather than just polynomial ones. Indeed, in this case x/y and y^2 are algebraically independent basic invariants. Can this be done in general? More precisely, we are asking whether or not the fraction field of $\mathbf{C}[V]^G$ is a purely transcendental extension of $\mathbf{C}$ or, equivalently, whether or not $V//G$ is a rational variety. This question is known as the *Noether Problem*. Emmy Noether conjectured in 1913 that $V//G$ is, indeed, rational for any representation of any finite group G; see [?].

In the next seventy years a great deal of work was done on the Noether conjecture over the rationals; see [Sw]. The first counterexample to the Noether conjecture in the form we stated it (i.e. for a finite group acting on a complex vector space) was constructed by David Saltman in 1984; see [Sa]. In the case of connected groups the question remains open to this day.

2.1 (Noether Conjecture, strong form) *Let G be a connected reductive complex algebraic group. Then for any finite-dimensional representation $r : G \longrightarrow \mathrm{GL}_n(V)$ the quotient variety $V//G$ is rational.*

For technical reasons it is often convenient to replace the notion of rationality by the slightly weaker notion of stable rationality. (The fact that the two are distinct is a relatively recent theorem; see [BCSS].) Let X and Y be complex algebraic varieties and let $\mathbf{C}(X)$ and $\mathbf{C}(Y)$ be their fields of rational functions. Then X and Y are called *stably birational* if $\mathbf{C}(X)(t_1,\ldots,t_a)$ is isomorphic to $\mathbf{C}(Y)(s_1,\ldots,s_b)$ for some $a, b \geq 0$. Here $t_1,\ldots,t_a,s_1,\ldots,s_b$ are indeterminates. A variety X is called *stably rational* if it is stably birational to an affine space, i.e. if $\mathbf{C}(X)(t_1,\ldots,t_a)$ is a purely transcendental extension of $\mathbf{C}$ for some $a \geq 0$. If $a = 0$ then X is rational.

Most of the positive results on the Noether Conjecture make use of the following important theorem. Recall that a representation $G \longrightarrow \mathrm{GL}(V)$ is called *generically free* if for x in general position $gx = x$ implies $g = 1$.

Theorem 2.2 (Bogomolov Transversality Theorem [BK]) *Let G be a reductive complex algebraic group and let $r_1 : G \longrightarrow \mathrm{GL}(V)$ and $r_2 : G \longrightarrow \mathrm{GL}(W)$ be generically free finite-dimensional representations. Then the quotients $V//G$ and $W//G$ are stably birational.*

2.3 (Noether Conjecture, weak form) *Let G be a connected reductive complex algebraic group. Then for any generically free finite-dimensional representation $r : G \longrightarrow \mathrm{GL}_n(V)$ the quotient variety $V//G$ is stably rational.*

Incorporating the assumptions of Bogomolov's theorem into the Noether Conjecture weakens it a little bit. While this weaker form of the Noether Conjecture is still open, it is much more manageble than the strong form because it depends only on the group G and not on the representation r. Indeed, it is enough to find one generaically free representation of G for which the ring of invariants can be easily computed. If the quotient $V//G$ is rational in this case then it is stably rational for every generically free representation of G. Here is an example of this approach.

Theorem 2.4 *Conjecture 2.3 holds for*
(a) $G = \mathrm{GL}_n$,
(b) $G = \mathrm{SL}_n$.

Proof. (a) Let V be the vector space of all complex $n \times n$-matrices and let $r : \mathrm{GL}_n \longrightarrow GL(V)$ be given by left multiplication, i.e. for $g \in \mathrm{GL}_n$ and $M \in V$ we have $g(M) = gM$. This representation is generically free because $gM = M$ implies $g = 1$ whenever M is non-singular. In fact, the non-singular matrices form a single orbit which is dense in V. Hence, this action has no non-constant invariants, $V//\mathrm{GL}_n$ is a point and, hence, is rational. Thus $W//\mathrm{GL}_n$ is stably rational for every generic representation of GL_n.

