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Vectorspace Categories Immersed in Directed Components

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VECTORSPACE CATEGORIES IMMERSED IN DIRECTED COMPONENTS

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Abstract

We know, after [P1], that, given a tame algebra Λ, the Tits form q_Λ is weakly non negative.

Moreover, the converse has been shown for some families of algebras, but it is not true in general. The purpose of this work is to show that for certain wild vectorspace categories $\mathbb{K} = Hom(M, B-mod)$, where B is tame and M is an indecomposable B-module, we have $q_{B[M]}$ strongly indefinite. This gives partial converses of the above theorem.

1 Preliminaries

Throughout this paper, k denotes an algebraically closed field. By an algebra Λ we mean a finite-dimensional, basic and connected k-algebra of the form $\Lambda \cong kQ/I$ where Q is a finite quiver and I an admissible ideal. We assume that Q has no oriented cycles. Let Λ-mod denote the category of finite-dimensional left Λ-modules, and Λ-ind a full subcategory of Λ-mod consisting of a complete set of non-isomorphic indecomposable objects of Λ-mod.

For each $i \in Q_\Lambda$ we denote by S_i (resp. P_i and I_i) the corresponding simple Λ-module (resp. the indecomposable projective, injective). The dimension-vector of a Λ-module X is the vector $\underline{dim}X = (dim_k Hom_\Lambda(P_i, X))_{i \in Q_0}$ in the Grothendieck group $\mathbb{K}_0(\Lambda)$. The support of a Λ-module X, supp(X) is the full subcategory of Λ defined by the set $\{i \in Q_0 | X(i) \neq 0\}$.

We shall use freely the known properties of the Auslander-Reiten translations, τ and τ^{-1}, and the Auslander-Reiten quiver of Λ-mod, Γ_Λ . Moreover, we shall identify the points of Γ_Λ with the corresponding indecomposable Λ-modules. A component Γ of Γ_Λ is called *standard* if Γ is equivalent to its mesh-category $k(\Gamma)$. A path $M_0 \to M_1 \to ... \to M_t$ is called *sectional* if $M_{i-1} \neq \tau(M_{i+1})$, for all i, $0 < i < t$. For basic notions we refer to [R2] and [ARS].

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The purpose of this work is to study wild finite vectorspace categories imersed in directed components. We can show that some of those categories contain full subcategories from those listed by N. Marmaridis in [M1]. Besides, as the category number VII of [M1] can be assumed not to be path incomparable, we obtain the following result:
Let B be a tame algebra with $gldimB \leq 2$ and M an indecomposable module in $\mathcal{C}$, where $\mathcal{C}$ is a directed, standard, tree type component of the Auslander-Reiten quiver of B such that the vectorspace category $\mathbb{K} = Hom(M, B-mod) = Hom(M, \mathcal{C})$ is finite. If $\mathbb{K}$ contains, as a full subcategory, one of the categories of the list below then $q_{B[M]}$ is strongly indefinite.

List

1)(1,1,1,1,1), (1,1,1,2), (2,2,3), (1,3,4), (1,2,6) (N,5) path incomparable

2) () , (), () or ()

3){$|X|, |Y|$}; where

$$\| X \| = \| Y \| = dim_k Hom(|X|, |Y|) = 2$$

with $Hom(|Y|, |X|) = 0$ and such that $Hom(|X|, |Y|)$ is given by one of the following bases:

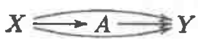
$$B_1 \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\} \quad \text{(category B)}$$

$$B_2 \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\} \quad \text{(category D)}$$

$$B_3 \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\} \quad \text{(category D}^{OP}\text{)}$$

with respect to some chosen bases of $|X|$ and $|Y|$.

4)

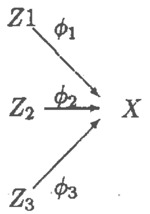


$\| A \| = 1$, $\| X \| = \| Y \| = 2$ with the morphisms given by:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

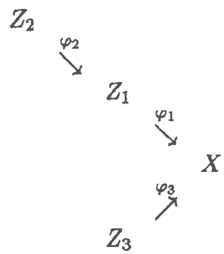
$X \to A(1,0)$ e $A \to Y \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, that is, there exist $U \in P^1(X)$, V in $P^1(Y)$ such that $|\varphi|(U) \subset V$ $\forall |\varphi| : |X| \to |Y|$ and $ker(X \to A) = U$, $Im(A \to Y) = V$.

5)



with $\| X \| = 2$, $\| Z_i \| = 1$, $Im|\varphi_i| = U \in P^1(X)$, $Hom(|Z_i|, |Z_j|) = 0$, $\forall i, j$.

Or



with $\| X \| = 2$, $\| Z_i \| = 1$, $\forall i$.
$Im\ |\varphi_1| = Im\ |\varphi_2| = U \in P^1(X)$
$Hom(|Z_2|,\ |Z_3|) = Hom(|Z_1|,\ |Z_3|) = Hom(|Z_3|,\ |Z_1|) =$
$Hom(|Z_3|,\ |Z_2|) = 0$ and $dim\ Hom(|Z_i|,\ |X|) = 1$.

We begin now to recall the concepts and results that are the background of this work.

Theorem 1.1 (Drozd)[CB] If Λ is a finite dimensional k-algebra then Λ is tame or wild, but not both.

See [CB] for the definitions.

Definition 1.2 [R1] A vectorspace category $(\mathbb{K}, |\ |)$ is given by a Krull-Schmidt k-category $\mathbb{K}$ and a faithful functor $|\ | \ : \mathbb{K} \to \text{mod } k$.

Given a vectorspace category $(\mathbb{K}, |\ |)$, its objects (resp. the morphisms) are usually considered to be the objects (resp. the morphisms) of the image of $|\ |$, and the subspace category $\mathcal{U}(\mathbb{K})$, is defined as follows: the objects are triples (X, U, φ) with $X \in \text{Obj}\mathbb{K}$, U a k-vector space and φ: $U \to |X|$, k-linear. The morphisms $(X, U, \varphi) \to (X', U', \varphi')$ are the pairs (α, β) with β: $X \to X'$ in $\mathbb{K}$, $\alpha : U \to U'$ k-linear and such that $|\beta|\varphi = \varphi'\alpha$. Clearly, any object of $\mathcal{U}(\mathbb{K})$ is isomorphic to a direct sum of a triple (X, U, φ) with $\varphi : U \to |X|$ injective and copies of $(0, k, 0)$.

Definition 1.3 [R1] A k-category $\mathbb{K}$ is *schurian* if $\text{End}_{\mathbb{K}}(X) \cong k$ for any $X \in \text{Obj}\mathbb{K}$, X indecomposable.

Lemma 1.4 [R1] Let $\mathbb{K}$ be a schurian vectorspace category. If $\mathbb{K}$ is of finite representation type then every indecomposable object has dimension 1. If $\mathbb{K}$ is not wild then every indecomposable object has dimension at most 2, and moreover, if X, Y are indecomposables with $dim|X| = 2$ then $Hom(X,Y) \neq 0$ or $Hom(Y,X) \neq 0$.

A schurian vectorspace category whose indecomposable objects have dimension one, corresponds to the additive category of a poset *add kS*. In this case, we have:

Theorem 1.5 [R2] (Nazarova)The poset S is of wild representation type if and only if S contains a full subset of the form: (1,1,1,1,1), (1,1,1,2), (2,2,3), (1,3,4), (1,2,6) or (N,5). Moreover: if S is not wild then S is tame.

Definition 1.6 [R1]

The one-point extension of the algebra B by the module M is the algebra

$$A = B[M] \cong \begin{bmatrix} B & M \\ 0 & k \end{bmatrix}$$

with the usual operations of matrices.

The $B[M]$-modules can be identified with triples (X, U, φ) where $X \in B$-mod, U is a k vectorspace and $\varphi : U \to Hom(M, X)$ is k linear.

For more details we refer to [R1].

If B has finite global dimension then $gldim\ B[M] = max\{gldim\ B,\ pd_B M\ + 1\}$. Moreover, B-mod is a full, extension closed subcategory of $B[M]$-mod.

It is known ([R1]) that the representation type of $B[M]$ depends on the representation type of B and of $\mathcal{U}(\text{Hom}(M, B - \text{mod})$.

2 Comparing cuadratic forms

In this section, we assume that B is such that $gldimB \leq 2$, then for any B-module M we have $gldimB[M] \leq 3$. Hence we would be able to relate the Euler and the Tits form for $A = B[M]$

Definition 2.1 [R2]

Let C_B be the Cartan matrix of B and let x and y vectors in $\mathbb{K}_0(B)$. Then we have a bilinear form $<,>= xC_B^{-T}y^T$, where the corresponding quadratic form $\chi_B(x) =< x, x >$ is called the Euler form of B.

This bilinear form $<,>$ has the following homological interpretation:

Lemma 2.2 [R2] Let X and Y be B-modules then
$< \underline{dim}X, \underline{dim}Y >= \sum_{t\geq 0}(-1)^t dim_k Ext^t_B(X,Y)$.

Definition 2.3 [Bo] The Tits quadratic form is given by:
$q_B(x_1, x_2, ..., x_l) = \sum_{i\in Q_0} x_i^2 - \sum_{i,j\in Q_0} x_i.x_j.dim_k Ext^1_B(S_i, S_j)$
$+ \sum_{i,j\in Q_0} x_i.x_j.dim_k Ext^2_B(S_i, S_j)$

By [R2] the Euler form of $A = B[M]$ can be calculated in terms of χ_B: let X be a A-module and consider:

$\underline{dim}_A(X) = \underline{dim}_B(Y) + n.\underline{dim}_A(S_e)$, where e is the new vertex, then
$\chi_A(\underline{dim}X) = (\underline{dim}X)C_A^{-T}(\underline{dim}X)^{-T} =$
$\chi_B(\underline{dim}Y) + n^2 - n < \underline{dim}M, \underline{dim}Y >_B=$
$\chi_B(\underline{dim}Y) + n^2 - n(dim_k Hom_B(M,Y) - dim_k Ext^1_B(M,Y) +$
$dim_k Ext^2_B(M,Y))$

On the other hand, using Bongartz result (see [Bo]) that if $gldimB \leq 2$ then $\chi_B = q_B$, its Tits form is computed in following:

$q_A(x_1, x_2, ..., x_l, n) =$
$q_B(x_1, x_2, ..., x_l) + n^2 - \sum_{j\in Q_0} n.x_j(dim_k Ext^1_A(S_e, S_j) +$
$dim_k Ext^1_A(S_j, S_e)) + \sum_{j\in Q_0} n.x_j(dim_k Ext^2_A(S_e, S_j) +$
$dim_k Ext^2_A(S_j, S_e))$

Comparing, we have:

Proposition 2.4 With the above notation:
$\chi_A(\underline{dim}X) = q_A(\underline{dim}X) - n.dim_k Ext^2_B(M,Y)$

The main motivation for our work is the following theorem of De la Peña

Theorem 2.5 (De la Peña)[P1]
If B is a tame algebra, then q_B is weakly non negative.

