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ON THE LIMITING DISTRIBUTION OF THE FAILURE TIME OF COMPOSITE MATERIAL*

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ABSTRACT

A topic of current interest in mechanical engeneering is the reliability of mechanical systems. The reliability of these systems is usually given by the probability distribution of the system failure time and, in general, the problem is to determine relevant properties of system reliability from knowledge of the reliability of each of its components.

The kind of system we shall deal with in this paper consists of a bundle of n fibers connected in parallel wich are encased in a ductile matrix compound, thus forming a composite system. Since cables which are both long and have many filaments prevail, our problem consists in determining the asymptotic behavior of the failure time for a system subjected to a total constant load, when both the length of the cable and the number of filaments grow large simultaneously.

Key Words: Mechanical Systems; Parallel Fiber Bundles; Reliability; Composite Materials.

1. INTRODUCTION

A topic of current interest in mechanical engeneering is the reliability of mechanical systems. The reliability of these systems is usually given by the probability distribution of the system failure time and, in general, the problem is to determine relevant properties of system reliability from knowledge of the reliability of each of its components.

The kind of system we shall with in this paper consists of a bundle of n fibers connected in parallel wich are encased in a ductile matrix compound, thus forming a composite system. Since cables which are both long and have many filaments prevail, our problem consists in determining the asymptotic behavior of the failure time for a system subjected to a time varying load, when both the length of the cable and the number of filaments grow large simultaneously.

These composite structures are assumed to behave like a chain of parallel bundles of fibers, connected in series, each of which behave according to the pure death model proposed by Coleman (1957). In Coleman's model, all the surviving fibers at any instant share the load equally, and in a somewhat generalized way this assumption was also made by Birnbaum and Saunders (1958). Coleman's assumptions regarding the time to failure of single fibers were remarkably consistent with the experimental behavior of a variety of structural materials.

An analysis similar to the one in this paper was carried out in a static strength model by Gücer and Gurland (1962).

On approach will be to use the results developed by the author (1978) on the asymptotic tail behavior of row sums in triangular arrays [c.f. section 3.1].

2. THE MODEL

Consider first a bundle consisting of n filaments aligned in parallel which is subjected to a total tensile load of nL, constantly over time. As time passes individual filaments fail in a random maner which depends on their load histories. Denoting the random points in time at which the successive individual filament failures occur by

$$\tau_1 \le \tau_2 \le \dots \le \tau_n ,$$

the number of unfailed filaments at time $t \ge 0$ is given by the counting process

$$N(t) = \sum_{j=0}^{n} (n-j) 1_{[\tau_j, \tau_{j+1})}(t),$$

where $\tau_0 \equiv 0$ and $\tau_{n+1} \equiv +\infty$. As in Coleman (1957) we postulate that:

(A1) each single filament, subjected to the known load history $\ell(\cdot)$ has a random failure time S whose distribution function is of the form

$$\text{(2.1)} \qquad P\{S \le s\} = 1 - \exp\left\{-\int_0^s b(\ell(t))dt\right\}, \qquad s \text{ in } \mathbb{R}_+;$$

(A2) the equal load sharing rule is in force, i.e., at any time $t \ge 0$, the total system load of nL is equally shared by the unfailed filaments.

Termed breakdown rule, the function $b(\cdot)$ in (2.1) expresses how load changes affect the hazard or failute rate $r(\cdot) = b \circ \ell(\cdot)$.

Under these assumptions, the Coleman model for the bundle [c.f.Coleman (1957)] is in force so that the random variables $\tau_1, \tau_2-\tau_1, \ldots, \tau_n-\tau_{n-1}$ are independent and the distribution function of $\tau_{j+1}-\tau_j$ is given by

$$P\{\tau_{j+1}-\tau_j \le t\} = 1 - \exp\{-(n-j)b(nL/(n-j))t\}, \qquad t \text{ in } \mathbb{R}_+,$$

for $j=0,1,\ldots,n-1$. The stochastic process $\{N(t):t\geq 0\}$ is thus a pure death process with death parameters

$$\mu_j = j\cdot b(nL/j), \qquad j=1,2,\ldots,n,$$

and $\mu_0 = 0$.

The above model, which describes the behavior along time of a parallel element system subjected to a constant load, under the equal load sharing assumption, provides us a representation for the failure time S_n of the bundle as a sum

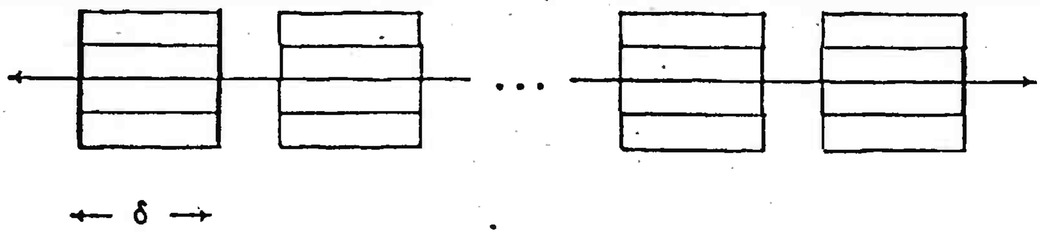
$$S_n = Y_{n,1} + Y_{n,2} + \ldots + Y_{n,n}$$

of independent random variables. More specifically, $Y_{n,j}$ is the sojourn time of $\{N(t):t\leq 0\}$ in state j so that

$$(2.2) \qquad P\{Y_{n,j} \le y\} = 1 - \exp\{-jb(nL/j)y\}, \qquad y \text{ in } \mathbb{R}_+,$$

for $j=0,1,\ldots,n-1$. We shall denote the distribution function in (2.2) by $F_{n,j}$.

