

ON THE FIBONACCI COMPLEX DYNAMICAL SYSTEMS

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ABSTRACT. We consider in this paper a sequence of complex analytic functions constructed by the following procedure $f_n(z) = f_{n-1}(z)f_{n-2}(z) + c$, where $c \in \mathbb{C}$ is a parameter. Our aim is to give a thorough dynamical study of this family, in particular we are able to extend the familiar notions of Julia sets and Green function and to analyze their properties. As a consequence, we extend some well-known results. Finally we study in detail the case where c is small.

1. Introduction. In this article we study a family of complex dynamical systems. This family is constructed using the so-called multiplicative Fibonacci procedure and we call it the complex Fibonacci dynamical systems. There are several motivations for studying these mappings.

On one hand, they are polynomials endomorphisms of \mathbb{C}^2 defined by $H_c(x, y) = (xy + c, x)$ and constitute an extension of the classical iteration of polynomials in one complex variable. From the dynamical point of view a natural set to study is $K^+(H_c) = \{(x, y) \in \mathbb{C}^2, H_c^n(x, y) \text{ is bounded}\}$ where most of the non-trivial dynamics concentrates, $H_c^n(x, y)$ is the n -th iterate of H_c . We will also show that H_c shares a lot of properties with the complex Hénon maps defined by $(x, y) \mapsto (y, P(y) + c - ax)$ for all $(x, y) \in \mathbb{C}^2$, where P is a complex polynomial and a, c are fixed complex numbers (see [2], [7], [16]).

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On the other hand, and this was the starting point of our investigations, complex Fibonacci systems naturally appeared in the context of stochastic adding machines. Indeed, Killeen and Taylor in [14] defined a stochastic adding machine in base 2 by considering that in the addition of 1, the carry is added with probability $p > 0$ and is not added with probability $1 - p$. They obtained a Markov chain and they proved that the spectrum in l^∞ of the transition operator associated to this Markov chain is equal to the filled Julia set of a quadratic map whose coefficients depend on p .

Messaoudi and Smania [15] gave a connection of adding machine with Julia sets in a higher dimensional complex setting. In particular, they proved that the spectrum Σ in l^∞ of the transition operator associated to a stochastic adding machine in an exotic base (given by Fibonacci numbers) is related to the set $K^+(H_c)$ where $H_c(x, y) = (xy + c, x)$ for all $(x, y) \in \mathbb{C}^2$ with c a fixed real number. More precisely (see [5]), the spectrum Σ contains the set $\mathcal{K}_c = \{z \in \mathbb{C}, (z, z) \in K^+(H_c)\}$, and the conjecture is that $\Sigma = \mathcal{K}_c$.

In this paper we will lay the basis of a Fatou-Julia theory of those Fibonacci dynamical systems. To be more specific, let us consider in all what follows $f : \mathbb{C} \rightarrow \mathbb{C}$ a non constant polynomial function of degree d .

Definition 1.1. Let us denote $f_0(z) = z$, $f_1(z) = f(z)$ and for any $n \geq 2$ define $f_n : \mathbb{C} \rightarrow \mathbb{C}$ by the relation

$$f_n(z) = f_{n-1}(z)f_{n-2}(z) + c. \quad (1)$$

where $c \in \mathbb{C}$ is a parameter.

We will call the system given by the above definition, a Complex Fibonacci System (CFS). Here we are mainly interested in the description of topological properties of those Julia sets and of the associated connected locus.

Although the initial problem is a one-dimensional, the behavior of the family (f_n) is closely related to the higher dimensional dynamics of H_c , since one has $H_c^n(z, z) = (f_{n+1}(z), f_n(z))$. Similarly, the filled-in Julia sets of Fibonacci complex dynamical systems can be seen as a slice of the filled-in Julia sets of $H_c : \mathcal{K}(f, H_c) = \{z \in \mathbb{C}, (f(z), z) \in K^+(H_c)\}$.

Ergodic properties of the dynamics of H_c were considered by Guedj in [9] and [8]. He mentioned there that H_c has a small topological degree (i.e. $\deg(H_c) = 1$) and shares this property with the Hénon maps (see [9, Chap.4]). Therefore, from the viewpoint of ergodic investigations, the dynamics is not rich and most interesting properties can be established directly, such as the mixing property, hyperbolicity, topological entropy, the existence of the unique measure of maximal entropy, equidistribution of periodic points. Nevertheless, the study of holomorphic dynamics of this family of maps seems to need delicate arguments.

In this paper, we prove that the set $\mathcal{K}(f, H_c) = \{z \in \mathbb{C}, (f(z), z) \in K^+(H_c)\}$ is a non empty compact set, such that $\mathbb{C} \setminus \mathcal{K}(f, H_c)$ is a connected set. Moreover, when c is real with $|c| > 2$ and $f(0) = f'(0) = 0$, then $\mathcal{K}(f, H_c)$ is not connected. In the case, where $f(z) = z$, we also show that $\mathcal{K}(f, H_c)$ is not connected, if c is a real number such that $c < -2$.

Our main theorem asserts that $\mathcal{K}(f, H_c)$ is a quasi-disk, when $|c|$ is small and when $f(z) = z$. In that particular case we will then prove that there exists a real number $\delta > 0$ such that if $|c| < \delta$ then $\mathcal{K}(f, H_c)$ is a quasi-disk.

Let us mention here that the classical one dimensional machinery used in the study of the usual Julia sets can not be applied in our setting since the graphs $\{f(z), z, z \in \mathbb{C}\}$ are not invariant under H_c .

All our results can be extended easily to the sets $\mathcal{K}(f, h_c)$ associated to the maps $h_c(x, y) = (x^a y^b + c, x)$ where a, b are positive real numbers.

In section 2 we shall prove some topological properties of the filled-in Julia set $\mathcal{K}(f, H_c)$. Precisely, we define an “escape radius” and prove that $\mathcal{K}(f, H_c)$ is a non empty compact and simply connected set. In section 3 we define the Green function associated to $\mathcal{K}(f, H_c)$ and prove that it has all the expected properties. Section 4 is devoted to the study of some aspect of the connected locus of the Fibonacci dynamical systems. In particular, we prove that if $f(0) = f'(0) = 0$ and $|c| > 2$, then $\mathcal{K}(f, H_c)$ is not connected. In the last three sections, we prove our main result that says that if $|c|$ is sufficiently small and $f(z) = z$, then $\mathcal{K}(f, H_c)$ is a quasi-disk. For $c = 0$, $\mathcal{K}(f, H_0)$ simply reduce to the unit disk, and we are able to define a holomorphic motion from this disk to $\mathcal{K}(f, H_c)$.

2. On the topological properties of the filled Julia sets of CFS. Let f be a non null complex polynomial and $(f_n)_{n \geq 0}$ be the sequence of functions defined by $f_0(z) = z$, $f_1(z) = f(z)$ and $f_n(z) = f_{n-1}(z)f_{n-2}(z) + c$, where $c \in \mathbb{C}$ is a fixed complex number.

Set

$$\mathcal{K}_c = \{z \in \mathbb{C} : \sup_{n \in \mathbb{N}} |f_n(z)| < +\infty\}.$$

Observe that

$$\mathcal{K}_c = \mathcal{K}(f, H_c) = \{z \in \mathbb{C}, \text{ such that } (f(z), z) \in K^+(H_c)\}.$$

where $H_c : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ is the function defined by $H_c(x, y) = (xy + c, x)$ for all $(x, y) \in \mathbb{C}^2$ and $K^+(H_c) = \{(x, y) \in \mathbb{C}^2, H_c^n(x, y) \text{ is bounded}\}$.

The first step is to find an “escape radius” for the family $(f_n)_{n \geq 0}$: this is the object of the following crucial proposition.

Proposition 1 (Escape Radius). *There exists $R = R(c) > 1$, such that*

$$\mathcal{K}_c = \bigcap_{n=0}^{+\infty} f_n^{-1}(\overline{\mathbb{D}(0, R)}).$$

Lemma 2.1. *Let $M > \max\{2, \sqrt{2|c|}\}$. If the set $\{n \in \mathbb{N} : |f_n(z)| > M\}$ contains two consecutive integers then the sequence $(f_n(z))_{n \geq 0}$ is unbounded.*

Proof. Let $k, k + 1$ be two consecutive integers in $\{n \in \mathbb{N} : |f_n(z)| > M\}$. Then, by the triangle inequality, we have

$$|f_{k+2}(z)| \geq M^2 - |c| > \frac{M^2}{2}.$$

Hence by the same reasoning we get

$$\begin{aligned} |f_{k+3}(z)| &\geq \frac{M^3}{2} - |c| \\ &> \left(\frac{M^2}{2} + |c|\right) \frac{M}{2} - |c| > 2 \left(\frac{M}{2}\right)^3. \end{aligned}$$

Applying the same procedure one may easily deduce by induction that

$$\forall n \in \mathbb{N}, |f_{k+n}(z)| > 2 \cdot \left(\frac{M}{2}\right)^{F_n}$$

where F_n is the n -th Fibonacci number with initial values $F_0 = F_1 = 1$. The proof of the Lemma is thus completed. □

Lemma 2.2. *Let P be a non constant polynomial function. There exist positive real numbers $\delta = \delta(P)$, $p = p(P) > 1$, $d = d(P) > 0$ such that for all $E \geq \delta$ and for all $z \in \mathbb{C}$ we have*

$$|z| > E \Rightarrow |P(z)| > dE \text{ and } |P(z)| > E^p \Rightarrow |z| > E.$$

Proof. The first inequality comes from the fact that the function $\frac{|P(z)|}{|z|}$ converges either to a positive constant or to infinity as $|z|$ goes to infinity. The second inequality is due to the fact that if $E > 1$ is a real number and $|z| \leq E$ then $|P(z)| \leq c_1 E^k$ where k is the degree of the polynomial P and c_1 is the sum of the modulus of the coefficients of P . □

Proposition 2. *Let M be as in Lemma 2.1 then there exists $R > M$ such that if $|f_k(z)| > R$, for some integer k , then the sequence $(f_n(z))_{n \geq 0}$ is not bounded.*

Proof. Consider δ, p , and d given by Lemma 2.2 applied to $P = f_1$. Let $M > \max\{2, \sqrt{2|c|}, \delta\}$ and $R > \max\{M + 2|c|, \frac{M}{d}, M^p\}$. If $|f_k(z)| > R$ for $k = 0$ or 1 , then by Lemmas 2.1 and 2.2, we have $(f_n(z))_{n \geq 0}$ is unbounded.