3 Directed Modules

In this section, we assume that $gldimB \leq 2$ and we suppose that $\mathcal{C}$ is a directed, standard, connected component of tree type (see below) of the Auslander-Reiten quiver of B and M is an indecomposable B-module in $\mathcal{C}$, such that $Hom(M, I) = 0$, for all indecomposable injectives I that are not in $\mathcal{C}$. In this case, the vectorspace category $\mathbb{K} = Hom(M, B - mod) = Hom(M, \mathcal{C})$ is finite and schurian.

We will study cases in which $\mathbb{K}$ is wild. Our results extend, in some sense, the list given in [M1]. For M a *good* decomposable B-module, the result follows from [MP].

Definition 3.1 [MP] Let $\mathcal{C}$ be a preinjective component of Γ_A. The orbit quiver $O_\tau(\mathcal{C})$ of $\mathcal{C}$ is defined as follows: The vertices of $O_\tau(\mathcal{C})$ are the τ-orbits I_x^τ of the indecomposable injectives of $\mathcal{C}$. The arrows are determined as follows: consider $I_x \in \mathcal{C}$, with $I_x/S_x \cong \oplus Y_i$. If Y_i belongs to the τ-orbit of the injective I_i, we put an arrow from I_x^τ to I_i^τ.

We say that $\mathcal{C}$ is of tree type if the graph of the orbit quiver is a tree.

Definition 3.2 [M1] Two subsets L and L' of $ind\mathbb{K}$ are said to be path-incomparable if for every $|X| \in L$ and $|Y| \in L'$, there is no path in $\mathcal{C}$ from X to Y or from Y to X.

Initially, let us write down some preliminaries results.

As in [M1] we denote by $\square$ the objects of $\mathbb{K}$ that have dimension two and by $\blacksquare$ the objects of dimension greater than or equal to three. Let us recall the list A of vectorspace categories given in [M1]

A:(1,1,1,1,1), (1,1,1,2), (2,2,3), (1,3,4), (1,2,6) (N,5) or ($\square$ $\bullet$) , ($\blacksquare$)

Finally, let us denote by $\| X \|$ the dimension of the object $|X| \in \mathbb{K}$, i.e $\| X \| = dim_k Hom_B(M, X)$.

Theorem 3.3 [M1] If the preinjective component $\mathcal{C}$ of the Auslander-Reiten quiver of B is of tree type and if the category ind $\mathbb{K}$ contains, as a full subcategory, one of the categories L of the list A with its connected components of L being path-incomparable, then the Euler form χ_A is strongly indefinite.

Proposition 3.4 [MP] Let $\mathcal{C}$ be a preinjective component of Γ_B of tree type. Let $\gamma : X_0 \to \cdots \to X_s$ be a sectional path in $\mathcal{C}$. Let $\delta : \tau^{-n} X_0 \to \tau^m X_s$ be a path with $n, m \geq 0$. Then $n = 0 = m$ and $\delta = \gamma$.

Lemma 3.5 [MP] Let $\mathcal{C}$ be a preinjective component of Γ_B of tree type. Let $\gamma : X_0 \to \cdots X_{s-1} \to X_s$ be a sectional path in $\mathcal{C}$ with $s \geq 1$.

Assume that there exist a module $M \in \mathcal{C}$ and paths:

$M \overset{\delta}{\rightsquigarrow} X_0$ and $M \overset{\gamma}{\rightsquigarrow} Y \longrightarrow X_s$ with $Y \neq X_{s-1}$. Then X_i is not projective, for $0 < i \leq s$.

The above results, that were stated for preinjective components, remain true in this context, that is, for $\mathcal{C}$ a directed, standard, connected component of tree type.

Theorem 3.6 [MP] Let B a tame algebra and M a B-module. Suppose that M is a *good relative preinjective of tree type* and that $\dim_k \operatorname{Hom}(M, N) \leq 1$ $\forall N \in \Gamma_B$. Then $A = B[M]$ is tame if and only if q_A is weakly non negative.

We begin to prove:

Theorem 3.7 Let B be a tame algebra with $gldimB \leq 2$, M a indecomposable B-module in $\mathcal{C}$ where $\mathcal{C}$ is a directed, standard component of tree type of the Auslander-Reiten quiver of B and suppose that $\mathbb{K} =$

$\mathrm{Hom}(M, B-\mathrm{mod}) = \mathrm{Hom}(M, \mathcal{C})$ is finite. If $\mathbb{K}$ contains as a full subcategory, one of the following categories:
a) $\{|X|\}$ with $\| X \| = \dim_k |X| \geq 3$
b) $\{|X|, |Y|\}$ with $\dim_k |X| = 2$, $|Y| \neq 0$ and
$\dim_k \mathrm{Hom}_{\mathbb{K}}(|X|, |Y|) + \dim_k \mathrm{Hom}_{\mathbb{K}}(|Y|, |X|) = 0$
c) $\{|X|, |Y|\}$ with $\dim_k |X| = 2$, $\dim_k |Y| = 2$, and
$\dim_k \mathrm{Hom}_{\mathbb{K}}(|X|, |Y|) + \dim_k \mathrm{Hom}_{\mathbb{K}}(|Y|, |X|) = 1$
then $q_{B[M]}$ is strongly indefinite.

To prove the theorem we need some preliminaries results:

Lemma 3.8 Let $\mathbb{K}$ be a category as above and suppose that $\mathbb{K}$ contains as a full subcategory L one of the categories of the list A. Assume that the connected components de L are path-incomparable. Then $q_{B[M]}$ is strongly indefinite.

Proof. Consider the dimension vector z given in [M1].□

Proposition 3.9 Let $\mathbb{K}$ be a vectorspace category as above and X_0, Y_0 in $\mathcal{C}$ such that $\| X_0 \| = 2$, $\| Y_0 \| \neq 0$, $\mathrm{Hom}(|X_0|, |Y_0|) = 0$ and suppose that there exists a path in $\mathcal{C}$, $\delta : X_0 \longrightarrow Y_0$. Then $\mathbb{K}$ contains a full subcategory L of the list A'.

Notation We call list A' the list given by the categories of A plus (▯ ▯) and such that the connected components are path incomparable.
Proof. (by induction on the length of the paths). Suppose that there exists a path $\delta : X_0 \longrightarrow Y_0$, with $\mathrm{Hom}(M, \delta) = 0 = |\delta|$. Let $n = l(\delta)$. For $n = 1$, δ is an arrow. As $|\delta| = 0$, δ is an epimorphism, and we can consider the ARS that ends at Y_0. Let us call $Z = X_0 \oplus Z_i$ the middle term of the sequence. Since we have $g : M \to Y_0$, $|g| \neq 0$, and g is not a split epimorphism, g factors through a direct summand of Z, different of X_0, say Z_i for some i, in this case $|Z_i| \neq 0$, and we can consider the path incomparable subcategory $\{|X_0|, |Z_i|\}$.

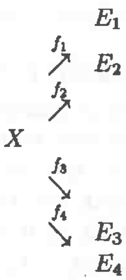
Assume the result is true for length of δ, $l(\delta)$ less than or equal to n. We can assume that $g : M \to Y_0$ is an epimorphism, in fact if $M \longrightarrow \oplus Y_i \overset{\eta}{\hookrightarrow} Y_0$ with $\mathrm{Im}\ g = \oplus Y_i$ then, by induction hypothesis we can assume that $\mathrm{Hom}(|X_0|, |Y_i|) \neq 0$ for some i. But, in this case, $\mathrm{Hom}(|X_0|, |Y_0|) \neq 0$, a contradiction. Now, considering the ARS ending in Y_0. As g is not a split epimorphism, there exists Z_0 an indecomposable direct summand of the middle term of the ARS ending in Y_0, such that $|Z_0| \neq 0$. So, if there is no path $X_0 \longrightarrow Z_0$, we have X_0 and Z_0 path-incomparable. If there exist a path $\delta' : X_0 \longrightarrow Z_0$ with $\mathrm{Hom}(|X_0|, |Z_0|) = 0$, we can apply the induction hypothesis. We can assume that there exists a path $\delta' : X_0 \to Z_0$, with $|\delta'| \neq 0$ and $|\delta| = |\beta\delta'| = 0$, since $\beta : Z_0 \to Y_0$. Then $|\beta|$ is an epimorphism and $\dim |Z_0| \geq 2$ because we have two morphisms from M