At this point we can turn our attention to the mechanical systems of interest as described in section 1. Recall that these consist of cables of filaments aligned in parallel wich are encased in a matrix compound. An important model for these systems is to consider such cables as a chain of short bundles connected in series, each of then having length δ. This idea was introduced by Rosen (1964) as an adaption to a model of Gücer and Gurland (1962).



The section length δ is taken to be the ineffective length, which is defined as the length surrounding a break over which a fiber is ineffective in arrying load. We will also postulate that the ineffective length remains constant over time, which may not suit viscous matrix compounds. Furthermore, when loaded, each section will be assumed to operate like a bundle of n filaments placed in parallel, independently of each other, with the assumptions (A1) and (A2) in force.

Introduce now a subscript i to indicate the i-th blundle in the chain and let $S_{n,i}$ denote its failure time, for $i=1,2,\ldots,k$. The subscript n indicates the number of filaments in the blundle. Under the assumptions of our model the composite structure failure time $T_{n,k}$ is given by

$$T_{n,k} = \min\{S_{n,1}, S_{n,2}, \ldots, S_{n,k}\}$$

where $S_{n,1}, S_{n,2}, \ldots, S_{n,k}$ are independent random variables such that

$$P\{S_{n,i} \le t\} = F_n(t), \qquad t \text{ in } \mathbb{R},$$

for $i=1,2,\ldots,k$, where $F_n = F_{n,1} * \cdots * F_{n,n}$.

3. THE ASYMPTOTICS

In our analysis we will be concerned with the power law breakdown rule, in which $b(\cdot)$ is a power. We add then the following assumption:

(A3) $b(x) = x^\rho$, x in $\mathbb{R}_+$, where $\rho \geq 1$.

The restriction to $\rho \geq 1$ is not-severe in practice for some material types as shown in Phoenix (1976) who gives the following estimates for ρ:

Standard Type	ρ
Kevlar/Epoxy	42
Graphite Fiber/Epoxy	78
S-Glass/Epoxy	30
Beryllium Wire/Epoxy	26

In the sequel we shall refer to (A1) - (A3) as the assumption A.

As we mentioned earlyer we shall allow $k = k(n)$ to depend on n, and abbreviate $T_n = T_{n,k(n)}$. We also assume that $k(n)$ grows without bounds with n. Our set up now consists of a given triangular array

$$S = \{S_{n,i}; 1 \leq i \leq k(n), n \geq 1\}$$

such that for each $n \geq 1$ the random variables $S_{n,1}, \ldots, S_{n,k(n)}$ in the n-th row are independent and identically distributed but the common distribution within each row changes as the row changes. We are particularly interested in the asymptotic behavior of the row-minimum

$$T_n = \min_{1 \leq i \leq k(n)} S_{n,i}$$

as n gets large.

Extreme value theory in triangular arrays was invetigated by Loève (1956), who extended some of the earlyer results of Gnedenko

(1946) and Smirnov (1952) in classical Extreme Value Theory.

From theorem 1 in Loève (1956, p.), the problem of examining the existence of real constants $a_n > 0$ and b_n, and a non-degenerate distribution function G such that

$$(3.0.1) \qquad L(a_n^{-1}[T_n - b_n]) \Longrightarrow G$$

reduces to that of examining the limit behavior of

$$(3.0.2) \qquad k(n)P\{S_{n,1} \le a_n x + b_n\} = k(n)F_n(a_n x + b_n)$$

as $n \to \infty$. More explicitly, (3.0.1) holds if and only if, as $n \to \infty$,

$$(3.0.3) \qquad k(n)F_n(a_n x + b_n) \longrightarrow V(x)$$

for every x which is a continuity point of V, where $V\colon \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ is the non-negative monotone increasing function defined by the equation

$$G(x) = 1 - e^{-V(x)}, \qquad x \text{ in } \mathbb{R}.$$

Limit behavior of sequences as (3.0.2) was first discussed by Ivchenko (1973) in a simpler set up, and their extention to a general setting, so as to include the case at hand, have been derived by the author (1978). In particular the following results hold.

