Torsion units in integral group rings

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Introduction

Let G be a group and let $V\mathbb{Z}G$ be the group of units of augmentation one of the integral group ring $\mathbb{Z}G$. Given an element $x = \sum x(g)g \in \mathbb{Z}G$ we set $T^{(k)}(x) = \sum_{g \in G(k)} x(g)$, called the *k-generalized* trace of x. Here $G(k) = \{g \in G : o(g) = k\}$. We also set $\tilde{x}(g) = \sum_{h \sim g} x(h)$, where $\sim$ denotes conjugacy.

In 1987 A. Bovdi made the follwing conjecture which we shall call BC1 ([1]).

$$T^{(p^n)}(x) = 1 \quad \text{and} \quad T^{(p^j)}(x) = 0 \quad \text{for} \quad j < n \; .$$

In 1987 M. Dockuchaev proved this conjecture for metabelian nilpotent groups [3]. In 1993 A. Bovdi, Z. Marcianik and S. K. Sehgal confirmed this conjecture for nilpotent groups [2]. In the same year M. Dockuchaev and S. K. Sehgal proved this conjecture for metabelian groups [4]. We prove the conjecture for other classes of groups. Proofs will appear elswhere.

I - The Results

Theorem 1.1: ***Bovdi's conjecture holds for any finite solvable group G such that every Sylow subgroup of G is abelian.***

Theorem 1.2: *Let G be a finite soluble group such that, for every prime p, if $p \mid |G|$ then $p^4 \not| \; |G|$. Then, Bovdi's Conjecture holds for G.*

In particular BC1 holds for any group whose order is not divisable by the fourth power of any prime.

We introduce a definition that will be needed in the sequel. Let G be a group and m, n positive integers. We shall say that G is (m, n)-*absorvent* if the subgroup $H = \langle g \in G : o(g) | m^n \rangle$ has exponent less than or equal to m^n. If G is (m, n)-absorvent for all pairs (m, n) then G is called *absorvent*. Note that abelian and regular p-groups are absorvent.

Theorem 1.3: *BC1 holds for absorvent groups.*

Theorem 1.4: *BC1 holds for supersolvable groups.*

Theorem 1.5: *BC1 holds for finite Frobenius groups.*

Theorem 1.6: *Let G be a finite group whose commutator subgroup is nilpotent. Then BC1 holds for G.*

As a consequence we have the following result

Theorem 1.6: *Let G be a polyciclic group whose commutator subgroup is nilpotent. Then BC1 holds for G.*

Note that Theorem 1.4 is a consequence of the last Theorem.

II - Main Ingredients

We shall now quote the main ingredient used to prove the results mencioned in the previous section.

The following results can be found in [5].

Lemma 2.1: *Let G a finite group and $\alpha \in V\mathbb{Z}G$ a unit of finite order. Then $\beta^{-1}\alpha\beta \in G$ for some $\beta \in U(QG)$ if and only if for every element γ in the subgroup generated by α there is an element $g_0 \in G$, unique up to* *conjugacy, such that $\tilde{\gamma}(g_0) \neq 0$.*

Lemma 2.2: *Let $G = P \rtimes X$ where P is a Sylow p-subgroup of G. Let $H \subseteq U(1+\Delta(G,P))$ be finite. Then there exists $\alpha \in QG$ such that $H^\alpha \subseteq G$.*

Lemma 2.3: *Let G be a noetherian group and $u \in V\mathbb{Z}G$ a torsion element. Let $x \in G$ be of infinite order. Then $\tilde{u}(x) = 0$.*

The next Lemmas give us an induction argument.

Lemma 2.4: *Let G be a finite group and $H \lhd G$ a normal subgroup of G. Let $\psi : \mathbb{Z}G \to \mathbb{Z}(G/H)$ be the natural projection and $\alpha \in V\mathbb{Z}G$ such that $(o(\alpha), |H|) = 1$. If $\beta = \psi(\alpha)$ then $T^{(k)}(\alpha) = T^{(k)}(\beta)$ for every k such that $(k, |H|) = 1$ and $T^{(k)}(\alpha) = 0$ if $(k, |H|) \neq 1$.*

Lemma 2.5: *Let p be a prime and G a finite group. Suppose that G has unique subgroup H of order p. Let $\alpha \in V\mathbb{Z}G$ be such that $o(\alpha) = p^n$. Then, with the notation of Lemma I.4, we have that $T^{(p^j+1)}(\alpha) = T^{(p^j)}(\beta)$.*

Lemma 2.6: *Let G be a noetherian group containing $H \lhd G$ with H torsion free. If $\alpha \in V\mathbb{Z}G$ is a torsion element then, with the notation of Lemma I.4, we have that $T^{(k)}(\alpha) = T^{(k)}(\beta)$.*

The next Lemma deals with the absorvent case

Lemma 2.7: *Let G be group and $\alpha \in V\mathbb{Z}G$ an element such that $o(\alpha) = p^n$, p a prime. If G is (p,k)-absorvent for all $k \leq n$ then $T^{(p^j)}(\alpha) = \delta_{nj}$.*

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