to Z_0; g' with $\beta g' \neq 0$ and $\delta' \tilde{f} i$ with $\beta\delta' \tilde{f} i = 0$, linearly independent, for $\tilde{f}_i : M \to X_0$. If there exist another Z_i, with $|Z_i| \neq 0$, then Z_0 and Z_i are path-incomparable. Assume that $|Z_i| = 0\ \forall i \neq 0$ and so $\alpha_i : \tau Y_0 \to Z_i$ is an epimorphism for all $i \neq 0$. Observe that $g' : M \to Z_0$ cannot factor through τY_0. Indeed, suppose that there exists $\varphi : M \to \tau Y_0$ with $\alpha\varphi = g'$. In this case $\beta\alpha\varphi = \beta g' = g$ but $|\beta\alpha| = \sum_{i\neq 0} |\beta_i \alpha_i|$ and since $\mathrm{Hom}(M, Z_i) = 0$ ($\forall i \neq 0$), $|\beta\alpha\varphi| = |(\sum_{i \neq 0} \beta_i\alpha_i)\varphi| = 0$, a contradiction. Then there exists a direct predecessor of Z_0, say T with $|T| \neq 0$, $T \neq \tau Y_0$ and $h : M \to T$ such that $g' = \beta' h$. On the other hand, $\delta' : X_0 \to Z_0$ is such that $|\delta'| \neq 0$ but $|\beta\delta'| = 0$ and the function $\delta'(\tilde{f}_i)$ factors through the kernel of $(\beta, \beta_1, \cdots, \beta_t)$, that is, $|\tau Y_0| \neq 0$. We can assume that there exists $\gamma : X_0 \to T$, $|\gamma| \neq 0$, by our induction hypothesis. Consider first the case where T is indecomposable. There are two morphisms from M to T: $\gamma \tilde{f}_i$, with $\beta\beta'\gamma\tilde{f}_i = 0$, and h with $\beta' h = g'$. Since $\beta(\beta' h) = \beta g' = g \neq 0$, $\gamma\tilde{f}_i$ and h are linearly independent. We have also $\| T \| \geq 2$ and T and τY_0 are path-incomparable. Consider the case $T = T_1 \oplus T_2$ and $\| T_1 \| = \| T_2 \| = 1$. As $|\tau Y_0| \neq 0$, we can assume that $\| Y_0 \| = 1$. In this case $\| \tau Y_0 \| = 1$, and suppose that g' factors through T_1. Assume that there exists $\gamma : X_0 \to T_1$ with $|\gamma| \neq 0$, but $\mathrm{Im}|\gamma| = < h > = |T_1|$ and $\beta' h = h'$. In this case $\beta\beta' h = g \neq 0$ then $\mathrm{Hom}(|X_0|, |Y_0|) \neq 0$ a contradiction, so $\mathrm{Hom}(|X_0|, |T_1|) = 0$ and we can apply the induction hypothesis.□

Corollary 3.10 With the above hypothesis, $q_{B[M]}$ is strongly indefinite.

Proof. It is enough to consider the case where X, Y are path-incomparable and $\| X \| = \| Y \| = 2$, but, in this case $q_{B[M]}(\underline{\dim}\, X + \underline{\dim}\, Y + \underline{\dim}\, S_e) = -1$.□

Lemma 3.11 Let $\mathbb{K}$ be a category as above and suppose that $\mathbb{K}$ contains the following subcategory where $\| X \| = 2$, $\| E_i \| = 1$ and f_i is an irreducible morphism.



Then $\mathbb{K}$ contains a full subcategory path incomparable of the list A'.

Proof. We are going to show that if $\mathbb{K}$ does not contain one of the categories, path incomparable, of the list A', then this subcategory repeats itself. Suppose that $\| X \| = 2$ and $\| E_i \| = 1$. Then $|f_i|$ is surjective or zero. By 3.9, we can assume that $|f_i| \neq 0 \ \forall i$. Then f_i is an epimorphism, E_i is not projective and we can consider the ARS that ends in E_i: $0 \longrightarrow \tau E_i \longrightarrow X \oplus W_i \longrightarrow E_i \longrightarrow 0$. If, for some i, there exists W_i with $|W_i| \neq 0$, then $\{|X|, |W_i|\}$ is a category of the list A', path-incomparable. So, we assume that $|W_i| = 0$ for all i. But, in this case $\| \tau E_i \| = 1$ and by 3.4, X is not projective and we consider the ARS that ends in X. Again: if there exists W with $|W| \neq 0$, $0 \longrightarrow \tau X \longrightarrow W \oplus \tau E_1 \oplus \tau E_2 \oplus \tau E_3 \oplus \tau E_4 \longrightarrow X \longrightarrow 0$ then $\| \tau X \| \geq 3$ and $\{|\tau X|\}$ is a path incomparable subcategory of the list A'. But, if there does not exist W in this conditions, the subcategory repeats itself.□

Proposition 3.12 Let $\mathbb{K}$ be as above and suppose that $\mathbb{K}$ contains, as a full subcategory, $L = \{|X|, |Y|\}$ with $\| X \| = \| Y \| = 2$ and $\dim \operatorname{Hom}(|X|, |Y|) = 1$, then $q_{B[M]}$ is strongly indefinite

Proof. We want to prove that either $\mathbb{K}$ contains a subcategory of the list A' path-incomparable or there exist $X, Y \in \mathcal{C}$ such that $\operatorname{Hom}(M, X) \neq 0$ and $\operatorname{Hom}(M, Y) \neq 0$ and
$\dim_k \operatorname{Hom}_B(X, Y) - \dim_k \operatorname{Ext}^1(Y, X) + \dim_k \operatorname{Ext}^2(Y, X) = 1$, and that, in this case, the Tits forms is strongly indefinite.
1) Suppose that $X, Y \in \mathcal{C}$ are such that $\| X \| = \| Y \| = 2$ and $\dim_k \operatorname{Hom}(X, Y) - \dim_k \operatorname{Ext}^1(Y, X) + \dim_k \operatorname{Ext}^2_B(Y, X) = 1$ where S_e is the new simple injective in $B[M]$. Then $q_{B[M]}(\underline{\dim}\ (X^2 \oplus Y \oplus S_e^2)) = -1$
2) We will see that either we have 1) or $\mathbb{K}$ contains a full subcategory path-incomparable of the list A'. Let us consider $L = \{|X|, |Y|\}$ with $\| X \| = \| Y \| = 2$ and $\dim \operatorname{Hom}_k(|X|, |Y|) = 1$, then there exist a path $\delta : X \to Y$ with $|\delta| \neq 0$. We can assume, by 3.4 that δ is non sectional. If $l(\delta) = 2$ then $X = \tau Y$ and considering the ARS $0 \longrightarrow X \longrightarrow E \longrightarrow Y \longrightarrow 0$ since $|\delta| \neq 0$, $|E|$ is not indecomposable (in $\mathbb{K}$). Counting dimensions, if E has three or less summands, $E = E_1 \oplus E_2 \oplus E_3$ with $|E_i| \neq 0$, there exist one of them with dimension 2 or 3 (say E_1) then $\{|E_1|, |E_2|\}$ ($|E_2| \neq 0$) is a category of the list A' path-incomparable.

We can assume $E = E_1 \oplus E_2 \oplus E_3 \oplus E_4$ with $\| E_i \| = 1$, but in this case by 3.11, $\mathbb{K}$ contains a subcategory path incomparable of the list A'. Now we proceed by induction in n, with $n = l(\delta) \geq 2$, with $\delta : X \longrightarrow Y$ given by a non sectional path, $|\delta| \neq 0$, $\| X \| = \| Y \| = 2$ and $\dim_k \operatorname{Hom}(|X|, |Y|) = 1$.

Let us consider:

$$\begin{array}{ccc} & X_j & \\ & \nearrow \quad \searrow & \\ \delta : X = X_0 \to X_1 \to \cdots X_{j-1} & & X_{j+1} \to \cdots \to X_{n-1} \to Y \end{array}$$

and let us assume that $X_{j-1} = \tau X_{j+1}$ and $X_{j+1} \to \cdots \to Y$ is sectional. We can assume that there exists Y', predecessor of Y with $Y' \neq X_{n-1}$ and $\operatorname{Hom}(M, Y') \neq 0$. In this case $\| X_{n-1} \| < 2$ and we have the path

$$\begin{array}{ccc} & \nearrow \searrow & \\ X \longrightarrow \tau Y & & Y \\ & \searrow \nearrow & \end{array}$$

. Let us consider the ARS that ends in Y:

$0 \longrightarrow \tau Y \longrightarrow E \longrightarrow Y \longrightarrow 0$ with $E = \oplus E_i$ a decomposition into indecomposables. We can assume that $\| E_i \| = 1$ for any i, so there exist at least two different direct summands with $|E_i| \neq 0$, $g_i : E_i \rightarrow Y$, $|g_i| \neq 0$. Moreover $|E| \xrightarrow{|g|} |Y|$ is surjective, and we can assume that $\mathrm{Im}|g_1| \neq \mathrm{Im}|g_2|$. By 3.9 we can assume that $\mathrm{Hom}(|X|, |E_i|) \neq 0$,for $i = 1, 2$. But, in this case $\dim\mathrm{Hom}(|X|, |Y|) > 1$, a contradiction.□

Notation: Denote by A^* the list given by A, plus the categories

(□ → □), (□ □) and (□•)

Lemma 3.13 (Merklen) Let $\mathbb{K}$ be as above and assume that $\mathbb{K}$ contains the following subcategory $A \xrightarrow{\alpha} C$ where $\| A \| = \| C \| = 1$, $|\alpha| = 0$, and α an irreducible morphism. Then $\mathbb{K}$ contains a full subcategory of the list A^*, path-incomparable.

Proof. We are going to prove that if $\mathbb{K}$ does not contain a subcategory of the list A^*, path incomparable, then the category repeats itself.

As $\alpha : A \rightarrow C$ is an irreducible morphism, α is an epimorphism and C is not projective.