3.1 - Auxiliary Limit Results

Let $W = \{W_{n,j} : 1 \le j \le n, n \ge 1\}$ be a given family of distribution functions and let

$$W_n = W_{n,1} * \cdots * W_{n,n}$$

Assume further that the Cramer-Petrov conditions hold, i.e.:

(3.1.1) $E(W_{n,j}) = 0$ and $\sum_{j=1}^{n} \operatorname{Var}(W_{n,j}) = 1$ for all n and j;

(3.1.2) There exist positive real numbers $\{c_{n,j}: 1 \le j \le n,\ n \ge 1\}$, A and B such that

$$L_{n,j}(h) = \log \int \exp\{hy\} W_{n,j}(dy) \text{ is analytic in } |h| \le Bn^{1/2}$$

$$|L_{n,j}(h)| \le c_{n,j} \text{ for } |h| \le Bn^{1/2} \text{ and}$$

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} c_{n,j}^{3/2} = A < \infty.$$

[In (3.1.2) log denotes the principal value of the logarithm]. It has also been shown by the author (1978) that if we let

$$\psi_n(h) = \sum_{j=1}^{n} L_{n,j}(h), \qquad |h| < Bn^{1/2},$$

and

$$t_n(h) = n^{-1/2}\psi_n'(n^{1/2}h), \quad |h| < B,$$

there exist positive constants B* and C* such that for all n the inverse function of t_n, denoted by ℓ_n, is defined and analytic in the circle $|t| \le C^*$, satisfying the inequality

$$|\ell_n(t)| \le B^* \quad \text{for} \quad |t| \le C^*,$$

uniformily in n. Furthermore, if we let $\tau_n(y)$ denote the solution of the equation

(3.1.3) $$y = \int_0^{\tau} \ell_n(t)dt,$$

the function $\tau_n(\cdot)$ is two valued in the interval $(0, L_n^+ \wedge L_n^-)$, where $L_n^{\pm} = \int_0^{\pm C^*} \ell_n(t)dt$. Denoting by $\tau_{n,0}(y)$ and $\tau_{n,1}(y)$ the non-positive and non-negative branches, respectively, we also have that

(3.1.4) $$\tau_{n,0}(y) = -\sqrt{2y}[1+g_n(-\sqrt{2y})], \qquad 0 \le y < M^*,$$

and

$$\tau_{n,1}(y) = \sqrt{2y}[1+g_n(\sqrt{2y})], \qquad 0 \le y < M^*,$$

where $M^* > 0$ is a constant independent of n, and $g_n(z) \to 0$ as $z \to 0$ uniformily in n. Finally, as $n \to \infty$ we have that

(3.1.5) $$k(n)W_n(z_{n,0}) \to e^z$$

and

$$k(n)[1-W_n(z_{n,1})] \to e^{-z}$$

provided that $\{k(n): n \ge 1\}$ is a sequence of positive integers such that $k(n) \to \infty$, $\log k(n) = o(n)$ as $n \to \infty$, where

(3.1.6) $$z_{n,0} = \sqrt{n}\, \tau_{n,0} + (2 \log k(n))^{-1/2}(z + \tfrac{1}{2} \log 4\pi)$$

and

$$z_{n,1} = \sqrt{n}\, \tau_{n,1} + (2 \log k(n))^{-1/2}(z - \tfrac{1}{2} \log 4\pi)$$

for any fixed real number z, and

$$\tau_{n,i} = \tau_{n,i}((\log k(n) - \tfrac{1}{2} \log \log k(n))/n), \qquad i=0,1.$$

For definitess we define $\tau_{n,i}(y) = 0$ for $y \geq M^*$.

3.2. Limiting Distribution of T_n

The limit behavior of the failure time T_n of the composite structure, will now follow from these results provided we can find real constants $\alpha_n > 0$ and $\beta_{n,j}$ such that the family $\mathcal{W} = \{W_{n,j}: 1 \leq j \leq n, n \geq 1\}$ of distribution functions defined by

$$(3.2.1) \qquad W_{n,j}(x) = F_{n,j}(\alpha_n x + \beta_{n,j}), \qquad x \text{ in } \mathbb{R},$$

satisfies conditions (3.1.1) and (3.1.2). Observe that if this is possible, we have that

$$k(n) W_n(\gamma_n x + \delta_n) = k(n) F_n(a_n x + b_n), \qquad x \text{ in } \mathbb{R},$$

where

$$(3.2.2) \qquad a_n = \gamma_n \delta_n \quad \text{and} \quad b_n = \delta_n \alpha_n + \sum_{j=1}^{n} \beta_{n,j},$$

and from (3.1.5) and the equivalence between (3.0.1) and (3.0.3) we have that

$$L(a_n^{-1}[T_n - b_n]) \Longrightarrow \Lambda_1$$

where

$$\Lambda_1(x) = 1 - e^{-e^x}, \qquad x \text{ in } \mathbb{R},$$

and real constants γ_n, δ_n, a_n and b_n can be obtained from (3.1.6) and 3.2.2).

We now proceed to establish the validity of conditions (3.1.1) and (3.1.2).

