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ON SETS OF INTEGERS WITHOUT ARITHMETIC PROGRESSIONS

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On sets of integers without arithmetic progressions

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Abstract

In this note we consider a sequence (b_n) of positive integers without arithmetic progressions of a given lenght $k \geq 3$. We investigate the conjecture that states that in this situation $\sum \frac{1}{b_n} < \infty$. We show that this statment is equivalent to some seemingly stronger results. We also consider the problem of partitioning the set $[1..n]$ in two sets, one of them of size s, seeking the minimization of the number of arithmetic progressions of lenght 3 contained in it, and we proove a min-max result for this problem.

1 Introduction

Questions concerning sets of integers free of arithmetic progressions of a given length have been subject of a considerable amount of research since Van der Waerden famous result appeared in 1928 (see [W]). He showed that if the set of integers is splited in a finite number of pieces, then at least one of them should have arithmetic progressions of arbitrary length.

A natural question arose, and it was: Let S_n^k be the cardinality of the largest set of integers free of k-arithmetic progressions contained in [1..n]. How fast does S_n^k grow?

In [ET], Erdös and Turan proposed two nice conjectures, namely, that $\lim_{k\to\infty} \frac{S_n^k}{n} = 0$ and that $\forall k \in N$ there are positive reals ε_k and C_k such that $S_n^k < C_k n^{1-\varepsilon_k}$. The second conjecture was proven false by Salem and Spencer in [SS] who showed that $S_n^3 > n^{1-\frac{c}{\log\log n}}$, where $c > 0$ is a constant. Subsequently, Behrend improved this bound and showed that

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$S_n^3 > n^{1-\frac{c}{\sqrt{\log n}}}$ (see [B]). But the first conjecture was positively answered in 1975 by Szmerédi in [S], using purely combinatorial methods. Shortly after that, Furstenberg achieved that same result by means of ergodic theory.

Following this, Erdös conjectured (see [E]) that $\lim\limits_{n\to\infty} \dfrac{S_n^k}{n(\log n)^\beta} = 0$ for every $k \in N$ and every $\beta \in \mathrm{R}^+$. He also proposed that if (b_n) is a growing sequence of positive integers, no k of those are in progression, then $\sum\limits_{n\in N} \frac{1}{b_n} < \infty$. Both these statements remain unproved. It's worth noticing that a positive answer to any of this conjectures would imply that there are arbitrarily large arithmetic progressions among the prime numbers, a very old problem of number theory. In the same article, Erdös proposes some *other* problems related with this, in particular, if is it true that exists, for every > 0 and $k \in \mathrm{N}^*$, a natural n_0 such that, if (b_n) is a sequence of integers without arithmetic progressions of length k and $n_0 \leq b_1 < b_2 < \cdots$, then $\sum\limits_{n=1}^{\infty} \frac{1}{b_n} \leq \varepsilon$? It's clear that this implies the first conjecture, and we prove that they are equivalent. In fact we show that the first conjecture is equivalent to four other seemingly stronger statements.

In Section 2 of this work we prove the equivalences and in Section 3 we show a min-max principle for S_n^3.

2 Some equivalences

We will denote $\mathcal{F}_k$ the class of subsets of N (finite or not) that do not have k elements in arithmetic progressions, and we will represent by $[a..b]$ the set $\{n \in \mathrm{Z}: a \leq n \leq b\}$.

Remember that, for positive integers, n and k, $k \geq 3$, we will denote by $S_n^k = \max \#B$ such that $B \subset [1..n]$ and $B \in \mathcal{F}_k$, and let a_n^k be the minimum of $\{t \in \mathrm{N}: \exists B \subset [1..t]$, with $\#B = n$ and $B \in \mathcal{F}_k\}$. We note that $S_{a_n^k}^k = n$.

From this point until the end of this section we will fix a integer $k \geq 3$.