Now, assume that the property is true for $1 \leq n < k$ and that $|f_k(z)| > R$. It follows from Lemma 2.1 that

$$\max\{|f_{k-1}(z)|, |f_{k+1}(z)|\} \leq M.$$

Applying the triangle inequality we obtain

$$|f_{k-1}(z)| < \frac{|f_{k+1}(z)| + |c|}{R} \leq \frac{M + |c|}{R}. \tag{2}$$

On the other hand, we have

$$R < |f_k(z)| < \left(\frac{M + |c|}{R}\right) |f_{k-2}(z)| + |c|.$$

It follows that $|f_{k-2}(z)| > R \cdot \frac{R - |c|}{M + |c|} > R$. Hence $(f_n(z))_{n \geq 0}$ is unbounded. □

The proof of Proposition 1 is now an immediate consequence of Proposition 2.

Corollary 1. \mathcal{K}_c is a compact subset of \mathbb{C} .

Proof. This corollary is straightforward from Proposition 1. □

Proposition 3. *The set $\mathbb{C} \setminus \mathcal{K}_c$ is a connected set.*

Proof. By Proposition 1, there exists a real number $R > 0$ such that $\mathbb{C} \setminus \mathcal{K}_c = \bigcup_{n=0}^{+\infty} \mathbb{C} \setminus f_n^{-1}(D(0, R))$. Now, for all $n \geq 0$, $\mathbb{C} \setminus f_n^{-1}(D(0, R))$ contains a neighborhood of infinity and is a connected set: indeed its complement $\{z; |f_n(z)| \leq R\}$ is necessarily simply connected (by the maximum principle applied to f_n). □

Proposition 4. *There exists a real number $R > 0$ such that*

$$\mathcal{K}_c = \bigcap_{n=1}^{+\infty} f_n^{-1}(D(0, R)) \text{ and } f_{n+1}^{-1}(D(0, R)) \subset f_n^{-1}(D(0, R)), \forall n \geq 1.$$

Lemma 2.3. *There exists a real number $R_0 > 0$ such that for $R > R_0$, and for all $k \in \{1, 2, 3, 4\}$, we have*

$$|f_{k+1}(z)| < R \implies |f_k(z)| < R. \tag{3}$$

Proof. Consider $\delta(f_i)$, $p(f_i)$ and $d(f_i)$ the real numbers defined in Lemma 2.2 corresponding to f_i for $i = 1, \dots, 4$. Let $R_1 > \delta(f_1)$ such that for all $R > R_1$, we have $R^{\frac{1}{p(f_1)}+1} - |c| \geq R$. Fix $R > R_1$, and $z \in \mathbb{C}$ such that $|f_1(z)| \geq R$, then by Lemma 2.2, $|f_0(z)| = |z| \geq R^{1/p(f_1)}$, hence $|f_2(z)| \geq R^{\frac{1}{p(f_1)}+1} - |c| \geq R$, then the result is true for $k = 1$.

Let $R_2 > \delta(f_2)$ such that for all $R > R_2$, we have $d(f_2)R^{\frac{1}{p(f_2)}+1} - |c| \geq R$.

Consider $R > R_2$ and $z \in \mathbb{C}$ such that $|f_2(z)| \geq R$, then by Lemma 2.2 we obtain that $|z| \geq R^{1/p(f_2)}$ hence again by Lemma 2.2, we have $|f_1(z)| \geq d(f_2)R^{1/p(f_2)}$. Thus $|f_3(z)| \geq d(f_2)|R^{\frac{1}{p(f_2)}+1} - |c| \geq R$, then we have the result for $k = 2$. By the same way, if we choose for $i = 3, 4$, $R_i > \delta(f_i)$ such that for all $R > R_i$, we have $d(f_i)R^{\frac{1}{p(f_i)}+1} - |c| \geq R$. We prove that the result holds for $k = 3$ and 4. Taking $R_0 = \max\{R_1, R_2, R_3, R_4\}$, we conclude the proof of Lemma 2.3. \square

Proof of Proposition 4. Let $R > R_0$ be a large number such that $\mathcal{K}_c = \bigcap_{n=0}^{+\infty} f_n^{-1}(D(0, R))$. Let us prove by induction that for all $k \in \mathbb{N}$, $f_{k+1}^{-1}(D(0, R)) \subset f_k^{-1}(D(0, R))$.

By Lemma 2.3, the previous inclusion is true for $k = 0, 1, 2, 3$. Now assume that it is also true for all $k = 0, \dots, n - 1$, $n \geq 4$.

Let $z \in f_{n+1}^{-1}(D(0, R))$, then $|f_{n+1}(z)| < R$. Assume that $|f_n(z)| \geq R$. By using relation (1) and a triangle inequality, we have

$$|f_{n-1}(z)| < \frac{R + |c|}{R} = 1 + \frac{|c|}{R} = O(1). \tag{4}$$

We choose R sufficiently large such that $\frac{R+|c|}{R} \leq R$, then $|f_{n-1}(z)| < R$. Hence, by induction hypothesis

$$|f_k(z)| < R, \forall k = 0, \dots, n - 2. \tag{5}$$

On the other hand, by using relations (1) and (4), we have

$$|f_{n-2}(z)| > R \frac{R - |c|}{R + |c|} := h_1(R). \tag{6}$$

By relations (1), (4) and (6), we deduce that

$$|f_{n-3}(z)| < \frac{(R(1 + |c|) + |c|)(R + |c|)}{R^2(R - |c|)} := h_2(R). \tag{7}$$

By using relations (1), (6) and (7), we deduce

$$|f_{n-4}(z)| \geq \frac{h_1(R) - |c|}{h_2(R)} = O(R^2) > R. \tag{8}$$

This contradicts relation (5), and we obtain the result. \square

Remark 1. 1. It is easy to see that the Julia set $J_c := \partial\mathcal{K}_c$ is exactly the set of points $z \in \mathbb{C}$ such that $(f_n)_{n \in \mathbb{N}}$ does not form a normal family in a neighborhood z .

2. If f is a constant non null polynomial, then the previous results are still true, in particular \mathcal{K}_c is a non empty compact and simply connected set.

If $f = 0$, then $\mathcal{K}_c = \mathbb{C}$ if $0 \in \mathcal{K}_c$ and $\mathcal{K}_c = \emptyset$ if $0 \notin \mathcal{K}_c$.

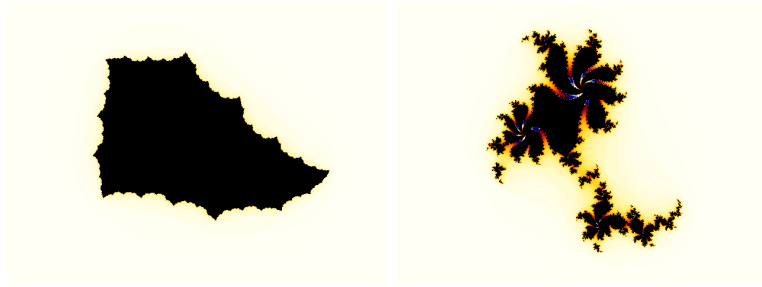


FIGURE 1. Filled-in Julia set intersected with the diagonal, for $c = -0.5 + i0.5$ in the left and $c = 0.36 + i0.575$ in the right

3. Green’s function. In this section we introduce the Green’s function of the filled-in Julia set, which is the main tool to relate potential theory and dynamics. According to the theory of polynomial endomorphisms in \mathbb{C}^2 , the Green’s function of $H_c(w, z) = (wz + c, w)$ is well defined and has nice properties (see for example [11], Remark 2.2). But here we will provide a new characterization and be more specific on the Green’s function associated to our Fibonacci system.

Let \log_+ be the real function defined by $\log_+(x) = \max(\log |x|, 0)$. For all $(w, z) \in \mathbb{C}^2$, we denote $H_c^n(w, z) := (h_{n+1}(w, z), h_n(w, z))$ for all $n \geq 0$.

With this notation $f_n(z) = h_n(f(z), z)$. Let us denote d_n the degree of f_n . These numbers satisfy $d_0 = 1$, $d_1 = \deg(f)$ and the well known Fibonacci relation $d_{n+1} = d_n + d_{n-1}$ for all $n \geq 1$.

Before introducing the Green’s function we need to establish some control on h_n and f_n . The first tools that we will use are the following technical Lemmas.

Lemma 3.1. *There exists a real number $R > 1$ such that for all $|w|, |z| > R$ and for all $n \geq 0$,*

$$R \leq |h_n(w, z)| \leq \frac{e^{d_n}}{2} |z|^{d_n} |w|^{d_n}. \tag{9}$$

Proof. Let $R > 1$ such that $R^2 - |c| > R$, then the lower bound follows by the same reasoning as in Lemma 2.1. The upper bound is proved by induction. Since $h_0(w, z) = z$ and $h_1(w, z) = w$ the claim is straightforward for $n = 0$ and 1. Now assume that (9) holds for the steps $n - 1$ and n ,

$$\begin{aligned} |h_{n+1}(w, z)| &\leq |h_n(w, z)h_{n-1}(w, z)| + |c| \\ &\leq \frac{e^{d_{n+1}}}{4} |z|^{d_{n+1}} |w|^{d_{n+1}} + |c| \\ &\leq \frac{e^{d_{n+1}}}{2} |z|^{d_{n+1}} |w|^{d_{n+1}}. \end{aligned} \tag{10}$$

For this last estimate we used $2|c| \leq R^2 \leq |z||w|$. □

Lemma 3.2. *There exist positive real numbers A and R such that for all $n > 0$ and for all $z \in \mathbb{C}$*

$$|\log_+ |f_{n+1}(z)| - \log_+ |f_n(z)| - \log_+ |f_{n-1}(z)|| \leq A, \tag{11}$$

and similarly, for all $|z| \geq R$ and $|w| \geq R$:

$$|\log_+ |h_{n+1}(w, z)| - \log_+ |h_n(w, z)| - \log_+ |h_{n-1}(w, z)|| \leq A. \tag{12}$$

Proof. Obviously (11) and (12) are proved in a very similar way. First we focus on (11). Fix $z \in \mathbb{C}$ and consider $R > 1$ as in Proposition (4). If $|f_{n-1}(z)| \geq R$ for some integer $n \geq 1$, then $|f_n(z)| \geq R$. There are only 3 cases to consider.