Lemma 1 - Under assumption (A3) the family $W = \{W_{n,j} : 1 \le j \le n,\ n \ge 1\}$ of distribution functions, defined by (3.2.1) satisfies the Cramer-Petrov conditions provided that:

$$\alpha_n = \left[\sum_{j=1}^{n} \mathrm{Var}(F_{n,j}) \right]^{1/2}, \quad \beta_{n,j} = E(F_{n,j}).$$

Proof: It is immediate that condition (3.1.1) holds. To prove that condition (3.1.2) holds, observe that for each n and j, $F_{n,j}$ is an exponential distribution function with parameter

$$\lambda_{n,j} = jb(nL/j),$$

so that

$$\alpha_n = \left[\sum_{j=0}^{n-1} \lambda_{n,j}^{-2} \right]^{1/2}, \quad \beta_{n;j} = \lambda_{n,j}^{-1}$$

and

$$\int \exp\{hy\} W_{n,j}(dy) = \exp\{-h\alpha_n^{-1}\beta_{n,j}\} \int \exp\{h\alpha_n^{-1}y\} F_{n,j}(dy)$$

$$= \exp\{-h\alpha_n^{-1}\beta_{n,j}\}\lambda_{n,j}/(\lambda_{n,j} - h\alpha_n^{-1})$$

for $\mathrm{Re}(h\alpha_n^{-1}) < \lambda_{n,j}$. Now, since

$$\lim_{n\to\infty} n^{1/2}\alpha_n = L^{-\rho}\left[\int_0^1 (1-x)^{2\rho-2}dx\right]^{1/2} > 0$$

we can find $\delta > 0$ such that

$$n^{1/2}\alpha_n \ge \delta \qquad \text{for all } n,$$

and hence $\int \exp\{hy\} W_{n,j}(dy)$ is analytic in the circle $|h| < L^{\rho}\delta n^{1/2}$.

Finally, in the circle $|h| < L^{\rho}\delta n^{1/2}$

$$\left|\int \exp\{hy\}W_{n,j}(dy) - 1\right| \le \frac{|\alpha_n^{-1}h|}{\lambda_{n,j} - |\alpha_n^{-1}h|}\left(\exp\left\{\left|h\alpha_n^{-1}\beta_{n,j}\right|\right\} + 1\right),$$

so that

$$\left|\int \exp\{hy\}W_{n,j}(dy) - 1\right| < 1$$

for $|h| < Bn^{1/2}$, where $B = L^{\rho}\delta\min\{1/4, \log 2\}$. It follows from the last inequality that for all n and j

$$L_{n,j}(h) = \log\int \exp\{hy\}W_{n,j}(dy)$$

is analytic and uniformly bounded in the circle $|h| < Bn^{1/2}$ from which condition (3.1.2) follows.

From lemma 1 we can apply the results in section 3.1 to the family $\mathcal{W} = \{W_{n,j}: 1 \le j \le n,\ n \ge 1\}$ defined by (3.2.1) to obtain the limit behavior in distribution of the failure time T_n of the composite structure, which can be summarized in the following theorem.

Theorem 1 - Under assumption A,

$$\mathcal{L}(a_n^{-1}[T_n - b_n]) \Longrightarrow \Lambda_1$$

provided that $\log k(n) = o(n)$, where, for each $n \ge 1$

$$a_n = (nL^{\rho})^{-1}\left(\sum_{j=0}^{n-1}(1-j/n)^{2\rho-2}/2\ \log k(n)\right)^{1/2},$$

$$b_n = (n^{1/2}L^{\rho})^{-1}\left(\sum_{j=0}^{n-1}(1-j/n)^{2\rho-2}\right)^{1/2}\tau_{n,0} + \gamma_n \frac{1}{2}\log 4\pi + (nL^{\rho})^{-1}\sum_{j=0}^{n-1}(1-j/n)^{\rho-1},$$

and

$$\tau_{n,0} = \tau_{n,0}\left(\frac{1}{n}[\log k(n) - \frac{1}{2}\log\log k(n)]\right).$$

[Recall from Section 3.1 that we agree to define $\tau_n(y) = 0$ for $y \geq M^*$].

Under the assumptions of Lemma 1,

$$L_{n,j}(h) = \log\int \exp\{hy\} W_{n,j}(dy)$$

$$= \log[\exp\{-h\alpha_n^{-1}\beta_{n,j}\}(1-h\alpha_n^{-1}\beta_{n,j})^{-1}] .$$

$$= -\log[1-h\alpha_n^{-1}\beta_{n,j}] - h\alpha_n^{-1}\beta_{n,j}, \qquad |h| < L^{\rho}\frac{\delta}{4}n^{1/2},$$

where $\delta > 0$ is any positive real constant such that $n^{1/2}\alpha_n \geq \delta$ for all n, so that the function $t_n(h)$ defined in section 3.1 is given by

$$t_n(h) = n^{-1/2}\sum_{j=0}^{n-1} n^{1/2}h/\alpha_n\beta_{n,j}^{-1}(\alpha_n\beta_{n,j}^{-1}-n^{1/2}h)$$

$$= \sum_{j=0}^{n-1} h/\alpha_n\beta_{n,j}^{-i}(\alpha_n\beta_{n,j}^{-1}-n^{1/2}h), \qquad |h| < L^{\rho}\delta/4,$$

and its inverse $\ell_n(t)$ can be found by solving for h the equation

$$t\alpha_n \prod_i \beta_{n,j}^{-1}(\alpha_n\beta_{n,j}^{-1}-n^{1/2}h) = \sum_{j=0}^{n-1} h \prod_{j\neq i} \beta_{n,j}^{-1}(\alpha_n\beta_{n,j}^{-1}-n^{1/2}h).$$