Let us consider the ARS that ends in C: $0 \rightarrow \tau C \rightarrow A \oplus B \rightarrow C \rightarrow 0$. Then for any $h : M \rightarrow C$, h factors through $A \oplus B$. As $|\alpha| = 0$, then $\| B \| \neq 0$. If $|B|$ is indecomposable and $\dim|B| \geq 2$, then $\{|B|, |A|\}$ is a full subcategory of the list A^*, path-incomparable. If $|B|$ is indecomposable and $\dim|B| = 1$, considering the morphisms: $\beta : B \rightarrow C$, $\gamma : \tau C \rightarrow A$, $\delta : \tau C \rightarrow B$, by the mesh commutativity, then: $|\alpha\gamma| = |\beta\delta| = 0$. But $|\beta| \neq 0$, $|\beta|$ is an isomorphism, so $|\delta| = 0$. But in this case, δ is an epimorphism, B is not projective, and we begin again. Let us suppose that B is decomposable. We can assume that $B = B_1 \oplus B_2$ with $|B_i| \neq 0$, then considering the morphisms: $\beta_0 : \tau C \rightarrow A$, $\beta_i : \tau C \rightarrow B_i$, $\alpha_0 : A \rightarrow C$, $\alpha_i : B_i \rightarrow C$, we have: $\| \tau C \| = 2$ and by the mesh commutativity $|\alpha_1\beta_1| = |\alpha_2\beta_2|$ then $\ker|\beta_1| = \ker|\beta_2|$. Since $\| \tau C \| = 2$, and $\| B_i \| = 1$, two situations can occur: either $|\beta_i| = 0$ and in this case, we have the result by 3.9 or $|\beta_i|$ is surjective, and we consider the ARS that end in A and in B_i: $0 \rightarrow \tau A \rightarrow \tau C \oplus W_A \rightarrow A \rightarrow 0$, $0 \rightarrow \tau B_i \rightarrow \tau C \oplus W_i \rightarrow B_i \rightarrow 0$. If there exists W with $|W| \neq 0$, then $\{|\tau C|, |W|\}$ is a category path incomparable. Let us suppose that there does not exist such a W, then

$$
\begin{array}{ccccccccc}
 & & \tau A & & & & A & & \\
 & \overset{\delta_0}{\nearrow} & & \overset{\gamma_0}{\searrow} & & \overset{\beta_0}{\nearrow} & & \overset{\alpha_0}{\searrow} & \\
\tau^2 C & \xrightarrow{\delta_1} & \tau B_1 & \xrightarrow{\gamma_1} & \tau C & \xrightarrow{\beta_1} & B_1 & \xrightarrow{\alpha_1} & C \\
 & \underset{\delta_2}{\searrow} & & \underset{\gamma_2}{\nearrow} & & \underset{\beta_2}{\searrow} & & \underset{\alpha_2}{\nearrow} & \\
 & & \tau B_2 & & & & B_2 & &
\end{array}
$$

Let us observe that $\mathrm{Hom}(|\tau B_i|, |C|) = 0$ because $\ker|\beta_j| = \mathrm{Im}|\gamma_i|$ and $|\alpha| = 0$. So the path from $\tau^2 C$ to C passing through τA is equal to the path passing through $\tau B_1 \oplus \tau B_2$ and so is zero. But, moreover, $\mathrm{Im}|\gamma_1| = \mathrm{Im}|\gamma_2|$ and $\mathrm{Im}|\gamma_0| \neq \mathrm{Im}|\gamma_1|$, because there are two morphisms from M to τC that factor through $\tau A \oplus \tau B_1 \oplus \tau B_2$. But $|\gamma_0\delta_0| = |\gamma_1\delta_1 + \gamma_2\delta_2|$, $\mathrm{Im}|\gamma_0\delta_0| \subset \mathrm{Im}|\gamma_1\delta_1 + \gamma_2\delta_2| = \mathrm{Im}|\gamma_1|$ so $|\gamma_0\delta_0| = 0$. As $\| \tau^2 C \| = 1$ and $|\gamma_0| \neq 0$ by construction, then $|\delta_0| = 0$ and the process repeats.□

4 Proof of Theorem

It is enough to consider the case where there exists a path $Y \overset{\delta}{\rightsquigarrow} X$ with $|\delta| = 0$ and we can assume that $\| X \| = 2$ and (by 3.9) $\| Y \| = 1$. Again, we prove it by induction on $n=$ maximal length of paths between Y and X such that $\| Y \| = 1$, $\| X \| = 2$ and $\mathrm{Hom}(|Y|, |X|) = 0$. Let $n = 1$, $\delta : Y \to X$ an irreducible morphism. As $|\delta| = 0$, then X is not projective and we can consider the ARS: $0 \longrightarrow \tau X \longrightarrow Y \oplus Z \longrightarrow X \longrightarrow 0$. The morphisms $M \to X$ factor through Z (Z indecomposable or not) and then $\dim|Z| \geq 2$ so $\| Y \oplus Z \| \geq 3$ and $|\tau X| \neq 0$. If for some Z_i indecomposable direct summand of Z we have $\| Z_i \| = 2$, the category $\{|Y|, |Z_i|\}$ is path incomparable. So we assume $Z = Z_1 \oplus Z_2 \cdots$ with $\dim|Z_i| = 1$. Let us consider $g : M \to Y$, since $|\delta| = 0$, g factors through τX. On the other hand, since $|\tau X| \neq 0$ and $\| Z_i \| = 1$, then the $|\alpha_i| : |\tau X| \to |Z_i|$ are such that either $|\alpha_i| = 0$ for some i or $|\alpha_i|$ is surjective for all i. Let us suppose that $|\alpha_i| = 0$ for some i. Then by 3.9 or by 3.13 we have the result. Let us suppose that we have $|\alpha_i|$ surjective for all i. By 3.11 we can assume that $\| \tau X \| = 1$, in this case any morphism $M \longrightarrow Z_i$ factors through τX. Let $\beta_i : Z_i \to X$ be an arrow. By the commutativity of the mesh, $|\delta\gamma + \beta_1\alpha_1 + \beta_2\alpha_2| = 0 = |\beta_1\alpha_1 + \beta_2\alpha_2|$, and in this case, $\mathrm{Im}|\beta_1| = \mathrm{Im}|\beta_2|$. But then, there are two linearly independents morphisms from M to X that factor through $Y \oplus Z$, a contradiction. Now, we proceed by induction, let us suppose it is true for a maximal path of length less than n, and let us assume δ is a maximal path of length n: $\delta : Y = Y_0 \to Y_1 \to \cdots \to Y_n \xrightarrow{f_n} X$ with $|\delta| = 0$. If $\| Y_n \| = 2$, and $|f_n| \neq 0$, we have the result by 3.12 and if $|f_n| = 0$ by 3.9. So we can assume that $\| Y_n \| \leq 1$ and, by 3.5 we can consider the ARS: $0 \longrightarrow \tau X \longrightarrow Y_n \oplus E_1 \oplus \cdots \longrightarrow X \longrightarrow 0$. By 3.12, or by 3.11 we can assume that $\| \tau X \| \leq 1$. Consider now the origin of the path. We have the following possibilities:

1) $\| Y_1 \| = 1$.
2) $\| Y_1 \| = 2$.
3) $\| Y_1 \| = 0$.

Let us consider 1) $Y = Y_0 \xrightarrow{f_0} Y_1$, if $|f_0| = 0$ we have the result by 3.13. If $|f_0| \neq 0$, then $|f|$ is an isomorphism and so $\mathrm{Hom}(|Y_1|, |X|) = 0$ and we apply the induction hypothesis.

Case 2) $\| Y_1 \| = 2$. If $\dim \mathrm{Hom}(|Y_1|, |X|) \leq 1$ we have the result by 3.9 or 3.12, so we can assume $\dim \mathrm{Hom}(|Y_1|, |X|) \geq 2$. Let Y_2 be a direct successor of Y_1, if $\| Y_2 \| = 2$ we have the result by 3.12. So, let us assume that all

successors of Y_1 have dimension less or equal 1.
Let us consider two situations:
2A) Y_1 non injective
2B) Y_1 injective

2A) We can consider the ARS: $0 \longrightarrow Y_1 \xrightarrow{\alpha} T_1 \oplus T_2 \oplus \cdots \longrightarrow \tau^{-1}Y_1 \longrightarrow 0$ where $|\alpha|$ is injective, and there exists some summand T_i with $\| T_i \| = 1$, and we apply the induction hypothesis.

Let us see 2B, Y_1 is injective, but Y_1 is not projective. Let us suppose first that $|\tau Y_1| = 0$. In this case, there exist W and T direct predecessors of Z and Y_0 respectively with $|W| \neq 0$, $|T| \neq 0$. If $\| T \| = 2$, $\{|T|, |Z|\}$ is a category path incomparable. Let us suppose that $\| T \| = 1$. Since the path δ is maximal, there exists $\delta' : T \to X_0$ a non zero path and then, there exists a direct successor of T, different of Y, say T_1 with $|T_1| \neq 0$ and T_1 being path incomparable with Y_1. Let us suppose that $|\tau Y_1| \neq 0$. By 3.11, we can assume that $\| \tau Y_1 \| = 1$. As dim $\mathrm{Hom}(|Y_1|, |X|) \geq 2$, we have at least one direct successor of Y_1, Y_2, with $\| Y_2 \| = 1$, moreover, if $\mathrm{Hom}(|Y|, |Y_2|) \neq 0$, then $\mathrm{Hom}(|Y_2|, |X|) = 0$ and we apply the induction hypothesis. So, we can consider that $\mathrm{Hom}(|Y|, |Y_2|) = 0$, and that any other direct successor of Y_1 has this property. We will see that in this case, Y_2 is the unique direct successor of Y_1 and $Y = \tau Y_2$. Moreover, the other direct predecessors of Y_1 in $\mathcal{C}$ are injectives. Let us see first that Y_2 is the unique non zero successor in $\mathbb{K}$. Suppose that Y_1 has another successor, V with $\| V \| = 1$. If $\mathrm{Hom}(|V|, |X|) = 0$ we apply the induction hypothesis. If $\mathrm{Hom}(|Y_2|, |X|) \neq 0$ we have the figure

$$\begin{array}{ccccccc} & & Z_1 & & & & V \\ & \overset{h'_1}{\nearrow} & & \overset{h_1}{\searrow} & & \overset{\varphi'}{\nearrow} & \\ \tau Y_1 & \xrightarrow{h'_2} & Z_2 & \xrightarrow{h_2} & Y_1 & & \\ & \overset{g}{\searrow} & & \overset{f}{\nearrow} & & \overset{\varphi}{\searrow} & \\ & & Y & & & & Y_2 \end{array}$$