Either, $|f_{n-1}(z)| \geq R$, then in this case $|f_n(z)| \geq R$. Hence

$$\frac{|f_{n+1}(z)|}{|f_n(z)||f_{n-1}(z)|} \leq \left| 1 + \frac{|c|}{|f_n(z)||f_{n-1}(z)|} \right| \leq 1 + \frac{|c|}{R^2}.$$

Hence

$$\begin{aligned} |\log_+ |f_{n+1}(z)| - \log_+ |f_n(z)| - \log_+ |f_{n-1}(z)|| &= \left| \log\left(\frac{|f_{n+1}(z)|}{|f_n(z)||f_{n-1}(z)|}\right) \right| \\ &\leq \log\left(1 + \frac{|c|}{R^2}\right) := A_1. \end{aligned}$$

The second case is $|f_{n-1}(z)| < R$ and $|f_n(z)| \leq R$, then $|f_{n+1}(z)| < R^2 + |c| \leq R^3$. Therefore

$$|\log_+ |f_{n+1}(z)| - \log_+ |f_n(z)| - \log_+ |f_{n-1}(z)|| \leq 2 \log R + \log(R^3) = 5 \log R := A_2.$$

The last case to consider is $|f_{n-1}(z)| < R$ and $|f_n(z)| > R$, then $|f_{n+1}(z)| \geq R$, then

$$R \leq |f_n(z)f_{n-1}(z)| + |c|$$

thus we obtain $|f_n(z)f_{n-1}(z)| \geq R - |c|$, then

$$\frac{|f_{n+1}(z)|}{|f_n(z)||f_{n-1}(z)|} \leq 1 + \frac{|c|}{R - |c|}.$$

Taking the \log_+ we conclude.

Regarding (12), we assume now that both $|z|$ and $|w|$ are greater than R . According to Lemma 3.1 $|h_k(w, z)| \geq R$. Thus we are precisely in the first case above and we can proceed in the same way. \square

Proposition 5. *There exists a constant $C > 0$ such that, for all $n \geq 0$ and $z \in \mathbb{C}$*

$$\left| \frac{1}{d_{n+1}} \log_+ |f_{n+1}(z)| - \frac{1}{d_n} \log_+ |f_n(z)| \right| \leq \frac{C}{d_{n+1}}.$$

Proof. This key proposition is based on the following identity.

$$\begin{aligned} \log_+ |f_{n+1}(z)| - \frac{d_{n+1}}{d_n} \log_+ |f_n(z)| &= \log_+ |f_{n+1}(z)| - \log_+ |f_n(z)| - \log_+ |f_{n-1}(z)| \\ &\quad + \sum_{k=0}^{n-2} (-1)^{k+1} \frac{d_{n-k-1}}{d_n} (\log_+ |f_{n-k}(z)| \\ &\quad - \log_+ |f_{n-k-1}(z)| - \log_+ |f_{n-k-2}(z)|) \\ &\quad + \frac{(-1)^n}{d_n} (d_0 \log_+ |f_1(z)| - d_1 \log_+ |f_0(z)|). \end{aligned} \tag{13}$$

It's a simple calculation to check this equality. Indeed, if we re-arrange the terms of $\log_+ |f_{n-k}|$ for $1 < k < n - 1$, we obtain

$$\frac{(-1)^k}{d_n} (-d_{n-k-1} - d_{n-k} + d_{n-k+1}) = 0.$$

But $(d_n)_{n \geq 0}$ is a Fibonacci sequence, so we obtain :

$$\left| \frac{d_{n-k-1}}{d_n} \right| \leq \frac{D}{\rho^k},$$

where $\rho = \frac{1+\sqrt{5}}{2}$ is the golden mean and $D > 0$ is a real constant.

From this estimation combined with Lemma 3.2 and the identity above we deduce

$$\begin{aligned} \left| \log_+ |f_{n+1}(z)| - \frac{d_{n+1}}{d_n} \log_+ |f_n(z)| \right| &\leq \sum_{k=0}^n \frac{DA}{\rho^k} \\ &+ \left| \frac{1}{d_n} (d_0 \log_+ |f_1(z)| - d_1 \log_+ |f_0(z)|) \right|. \end{aligned} \tag{14}$$

Let $I(z) := d_0 \log_+ |f_1(z)| - d_1 \log_+ |f_0(z)|$. For $|z| \leq R$, $|I(z)|$ is clearly bounded from above. On the other hand, whenever $|z| > R$ then $I(z) = \log |f_1(z)| - d_1 \log |z|$, Let

$$f_1(z) = a_{d_1} z^{d_1} + \dots + a_1 z + a_0 = z^{d_1} (a_{d_1} + \dots + a_1 z^{1-d_1} + a_0 z^{-d_1}),$$

then

$$I(z) = \log(|a_{d_1} + a_{d_1-1} z^{-1} + \dots + a_1 z^{1-d_1} + a_0 z^{-d_1}|).$$

Thus, if $|z| > R$ we obtain $|I(z)| \leq \log(\max(|a_i|)(1 + d_1/R))$.

Using (14), this proves immediately that $\left| \log_+ |f_{n+1}(z)| - \frac{d_{n+1}}{d_n} \log_+ |f_n(z)| \right|$ is bounded above by a constant. We obtain therefore the expected result dividing this inequality by d_{n+1} . \square

In the same spirit as the previous proposition one can show, the following:

Proposition 6. *There exist real numbers $C > 0$ and $R > 1$ (as in Proposition 4) such that for all $n \geq 0$ and all $|w|, |z| > R$,*

$$\left| \log_+ |h_{n+1}(w, z)| - \frac{d_{n+1}}{d_n} \log_+ |h_n(w, z)| \right| \leq C \tag{15}$$

and

$$\left| \log_+ |h_{n+1}(w, z)| - \rho \log_+ |h_n(w, z)| \right| \leq C_1 + C_2 \rho^{-n} (\log |z| + \log |w|), \tag{16}$$

where $C_1, C_2 \in \mathbb{R}$.

Proof. The first inequality is proved in the same way as Proposition 5 and we leave it to the reader. Now we deduce (16) from inequality (15). Recall $d_n = \lambda_0 \rho^n + \lambda_1 (-\rho)^{-n}$ where λ_0, λ_1 are real numbers, Thus

$$\frac{d_{n+1}}{d_n} = \frac{\lambda_0 \rho^{n+1} + \lambda_1 (-\rho)^{-n-1}}{\lambda_0 \rho^n + \lambda_1 (-\rho)^{-n}} = \rho + O(\rho^{-2n}).$$

Replacing this expression of $\frac{d_{n+1}}{d_n}$ in the estimate (15), we deduce

$$\left| \log_+ |h_{n+1}(w, z)| - \rho \log_+ |h_n(w, z)| \right| \leq C + O(\rho^{-2n}) \log_+ |h_n(w, z)|.$$

From Lemma 3.1 one deduce for all $|z|, |w| \geq R$,

$$\log_+ |h_n(w, z)| \leq d_n + d_n (\log |z| + \log |w|).$$

This yields

$$\left| \log_+ |h_{n+1}(w, z)| - \rho \log_+ |h_n(w, z)| \right| \leq C + O(\rho^{-n}) (\log |z| + \log |w|).$$

\square

Now, let us define the (dynamical) Green’s function of the compact \mathcal{K}_c , for all $z \in \mathbb{C}$:

$$g(z) = g_c(z) := \lim_{n \rightarrow +\infty} \frac{1}{d_n} \log_+ |f_n(z)|.$$

This definition makes sense. Indeed,

Theorem 3.3. *The function $g_c : \mathbb{C} \rightarrow \mathbb{R}_+$ satisfies the following properties:*

1. g_c is harmonic in $\mathbb{C} \setminus \mathcal{K}_c$;
2. \mathcal{K}_c is exactly the set $g_c^{-1}(\{0\})$;
3. $g_c(z) - \log |z|$ tends to the constant $(\log |a_{d_1}|) \sum_{n=0}^{+\infty} \frac{(-1)^n}{d_n d_{n+1}}$ as $|z|$ tends to $+\infty$, where $f(z) = f_1(z) = a_{d_1} z^{d_1} + \dots + a_0$;
4. $(c, z) \mapsto g_c(z)$ is continuous.

Proof. We forget obvious dependencies on c when not necessary. The first point is more or less obvious, it suffices to write g as the series

$$g(z) - \log_+ |z| = \sum_{n=0}^{+\infty} g_{n+1}(z) - g_n(z) \tag{17}$$

and in view of Proposition 5 the functions g_n :

$$g_n(z) = \frac{1}{d_n} \log_+ |f_n(z)|,$$

form a sequence of harmonic functions that converge uniformly to g .

Clearly, if $(f_n(z))_{n \in \mathbb{N}}$ is bounded then $g(z) = 0$. The converse requires to be a little more cautious. Assume $z \notin \mathcal{K}_c$, according to Proposition 4 there exists n_0 such that both $|f_{n_0}(z)|$ and $|f_{n_0+1}(z)| > R$. Let $\lambda = \frac{\min\{|f_{n_0}(z)|, |f_{n_0+1}(z)|\}}{R}$. Obviously $\lambda > 1$.