However, this task may get very complicated and solving

$$-n\int_{\tau_{n,0}}^{0} \ell_n(t)\,dt = \log k(n) - \frac{1}{2}\log\log k(n)$$

for $\tau_{n,0}$ may get even more complex. However, this difficulty in the computation of the normalizing constants a_n and b_n can be reduced provided we make some adjustments on the rate of convergence of k(n). It is also shown by the author (1978, proof of lemma 3.2.3) that

$$\tau_{n,0}(y) = \gamma_n^{-1}(-\sqrt{2y}), \qquad 0 \leq y < S^2,$$

where

$$\gamma_n^{-1}(s) = \sum_{k=1}^{\infty} b_{n,k} \frac{s^k}{k!}, \qquad |s| < S,$$

with $b_{n,1} = 1$. Furthermore, $|\gamma_n^{-1}(s)| \le C^{**}$ on the circle $|s| < S$ and the positive constants S and C^{**} do not depend on n.

Lemma 2 - If for some $m \ge 1$ $(\log k(n))^{(m+2)/2} = o(n)$ as $n \to \infty$, then

$$\lim_{n\to\infty} \sqrt{n \log k(n)} \sum_{k=m+1}^{\infty} |b_{n,k}| (\sqrt{2y_n})^k / k! = 0$$

where $y_n = \frac{1}{n}[\log k(n) - \frac{1}{2} \log \log k(n)]$.

Proof:

From Cauchy's estimates,

$$|b_{n,k}| \le \frac{k! C^{**}}{S^k}$$

for all n and k. Hence, for any $\varepsilon > 0$

$$\sqrt{n \log k(n)} \sum_{k=m+1}^{\infty} |b_{n,k}| (\sqrt{2y_n})^k / k! \le$$

$$\le |C^{**}| \sqrt{n \log k(n)} \sum_{k=m+1}^{\infty} \left[\frac{\log k(n)(1 - \log\log k(n)/2 \log k(n))}{nS^2}\right]^{\frac{k}{2}}$$

$$\le |C^{**}| \sum_{k=m}^{\infty} \frac{\log k(n)^{\frac{k+2}{2}}}{n^{k/2}} \frac{1}{S^k}$$

$$\le |C^{**}| \sum_{k=m}^{\infty} (\sqrt{\varepsilon}/S)^k$$

for all sufficiently large n, since by our assumptions

$$(\log k(n))^{\frac{k+2}{2}} / n^{k/2} = [(\log k(n))^{\frac{m+2}{m}} / n]^{\frac{m}{2}} [\log k(n)/n]^{\frac{s}{2}} \le \varepsilon^{\frac{k}{2}}$$

for $k = m + s$, $s \ge 0$ and all sufficiently large n.

Now, for $m \geq 1$ let

$$\tau_{n,0}^{[m]} = \begin{cases} 0 & \text{if } y_n \geq S^2 \\ \sum_{k=1}^{m} b_{n,k} \dfrac{(-\sqrt{2y_n})^k}{k!} & \text{otherwise} \end{cases}$$

where $y_n = \frac{1}{n}[\log k(n) - \frac{1}{2} \log \log k(n)]$.

Theorem 2 - Under assumption A

$$L(\tilde{a}_n^{-1}[T_n - \tilde{b}_n]) \Longrightarrow \Lambda_1$$

where

$$\tilde{a}_n = (nL^\rho)^{-1}\left(\sum_{j=0}^{n-1}(1-j/n)^{2\rho-2}/2 \log k(n)\right)^{1/2}$$

and

$$\tilde{b}_n = (n^{1/2}L^\rho)^{-1}\left(\sum_{j=0}^{n-1}(1-j/n)^{2\rho-2}\right)^{1/2} \cdot \tau_{n,0}^{[m]} + \tilde{a}_n \frac{1}{2}\log 4\pi + (nL^\rho)^{-1} \sum_{j=0}^{n-1}(1-j/n)^{\rho-1}$$

provided that $(\log k(n))^{\frac{m+2}{2}} = o(n)$ as $n \to \infty$.