with $Im|f| = ker|\varphi'| = ker|\varphi|$ and then, as $\| V \| = 1$, V is not projective, similarly for Y_2, τV and τY_2 are direct predecessors of Y_1 with dimension 1. Then, for some i, we have $Im|h_i| = Im|f|$. Let us suppose $i = 1$, since any morphism $M \to Y_1$ factors through $Z_1 \oplus Z_2 \oplus Y$, $Im|h_2|$ is linearly independent with $Im|f|$. So, we can consider $\{y_1, y_2\}$ a basis of $|Y_1|$ of the following form $Im|h_1| = Im|f| = < y_1 >, Im|h_2| = < y_2 >$. Since $fg + h_1 h'_1 + h_2 h'_2 = 0$, then $-|h_2 h'_2| = |fg + h_1 h'_1|$ and $Im|h_2 h'_2| \subset < y_1 >$ and then, $|h_2 h'_2| = 0$, but $|h'_2| : |\tau Y_1| \to |Z_2|$ is either zero and in this case, we have the result by 3.13 or is an isomorphism and in this case $|h_2| = 0$ and we apply the induction hypothesis. Hence, we can assume that $Y = \tau Y_2$ and that Y_2 is the unique non zero successor of Y_1 in $\mathbb{K}$. Let us consider the ARS that ends in Y_2: $0 \longrightarrow Y \longrightarrow Y_1 \oplus W \longrightarrow Y_2 \longrightarrow 0$. If there exists W in $\mathcal{C}$ such that $|W| \neq 0$, then for some W_i direct summand of W, $|W_i| \neq 0$

and $\{|W_i|, |Y_1|\}$ is a category path incomparable. If there exists W in C such that $|W| = 0$, then for all W_i direct summand of W, the morphisms $\beta_i : Y \to W_i$ are epimorphisms, and we can consider the ARS that ends in W_i: $0 \longrightarrow \tau W_i \longrightarrow Y \oplus L_i \longrightarrow W_i \longrightarrow 0$ and where, counting dimensions, we get $|\tau W_i| \neq 0$. If $\| \tau W_i \| = 1$, we have $|L_i| = 0$ e $\mathrm{Hom}(|\tau W_i|, |X|) = 0$. On the other hand, if $\| \tau W_i \| = 2$, there is a sectional path from τW_i to Y_1 and the result follows by 3.12. So, we can assume that such W does not exist in C and that the ARS: $0 \longrightarrow Y \xrightarrow{f} Y_1 \longrightarrow Y_2 \longrightarrow 0$ is such that f is an injective envelope and that Y_2 is a simple. Let us see that Y_1 does not have any other successor in C, indeed, suppose that there exists L direct successor of Y_1 with $|L| = 0$ then, L is not projective, τL is a direct predecessor of Y_1 and $\| \tau L \| = 2$ and again, the result follows by 3.12. On the other hand, let N be a direct predecessor of Y_1 different from Y, if N is not injective, then $|\tau^{-1} N| \neq 0$. So, we conclude that Y is a simple module and any other summands of the middle term of the ARS that ends in Y_1 are injective. But, the ARS $0 \longrightarrow \tau Y_1 \longrightarrow Y \oplus Z_1 \oplus Z_2 \oplus N \longrightarrow Y_1 \longrightarrow 0$ is such that Y_1 is not simple, then the sequence: $0 \longrightarrow soc(\tau Y_1) \longrightarrow soc(Y \oplus Z_1 \oplus Z_2 \oplus N) \longrightarrow soc(Y_1) \longrightarrow 0$ is split-exact. Since $Y \cong soc(Y_1)$ and $Z_1 \oplus Z_2 \oplus N = I^*$ is injective, and we have that $soc(\tau Y_1) \cong soc(I^*)$ and $\tau Y_1 \to I^*$ is the injective envelope of τY_1. Moreover, the sequence : $0 \longrightarrow \tau Y_1 \xrightarrow{h'} Y \oplus I^* \xrightarrow{h} Y_1 \longrightarrow 0$ is exact and

$$\frac{I^* \oplus Y}{h'(\tau Y_1)} \cong \frac{I^* \oplus Y}{(h'(\tau Y_1) \cap I^*) \oplus (h'(\tau Y_1) \cap Y)} \cong \frac{I^*}{h'(\tau Y_1) \cap I^*} \oplus Y \cong Y_1$$

because $h'(\tau Y_1) \cap Y = 0$ as Y_1 is not a summand of I^*, but in this case Y_1 is decomposable, a contradiction.

Case 3) Suppose that $|Y_1| = 0$. We can assume that all successors de Y are such that $|Y_1| = 0$. In this case $Y \xrightarrow{f} Y_1$, $|f| = 0$, f is an epimorphism and considering the corresponding ARS, then we have that $|\tau Y_1| \neq 0$ and $Y_0 \oplus Z$ is the middle term. If $\| \tau Y_1 \| = 1$, then $|Z| = 0$ and $\mathrm{Hom}(|\tau Y_1|, |X|) = 0$ which contradicts the maximality of the path. If $\| \tau Y_1 \| = 2$, we can assume that Z is indecomposable with $\| Z \| = 1$ and by 3.9 we can assume that $\mathrm{Hom}(|\tau Y_1|, |X|) \neq 0$, and there exists $\varphi : \tau Y_1 \to X$, φ is not a split monomorphism and $|\varphi| \neq 0$ so, we can assume that there exists $\overline{\varphi} : Z \to X$ such that $\alpha_1 \overline{\varphi} = \varphi$, then $|\alpha_1 \overline{\varphi}| = |\varphi| \neq 0$ and so $|\overline{\varphi}| \neq 0$ and any morphism from $|\tau Y_1|$ to $|X|$ factors through $|Z|$. Moreover, there exists a path from Z to X that does not factor through Y_1 and so, none of these paths is sectional. Moreover, the arrow $\alpha_1 : \tau Y_1 \to Z$ is such that $|\alpha_1| : |\tau Y_1| \to |Z|$ is not injective and we consider the ARS that ends in Z. If there exists T, direct predecessor of Z, with $|T| \neq 0$, then $\{|T|, |\tau Y_1|\}$ is a category path incomparable. Let us see that the hypothesis $|T| = 0$ will lead to a contradiction. If $|T| = 0$ then $\| \tau Z \| = 1$. Since $\mathrm{Hom}(|\tau Z|, |X|) = 0$ contradicts the maximality of the path, we can assume that $\mathrm{Hom}(|\tau Z|, |X|) \neq 0$, so that there exists $\psi : \tau Z \longrightarrow X$, with $|\psi| \neq 0$ and ψ does not factor through Y, or Y_1. If the ARS that ends in Z does not have other direct summands in

B-mod, then $\alpha_1\eta = 0$, so $|\alpha_1\eta| = 0$ and $|\psi| = 0$, contradiction, because we are assuming that $\mathrm{Hom}(|\tau Z|, |X|) \neq 0$. Let us assume that the ARS that ends in Z has other summands T_i with $|\oplus T_i| = 0$

$$\begin{array}{ccccc} & & \oplus T_i & & \\ & \bar{\eta} \nearrow & & \searrow \bar{\alpha} & \\ \tau Z & & & & Z \\ & \eta \searrow & & \nearrow \alpha & \\ & & \tau Y_1 & & \end{array}$$

In this case:

$$\alpha_1\eta + \overline{\alpha\eta} = 0 \quad \text{so} \quad |\alpha_1\eta + \overline{\alpha\eta}| = 0$$

but $|\overline{\alpha\eta}| = 0$ because $|T| = 0$ so $|\alpha_1\eta| = 0$, a contradiction to the fact that $\mathrm{Hom}(|\tau Z|, |X|) \neq 0$.□

5 Roiter's categories

In this section we will analyse other finite vectorspace categories that are immersed in directed, standard components of tree type. As there is not a classification of the wild vectorspace categories, but there is a classification of the polynomial growth schurian categories given by V. Bekkert in [B1], our work goes in this direction. Some of these categories appear in [Ro]. Let us write down the results obtained by Bekkert. Let $\mathbb{K}$ be a schurian vectorspace category. Denote by $S\mathbb{K}$ the set of indecomposable objects of dimension one and by $L\mathbb{K}$ the set of indecomposable objects of dimension two. For an object $|X| \in L\mathbb{K}$, we identify the set of the subspaces of $|X|$ of dimension one with the projective line and denote it by $P^1(X)$. For $|X| \in L\mathbb{K}$ and $V \in P^1(X)$ we denote $S_{X,V}$ (resp. $S^{X,V}$) the set of objects $Z \in S\mathbb{K}$ such that $Hom(|X|, |Z|) \neq 0$ (resp. $Hom(|Z|, |X|) \neq 0$) and $V = ker\varphi$ (resp. $V = Im\varphi$) for all $\varphi \in Hom(|X|, |Z|)$. For an object $|X| \in L\mathbb{K}$, we denote by Ω_X (resp. Ω^X)the set of $V \in P^1(X)$ such that $S_{X,V} \neq \emptyset$ (resp. $S^{X,V} \neq \emptyset$). Let us define $S_X = \cup_{V\in\Omega_X} S_{X,V}$ (resp. $S^X = \cup_{V\in\Omega^X} S^{X,V}$). Since we are assuming $\mathbb{K}$ wild, $\mathbb{K}$ is not of polynomial growth and we write the theorem in the following form:

Theorem 5.1 [B1] A schurian vectorspace category $\mathbb{K}$ is not of polynomial growth if and only if one of the following condition holds:
($\hat{A}$) There exists $|X|$ indecomposable in $\mathbb{K}$, with $dim_k|X| \geq 3$
($\hat{B}$) $dim_k Hom_{\mathbb{K}}(|X|, |Y|) + dim_k Hom_{\mathbb{K}}(|Y|, |X|) = 0$ for some $|X|$ in $L\mathbb{K}$, and $|Y|$ in $S\mathbb{K}$
($\hat{C}$) $dim_k Hom_{\mathbb{K}}(|X|, |Z|) + dim_k Hom_{\mathbb{K}}(|Z|, |X|) \leq 2$ for some $|X|$ and $|Z|$ in $L\mathbb{K}$
($\hat{D}$) There exist $|X|$ and $|Y|$ in $L\mathbb{K}$, $V \in P^1(X), U \in P^1(Y)$ such that $Hom(|X|, |Y|)(V) = U$ and $S_{X,V} \cap S^{Y,U} \neq \emptyset$.
($\hat{E}$) $S_{X,V}$ or $S^{X,V}$ contains a poset $(1,1,1)$ or $(1,2)$ for some $X \in L\mathbb{K}$ and some $V \in P^1(X)$.