We claim that for all $k \geq 0$, $|f_{n_0+k}(z)| > \lambda^{F_k} R$ where F_k is the Fibonacci number with initial terms $F_0 = F_1 = 1$. For $k = 0$ and 1 this is just the definition of λ . Now by induction, assume for all $p \leq k$, $|f_{n_0+p}(z)| > \lambda^{F_p} R$. Then

$$\begin{aligned} |f_{n_0+k+1}(z)| &\geq |f_{n_0+k}(z)f_{n_0+k-1}(z)| - |c| \\ &> \lambda^{F_{k+1}} R^2 - |c| \geq \lambda^{F_{k+1}} R. \end{aligned}$$

For this last estimate we use the fact that $R^2 - |c| \geq R$, and the claim follows.

Now we return to the Green’s function:

$$g_{n_0+k}(z) \geq \frac{F_k}{d_{n_0+k}} \log \lambda$$

and this guarantees that $g(z) = \lim_{k \rightarrow \infty} g_{n_0+k}(z) > 0$ whenever $z \notin \mathcal{K}_c$.

Concerning the third point, each term in (17) goes to zero as $n \rightarrow +\infty$. Indeed, let us write $f_n(z) = a_{d_n, n} z^{d_n} + a_{d_n-1, n} z^{d_n-1} + \dots + a_{0, n}$ for all $n \geq 0$. Since $f_0(z) = z$, it is easy to see by induction that $a_{d_n, n} = a_{d_1, 1}^{F_n-1}$ for all integer $n \geq 1$. Therefore

$$\begin{aligned} g_{n+1}(z) - g_n(z) &= \left(\frac{F_n}{d_{n+1}} - \frac{F_{n-1}}{d_n} \right) \log |a_{d_1, 1}| \\ &\quad + \frac{1}{d_{n+1}} \log_+ |f_{n+1}(z)/a_{d_{n+1}, n+1} z^{d_{n+1}}| \\ &\quad - \frac{1}{d_n} \log_+ |f_n(z)/a_{d_n, n} z^{d_n}| \end{aligned} \tag{18}$$

On the other hand, for all integer $n \geq 2$,

$$\begin{pmatrix} F_{n-1} & F_n \\ d_n & d_{n+1} \end{pmatrix} = \begin{pmatrix} F_{n-2} & F_{n-1} \\ d_{n-1} & d_n \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

Hence

$$\begin{pmatrix} F_{n-1} & F_n \\ d_n & d_{n+1} \end{pmatrix} = \begin{pmatrix} F_0 & F_1 \\ d_1 & d_2 \end{pmatrix} M^{n-1}$$

where $M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Thus $F_n d_n - F_{n-1} d_{n+1} = (F_1 d_1 - F_0 d_2)(-1)^{n-1} = (-1)^n$.

Therefore, as $|z|$ converges to $+\infty$, $g_{n+1}(z) - g_n(z)$ converges to $\frac{(-1)^n}{d_n d_{n+1}} \log |a_{d_1,1}|$. Since the series converges normally we deduce that $g(z) - \log(|z|)$ tends to $\log |a_{d_1,1}| \sum_{n=0}^{+\infty} \frac{(-1)^n}{d_n d_{n+1}}$ as $|z| \rightarrow +\infty$. Each g_n depends continuously on (z, c) , thus the sum of the series is also continuous. \square

Remark 2. According to Proposition 6, we can also define the 2-dimensional Green’s function associated to ϕ . For all $|z|, |w| > R$, let

$$G(w, z) := \lim_{n \rightarrow +\infty} \frac{1}{d_n} \log_+ |h_n(w, z)|.$$

This formula makes sense and G and g are related by $G(f(z), z) = g(z)$. Moreover, G satisfies the functional equation $G \circ H_c(w, z) = \rho G(w, z)$. This fact can be interpreted in connection with the existence of invariant measure on the Julia set for which the action of H_c on the Julia set is mixing and with maximal entropy given here by $\log(\rho)$, such probability measure exists by virtue of Dinh-Sibony Theorem (see [9] and the references therein). One can obtain an alternative proof of Proposition 2.6 using the properties of G . We shall include the proof here for the convenience of the reader (see Proposition 3.8).

Remark 3. Recall that we denote $\rho = \frac{1+\sqrt{5}}{2}$, d_n is equal to $\lambda_0 \rho^n + \lambda_1 ((-1/\rho)^n)$ with λ_0, λ_1 constants which essentially depends on d_1 . Thus $\frac{d_n}{\rho^n}$ tends to λ_0 and we could replace d_n by $\lambda_0 \rho^n$ in the definition of the Green’s function

$$g(z) = \lim_{n \rightarrow +\infty} \frac{1}{\lambda_0 \rho^n} \log_+ |f_n(z)|.$$

Theorem 3.3 admits several consequences.

Proposition 7. *The filled-in Julia set is a full compact subset of \mathbb{C} i.e. its complement has no bounded component. The logarithmic capacity of the filled-in Julia set is equal to $e^{-\sigma}$ where $\sigma = \log |a_{d_1}| \sum_{n=0}^{+\infty} \frac{(-1)^n}{d_n d_{n+1}}$.*

Proof. The proof is word to word the same as for polynomial dynamical systems. Indeed, assume there exists \mathcal{O} a bounded connected component of $\mathbb{C} \setminus \mathcal{K}_c$. Then by the maximum principle

$$\max_{z \in \mathcal{O}} g(z) = \max_{z \in \partial \mathcal{O}} g(z) = 0$$

as $\partial \mathcal{O} \subset \mathcal{K}_c$. This provides a contradiction since \mathcal{O} is not included in \mathcal{K}_c .

The logarithmic capacity is defined as $e^{-\sigma}$ where σ is the constant in

$$g(z) = \log(|z|) + \sigma + o(1),$$

and according to Theorem 3.3 we obtain $\sigma = \log |a_{d_1}| \sum_{n=0}^{+\infty} \frac{(-1)^n}{d_n d_{n+1}}$. \square

An other consequence of Theorem 3.3 is the following. Let $Comp^*(\mathbb{C})$ denotes the set of non empty compact Hausdorff subsets of \mathbb{C} .

Proposition 8. *The map $c \mapsto J_c$ from \mathbb{C} to $\text{Comp}^*(\mathbb{C})$ is lower semi-continuous.*

Proof. The standard proof works in our setting. Indeed, the map $c \mapsto g_c$ is continuous in L^1 according to Theorem 3.3, and $c \mapsto \Delta g_c$ is also continuous in the sense of distributions. Hence the support of the measure Δg_c is lower semi-continuous with respect to c . Moreover, g_c is harmonic in $\mathbb{C} \setminus \mathcal{K}_c$ and $g_c = 0$ on the interior of \mathcal{K}_c , thus the support of Δg_c coincides with J_c . \square

4. Properties of the connectedness locus of CFS.

Proposition 9. *Assume that $f_1(z) = f(z)$ satisfies $f(0) = f'(0) = 0$, then for all complex numbers c such that $|c| > 2$, we have that \mathcal{K}_c is a disconnected set.*

Proof. Since $f(0) = f'(0) = 0$, it is easy to check that $f'_n(0) = 0$ for all $n \geq 1$. Using Riemann-Hurwitz Formula and Proposition 4, we deduce that if $0 \notin \mathcal{K}_c$, then \mathcal{K}_c is not connected.

Assume, now that $|c| > 2$, then we have by induction the following claim:

Claim. $|f_n(0)| \geq (|c| - 1)|f_{n-1}(0)|, \forall n \geq 4$.

Indeed, $|f_4(0)| = |c^2 + c| \geq (|c| - 1)|f_3(0)|$. Suppose that the claim is true for all integer $4 \leq k \leq n$, since $|c| > 2$ and $f_3(0) = c$, we deduce that $|f_k(0)| \geq |c|$ for all $3 \leq k \leq n$. Hence

$$|f_{n+1}(0)| \geq |f_n(0)f_{n-1}(0)| - |c| \geq (|c| - 1)|f_{n-1}(0)|.$$

Then, we obtain the claim. \square

Question.

1. *The condition that 0 is a zero of multiplicity two is crucial in our proof. One may ask if this condition can be relaxed.*
2. *In our proof we have proved that \mathcal{K}_c is disconnected for $|c| > 2$. One may ask what is the minimal constant d such that for all $|c| > d$, \mathcal{K}_c is still disconnected.*

Proposition 10. *Assume that $f_1(z) = z$, then for all real numbers $c < -2$, \mathcal{K}_c is a disconnected set.*

Proof. Since $H_c^3(-1, -1) = (-1, -1)$, then -1 belongs to \mathcal{K}_c . On the other hand $z_0 = \frac{1+\sqrt{1-4c}}{2} > 0$ belongs to \mathcal{K}_c since (z_0, z_0) is a fixed point of H_c .

Claim. *The line $i\mathbb{R}$ doesn't intersect \mathcal{K}_c .*

Indeed: let $x \in \mathbb{R}$ and $c < -2$, then $f_2(ix) = -x^2 + c < c < -2$ and

$$|f_3(x)| = \sqrt{x^2(-x^2 + c)^2 + c^2} > |c|,$$

then by Lemma 2.1, we deduce $ix \notin \mathcal{K}_c$. Then we obtain the claim. \square

The connectedness locus of CFS $(f_n)_{n \in \mathbb{N}}$ is the set

$$\mathcal{M} = \{c \in \mathbb{C} : \mathcal{K}_c \text{ is connected}\}.$$

Another important set is the set \mathcal{M}_0 defined by

$$\mathcal{M}_0 = \{c \in \mathbb{C} : (f_n(0))_{n \in \mathbb{N}} \text{ is bounded}\}.$$

It is clear that under the assumption $f_1(0) = f'_1(0) = 0$ the set \mathcal{M} is a subset of \mathcal{M}_0 .