(b) Consider the restriction of the representation r of part (a) to SL_n. It is easy to see that the determinant function on V forms a one-element system of basic invariants. This the quotient variety for this action is $\mathbf{C}$. Now apply Bogomolov's theorem as in part (a). □

The above argument is so simple that one wonders if it can be carried out for other groups G. This has, indeed, been done for some groups; see [B]. However, there are other groups which elude this kind of analysis; Conjecture 2.3 is still open for them. The most interesting ones among them are the groups PGL_n. In this case the Noether Conjecture reduces to the classical rationality problem for matrix invariants. Indeed, let V be the vector space of m-tuples of complex $n \times n$-matrices. Consider the representation of PGL_n on V given by simultaneous conjugation. That is, for $g \in \mathrm{PGL}_n$ and $x = (X_1, \ldots, X_m)$ we have

$$g(x) = (gX_1g^{-1}, \ldots, gX_mg^{-1}) .$$

For $m \geq 2$ this representation is easily seen to be generically free. The quotient $Q_{m,n} = V//\mathrm{PGL}_n$ is a beautiful variety whose structure is rather poorly understood; see [LP], [R1], [R2], [R3]. For example, it is known that the ring of invariants $\mathbf{C}[V]^{\mathrm{PGL}_n}$ is generated by elements of the form

$$\mathrm{tr}(X_{i_1} \ldots X_{i_N}) \tag{1}$$

where $1 \leq i_1, \ldots, i_N \leq m$ and N ranges over the positive integers; see [Si], [Pr2]. However, the relations among these generators are unknown.

We are interested in the following question.

2.5 (Noether Conjecture for PGL_n) *$Q_{m,n}$ is (stably) rational.*

We summarise the known rationality results for $Q_{m,n}$ below.

Theorem 2.6 *Let $Q_{m,n}$ be the variety of m-tuples of $n \times n$-matrices modulo the simultaneous conjugation action of* PGL_n. *Assume $m, n \geq 2$. Then*

(a) (Procesi [Pr1]) *$Q_{m+1,n}$ is birational to $Q_{m,n} \times C^{n^2}$. In particular, for a fixed n the varieties $Q_{m,n}$ are all stably birational to each other.*

(b) (Procesi [Pr1]) *$Q_{m,n}$ is rational for $n = 2$.*

(c) (Formanek [F1], [F2]) *$Q_{m,n}$ is rational for $n = 3, 4$.*

(d) (Bessenrodt-LeBruyn [BL]) *$Q_{m,n}$ is stably rational for $n = 5, 7$.*

(e) (Schofield [Sc], Katsylo [Ka]) *If n_1 and n_2 are relatively prime then Q_{m,n_1n_2} is stably birational to $Q_{m,n_1} \times Q_{m,n_2}$.*

Combining part (e) with the previous parts we see that $Q_{m,n}$ is stably rational for any n which divides 420. For other values of n the Conjecture 2.5 remains open.

3 The Crossed Product Problem

In this section we discuss some well-known facts about division algebras and crossed products, including the Crossed Product Problem. A more detailed and systematic exposition of this theory can be found in [Pi] or [Ro].

Let D be a division algebra which is finite-dimensional over its center K. If L is the algebraic closure of K then

$$D \otimes_K L \simeq M_{n \times n}(L). \tag{2}$$

The integer n is called the degree of D. In particular, $\dim_K(D) = n^2$. A subfield L of D is *maximal* (i.e. is not contained in any other subfield of D) if it contains K and $\dim_K(L) = n$. Using the isomorphism (2) one defines the (reduced) trace and determinant of an element $x \in D$. This definition is intrinsic, i.e. independent of the choice of the isomorphism (2). Moreover, $\mathrm{tr}(x), \det(x) \in K$; see [Pi, 16.1]. In other words, a division algebra is closed under tr and det.

Definition 3.1 *A division algebra D is called a* crossed product *if it contains a maximal subfield L which is Galois over the center K of D.*

The structure of such a division algebra can be described in a particularly simple way. Indeed, let $g \in G = Gal(L, K)$. By the Skolem-Noether theorem there exists a non-zero element $x_g \in D$ such that $g(l) = x_g l x_g^{-1}$ for every $l \in L$. The elements x_g are easily seen to form a basis for D as a left L-vector space as g ranges over G. Moreover, since conjugation by x_{gh} and conjugation by $x_g x_h$ coincide on L, these two elements differ by a factor of $l(g, h) \in L$. Comparing x_{ghk} to $x_g x_h x_k$ one can now check that the resulting cochain $\{l(g, h)\} \in C^2(G, L^*)$ is, in fact, a cocycle. The element of $H^2(G, L^*)$ it defines is independent of the ambiguity involved in the choice of x_g. The structure of D is then completely determined by this cohomology class.

The question of whether or not every division algebra of a given degree is a crossed product can be reduced to studying division algebras of a particular type which we now introduce.