In this section we will show that the following statement

(i) *If (b_i) is a sequence in $\mathcal{F}_k$ then* $\sum\limits_{i=1}^{\infty} \frac{1}{b_i} < +\infty$

is equivalent to some other seemingly stronger propositions, namely, (i) is equivalent to

(ii) *There is a $L = L(k)$ such that, for all $(b_i) \in \mathcal{F}_k$,* $\sum\limits_{i=1}^{\infty} \frac{1}{b_i} < L$.

(iii) *The series* $\sum_{i=1}^{\infty} \frac{1}{a_i^{\frac{1}{k}}}$ *is convergent.*

(iv) If $\varepsilon > 0$ then there is a $n_0 = n_0(k;\varepsilon) \in \mathbb{N}^*$ such that if $(b_i) \in \mathcal{F}_k$ with $n_0 \leq b_1 < b_2 < \ldots$ then $\sum_{i=1}^{\infty} \frac{1}{b_i} < \varepsilon$.

(v) *There is a sequence* (c_k) *in* $\mathcal{F}_k$ *such that* $\sum_{i=1}^{\infty} \frac{1}{b_i} \leq \sum_{i=1}^{\infty} \frac{1}{c_i}$, *for all* $(b_i) \in \mathcal{F}_k$.

It's obvious that each one of these assertions implies (i), but the reciprocals are not evident. In [E], for example, Erdös states (iv) as a *"...related problem (with (i)) which seems interesting"*.

Lemma 1 *If* i *and* j *are positive integers and* $i < j$ *then* $\frac{S_j^k}{j} < 2\frac{S_i^k}{i}$

Proof: It's clear that if $A \in \mathcal{F}_k$ and ℓ is an integer then $A + \ell$ is in $\mathcal{F}_k$. So, if A is a subset of $[a..a+n]$ and is free of arithmetic progressions of length k then $\#A \leq S_n^k$.

Since $i < j$ there are integers q and r, with $q \geq 1$ and $0 \leq r \leq i-1$ such that $j = qi + r$.

Then $S_j^k \leq qS_i^k + r < (q+1)S_i^k$ and this shows that

$$S_l^k < \frac{j}{qi}(q+1)S_i^k < 2\frac{j}{i}S_i^k. \qquad \blacksquare$$

Now we will construct a special sequence (η_i) in $\mathcal{F}_k$.

First of all, consider for $n \in \mathbb{N}^*$ a $K_n \subset J_n = [2(3)^n..3^{n+1}[$ of $S_{3^n}^k$ elements, such that $K_n \in \mathcal{F}_k$ (this is possible, since $\#J_n = 3^n$).

Now take $K = \bigcup_{n=1}^{\infty} K_n$ and ordene this set obtaining by this way the sequence (η_i).

Lemma 2 *The sequence* (η_i) *is in* $\mathcal{F}_k$.

Proof: It's obvious that $K_n \in \mathcal{F}_k$.

Now, suppose, by contradiction, that $K \notin \mathcal{F}_k$ and take $a_1 < a_2 < \ldots < a_k$ in K such that (a_n), $1 \leq n \leq k$ is an arithmetic progression.

Then $a_1 \in K_{n_1}$ for some n_1 and there is a smaller j, $2 \leq j \leq n$ such that $a_j \notin K_{n_1}$. If $j \leq 3$ it's clear that $a_2 - a_1 < a_{j-1} - a_j$ and we have a contradiction.

Then we must have $a_2 \in K_{n_2}$, with $n_2 > n_1$. from this it's clear that $3^{n_2} < a_2 - a_1 < 2(3^{n_2})$.

So, if $a_3 \in K_{n_2}$, it's clear that $a_2 - a_1 > a_3 - a_2$ and we have a contradiction.

This implies that $a_3 \in K_{n_3}$ and $n_3 > n_2$.