Proposition 11. *Assume that $f_1(0) = f'_1(0) = 0$, then the set \mathcal{M}_0 satisfies the following properties*

1. $D(0, \frac{1}{4}) \subset \mathcal{M}_0 \subset D(0, 2)$.
2. \mathcal{M}_0 is a compact and simply connected set.

Proof. By Proposition (9), we have $\mathcal{M}_0 \subset D(0, 2)$. Now, let $A > 1$, then we have the following claim

Claim. $D(0, \frac{A-1}{A^2}) \subset \mathcal{M}_0$.

Indeed, put $r = \frac{A-1}{A^2}$ and assume that $|f_1(0)| = |c| < r < rA$, then $|f_2(0)| \leq r^2 + |c| < r^2A^2 + r = rA$. We deduce easily by induction on n that

$$|f_n(0)| < rA, \forall n \in \mathbb{N}.$$

Hence, we obtain the claim.

Since $\max\{\frac{A-1}{A^2}, A > 1\} = \frac{1}{4}$, we have $D(0, \frac{1}{4}) \subset \mathcal{M}_0$.

On the other hand, since $\mathcal{M}_0 \subset D(0, 2)$, there exists a constant R (which does not depend on c) such that if $|f_k(0)| > R$, for some integer k , then the sequence $(f_n(0))_{n \geq 0}$ is unbounded. Putting $f_n(0) = \phi_n(c)$, we have

$$\mathcal{M}_0 = \{c \in \mathbb{C}, |\phi_n(c)| \leq R, \forall n \in \mathbb{N}\}.$$

Hence \mathcal{M}_0 is a compact and simply connected set. □

5. Main cardioid: Preliminary results. Here we focus on a subset of the parameter space that generalizes the main cardioid of the Mandelbrot set. We aim to study the dynamics of CFS with small values of c .

For technical reasons we will assume that $f(z) = f_1(z) = z$. Under this restriction the filled-in Julia set of the CFS corresponds to $\mathcal{K}_c = K_c^+ \cap \Delta$ where Δ is the diagonal of \mathbb{C}^2 and $K_c^+ = \{(x, y) \in \mathbb{C}^2, H_c^n(x, y) \text{ is bounded}\}$.

The goal of the four last sections is the following result:

Theorem 5.1. *There exists $a > 0$ such that for all $0 \leq |c| < a$, the Julia set $\mathcal{K}_c = K_c^+ \cap \Delta$ is a quasi-disk, where Δ is the diagonal $\{(z, z) \mid z \in \mathbb{C}\}$. Moreover $c \mapsto \mathcal{K}_c$ is continuous for the Hausdorff topology.*

For a good introduction to quasi-conformal mappings and its applications to dynamics see [4] and [1].

Idea of the proof. When $c = 0$ the dynamics is simple to understand: there is an invariant torus $|x| = |y| = 1$ that cuts the diagonal Δ into a circle. Each point of that circle \mathcal{C} has a local stable manifold passing through, which is transverse to Δ . If one can show that these disks are analytic graphs that move holomorphically with c , their intersection points with Δ will define a holomorphic motion of the circle \mathcal{C} , hence define a family of quasi-circles \mathcal{C}_c . Then it remains to show that the interior of those quasi-circles are in K_c^+ (which will be a simple consequence of Liouville's theorem) and that the exteriors are in the escaping sets.

Invariant torus and adapted coordinates. In this section we consider the (monomial) map $H_0 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} xy \\ x \end{pmatrix}$ defined on \mathbb{C}^2 .

Lemma 5.2. *When $c = 0$, the torus $\mathbf{T}_0 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2; |x| = |y| = 1 \right\}$ is a hyperbolic invariant set. Moreover, around each point $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbf{T}_0$ there exists an open set with*

local branches of $y \mapsto y^{\beta_i}$ such that the two maps

$$\phi_1 : \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \frac{x}{y^{\beta_1}} \text{ and } \phi_2 : \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \frac{x}{y^{\beta_2}}$$

are well-defined and semi-conjugate H_0 to the map

$$L : \begin{pmatrix} u \\ v \end{pmatrix} \rightarrow \begin{pmatrix} u^{1-\beta_1} \\ v^{1-\beta_2} \end{pmatrix},$$

where $\beta_1 := \frac{1+\sqrt{5}}{2} \approx 1.61$ and $\beta_2 := \frac{1-\sqrt{5}}{2} \approx -0.61$ are the two eigenvalues of the matrix $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

Using coordinates $(e^{i\alpha}, e^{i\beta})$ on the torus \mathbf{T}_0 , the restriction of H_0 to \mathbf{T}_0 coincides with the linear map $M : \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \rightarrow \begin{pmatrix} \alpha+\beta \\ \alpha \end{pmatrix}$.

Proof. The stable and unstable manifolds of the invariant torus are 3-dimensional real analytic manifolds given by the equations $|x| = |y|^{\beta_i}$, where the β_i are the roots of $X^2 - X - 1$. The formulas for the semi-conjugacy come from a simple computation:

$$\phi_1 \circ H_0 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{x \cdot y}{x^{\beta_1}} = x^{1-\beta_1} \cdot y = \left(\frac{x}{y^{\beta_1}} \right)^{1-\beta_1} = \left(\phi_1 \begin{pmatrix} x \\ y \end{pmatrix} \right)^{1-\beta_1}.$$

□

Dynamics in the non-perturbated case.

We will need to visualize four-dimensional neighborhoods of the invariant torus. In order to do this we will draw the relevant domains in the plane $(|x|, |y|)$.

By taking absolute values of the functions $\phi_i, i = 1, 2$, we can easily prove:

Lemma 5.3. Let $R(x, y) := \frac{|x|}{|y|^{\beta_1}}$ and $S(x, y) := \frac{|x|}{|y|^{\beta_2}}$. Then one has

$$R \circ H_0 = R^{1-\beta_1} \text{ and } S \circ H_0 = S^{1-\beta_2}.$$

The dynamics in the plane $(R, S) \in \mathbb{R}^+ \times \mathbb{R}^+$ is given simply by $(R, S) \mapsto (R^{1-\beta_1}, S^{1-\beta_2})$. The level curves $(R = cst)$ look like graphs $y = \sqrt{x}$ and level curves $(S = cst)$ look like hyperbolas $x \cdot y = cst$.

Remark 4. Since $H_0^{(n)}(x, y) = (x^{F_n} y^{F_{n-1}}, x^{F_{n-1}} y^{F_{n-2}})$ for all integer $n \geq 2$, we deduce that

$$K_0^+ = \{(x, y) \in \mathbb{C}^2; |x^{\beta_1} y| \leq 1\} = \{(x, y) \in \mathbb{C}^2; S(x, y) \leq 1\}.$$

We recall that K_c^+ is the set of points of \mathbb{C}^2 with bounded forward orbit for the map H_c , the escape locus is defined as $U_c^+ = \mathbb{C}^2 \setminus K_c^+$. We denote by $\mathbb{B}(0, r) = B_r \times B_r$ the **open bidisk** centered at the origin in \mathbb{C}^2 , with radius $r > 0$.

Definition 5.4 (Neighborhood in “good position”). Let $0 < a < 1, 0 < b < 1$ and $0 < r < 1$ and let H_c be a fixed map. Let us consider the neighborhood of the torus $T_0 := \{(x, y) \in \mathbb{C}^2; |x| = |y| = 1\}$ defined by

$$\mathcal{N}_{a,b} = \{(x, y); 1 - a \leq R(x, y) \leq 1 + a \text{ and } 1 - b \leq S(x, y) \leq 1 + b\}.$$

Then we say that $\mathcal{N}_{a,b}$ is **in good position with respect to the open bidisk** $\mathbb{B}(0, 1 - r)$ if the following condition is satisfied:

$$H_c(\mathcal{N}_{a,b}) = \mathcal{M}_1 \amalg \mathcal{M}_2 \amalg \mathcal{M}_3 \text{ (disjoint union)}$$

where:

1. $\overline{\mathcal{M}}_1 \subset \mathbb{B}(0, 1 - r)$,
2. $\mathcal{M}_2 = \mathcal{N}_{a,b} \cap H_c(\mathcal{N}_{a,b})$,
3. $\overline{\mathcal{M}}_3 \subset U_c^+$ where U_c^+ is the set of points with unbounded forward orbit.

Definition 5.5 (Neighborhoods W_c^n). Let $W_c^0 := \mathcal{N}_{a,b}$ be a fixed neighborhood (as above) of the torus T_0 . Let us define inductively

$$W_c^{n+1} := W_c^n \cap H_c^{-1}(W_c^n).$$

Lemma 5.6 (Case ($c = 0$)). For any $W := W_0^0 = \mathcal{N}_{a,b}$, with a in $[0, 0.4]$ the sequence $(W_0^n)_{n \geq 0}$ is decreasing, all the W_0^n are homeomorphic to W , and the intersection

$$W_0^\infty := \bigcap_{n \geq 0} W_0^n$$

is homeomorphic to a direct product $S^1 \times (\text{Annulus})$.

Moreover, W_0^∞ coincides with $W \cap W^s(T_0)$, where $W^s(T_0)$ is the stable manifold of the torus T_0 invariant under H_0 .

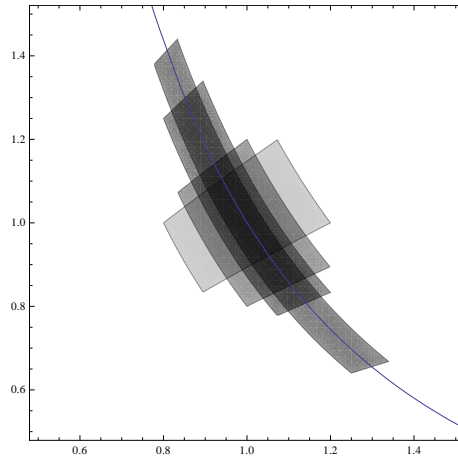


FIGURE 2. Preimages of W and locus $S=1$

Proof. We work in the plane (R, S) . Then for any rectangle \mathcal{R} of the form $\mathcal{R} := \{1 - a < R < 1 + a, \text{ and } 1 - b < S < 1 + b\}$ (with $0 \leq b \leq 1$ and $0 \leq a \leq 1$) we have $H_0^{-1}(\mathcal{R}) = \{(1 + a)^{-\beta_1} < R < (1 - a)^{-\beta_1}, \text{ and } (1 - b)^{-\beta_2} < S < (1 + b)^{-\beta_2}\}$.