($\hat{F}$) For some X with $1 < ord\Omega_X < 4, S_X$ (resp. S^X) contains a poset $(1,1,1,1)$
($\hat{G}$) There exist $X_1, X_2, \ldots, X_n \in L\mathbb{K}$, $V_i \in P^1(X_i)$, $1 \leq i \leq n$, $n \in \mathbb{N}$, such that $|Hom(|X_i|, |X_{i+1}|)|(V_i) = V_{i+1}$ for $1 \leq i < n$ and S_{X_n, V_n} and S^{X^1, V^1} contains the poset $(1,1)$.
($\hat{H}$) $S\mathbb{K}$ contains one of the following posets $(1,1,1,1,1)$, $(1,1,1,2)$, $(2,2,3)$, $(1,3,4)$, $(1,2,6)$, $(N,5)$ and $\{A < B > C < D > A, S, T\}$

As the conditions Â and B̂ have been studied in the last section, we consider now condition Ĉ, that is, there exist $|X|$ and $|Y|$ in $\mathbb{K}$, with $\| X \| = \| Y \| = 2$ and $dimHom(|X|,|Y|) + dimHom(|Y|,|X|) \leq 2$. In this case, if $dimHom(|X|,|Y|) + dimHom(|Y|,|X|) \leq 1$, again by 3.7, then $q_{B[M]}$ is strongly indefinite. If, on the other hand, $dimHom(|X|,|Y|) + dimHom(|Y|,|X|) = 2$, $\mathbb{K}$ will not always be wild.

Lemma 5.2 ([B1]) Let $\mathbb{K}$ be a schurian vectorspace category and let $|X|$ and $|Y|$ be in $\mathbb{K}$, with $\| X \| = \| Y \| = 2$. Then
a) If $dimHom(|X|,|Y|) = 3$ then $\mathbb{K}\{|X|,|Y|\}$ is equivalent to one of the categories:

I) $\square \substack{\to \\ \to \\ \to} \square$, with the morphisms given by $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$

J) $\square \substack{\to \\ \to \\ \to} \square$, with the morphisms given by $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$

b) If $dimHom(|X|,|Y|) + dimHom(|Y|,|X|) = 2$, then $\mathbb{K}\{|X|,|Y|\}$ is equivalent to one of the categories:

A) $\square \rightrightarrows \square$, with the morphisms given by $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$

B) $\square \rightrightarrows \square$, with the morphisms given by $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$

C) $\square \leftrightarrows \square$, with the morphisms given by $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$

D) $\square \rightrightarrows \square$, with the morphisms given by $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$

D^{op}) $\square \rightrightarrows \square$, with the morphisms given by $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

We maintain the notation of [B1]. The category of type A is tame, and since $\mathbb{K}$ is immersed in a directed component, the category of type C cannot occur. Let us analyse the others.

Lemma 5.3 Let $\mathbb{K}$ be a category as above and suppose that $\mathbb{K}$ contains a subcategory of the form $\{|X|, |Z_1|, |Z_2|\}$

$$\begin{array}{ccccccc} & & Z_1 & & Z_1 & & \\ & \overset{\varphi_1}{\nearrow} & & & & \overset{\psi_1}{\searrow} & \\ X & & & \text{or} & & & X \\ & \underset{\varphi_2}{\searrow} & & & & \underset{\psi_2}{\nearrow} & \\ & & Z_2 & & Z_2 & & \end{array}$$

with $\| X \| = 2$, $\| Z_i \| = 1$, φ_i irreducible morphisms, $|\varphi_i| \neq 0$, but $ker|\varphi_1| = ker|\varphi_2|$ or: ψ_i irreducible morphisms and $Im\,|\psi_1| = Im|\psi_2|$. Then $\mathbb{K}$ contains, as a full subcategory, one of the categories (path incomparable) of the list A^*.

Proof Let us suppose that

$$\begin{array}{ccc} & & Z_1 \\ & \nearrow & \\ X & & \\ & \searrow & \\ & & Z_2 \end{array}$$

then $|\varphi_i|$ is surjective and we can consider the ARS that ends in Z_i: $0 \to \tau Z_i \to X \oplus W_i \to Z_i \to 0$. If, for some i, $|W_i| \neq 0$, then $\{|X|, |W_i|\}$ is a path incomparable category. So, if $|W_i| = 0\ \forall i$, $0 \to |\tau Z_i| \xrightarrow{|\psi_i|} |X| \xrightarrow{|\varphi_i|} |Z_i| \to 0$ is an exact sequence in $\mathbb{K}$, $ker|\varphi_i| = Im|\psi_i|$ and $\| \tau Z_i \| = 1$.
Since, by 3.5, X is not projective, we consider the ARS that ends in X. Since the morphisms from M to X are not a split epimorphism, and since $Im|\psi_1| = Im|\psi_2|$, there exists T a predecessor of X, $T \not\cong \tau Z_i$ with $|T| \neq 0$, and $\psi_0 : T \to X$ with $Im|\psi_0| \neq Im|\psi_1|$. If T is indecomposable, and $\| T \| = 2$, then the category $\{|T|, |\tau Z_i|\}$ is path incomparable. We can assume that T is indecomposable (by 3.11) and that $\| T \| = 1$. In this case, let us consider $\alpha_0 : \tau X \to T$ $\alpha_i : \tau X \to \tau Z_i$, $\psi_0 : T \to X$ and $\| \tau X \| = 1$. Let us consider a basis of $|X|$, given by $\{x_0, x_1\}$ where $< x_0 >= Im|\psi_0|$, $< x_1 >= Im\ |\psi_1|$ and $\{g\}$ a basis of $|\tau X|$. Then $|\psi_1\alpha_1| + |\psi_2\alpha_2| + |\psi_0\alpha_0| = 0$ and in this case $-|\psi_0\alpha_0| = |\psi_1\alpha_2| + |\psi_2\alpha_2|$ and $\lambda x_0 = -|\psi_0\alpha_0|(g) = (|\psi_1\alpha_2| + |\psi_2\alpha_2|)(g) = \eta x_1$ so $|\psi_0\alpha_0| = 0$. Since $|\psi_0|$ is injective, $|\alpha_0| = 0$, and by 3.13, $\mathbb{K}$ contains as a full subcategory path incomparable one of the categories of the list A^*.$\square$

Proposition 5.4 Let $\mathbb{K}$ be as above and let us suppose that $\mathbb{K}$ contains $\{|X|, |Y|\}$, as a full subcategory where $\| X \| = \| Y \| = dim_k Hom(|X|, |Y|) = 2$ with $Hom(|Y|, |X|) = 0$.

Let $\{x_1, x_2\}$ and $\{y_1, y_2\}$ be bases, respec. of $|X|$ and of $|Y|$ such that $Hom(|X|, |Y|)$ is given by one of the following bases:

$$B_1 \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\} \quad \text{(category B)}$$

$$B_2 \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\} \quad \text{(category D)}$$

$$B_3 \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\} \quad \text{(category } \mathrm{D}^{OP}\text{)}$$

Then $q_{B[M]}$ is strongly indefinite.

Proof. Again, we prove by induction on the maximal length $n(\delta)$ of the paths δ from X to Y, that $\mathbb{K}$ contains a full subcategory (path incomparable) of the list A^*.

As we have two morphisms that are linearly independents from X to Y none of this paths can be sectional (see 3.4).

If $n(\delta) = 2$, the result follows from 3.11 or by the same argument done in 3.12. Let us suppose that δ is a maximal path of length n and let us assume the statement for maximal paths of length less than n. Let us consider $\delta : X = X_0 \to X_1 \to \cdots \to X_n \to Y$. If $\| X_n \| = 2$, the result follows from 3.7. So, we can assume that $\| X_n \| \leq 1$ and there exists a predecessor of Y, say Y' with $Y' \neq X_n$ and $|Y'| \neq 0$. If $Hom(|X|, |Y'|) = 0$, by 3.7, we have the result. Since this path cannot be sectional, there exists a path δ' from X_0 to τY. Dividing in cases:
1) δ' is a sectional path.
2) δ' is not a sectional path.

We begin with 1) Again, dividing in cases:
A) If $dim|\tau Y| = 2$ the result follows by 3.11.
B) If $|\tau Y| = 0$, at least for one i, the path $X_0 \to \tau Y \to E_i$ is sectional. Since $\| E_i \| = 1$, there exists a predecessor of E_i, say T_i with $|T_i| \neq 0$ and there does not exist a path from X to T_i by 3.4. Then, $\{|X|, |T_i|\}$ is a category path incomparable.
C) If $\| \tau Y \| = 1$, we can assume that there are three summands with $\| E_i \| = 1$ in the middle term of the ARS that ends in Y. Also, we can assume that any morphism from X to Y factors through τY: indeed, let us suppose that this not happen, then there exists another non zero path from X to E_1. In this case, there exists T_1 a direct predecessor of E_1 and by 3.5, E_1 is not projective
C1) If $X \not\cong \tau E_1$, then, counting dimensions, $|\tau E_1| \neq 0$, and there exists $\psi : X_0 \to E_1$ that is not a split epimorphism, so that there exists ψ_T :

$X \to T_1$ such that $\psi : h_1\psi_T$. If there does not exist a path from X to τE_1, then $\{|X|, |\tau E_1|\}$ is a category path incomparable. If there exists a path from X to τE_1, then i,f $\|\ \tau E_1\ \| = 2$, the path being sectional, the result follows from 3.7. If $\|\ \tau E_1\ \| = 1$, any morphism from X to Y is of the form: $|(\sum f_i g_i)\delta'|$. But $|\delta'|$ is surjective and not injective. We consider a basis of X, $\{x_1, x_2\}$ such that $|\delta'|(x_1) = 0$ (in this case, the category is of the form: D^{OP}). Since the path from X to τY is unique, there is a unique X_1 direct successor of X with: $X \overset{\varphi}{\to} X_1 \overset{\delta''}{\to} \tau Y$. If $\|\ X_1\ \| = 2$, or $\|\ X_1\ \| = 0$, the result follows from 3.7. Let us assume that $\|\ X_1\ \| = 1$. In this case $|X| \overset{|\varphi|}{\to} |X_1|$ is not injective, and the ARS is $\tau X_1 \overset{\psi}{\to} X \oplus A \overset{\varphi}{\to} X_1$. If $|A| \neq 0$, then $\{|X|, |A|\}$ is a category path incomparable. If $|A| = 0$, then $\|\ \tau X_1\ \| = 1$ and $ker|\varphi| = Im|\psi|$. Moreover, $\delta' = \delta''\varphi$ and $\delta'' : X_1 \to \tau Y$ is such that $|\delta''|$ is isomorphism then $ker|\delta'| = ker|\delta''\varphi| = ker|\varphi| = Im|\psi| =< x_1 >$. Since there is not another path from τX_1 to Y, then $Hom(|\tau X_1|, |Y|) = 0$ and the result follows from 3.7.