Let $X_1, \ldots, X_m$ be m generic $n \times n$ matrices. This means that $X_l = (x_{ij}^{(l)})$ where $x_{ij}^{(l)}$ are mn^2 commuting indeterminates. Here $m, n \geq 2$. We can think of $X_1, \ldots, X_m$ as elements of the matrix ring $M_n(R)$ where R is the polynomial ring $\mathbf{C}[x_{ij}^{(l)}]$. The $\mathbf{C}$-subalgebra of $M_n(R)$ generated by these m elements is called *the algebra of m generic $n \times n$-matrices.* We will denote it by $G_{m,n}$. By a theorem of Amitsur $G_{m,n}$ is a domain and has a division the division algebra of fractions; see [Pi, Ch. 20]. We will denote this algebra by $D_{m,n}$ and its center by $K_{m,n}$. The algebras $D_{m,n}$ are called *universal division algebras.* Note that in order to invert $z \in G_{m,n}$ we only need to invert $\det(z)$. This is a direct consequence of the Cayley-Hamilton Theorem. Thus $D_{m,n}$ is contained in $M_n(K)$ where K is the fraction field of the polynomial ring R. In particular, the degree of $D_{m,n}$ is n.

Universal division algebras are important because they carry "generic" information about all division algebras of the same degree. Given a division algebra of degree n, we can use (2) to embed it in a matrix algebra and then "specialize" $D_{m,n}$ to it. In particular, if $D_{m,n}$ is a crossed product then every division algebra is a crossed product. In view of this the following question is of fundamental importance in the structure theory of division algebras.

3.2 (The Crossed Product Problem) *Which universal division algebras $D_{m,n}$ are crossed products?*

We summarize the known results in the following theorem.

Theorem 3.3 *Let $D_{m,n}$ be the universal division algebra of m-tuples of $n \times n$-matrices. Here $m, n \geq 2$. Then*

(a) The answer to the the Crossed Product Problem 3.2 is independent of m.

(b) (Wedderburn-Albert [Ro, Thm. 3.2.29]) *$D_{m,n}$ is a crossed product for* $n = 2, 3, 4, 6, 12$.

(c) (Albert [Ro, Thm 3.1.40]) *If n_1 and n_2 are relatively prime and D_{m,n_1}, D_{m,n_2} are crossed products then so is $D_{m,n_1 n_2}$.*

(d) (Amitsur [A] or [Ro, 3.3.12]), *$D_{m,n}$ is not a crossed product if n is divisible by a cube.*

To the best of my knowledge, Problem 3.2 remains open for all other values of n. It is of particular interest in the case where $n \geq 5$ is a prime.

4 Rational Division Algebras

Looking at the statements of Theorems 2.6 and 3.3 it is easy to notice a certain degree of similarity. Lieven LeBruyn [lB] recently asked if his Theorem 2.6(d) has any bearing on the crossed product problem for $D_{m,5}$ and $D_{m,7}$. One can also ask whether or not Amitsur's Theorem 3.3(d) has any bearing on the (stable) rationality of $Q_{m,n}$ when n is divisible by a cube. Answering these questions depends on establishing a formal connection between Problems 2.5 and 3.2. In [RV2] Nikolaus Vonessen and I attempted to link the two problems by introducing the notion of rationality for division algebras.

Our starting point are the following theorems of Procesi and Demazure.

Theorem 4.1 (Procesi [Pr1]) *The function field of $Q_{m,n}$ is isomorphic to $K_{m,n}$, the center of the universal division algebra $D_{m,n}$.*

To see how these two fields are related, recall that as we observed in the beginning of Section 3, every division algebra is closed under tr. That is, for any $z \in D_{m,n}$, we have $\mathrm{tr}(z) \in K_{m,n}$ where $\mathrm{tr}(z)$ is viewed as an $n \times n$-scalar matrix. Taking z to be a monomial in the generic matrices $X_1, \ldots, X_m$ we see that the invariants (1) are all contained in $K_{m,n}$. Since they generate the coordinate ring of $Q_{m,n}$, we see that the function field of $Q_{m,n}$ is, in fact, contained in $K_{m,n}$. Procesi's theorem asserts that the two are, indeed, equal.

Theorem 4.2 (Demazure [D, p. 521, Cor. 1]) *Let X be an irreducible algebraic variety of dimension d. Then X is rational if and only if it admits a faithful rational action of a d-dimensional torus $T_d = (\mathbf{C}^*)^d$.*

If X is rational, i.e. $K(X) = \mathbf{C}(x_1, \ldots, x_d)$ then the desired action of T_d is given by $t(x_i) = t_i x_i$ for any $t = (t_1, \ldots, t_d) \in T_d$. The theorem asserts that the converse is also true.