Proof: Since our assumption implies that $\log k(n) = o(n)$ as $n \to \infty$, we have that

$$L(a_n^{-1}[T_n - b_n]) \Longrightarrow \Lambda_1$$

where a_n and b_n are defined as in Theorem 1. Consequentely in order that

$$L(\tilde{a}_n[T_n - \tilde{b}_n]) \Longrightarrow \Lambda_1$$

it is necessary and sufficient [c.f. Feller (1966) p. 253] that

$$\frac{\tilde{a}_n}{a_n} \to 1 \quad \text{and} \quad \frac{\tilde{b}_n - b_n}{a_n} \to 0.$$

The first condition as immediately satisfied since $a_n = \tilde{a}_n$. To check the second, observe that for all sufficiently large n

$$|(\tilde{b}_n - b_n)/a_n| = \sqrt{2n \log k(n)}\,|\tau_{n,0}^{[m]} - \tau_{n,0}|$$

$$\leq \sqrt{2n \log k(n)} \sum_{k=m+1}^{\infty} |b_{n,k}|(\sqrt{2y_n})^k/k!$$

where $y_n = \frac{1}{n}[\log k(n) - \frac{1}{2}\log\log k(n)]$, since

$\tau_{n,0} = \sum_{k=1}^{\infty} b_{n,k}(-\sqrt{2y_n})^k/k!$ for all sufficiently large n.

In particular, if $(\log k(n))^3 = o(n)$ the normalizing constants a_n and b_n assume their simplest form since in this case.

$$\tau_{n,0}^{[1]} = -\sqrt{2[\log k(n) - \log\log k(n)]/n}.$$

4. FURTHER REMARKS

Under Assumption A, the failure time S_n of a system comprised of n fibers aligned in parallel and subjected to a constant load of nL is such that

$$P\{S_n \leq t\} = \sum_{j=1}^{n} c_{n,j}(1 - \exp\{-\lambda_{n,j} t\}), \qquad t \geq 0, \tag{4.1}$$

[c.f. Feller (1966) p. 40] where

$$\lambda_{n,j} = jb(nL/j) \quad \text{and} \quad c_{n,j} = \prod_{i \neq j} \frac{\lambda_{n,i}}{\lambda_{n,i} - \lambda_{n,j}}.$$

It is rather inconvenient, however, to use formula (4.1) in practical calculations, specially when n is not small, so that the availability of approximate expressions would be desirable. The same is true for the distribution function of the time to failure $T_{n,k}$ of composite materials, when modeled as a chain of k independent parallel n-element systems connected in series, in which case

$$P\{T_{n,k}>t\} = [P\{S_n>t\}]^k.$$

Since

$$\lim_{t\downarrow 0} t^{-n}P\{S_n\le t\} = \frac{1}{n!}\prod_{j=1}^{n}\lambda_{n,j} > 0,$$

a fact that holds even under more general assumptions concerning the distribution function of the time to failure of a single fiber as shown by Harlow, Smith and Taylor (1977). It follows from the classical results of Gnedenko (1946) that for fixed n,

$$\text{(4.2)}\qquad \lim_{k\to\infty} P\{k^{1/n}T_{n,k}>t\} = \exp\{-\left(\prod_{j=1}^{n}\lambda_{n,j}\right)t^n/n!\}$$

for t≥0, so that for large k we have the following approximation formula:

$$P\{T_{n,k}>t\} = \exp\{-k\left(\prod_{j=1}^{n}\lambda_{r,j}\right)t^n/n!\}.$$

The limiting result (4.2), as we can see, is the result of the approxmation formula

$$\text{(4.3)}\qquad P\{S_n\le t\} = \frac{1}{n!}\left(\prod_{j=1}^{n}\lambda_{n,j}\right)t^n$$

for small positive values of t, which numerical results have shown unsuitable from the application standpoint, since the deviation from its true value increase as n increases, for each fixed value of t. A search for better approximations led us to consider the development of asymptotic results allowing both n and k in (4.2) to grow simultaneously without bounds. In doing so, we have been able to combine both the fact that under assumption A S_n is asymptotically normal and the fact that Φ, the normal distribution function, is in the minimum domain of attraction of Λ_1.

The asymptotic normality of S_n, provides automatically another approxmation formula for its distribution function, namely,

$$P\{S_n \le t\} \approx \Phi((t-\mu_n)/\sigma_n)$$

where $\mu_n = E(S_n)$ and $\sigma_n^2 = Var(S_n)$. In Figure 4.1 we depict the result of this approximation by plotting

$$\log P\{S_n \le t\} \text{ vs. } \log \Phi((t-\mu_n)/\sigma_n),$$

and observe that, contrary to what happened with the approximation formula (4.3), the deviations decrease for each fixed value of t, as n increases. These deviations can be made even smaller if we make use of estimates of the Cramer-Petrov type for the probabilites

$$P\{[S_n-\mu_n]/\sigma_n \le t_n\}$$

of large deviations for the sum S_n. Following Book (1976) it is possible to show that there exists a positive constant $\tau > 0$ such that in the interval $0 \le x \le \tau n^{1/2}$

$$(4.4) \qquad P\{S_n \le \mu_n - \sigma_n x\} = \Phi(-x)\exp\{-x^3 n^{-1/2}\lambda_n(-xn^{-1/2})\}(1+\ell_1\tau),$$