C2) If $X \cong \tau E_1$:

$$\begin{array}{ccccccc}
 & & T_1 & & & & \\
 & \nearrow & & \searrow & & & \\
X & \to & T_2 & \to & E_1 & & \\
 & \searrow & & \nearrow & & \searrow & \\
 & & \tau Y & \to & E_2 & \to & Y \\
 & & & \searrow & & \nearrow & \\
 & & & & E_3 & &
\end{array}$$

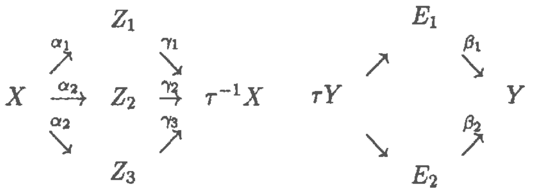
we see that $dim\ Hom(|X|, |Y|) \geq 3$, a contradiction.

Let us consider 2) δ' is not a sectional path. Dividing in cases:

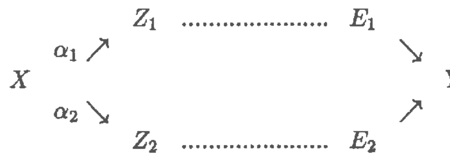
A) If $\|\ \tau Y\ \| = 2$, the result follows from 3.11.

B) If $\|\ \tau Y\ \| = 1$, since δ' is not sectional, by 3.7 we can assume that $dim\ Hom(|X|, |\tau Y|) = 1$ or 2. By the same argument above we can assume that the morphisms from X to Y factor through τY and that $\|\ Hom(|X|, |\tau Y|)\ \| = 1$. In this case, $\delta' : X \to \tau Y$ is such that $|\delta'|$ is not injective and again the subcategory is of type D^{OP}. We can assume that the direct successors of X, Z_i, are such that $\|\ Z_i\ \| \leq 1$ and are at most three. If there is only one, Z with $\|\ Z\ \| \neq 0$, then $\|\ \tau Z\ \| = 1$, $Hom(|\tau Z|, |Y|) = 0$ and the result follows from 3.7. Let us suppose that X has two or more direct successors Z_i with $|Z_i| \neq 0$. By 3.7, we assume that $Hom(|Z_i|, |\tau Y|) \neq 0$ and consider $\gamma_i : X \to Z_i$ and $\varphi_i : Z_i \to \tau Y$, in this case: $|\delta'| = |\varphi_1\gamma_1| = |\varphi_2\gamma_2|$, then $|\varphi_1|$ and $|\varphi_2|$ are isomorphisms and $ker|\delta'| = ker|\gamma_1| = ker|\gamma_2|$ and the result follows from 5.3.

C) If $|\tau Y| = 0$ and δ' is not sectional, we have: $\|\ E_1\ \| = \|\ E_2\ \| = 1$ and $\beta_i : E_i \to Y$. Considering 3.7, let us suppose that $Hom(|X|, |E_i|) \neq 0$, $i = 1, 2$ and considering 5.3, let us assume that $Im|\beta_1| \neq Im|\beta_2|$. Let $\{y_1, y_2\}$ be a basis of Y such that $< y_1 >= Im|\beta_1|$, $< y_2 >= Im|\beta_2|$. If, for some i, $dim\ Hom(|X|, |E_i|) > 1$, we have $dim\ Hom(|X|, |Y|) > 2$, a contradiction. So, let us assume that $dim\ Hom(|X|, |E_i|) = 1\ \forall_i = 1, 2$. Let us see the direct successors of X. If there is only one direct successor Z with $|Z| \neq 0$ then either $\| Z \| = 2$ and the result follows by 3.7 or $\| Z \| = 1$, and we have the ARS $0 \to \tau Z \to X \oplus W \to Z \to 0$. If $|W| \neq 0$, $\{|X|, |W|\}$ is a category path incomparable. If $|W| = 0$, then $\| \tau Z \| = 1$ and $Hom(|\tau Z|, |Y|) = 0$, again, the result follows from 3.7. So, let us assume that we have two or three successors of X. If X is not injective and $|\tau^{-1}X| \neq 0$ then



By 5.3, the $|\alpha_i' s|$ have different kernels. By 3.7, there exists $f : \tau^{-1}X \to Y$ such that $|f| \neq 0$, and f factors through $E_1 \oplus E_2$. So, for some i, there exists $f_i : \tau^{-1}X \to E_i$ such that $|f_i|$ is an isomorphism. But
$dim\ Hom(|X|, |\tau^{-1}X|) = 2$, so $dim\ Hom(|X|, |E_i|) = 2$, wich contradicts $dim\ Hom(|X|, |Y|) = 2$. Assume that $|\tau^{-}X| = 0$, and X has two non zero direct successors in $\mathbb{K}$

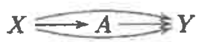


Since $dim Hom(|X|, |E_i|) = 1$ so $Hom(|Z_i|, |E_i|) \neq 0$, $i, j = 1, 2$, $Hom(|Z_i|, |E_j|) = 0$ so we can consider bases $\{x_1, x_2\}$ and $\{y_1, y_2\}$ such that

$$[\varphi_1] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad [\varphi_2] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \square$$

Let us suppose now that $\mathbb{K}$ satisfies condition $\hat{\text{D}}$. Then

Proposition 5.5 Let $|X|$ and $|Y|$ in $L\mathbb{K}$, $V \in P^1(X)$,
$U \in P^1(Y)$ be such that $Hom(|X|, |Y|)(V) = U$ and $S_{X,V} \cap S^{Y,U} \neq \emptyset$. Then $\mathbb{K}$ contains as a full subcategory a category of the form:



with $\| A \| = 1$, $\| X \| = \| Y \| = 2$ and the morphisms given by:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$(1, 0) : |X| \to |A|$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix} : |A| \to |Y|$, that is, there exist $U \in P^1(X)$, $V \in P^1(Y)$ such that $|\varphi|(U) \subset V$ $\forall |\varphi| : |X| \to |Y|$ and $ker(X \to A) =$

U, $Im(A \to Y) = V$. Moreover, if $\mathbb{K}$ contains the above subcategory then $\mathbb{K}$ contains as a full subcategory one of the categories of the list A^*.

Proof: Again, we do the proof by induction in $n(\delta)$ = length of the paths δ (with $|\delta| \neq 0$) between A and Y. Let us suppose that $\delta : A \to Y$ is an arrow. By 3.4, the paths from X to Y cannot be sectional. Since $\| A \| = 1$ and $\| Y \| = 2$, Y has other direct predecessor B with $\| B \| \neq 0$. If $\| B \| = 2$ and B is indecomposable, the category $\{|B|, |A|\}$ is path incomparable. So, let us assume that the direct summands of B have dimension 1. By 3.5, Y is not projective and there is a path $X \to \tau Y$. Let us denote by β_0, β_1 respec. the arrows from A to Y and from B to Y. First, let us consider B indecomposable, with $\| B \| = 1$. If $Hom(|X|, |B|) = 0$, the result follows from 3.7. Let us assume now that $dim\ Hom(|X|, |B|) = 2$. Since $Im|\beta_0| = V$, by 5.3, then $Im|\beta_1| \neq V$. But then $|\psi_1|$, $|\psi_2|$ are linearly independent for $\psi_i : X \to B$ with $|\beta_1\psi_i|(U) \subset V$. Since $Im|\beta_1| \neq V$ and $|\beta_1|$ is injective then $U \subset ker|\psi_i|$ for $i = 1, 2$, but in this case, $|\psi_1|$ and $|\psi_2|$ are linearly dependent, a contradiction. If, on the other hand, $dim\ Hom(|X|; |B|) = 1$ then $dim\ Hom(|X|, |Y|) = 2$, a contradiction. Let us suppose B decomposable (with two summands, by 3.11) and by the same argument, $dim\ Hom(|X|, |B_i|) = 1$, so $\| \tau Y \| = 1$. If $Hom(|X|, |\tau Y|) = 0$, the result follows from 3.7. If $dim\ Hom(|X|, |\tau Y|) = 2$, since $\alpha_0 : \tau Y \to A$ is such that $|\alpha_0|$ is an isomorphism, then $dim\ Hom(|X|, |A|) = 2$, a contradiction. So $dim\ Hom(|X|, |\tau Y|) = 1$. But, in this case, $\{|X|, |Y|\}$ is a subcategory of type D. This finish the case $n(\delta) = 1$. Let us suppose it is true for $n(\delta) < n$ and consider $\delta : A = A_0 \to A_1 \to \cdots \to A_n \to Y$. Let us assume that $|A_i| \neq 0\ \forall_i$ and $\| A_n \| = 1$, so, there exists another direct predecessor of Y, say B. Since $\delta = \beta_0\delta_0$ and $Im|\delta| = V = Im|\beta_0\delta_0|$ where $\beta_1 : B \to Y$, $\beta_0 : A_n \to Y$, $\delta_0 : A \to A_n$ and $|\delta_0|$ is isomorphism, then $V = Im|\beta_0|$. By 5.3, we can assume that $Im|\beta_1| \neq V$. As $dim\ Hom(|A|, |Y|) = 1$ then $Hom(|A|, |B|) = 0$ (and so, $\| B \| = 1$).
Dividing in cases:
1) $\| A_1 \| = 2$
2) $\| A_1 \| = 1$.
Let us see 1). Since $\| A_1 \| = 2$, we can assume that $\| A_2 \| = 1$ and that $|gh| \neq 0$ where: $A_0 \xrightarrow{h} A_1 \xrightarrow{g} A_2 \xrightarrow{\varphi_i} Y$ so that $|gh|$ is an isomorphism. If $dim\ Hom(|A_2|, |Y|) > 1$, then $|\varphi_i gh| \neq 0$, $\forall \varphi_i : A_2 \to Y$ with $|\varphi_i| \neq 0$ so $dim\ Hom(|A|, |Y|) > 1$, a contradiction. Let us suppose that $dim\ Hom(|A_2|, |Y|) = 1$ and $\forall |\varphi| : |A_2| \to |Y|$ has image V. From $X \xrightarrow{\psi} A \xrightarrow{h} A_1 \xrightarrow{g} A_2$, we get that $|gh|\ |\psi|$ has kernel U. If $dim\ Hom(|X|, |A_2|) = 1$, the category $\{|X|, |A_2|, |Y|\}$ satisfies the induction hypothesis. Let us suppose that $dim\ Hom(|X|, |A_2|) = 2$ and so there exists $f' : X \to A_2$ with $|f'|(U) \neq 0$. Since f' is not a split epimorphism, and A_2 is not projective, let us consider the ARS: $0 \to \tau A_2 \to A_1 \oplus W \to A_2 \to 0$. If there exists W with $|W| \neq 0$, then $\{|A_1|, |W|\}$ is a category path incomparable. Let us suppose $|W| = 0$ and $\| \tau A_2 \| = 1$. If $A_0 = \tau A_2$, then $|gh| = 0$,