Therefore $3^{n_3} < a_2 - a_1 < 2(3^{n_3})$. From this and the previous inequality for $a_2 - a_1$ we see that $a_3 - a_2 > a_2 - a_1$ and this contradicts the hypothesis that (a_n) is an arithmetic progression. ■

Theorem 1 ((i)⇒(ii)) *If (i) stands then there is a real $L = L(k)$ such that for all $(b_i) \in \mathcal{F}_k$ we have $\sum_{i=1}^{\infty} \frac{1}{b_i} < L$.*

Proof: From (i) and lemma 2, $\sum_{i=1}^{\infty} \frac{1}{\eta_i} = M < \infty$.

Let $(b_i) \in \mathcal{F}_k$ and note that $\#\{b_i : i \in \mathbb{N}^*\} \cap [3^j..3^{j+1}[\leq S^k_{2(3^j)} \leq 2S^k_{3^j}$. Then,

$$\sum_{i=1}^{\infty} \frac{1}{b_i} = \sum_{j=0}^{\infty} \left(\sum_{b_i \in [3^j..3^{j+1}[} \frac{1}{b_i} \right) \leq \sum_{j=0}^{\infty} \frac{2S^k_{3^j}}{3^j}. \tag{1}$$

In the other hand, since $\#\{\eta_i : i \in \mathbb{N}^*\} \cap [3^j..3^{j+1}[= S^k_{3^j}$, we have

$$M = \sum_{i=1}^{\infty} \frac{1}{\eta_i} = \sum_{j=0}^{\infty} \left(\sum_{\eta_i \in [3^j..3^{j+1}[} \frac{1}{\eta_i} \right) > \sum_{j=0}^{\infty} \frac{S^k_{3^j}}{3^{j+1}}. \tag{2}$$

From (1) and (2) it follows that

$$\sum_{i=1}^{\infty} \frac{1}{b_i} \leq \sum_{j=0}^{\infty} \frac{2S^k_{3^j}}{3^j} = 6 \sum_{j=0}^{\infty} \frac{S^k_{3^j}}{3^{j+1}} < 6M.$$ ■

Theorem 2 ((i)⇒ (iii)) *If (i) stands then $\sum_{j=1}^{\infty} \frac{1}{a^k_j}$ converges.*

Proof: By the same arguments as in the previous demonstration, since $S^k_{a^k_j} = j$ and $a^k_j > j$, we have from lemma 1 $\frac{j}{a^k_j} < 2\frac{S^k_j}{j}$ and

$$\sum_{j=1}^{\infty} \frac{1}{a^k_j} = \sum_{i=0}^{\infty} \sum_{\ell=0}^{2(3^i)-1} \frac{1}{a^k_{3^i+\ell}} < \sum_{i=0}^{\infty} \frac{2(3^i)}{a^k_{3^i}}. \tag{3}$$

Then, by (2) and (3) we obtain $\sum_{j=1}^{\infty} \frac{1}{a_j^k} < 12M$. ■

It follows from theorem 1 that if (i) holds the set

$$\left\{ \sigma \in \mathbf{R} : \sigma = \sum_{j=1}^{\infty} \frac{1}{b_j}, \text{ for some sequence } (b_j) \in \mathcal{F}_k \right\}.$$

is bounded, then it has a supremum, that we will denote L_k.

Theorem 3 ((i)⇒ (iv)) *Suppose that (i) holds and let $\varepsilon > 0$, then there is a $n_0 \in \mathbf{N}^*$ such that if (b_i) is a strictly crescent sequence in $\mathcal{F}_k$ and $b_1 \geq n_0$ then $\sum_{i=1}^{\infty} \frac{1}{b_i} < \varepsilon$.*

Proof: Let $\varepsilon > 0$ and a sequence $(d_n) \in \mathcal{F}_k$ such that $L_k - \sum_{j=1}^{\infty} \frac{1}{d_j} < \frac{\varepsilon}{24}$.

Then, choose n_0 such that $L_k - \sum_{j=1}^{n_0} \frac{1}{d_j} \leq \frac{\varepsilon}{12}$.