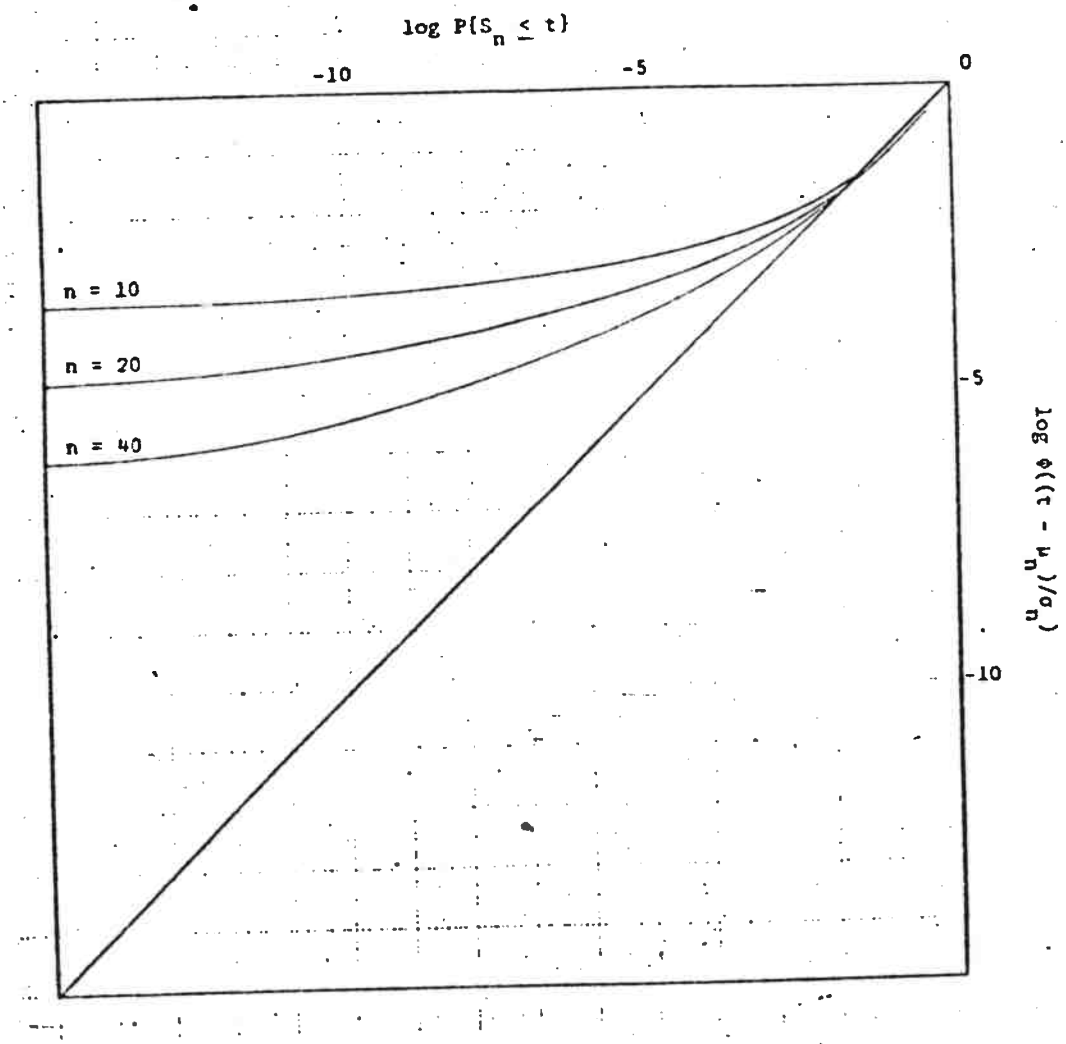
(4.5) $$P\{S_n > \mu_n + \sigma_n x\} = \Phi(-x)\exp\{x^3 n^{-1/2}\lambda_n(xn^{-1/2})\}(1+\ell_2\tau),$$

with $|\ell_1|-|\ell_2| \le \ell$ where ℓ is a constant [see also Statulevicius (1966 lemma 1)]. We recall that $\lambda_n(\cdot)$ is an absolutely convergent power species in a neighborhood of the origin, uniformly in n, whose coefficients can be defined in terms of the semi-invariants of the distribution of $[S_n-\mu_n]/\sigma_n$. More explicitly, we have

$$\lambda_n(t) = \frac{\Gamma_{3n}}{6\Gamma_{2n}^{3/2}} + \frac{\Gamma_{2n}\Gamma_{4n}-3\Gamma_{3n}^2}{24\Gamma_{2n}^3} t + \ldots$$

for sufficiently small t, where

Figure 4.1



$$\Gamma_{j,n} = \frac{1}{n}\sum_{k=1}^{n}\gamma_{n,k,j}, \qquad j\geq 1,$$

and $\gamma_{n,k,j}$ for $j\geq 1$ is the j-th semi-invariant of the distribution of $(Y_{n,k}-E(Y_{n,k}))/\sigma_n$.