Now, we have $(1 - b) < (1 - b)^{-\beta_2}$ and $(1 + b)^{-\beta_2} < 1 + b$. For a small enough (less than 0.4 is enough), we have $(1 + a)^{-\beta_1} < 1 - a$ and $1 + a < (1 - a)^{-\beta_1}$. Thus $\mathcal{R} \cap H_0^{-1}(\mathcal{R}) = \{1 - a < R < 1 + a, \text{ and } (1 - b)^{-\beta_2} < S < (1 + b)^{-\beta_2}\}$, which shows that the W_0^n are all homeomorphic and form a decreasing sequence converging to $\{1 - a < R < 1 + a\} \times \{1\}$. Now it suffices to remember that the stable manifold of the torus $\{R = 1, S = 1\}$ corresponds exactly to $\{S = 1\}$. \square

Existence of a neighborhood in good position for $c = 0$

First we show that, by taking c small enough we can find symmetric bidisks around the origin, of radius arbitrarily close to 1, that are contained entirely inside K_c^+ .

Lemma 5.7. *For any $0 < \delta < 1$ there exists $\epsilon > 0$ such that for all $0 \leq |c| < \epsilon$ the bidisk $\mathbb{B}(0, 1 - \delta)$ is included in K_c^+ .*

Proof. Since $|xy + c| \leq |xy| + |c| \leq (1 - \delta)^2 + |c| = (1 - \delta) + \delta^2 - \delta + |c|$, it is enough to take $\epsilon < \delta - \delta^2$. □

For the next Lemma, we keep the same notations as above.

Lemma 5.8. *Given any neighborhood \mathcal{N} of the torus T_0 , there exists $\delta > 0$, $\alpha > 0$ and $0 < a < 1$, $0 < b < 1$ such that the following conditions are satisfied:*

1. *the bidisk $\mathbb{B}(0, 1 - \delta)$ is included in K_c^+ for any c such that $|c| < \alpha$;*
2. *$\mathcal{N}_{a,b} \subset \mathcal{N}$;*
3. *the neighborhood $\mathcal{N}_{a,b}$ is in good position with respect to the bidisk $\mathbb{B}(0, 1 - \delta)$, for all maps H_c with $|c| < \alpha$.*

Proof. (1) comes from Lemma 5.7.

For the proof of (2) and (3), for any $0 < a < 1$, we denote by R_a the set

$$\{(x, y) \in \mathbb{C}^2, \text{ such that } 1 - a < R(x, y) < 1 + a\},$$

First we observe the following facts:

1. for any $a \in (0, 1)$ small enough (less than 0.4 is enough), we have $1 - a < (1 + a)^{\beta_2}$ and $(1 - a)^{\beta_2} < 1 + a$. By Lemma 5.3 and the fact that $\beta_2 = 1 - \beta_1$, we have $H_0(R_a) \subset \text{int}(R_a)$,
2. no bidisk $B_{1-\delta} \times B_{1-\delta}$ intersects a locus of the form

$$\{S > 1 + b\},$$

3. for any $b \in (0, 1)$ and $0 < a < b/2$ there exists $\delta_0 > 0$ such that for any $0 < \delta < \delta_0$ we have:

the bidisk $B_{1-\delta} \times B_{1-\delta}$ centered at the origin contains the locus

$$R_a \cap \{S < 1 - b\}.$$

For this last fact, it suffices to choose a and b such that

$$\left(\frac{1 - b}{1 - a}\right)^{1/1+\beta_1} < 1 - \delta, \quad (1 + a)\left(\frac{1 - b}{1 - a}\right)^{\beta_1/1+\beta_1} < 1 - \delta.$$

Now we observe that as $a \rightarrow 0$, $b \rightarrow 0$, the neighborhoods $\mathcal{N}_{a,b}$ converge to the torus \mathbf{T}_0 (corresponding to $|x| = |y| = 1$), thus given any \mathcal{N} one can certainly ensure that $\mathcal{N}_{a_0,b_0} \subset \mathcal{N}$ and notice that $\mathcal{N}_{a,b} \subset \mathcal{N}_{a_0,b_0}$ for all a, b satisfying $a \leq a_0$ and $b \leq b_0$. Now, once $\mathcal{N}_{a,b}$ is chosen ($a \leq a_0$, $b \leq b_0$), we choose a δ as above. Let us call $W := \mathcal{N}_{a,b}$. Then we see immediately that $H_0(W) \subset \text{int}(R_a)$. Also, the following partition of R_a into three parts will induce on $H_0(W)$ a corresponding partition:

1. First part: $R_a \cap \{S < 1 - b\} \subset \{S \leq 1\} = K_0^+$,
2. Second part: $R_a \cap \{1 - b \leq S \leq 1 + b\} = W$,
3. Third part: $R_a \cap \{S > 1 + b\} \subset U_0^+$.

On the other hand, we can prove that for δ small, the set $R_a \cap \{S > 1 + b\}$ is contained in the set $\{(x, y), |x| > 1 + \delta, |y| > 1 + \delta\}$.

For this, it suffices to choose $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$

$$\left(\frac{1 + b}{1 + a}\right)^{1/(\beta_1 - \beta_2)} > 1 + \delta \text{ and } (1 - a)\left(\frac{1 + b}{1 + a}\right)^{\beta_1/(\beta_1 - \beta_2)} > 1 + \delta.$$

We deduce that

$$H_0(W) \subset \text{int}(R_a) \subset B(0, 1 - \delta) \cup W \cup \{(x, y), |x| > 1 + \delta, |y| > 1 + \delta\}.$$

Lemma 5.9. *For any $0 < \delta < 1$ there exists $\epsilon > 0$ such that for all $0 \leq |c| < \epsilon$ the domain $\{|x| > 1 + \delta$ and $|y| > 1 + \delta\}$ is included in U_c^+ .*

Proof. Since $|xy + c| \geq |xy| - |c| \geq (1 + \delta)^2 - |c| \geq 1 + 2\delta + (\delta^2 - |c|)$, it is enough to take $\epsilon < \delta^2$ to ensure that the orbit diverges to (∞, ∞) . □

End of the proof of Lemma 5.8:

Existence of a neighborhood in good position for c small.

Let $W = \mathcal{N}_{a,b}$ is as defined in 5.8. Since for $c = 0$ we know that $H_0(\overline{W}) \subset \text{int}(\{1 - a < R < 1 + a\})$, the same is true for H_c with $|c|$ small enough. Thus $H_c(W) \subset \text{int}(\{1 - a < R < 1 + a\}) \subset \mathbb{B}(0, 1 - \delta) \cup W \cup \{|x| > 1 + \delta$ and $|y| > 1 + \delta\}$, and this is enough to ensure that W is in good position with respect to $\mathbb{B}(0, 1 - \delta)$ for all H_c with $|c|$ small enough.

This ends the proof of Lemma 5.8. □

As an immediate consequence of the partition of $H_0(W)$ into three parts we have the following useful Proposition:

Proposition 12 (Dynamics in W when $c = 0$). *Let $U_0^0(W)$ be the connected component of $H_0(W) - W$ that is inside U_0^+ and $K_0^0(W)$ be the connected component of $H_0(W) - W$ that is inside $\mathbb{B}(0, 1 - \delta)$. Then there is a partition*

$$W = (W \cap \text{int}(K_0^+)) \amalg \mathcal{W}^S(W) \amalg (W \cap U_0^+),$$

with the following properties:

1. $W \cap U_0^+ = \bigcup_{n \geq 1} U_0^n(W)$ where $U_0^n(W) := W \cap H^{-n}(U_0^0(W))$, for all $n \geq 0$.
2. $W \cap K_0^+ = \bigcup_{n \geq 1} K_0^n(W)$ where $K_0^n(W) := W \cap H^{-n}(K_0^0(W))$, for all $n \geq 0$.
3. $\mathcal{W}^s(W)$ is defined as $\{p = (x, y) \in W \mid H_0^{(n)}(p) \in W, \forall n \geq 0\}$.

Proof. From Lemma 5.8 we know the fate of points of W under forward iteration: they eventually fall in $\mathbb{B}(0, 1 - \delta) \subset \text{int}(K_0^+)$ or in U_0^+ , or stay forever in W (and hence, by definition, belong to $\mathcal{W}^s(W)$). From that same lemma, we know that a point $p \in W$ enters eventually $\text{int}(K_0^+)$ if and only if one of its forward iterates lands in the connected component of $(H_0(W) - W)$ that is inside $\mathbb{B}(0, 1 - \delta)$. Similarly a point $p \in W$ enters eventually U_0^+ if and only if one of its forward iterates lands in the connected component of $(H_0(W) - W)$ that is inside U_0^+ . □

An immediate consequence is:

Proposition 13. *Proposition 12 is also true for $|c|$ small, with H_0, K_0^+, U_0^+ respectively replaced by H_c, K_c^+, U_c^+ .*

The invariant torus \mathbb{T}_0 and its stable foliation. In this section we set $W := \mathcal{N}_{a,b}$, where $\mathcal{N}_{a,b}$ is as in Lemma 5.8.

Lemma 5.10. *When $c = 0$, the set $\mathcal{W}^s(W)$ defined in 12 coincides with the set*

$$\{1 - a < R < 1 + a\} \times \{S = 1\}.$$

It is also the part of the stable manifold of the invariant torus \mathbb{T}_0 lying in W , and is thus a disjoint union of analytic disks, each one being a piece of stable manifold of a point of the torus.