Now, for $m = 0, 1, \ldots$, consider the interval $I_m = [2(3^m)d_{n_0}..3^{m+1}d_{n_0}[$ and $K_m \subset I_m$, with $\#K_m = S^k_{3^m d_{n_0}}$ such that $K_m \in \mathcal{F}_k$.

Take $K = \bigcup_{m=0}^{\infty} K_m$ and proceeding as in the inequalities 1 and 2 of theorem 1 in order to show that, for each $(b_j) \in \mathcal{F}_k$, strictly crescent, with $b_1 \geq 2d_{n_0}$, we have

$$\sum_{j=1}^{\infty} \frac{1}{b_j} \leq 6 \sum_{i \in K} \frac{1}{i}. \tag{4}$$

Consider the sequence (g_n) obtained by ordering the set $\{d_1, \ldots, d_{n_0}\} \cup K$. It is easy to see that $(g_n) \in \mathcal{F}_k$ and so $\sum_{j=1}^{\infty} \frac{1}{g_j} \leq L_k$.

Since $L_k - \sum_{j=1}^{n_0} \frac{1}{d_j} < \frac{\varepsilon}{12}$, it follows from (4) that $\sum_{j=1}^{\infty} \frac{1}{b_j} \leq \frac{\varepsilon}{2}$. ■

In order to proof that (i) ⇒ (v) we will show a small lemma.

Lemma 3 *Suppose that (i) stands and let $x_n \in \mathcal{F}_k$, $x_n = (x_n)_j$, such that $\sigma_n = \sum_{j=1}^{\infty} \frac{1}{(x_n)_j}$ converges to L_k.*

Then there is a sequence (ℓ_j) such that $(x_n)_j \leq \ell_j$, for all j and n.

Proof: It is enough to show that, for $i \in \mathbf{N}$ fixed, the sequence $((x_n)_i : n \geq 1)$ is bounded.

Since $\sigma_n \to L_k$ it follows directly from theorem 3 that $((b_n)_1)$ is bounded.

Consider $m \in \mathbb{N}$, $m \geq 2$. Obviously, $\left\{\sigma = \sum_{i=1}^{m-1} \frac{1}{d_i} : \{d_1, \ldots, d_{m-1}\} \in \mathcal{F}_k\right\}$ has a maximum and let M_m be this maximum.

Of course, $M < L_k$ and since $\sigma_n \uparrow M$ there is a n_0 such that $L_k - \sigma_n < \frac{L_k - M}{2}$, for all $n \geq n_0$.

From theorem 3 we have that there is a $C > 0$ such that, if $(x_n)_m > C$ then $\sum_{i=m}^{\infty} \frac{1}{(x_n)_i} < \frac{L_k - M}{4}$.

This shows that, for $n \geq n_0$ we have $(x_n)_m \leq C$. ∎

Theorem 4 ((i)⇒(v)) *If (i) stands then there is a sequence $(\bar{b}_i) \in \mathcal{F}_k$ such that $\sum_{i=1}^{\infty} \frac{1}{c_i} \leq \sum_{i=1}^{\infty} \frac{1}{\bar{b}_i}$, for all $(c_i) \in \mathcal{F}_k$.*

Proof: From theorem 1 it follows that (ii) holds.

As in the demonstration of lemma 3, we consider M the supremum of the set given in and a sequence (x_n) of sequences in $\mathcal{F}_k$, $x_n = (x_n)_i : i \in \mathbb{N})$, such that $\sigma_n = \sum_{i=1}^{\infty} \frac{1}{(x_n)_i}$ converges to L_k.

It follows directly from lemma 3 that there is a sequence (c_i) such that each prefix $(c_1, c_2, \ldots, c_m)$ is the prefix of infinitely many (x_{n_j}) (i.e. there is a subsequence (x_{n_j}) of (x_n) such that $(x_{n_j})_s = c_s$, for $1 \leq s \leq m$).

It is clear that $(c_n) \in \mathcal{F}_k$ and $\sum_{j=1}^{\infty} \frac{1}{c_j} \leq L_k$.