Theorems 4.1 and 4.2 suggest the following question. Suppose the Noether Conjecture holds, i.e. $Q_{m,n}$ is rational. Then it admits an action of T_d which, in turn, induces a faithful T_d-action on its function field $K_{m,n}$. Here $d = \dim(Q_{m,n}) = (m-1)n^2 + 1$. By Procesi's theorem $K_{m,n}$ is the center of the universal division algebra $D_{m,n}$. Can we extend our torus action from the center to all of $D_{m,n}$? If so, what does it say about the division algebra structure of $D_{m,n}$?

Before we attempt to answer these questions, we first have to make them precise. The technical difficulty is that we have not specified what kind of a T_d-action on $D_{m,n}$ we allow. The action on $K_{m,n}$ we started out with is algebraic in the sense that it comes from an algebraic (rational) T_d-action on $Q_{m,n}$. (If this condition is not imposed, unexpected complications can arise; see [RV1, Ex. 4.1]). We now define an analogous notion for torus actions on division algebras.

Let D be a division algebra. We shall assume that D is finite-dimensional over its center K and that K is a finitely generated field extension of $\mathbf{C}$. Let $T = T_d = (\mathbf{C}^*)^d$ be a torus. Recall that a *T-action* on D is a group homomorphism $\phi : T \longrightarrow \mathrm{Aut}_{\mathbf{C}}(D)$. We say that this action is *faithful* if ϕ is injective. An element x of D is called *homogeneous* if it is an eigenvector for every $t \in T$, i.e. there exists a character $\chi : T \longrightarrow \mathbf{C}^*$ such that $t(x) = \chi(t)x$ for every $t \in T$.

Definition 4.3 *(a) An action $\phi : T \longrightarrow \mathrm{Aut}_{\mathbf{C}}(D)$ is called algebraic if the homogeneous elements generate D as a division algebra.*

(b) D is called rational if it admits a faithful algebraic T_d-action where d is the transcendence degree of the center of D over **C**.

It is easy to see that if D is, in fact, commutative then this notion of rationality coincides with the usual one. That is, a field is rational in the sense of Definition 4.3(b) if and only if it is a purely transcendental extension of **C**.

We can now state the main results of [RV2].

Theorem 4.4 *If D is a rational division algebra then the center of D is a rational field (i.e. a purely transcendental extension of* **C***).*

Theorem 4.5 *A rational division algebra is an abelian crossed product.*

It is thus natural to pose the following question which has a bearing on both the Noether Problem 2.5 and the Crossed Product Problem 3.2.

4.6 (Rationality Problem for Universal Division Algebras) *Which universal division algebras $D_{m,n}$ are rational?*

The only positive results we have been able to obtain are as follows.

Theorem 4.7 *(a) $D_{m,n}$ is rational for $n = 2$.*

(b) If $D_{m,n}$ is rational then so is $D_{m+1,n}$.

(c) $D_{m,n}$ is always unirational, i.e. is contained in a rational division algebra of degree n.

Combining Theorem 4.5 with Amitsur's Theorem 3.3(d), we see that $D_{m,n}$ is not rational when n is divisible by a cube. In fact, we can prove a stronger nonrationality result.

Theorem 4.8 *If n is divisible by a square then the universal division algebra $D_{m,n}$ is not rational.*

Theorem 4.8 implies, in particular, that converse statements to Theorems 4.4 and 4.5 are false. Indeed, we know that $K_{2,4}$ is rational (Theorem 2.6(c)) and $D_{2,4}$ is a crossed product (Theorem 3.3(b)). On the other hand $D_{2,4}$ is not a rational division algebra.

The situation is somewhat different in the case of universal division algebras whose degree n is square-free.

Theorem 4.9 *Suppose n is a product of distinct primes. Then the universal division algebra $D_{m,n}$ is a crossed product if and only if it contains a rational division subalgebra of degree n.*

Thus for square-free n the relationship between the crossed product problem and the rationality problem for universal division algebras can be summarized as follows:

$$\begin{array}{ccc} D_{m,n} \text{ is rational} & \Longrightarrow & \begin{array}{c} K_{m,n} \text{ is (stably) rational} \\ \text{(Noether Problem 2.5)} \end{array} \\ \Downarrow & & \\ \begin{array}{c} D_{m,n} \text{ contains a rational division} \\ \text{subalgebra of PI-degree } n \end{array} & \Longleftrightarrow & \begin{array}{c} D_{m,n} \text{ is a crossed product} \\ \text{(Crossed Product Problem 3.2)} \end{array} \end{array} \tag{3}$$

Can the implications in diagram (3) be reversed? A positive answer to this question would, in particular, imply that $D_{m,5}$ and $D_{m,7}$ are crossed products thus solving the Crossed Product Problem for $n = 5$ and $n = 7$.

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