In view of these estimates, the following approximation for the distribution function of S_n is available:

$$(4.6) \qquad P\{S_n\leq t\} = \begin{cases} \Phi(t_n)\exp\{\Gamma_{3n}t_n^3/6n^{1/2}\Gamma_{2n}^{3/2}\}, & t\leq\mu_n \\ 1-\Phi(-t_n)\exp\{\Gamma_{3n}t_n^3/6n^{1/2}\Gamma_{2n}^{3/2}\}, & t>\mu_n, \end{cases}$$

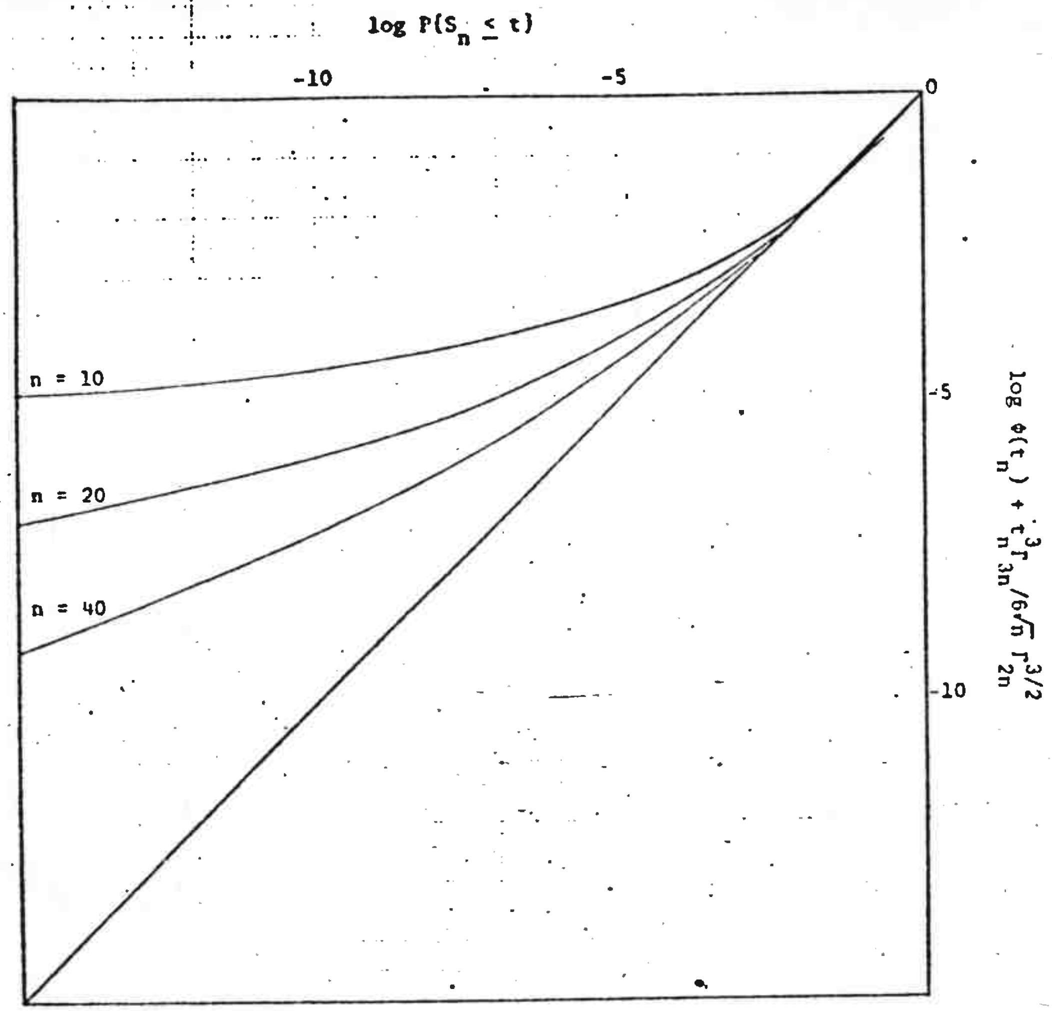
where $t_n = (t-\mu_n)/\sigma_n$.

In Figure 4.2, we plot

$$\log P\{S_n\leq t\} \text{ vs. } \log \Phi(t_n) + \Gamma_{3n}t_n^3/6n^{1/2}\Gamma_{2n}^{3/2}$$

for some values of t, and comparing with the normal approximation previously graphed in Figure (4.1) we can see the improvement provided by the correction. The approximation (4.6) seems to be very satisfactory from the engineering applications standpoint, and numerical results show that the inclusion of more terms in the power series expansion of $\lambda_n(\cdot)$ does not substantially improve the approximation. It is our feeling, however, that this can be achieved provided that estimates for ℓ_1 and ℓ_2 in (4.4) and (4.5) be developed.

Figure 4.2



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