Proof. The first assertions are immediate consequences of the fact that the functions R and S satisfy simple functional equations. That the stable manifolds are analytic manifolds comes from the standard Stable Manifold theorem for hyperbolic sets (see for example Theorem 6.4.9 in Katok-Hasselblatt [13]). \square

Lemma 5.11. *The analytic disks foliating $\mathcal{W}^s(W)$ intersect transversally the diagonal $\Delta := \{(z, z) \in \mathbb{C}^2, z \in \mathbb{C}\}$ along the unit circle.*

Proof. This comes from the fact that the intersection $\{S = 1\} \cap \Delta$ is given by the set $\left\{ (x, x) \mid \frac{|x|}{|x|^{\beta_2}} = 1 \right\}$, which is the unit circle inside the diagonal. Now the local stable manifolds of the torus are locally given by graphs of the form $\frac{x}{y^{\beta_2}} = \text{const}$ (where we use local branches of the function y^{β_1}), and those are transverse to the diagonal. \square

6. Small perturbations. The main idea of this section is that the invariant torus \mathbf{T}_0 will persist under perturbations when $c \neq 0$, and the local stable manifold of the torus will move analytically when c is changed, and will therefore intersect the diagonal along points that move analytically when c changes, thus defining a holomorphic motion of the initial circle $\mathcal{C} := \mathbf{T}_0 \cap \Delta$.

The general theory of Structural Stability of hyperbolic sets ensures the existence of an invariant torus \mathbf{T}_c for c small enough, such that the maps H_c and H_0 are conjugated on the tori.

The relevant Theorem is taken from Katok-Hasselblatt [13].

Theorem 6.1 (Structural stability of Hyperbolic sets). *Let M be a Riemannian manifold and $U \subset M$. Let $\Lambda \subset M$ be a hyperbolic set of the diffeomorphism $f : U \rightarrow M$. Then for any open neighborhood $V \subset U$ of Λ and every $\delta > 0$ there exists $\epsilon > 0$ such that if $f' : U \rightarrow M$ and $d_{C^1}(f|_V, f') < \epsilon$, there is a hyperbolic set $\Lambda' = f'(\Lambda) \subset V$ for f' and a homeomorphism $h : \Lambda' \rightarrow \Lambda$ with $d_{C^0}(Id, h) + d_{C^0}(Id, h^{-1}) < \delta$ (d_{C^i} , $i = 0, 1$ distances induced by C^i topologies) such that $h \circ f'_{|\Lambda'} = f_{|\Lambda} \circ h$. Moreover, h is unique when δ is small enough.*

Dependence of the local stable manifolds on the parameters.

We recall that for any $x \in K$ we can define a local stable manifold \mathcal{V}_x^- and a local unstable manifold \mathcal{V}_x^+ . The next theorem, taken from [18] (Section 15.2) describes the dependence of these local manifolds on the parameters.

Theorem 6.2 (Persistence). *The manifolds \mathcal{V}_x^\pm are of class C^r when f is of class C^r . Moreover the stable and unstable manifolds of Λ' depend continuously on f' for the C^r topologies.*

We will need a bit more, namely the analytic dependence of the stable manifold $\mathcal{W}^s(W)$:

Lemma 6.3. *The local stable manifolds in W depend analytically in (x, y, c) .*

Proof. One way to prove this Lemma would be to show the analytic dependence of the local stable manifolds of periodic points, and use the density of these periodic points on the invariant torus. Instead we use here the following result about persistence of complex laminations, due to Berger (see [3]).

Theorem 6.4 (Berger). *Let (L, \mathcal{L}) be a complex lamination immersed by i into a complex manifold M . Let f be a holomorphic endomorphism of M which preserves and 0-normally expands \mathcal{L} .*

Then for any complex k -ball B , and any complex analytic family $(f_t)_{t \in B}$ of endomorphisms of M , such that f_0 is f , there exist an open neighborhood B_0 of $0 \in B$ and a complex analytic family of laminations $(L, \mathcal{L}_t)_{t \in B_0}$ immersed by $(i_t)_t$ such that:

- $f_0 = f$, $i_0 = i$ and $\mathcal{L}_0 = \mathcal{L}$,
- f_t preserves the lamination (L, \mathcal{L}_t) holomorphically immersed by i_t into M , for any $t \in B_0$.

If moreover i is an embedding and f is forward plaque-expansive, then i_t is an embedding for every $t \in B_0$.

In this setting, we say that the endomorphism 0 -normally expands \mathcal{L} when the action of the dynamics on the normal bundle is expanding. This is the case in our situation, since here the normal bundle to the lamination coincides with the unstable bundle.

As a consequence of Berger's theorem one deduces that the local stable manifolds of points in W can be described as graphs of analytic functions of the form $f(y, c)$ (that is, they form a complex analytic family of laminations). \square

Remark 5. As observed in the introduction of this section the invariant torus and the hyperbolicity persist. Let us mention that, in addition, topological entropy, exponential mixing and the spectral structure persist under such small perturbations. For more details on the ergodic properties of H_c , we refer the reader to the survey by Guedj [9] and the references therein.

Now we are able to construct the quasi-disk.

Proposition 14. *Let \mathbf{T}_c be the invariant torus of H_c for c small enough, and $\mathcal{W}^s(\mathbf{T}_c)$ its stable manifold. Then there exists $\epsilon > 0$ such that for each $|c| < \epsilon$ we have: $\mathcal{W}^s(\mathbf{T}_c)$ intersects the diagonal Δ along a quasi-circle \mathcal{C}_c .*

Proof.

1. **Parameterization of the invariant torus:** When $c = 0$, the intersection $\Delta \cap \mathcal{W}^s(\mathbf{T}_0)$ coincides with the unit circle \mathcal{C}_0 in the diagonal Δ . Each point $p_0 \in \mathcal{C}_0$ has a *local stable manifold* that is passing through it, and that is an analytic disk transverse to the diagonal (see Lemma 5.11). We recall here that two smooth submanifolds L_1 and L_2 of equal dimension n , in a manifold M of dimension $2n$ intersect transversally if at every point of intersection p , the tangent space $T_p M$ can be written as a direct sum $T_p L_1 \oplus T_p L_2$. The invariant torus persists, and thus one can define a continuous function $c \mapsto p_c \in \mathbb{T}_c$ such that p_0 is the initial point on the circle \mathcal{C}_0 .
2. **Stability of transverse intersections:** Now we use the key fact of Stability of Transverse intersections (see [10], page 65) that says that small perturbations of transverse submanifolds L_1, L_2 stay transverse. Therefore, for c small enough, each local stable manifold of any point p_c intersects transversally the diagonal Δ into a single point.
3. **Holomorphic motion of the circle \mathcal{C}_0 :** From Berger's result we know that each local stable manifold of a point p_c is locally a graph of an analytic function in the parameter c . Thus the intersection of the local stable manifold $W_{\text{loc}}^s(p_c)$ with the diagonal Δ defines an analytic function $c \mapsto \phi(c, p_0)$ satisfying $\phi(0, p_0) = p_0$, hence a holomorphic motion of the entire circle \mathcal{C}_0 parameterized by c . Indeed, two distinct points p_0, q_0 have distinct local stable manifolds passing through them, that stay distinct for c small enough and therefore have distinct intersection points with the diagonal. By Slodkowski's

theorem, we can extend the holomorphic motion to the entire diagonal and show that the set \mathcal{C}_c obtained from \mathcal{C}_0 by the holomorphic motion is actually a quasi-circle, bounding a quasi-disk \mathcal{D}_c . \square

For completeness we recall here the definition of a holomorphic motion of a set in \mathbb{C} .

Definition 6.5 (Holomorphic motion of a set $E \subset \mathbb{C}$). Let E be a subset of the Riemann sphere having at least three points. Let $D(0, R)$ be the disk centered at the origin with radius R in \mathbb{C} . Then a holomorphic motion of E over $D(0, R)$ is a mapping

$$f : D(0, R) \times E \rightarrow \hat{\mathbb{C}}$$

with the following properties:

1. for a given $\lambda \in D(0, R)$, the map $f_\lambda : E \rightarrow \hat{\mathbb{C}}$ given by $f_\lambda(z) = f(\lambda, z)$ is injective;
2. the map f_0 is the identity;
3. for each $z \in E$, the map $\lambda \mapsto f_\lambda(z)$ is holomorphic.

The extension theorem that we used above is due to Slodkowski. The version given below is taken from the book of de Faria and de Melo [6]:

Theorem 6.6 (Slodkowski's theorem). Let $\Lambda \subset \mathbb{C}$ be a simply connected domain and E be a subset of the Riemann sphere with at least four points. Let $h : \Lambda \times E \rightarrow \hat{\mathbb{C}}$ be a holomorphic motion with basepoint $\lambda_0 \in \Lambda$. Then h admits a continuous extension $H : \Lambda \times \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ having the following properties:

- (i) H is a holomorphic motion of $\hat{\mathbb{C}}$;
- (ii) $H_\lambda : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ is K -quasiconformal with K bounded by the hyperbolic distance between λ and the basepoint λ_0 .

7. Proof of main result. The goal of this section is to prove that $K_c^+ \cap \Delta$ coincides with the closed quasi-disk $\overline{\mathcal{D}_c}$ bounded by the quasi-circle $\mathcal{W}^s(\mathbf{T}_c) \cap \Delta$ defined above, for c small enough.

In order to do this we will need several preliminary results, that will allow us to compare the dynamics in the case $c = 0$ to the dynamics when $c \neq 0$ is small enough.

One important tool that we will use in the proof is the theory of the *crossed mappings*, first developed by Hubbard and Oberste-Vorth in [12], and that we will now review.

Crossed mappings.

In the domain W , small perturbations of H_0 have expanding and contracting directions that allow the use of *graph transform* methods. A general framework for such methods is given by the theory of *crossed mappings* as described in [12].

Here we recall the main results of [12] concerning such mappings.

Let $B_1 = U_1 \times V_1$ and $B_2 = U_2 \times V_2$ be bidisks.