Now, since $\sigma_n \to L_k$, if $\varepsilon > 0$, it follows from theorem 3 that there are constants n_0 and C such that, for $n \geq n_0$

$$L_k - \sum_{i=1}^{C} \frac{1}{(x_n)_i} \leq \varepsilon.$$

Since we have shown that $(c_1, \ldots, c_C)$ is a prefix of all elements of a subsequence (x_{n_j}) of (x_n)we see that $L_k - \sum_{i=1}^{C} \frac{1}{c_i} < \varepsilon$. ∎

We will finish this section providing a way *to construct* a sequence (x_n) as in the previous result, *if (i) holds.*

Let $n \in \mathbb{N}^*$ and consider $\mathcal{F}_k^n$ the finite sequences of n elements in $\mathcal{F}_k$.

Of course, the set

$$\left\{\sigma \in \mathbf{R} : \sigma = \sum_{j=1}^{n} \frac{1}{b_j},\ (b_j) \in \mathcal{F}_k^n\right\}$$

is bounded, and moreover it has a maximum element.

Then we choose $x_n = ((x_n)_j : 1 \le j \le n) \in \mathcal{F}_k^n$ such that the sum of its reciprocals is maximal.

Clearly, the sequence (x_n) is in $\mathcal{F}_k$ for all n and, *if (i) holds*, (x_n) has the same proprieties as in the previous theorem.

3 A Min-Max Principle

In this section we deal with the problem of finding "large" sets free of arithmetic progressions of length 3. In order to do so, we start by posing a slightly different problem, that is, we seek sets free of arithmetic progressions of length 3 modulus n. A triplet $(b_i; b_j; b_k) \in [1..n]^3$ is an arithmetic progressions of length 3 modulus n if $b_i \neq b_j$ and $(b_j - b_i = b_k - b_j)_{mod n}$. A set B is free of arithmetic progressions of length 3 modulus n if it contains none. We define $\overline{S}_n^3$ as the maximum of the cardinality of these sets.

From now on n will not be divisible by 2 or 3. Since all "common" a. p. are also a. p. modulus n, it's clear that $\overline{S}_n^3 \le S_n^3$. Also, if a set $B \in [1..\frac{n-1}{2}]$ is free of regular a. p. , then it is also free of a. p. modulus n, therefore $S_{\frac{n-1}{2}}^3 \le \overline{S}_n^3$. This fact shows that our new problem is a good approximation for the previous one.

If B is a subset of $[1..n]$ we will denote $P_n^3(B)$ the number of arithmetic progressions of length 3 contained in B.

Our min-max theorem is

Theorem 5 *Let $B \subset [1..n]$ with s elements, $1 \le s \le n-1$, and consider $B^c = [1..n] \setminus B$. Then $P_n^3(B) + P_n^3(B^c) = n(n-1) - 3s(n-s)$.*

Proof: It's a simple counting result. There are exactly $n(n-1)$ arithmetic progressions modulus n, part of them is accounted for in $P_n^3(B)$.

Now, every $j \in [1..n]$ takes part in $3(n-1)$ distinct arithmetic progressions, and every pair $(j,k) \in [1..n] \times [1..n]$ participates together in 6 different progressions.

Now we begin to count the arithmetic progressions that have at least on element in B^c.

There are $n-s$ elements in B^c so we begin with $3(n-s)(n-1)$ arithmetic progressions, but we double-counted those progressions which contains a pair of B^c elements. Since there are $\frac{(n-s)(n-s-1)}{2}$ such pairs, we need to subtract $3(n-s)(n-s-1)$ from our previous total.

Finally, we note that we discounted all the arithmetic progressions completely contained in B^c.

All of this adds up to

$$n(n-1) - P_n^3(B) = 3(n-s)(n-1) - 3(n-s)(n-s-1) + P_n^3(B^c),$$

which gives us our desired result. ■

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