Definition 7.1. A crossed mapping from B_1 to B_2 is a triple (W_1, W_2, f) , where

1. $W_1 \subset U'_1 \times V_1$ where $U'_1 \subset U_1$ is a relatively compact open subset,
2. $W_2 \subset U_2 \times V'_2$ where $V'_2 \subset V_2$ is a relatively compact open subset,

3. $f : W_1 \rightarrow W_2$ is a holomorphic isomorphism, such that for all $y \in V_1$, the mapping

$$pr_1 \circ f|_{W_1 \cap (U_1 \times \{y\})} : W_1 \cap (U_1 \times \{y\}) \rightarrow U_2$$

is proper, and the mapping

$$pr_2 \circ f^{-1}|_{W_2 \cap (\{x\} \times V_2)} : W_2 \cap (\{x\} \times V_2) \rightarrow V_1$$

is proper.

Cone fields in the tangent space.

For any bidisk $B = U \times V$, consider the horizontal cone field $C_{(x,y)} \subset T_{(x,y)}B$ defined by

$$C_{(x,y)} = \{(\xi, \eta) \in T_{(x,y)}B \text{ such that } |(x, \xi)|_U \geq |(y, \eta)|_V\},$$

where $T_{(x,y)}B$ denotes the tangent space at the point (x, y) . One can also define a *vertical cone field* by reversing the inequality.

As in [12] we call smooth curves and surfaces in a bidisk $B = U \times V$ *horizontal-like* or *vertical-like* if their tangent spaces are in the horizontal or vertical cone respectively at each of their points.

The next proposition essentially says the following: if one has an infinite sequence of crossed-mappings, where the vertical direction is contracted and the horizontal is expanded, then the set of points inside one bidisk, say B_0 , whose entire forward orbit always stays within the B_i will be an almost vertical analytic graph that we should understand as a piece of stable manifold. A similar situation will occur for unstable manifolds. The next proposition is a formal description of this argument.

Given U a Riemann surface isomorphic to the disk and $U' \subset U$ a relatively compact open subset, we recall that the size of U' in U is defined as $1/M$ where M is the largest modulus of an annulus separating U' from the boundary of U (see [12] for more details).

Proposition 15. *Let*

$$\dots B_{-1} = U_{-1} \times V_{-1}, B_0 = U_0 \times V_0, B_1 = U_1 \times V_1, \dots$$

be a bi-infinite sequence of bidisks, and $f_i : B_i \rightarrow B_{i+1}$ be crossed mappings of degree 1, with U'_i of uniformly bounded size in U_i , where the U_i, U'_i, V_i, V'_i are as in the definition of crossed mappings. Then for all $m \in \mathbb{Z}$,

(1) *the set*

$$W_m^s = \{(x_m, y_m) \mid \text{there exist } (x_n, y_n) \in B_n \\ \text{for all } n \geq m \text{ such that } f_n(x_n, y_n) = (x_{n+1}, y_{n+1})\}$$

is a closed vertical-like Riemann surface in B_m , and $pr_2 : W_m^s \rightarrow V_m$ is an isomorphism;

(2) *the set*

$$W_m^U = \{(x_m, y_m) \mid \text{there exist } (x_n, y_n) \in B_n \\ \text{for all } n < m \text{ such that } f_n(x_n, y_n) = (x_{n+1}, y_{n+1})\}$$

is a closed horizontal-like Riemann surface in B_m , and $pr_1 : W_m^s \rightarrow U_m$ is an isomorphism.

(3) *Moreover, the sequence*

$$(x_m, y_m) := W_m^s \cap W_m^U, \quad m \in \mathbb{Z},$$

is the unique bi-infinite sequence with $(x_m, y_m) \in B_m$ for all $m \in \mathbb{Z}$, and $f_m(x_m, y_m) = (x_{m+1}, y_{m+1})$.

Application to the map H_0 and H_c for c small.

Lemma 7.2. *Let $z \in \mathbf{T}_0$. Then one can choose a bidisk $B_i, i \in \mathbb{Z}$ around each point $z_i, i \in \mathbb{Z}$ of the full orbit of z , such that $H_0 : B_i \rightarrow B_{i+1}$ is a crossed mapping of degree one. Moreover for any small enough c , the map $H_c : B_i \rightarrow B_{i+1}$ is also a crossed mapping of degree one.*

Proof. Locally around each z_i , we can find a local branches of the functions y^{β_1}, y^{β_2} , and therefore define functions u, v such that in a bidisk $|u| \leq \alpha, |v| \leq \beta$ the map H_0 is given by

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} u^{1-\beta_1} \\ v^{1-\beta_2} \end{pmatrix},$$

where one coordinate is expanded and the other is contracted, so the map is a crossed mapping that sends the “vertical boundary” $|u| = \alpha$ strictly outside of the bidisk. By compactness of the torus \mathbf{T}_0 we can ensure that the images of all vertical boundaries of the B_i are away from the initial bidisk by a fixed strictly positive distance. Therefore, by continuity, for c small enough, the restriction of H_c to the same family of bidisks will still be a crossed mapping of degree one. \square

Main Result. We recall the main result of the section.

Theorem 7.3. *The intersection $K_c^+ \cap \Delta$ coincides with the closed quasidisk $\overline{\mathcal{D}_c}$, for c small enough.*

Proof. **Easy inclusion:** $\mathcal{D}_c \subset (K_c^+ \cap \Delta)$. This is simply a matter of observing that the boundary of \mathcal{D}_c is included in K_c^+ since it is contained in the set $\mathcal{W}^s(\mathbf{T}_c)$ of points with forward orbit attracted by the torus \mathbf{T}_c . Then, an immediate application of the maximum principle yields that the points of the interior of the quasi-disk have also bounded forward orbits.

Second inclusion: $K_c^+ \cap \Delta \subset \mathcal{D}_c$. We will prove now by contradiction that the exterior of the quasi-disk does not intersect K_c^+ , and this alone will require several lemmas.

Recall that for all c , the stable manifold $\mathcal{W}^s(\mathbf{T}_c)$ is the set of all points in \mathbb{C}^2 whose forward orbit converges towards \mathbf{T}_c and that $\mathcal{W}_c^s(W)$ is the set of points with forward orbit entirely contained in W , where W is defined in 5.8.

Lemma 7.4. *We have $\mathcal{W}_c^s(W) = \mathcal{W}^s(\mathbf{T}_c)$.*

Proof. The inclusion $\mathcal{W}^s(\mathbf{T}_c) \subset \mathcal{W}_c^s(W)$ is immediate. Now let $p \in \mathcal{W}_c^s(W)$. Take the forward orbit $\mathcal{O} := (p, H_c(p), H_c^{(2)}(p), \dots)$. Then we observe that \mathcal{O} is an ε -pseudo orbit for H_0 , where ε depends only on $|c|$. By the Shadowing Lemma (see for example Theorem 18.1.3 in Katok-Hasselblatt [13]), there exists a true H_0 -orbit of a point p_0 which is ε -close to \mathcal{O} . Moreover $p_0 \in W_0^s(W) = W^s(T_0)$. Hence $p_0 \in W^s(\tilde{p}_0)$ for some point $\tilde{p}_0 \in T_0$.

Now by Lemma 7.2 we can take a bi-infinite sequence $B_i, i \in \mathbb{Z}$ of bidisks such that both $H_0 : B_i \rightarrow B_{i+1}$ are crossed mappings of degree 1. Then we know that p has a forward orbit that stays in all the B_i for $i \geq 0$, hence it belongs to $W^s(B_0)$ (which is the set of points of B_0 that has a forward orbit staying in all the B_i for $i \geq 0$). But $W^s(B_0)$ also contains the point $q \in W^s(B_0) \cap W^u(B_0)$ which by definition is a point of $\bigcap_{n \in \mathbb{Z}} H_c^n(W)$. Now we use a lemma which is a direct consequence of Theorem 7.4 in [17] “Stability of a hyperbolic invariant set”:

Lemma 7.5. $T_c = \bigcap_{n \in \mathbb{Z}} H_c^{(n)}(W)$

Thus we deduce that $q \in T_c$ and that finally $p \in W^s(q)$. \square

Lemma 7.6. $\mathcal{W}_c^s(W) \cap \Delta$ is equal to the boundary of the quasi-disk \mathcal{D}_c .

Proof. We know that \mathcal{D}_c is bounded by the quasi-circle $\mathcal{C}_c := \mathcal{W}^s(\mathbf{T}_c) \cap \Delta$. And we just proved that $\mathcal{W}^s(\mathbf{T}_c)$ coincides with $\mathcal{W}_c^s(W)$. \square

End of proof of theorem 7.3. Now it remains to show that the exterior of the quasi-disk does not contain points of $\text{int}(K_c^+)$. Let us consider the annulus \mathcal{A} defined as the complement in W of the quasi-disk. Since we know that W is the disjoint union

$$(W \cap \text{int}(K_c^+)) \amalg (\mathcal{W}_c^s(W)) \amalg (W \cap U_c^+),$$

the open annulus \mathcal{A} is the disjoint union of two open sets $\mathcal{A} \cap \text{int}(K_c^+)$ and $\mathcal{A} \cap U_c^+$. By connectedness, \mathcal{A} can only consist of points in $\text{int}(K_c^+)$ alone, or U_c^+ alone. Since very far from W in Δ we have only points in U_c^+ we deduce at once that \mathcal{A} contains no point in $\text{int}(K_c^+)$, which concludes the proof.

In summary we showed that $\Delta \cap K_c^+$ is exactly a quasi-disk, for c small enough, and this concludes the proof of theorem 7.3. \square

Remark 6. With minor changes in the proof above, one could also show that for $|c|$ small enough, the intersection $\mathcal{K}_c \cap \mathcal{P}$ is a quasi-disk, where \mathcal{P} is the parabola $\{(z^2, z); z \in \mathbb{C}\}$.

Generalization.

All results can be extended easily to the sets associated of the dynamics of $h(x, y) = (x^a y^b + c, x)$ where a, b are positive real numbers.

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