A_1 is not projective by 3.5 and the morphisms from X to A_2 factor through A_1. Again, let us consider the ARS: $0 \to \tau A_1 \to A_0 \oplus \tau A_2 \oplus T \to A_1 \to 0$. If $|T| = 0$, then the morphisms from X to A_2, are of the form: $X \to A_0 \to A_2$ with kernel U and $X \to \tau A_2 \to A_2$ wich is zero. Let us assume then, that $\| T \| = 1$, T indecomposable, and let us consider the diagram:

$$\begin{array}{ccccccccc} & & & & A_0 & & & & \\ & & & \nearrow & & \searrow & & & \\ X & --\to & \tau A_1 & \to & \tau A_2 & \to & A_1 & & \\ & & & \searrow & & \nearrow & & \searrow & \\ & & & & T & & & & A_2 \end{array}$$

Since $dim\ Hom(|X|, |A_0|) = 1$, $dim\ Hom(|X|, |\tau A_1|) = 1$. But $dim\ Hom(|X|, |T|) > 1$ because there exists $\hat{f} : X \to T$ such that $f' = gh\hat{f}$ and $ker|\hat{f}| \neq U$. If $A_2 = A_n$, and $A_1 \cong \tau Y$ the result follows from 3.11. If $A_1 \ncong \tau Y$ then A_1 and B are path incomparable.

Let us suppose that $A_n \ncong A_2$ and $Hom(|A_1|, |B|) \neq 0$. The paths from X to Y factoring through B are of the form:
$X \xrightarrow{\overline{\alpha}} A_0 \oplus \tau A_2 \oplus T \xrightarrow{\overline{\beta}} A_1 \xrightarrow{\overline{g}} B \xrightarrow{\beta} Y$. If $|\overline{g}| \neq 0$, $|\beta \overline{g}| \neq 0$ because $|\beta|$ is injective. Since $|\overline{\beta}|$ is surjective, $|\beta g \overline{\beta}| \neq 0$ and $|\overline{\beta}\ \overline{\alpha}|$ is non zero because there exists a non zero morphism from $|X|$ to $|A_1|$, then $|\beta \overline{g}\ \overline{\beta}\ \overline{\alpha}|(U) \subset V$. Since $Im|\beta| \neq V$, $|\beta \overline{g}\ \overline{\beta}\ \overline{\alpha}|(U) = 0$ but, $|\beta|$ being injective, $|\overline{g}\ \overline{\beta}\ \overline{\alpha}|(U) = 0$ that is $|\overline{\beta}\ \overline{\alpha}|(U) \subset ker\ \overline{g}$ and for all $\psi : X \to A_1$, $|\psi|(U) \subset ker|\overline{g}|$. Let $V_1 = ker|\overline{g}|$ then $(Hom|X|, |A_1|)(U) = V_1$ and $ker(|X| \to |A|) = U$. Hence $Im(|A| \to |A_1|) = V_1$ and the category $\{|X|, |A|, |A_1|\}$ satisfies induction hypothesis.

Let us see 2). $\| A_1 \| = 1$
with $|\psi| = (1, 0)$ and there exists another direct predecessor of Y, B with $\| B \| = 1$ and, by 5.3, $Im|\beta| \neq V$. If $dim\ Hom(|A_1|, |Y|) > 1$, since $|f_1|$ is an isomorphism, $dim\ Hom(|A|, |Y|) > 1$, a contradiction. So, $dim\ Hom(|A_1|, |Y|) = 1$ and $Im(|A_1| \to |Y|) = V$. If $dim_k Hom(|X|, |A_1|) = 1$, for any $g : X \to A_1$, $|g|$ is linearly dependent with $|f_1 \psi|$ and so has kernel U, hence the category $\{|X|, |A_1|, |Y|\}$ satisfies the induction hypothesis. If $dim\ Hom(|X|, |A_1|) > 1$ (so $=2$) then there exists $\overline{\psi} : X \to A_1$ such that $\overline{\psi}$ does not factor through A, and there exists another predecessor of A_1, B_1 such that $\overline{\psi}$ factors through B_1. If B_1 is indecomposable of dimension 2, the result follows from 3.7. If B_1 is indecomposable of dimension 1, then $dim\ Hom(|X|, |A_1|) = 1$. So let us consider B_1 decomposable and $\| \tau A_1 \| = 2$ and assume that for any $\hat{\psi} : X \to \tau A_1$, $|g\hat{\psi}|$ has kernel U, either $|\hat{\psi}|(U) = 0$ or $|\hat{\psi}|(U) \subset ker|g|$. Let $V_1 = ker|g|$. If $Hom(|\tau A_1|, |B|) = 0$, the result follows from 3.7. Let us suppose that $x : \tau A_1 \to B$, $|x| \neq 0$. Then $|\beta x| : |\tau A_1| \to |B| \to |Y|$ and $|\beta x \hat{\psi}| : |X| \to |Y|$. Since $Im|\beta| \neq V$, $|\beta x \hat{\psi}|(U) = 0$ and $|\hat{\psi}|(U) \subset V_1$, $|\beta x|(V_1) = 0$. But $|\beta|$ is injective, so $|x|$ has the same kernel as $|g|$. Since $Hom(|A|, |B|) = 0$, x cannot factor through A and there exists another direct successor of τA_1, B_1. But then

$\tau A_1 \xrightarrow{\alpha_1} B_1 \xrightarrow{t} B$, $x = t\alpha_1$ and since $\| B \| = \| B_1 \| = 1$, $|t|$ is an isomorphism and $ker|x| = ker|t\alpha_1| = ker|\alpha_1| = ker|g|$, a contradiction to 5.3.□

6 Categories of type Ê

Let us suppose that $\mathbb{K}$ satisfies condition Ê of the theorem 5.1, then $\mathbb{K}$ contains, as a full subcategory one of the categories below:

$$\hat{E}1 \quad \begin{array}{ccc} Z_1 & & \\ & \overset{\varphi_1}{\searrow} & \\ Z_2 & \overset{\varphi_2}{\rightarrow} & X \\ & \underset{\varphi_3}{\nearrow} & \\ Z_3 & & \end{array} \qquad \hat{E}2 \quad \begin{array}{ccccc} Z_2 & & & & \\ & \overset{\varphi_2}{\searrow} & & & \\ & & Z_1 & & \\ & & & \overset{\varphi_1}{\searrow} & \\ & & & & X \\ & & & \underset{\varphi_3}{\nearrow} & \\ & & Z_3 & & \end{array}$$

$$\hat{E}3 \quad \begin{array}{ccc} & & Z_1 \\ & \overset{\varphi_1}{\nearrow} & \\ X & \overset{\varphi_2}{\rightarrow} & Z_2 \\ & \underset{\varphi_3}{\searrow} & \\ & & Z_3 \end{array} \qquad \hat{E}4 \quad \begin{array}{ccccc} & & & & Z_2 \\ & & & \overset{\varphi_2}{\nearrow} & \\ & & Z_1 & & \\ & \overset{\varphi_1}{\nearrow} & & & \\ X & & & & \\ & \underset{\varphi_3}{\searrow} & & & \\ & & Z_3 & & \end{array}$$

with $\| X \| = 2$, $\| Z_i \| = 1$, $Im|\varphi_i| = U \in P^1(X)$;
(resp. $ker|\varphi_i| = U \in P^1(X)$;
$Hom(|Z_i|, |Z_j|) = 0 \ \forall i, j$ in the cases Ê1 and Ê3 and
$Hom(|Z_2|, |Z_3|) = Hom(|Z_1|, |Z_3|) = Hom(|Z_3|, |Z_1|) = Hom(|Z_3|, |Z_2|) = 0$ in the cases Ê2 and Ê4.
If φ_i are irreducible morphisms, then by 5.3, we have that $q_{B[M]}$ is strongly indefinite. Also for the categories of type Ê1 and Ê2 we have the same result.

Before the next result, we need the following:

Definition 6.1 [MP] Let $\rho(\mathbb{K})$ be the subcategory of $\mathbb{K}$ defined by the objects $\{|X| \in \mathbb{K} \mid dim_k|Y| \leq 1 \, \forall Y$ such that there exists a path in C from Y to $X\}$.

Proposition 6.2 [MP] Let C be a directed, standard component, of tree type, with $M \in C$ and $\mathbb{K} = Hom(M, C) = Hom(M, B-mod)$, finite. Let X and Y be in C such that $|X|, |Y| \in \rho(\mathbb{K})$. Then there exists a path from X to Y in C if and only if $|X| \leq |Y|$ in $\rho(\mathbb{K})$, that is, $Hom_{\rho(\mathbb{K})}(|X|, |Y|) \neq 0$.

Remark This proposition was proved for C a pre injective component of tree type, but it remains valid in the present context.

Proposition 6.3 Let $\mathbb{K}$ be as above and let us suppose that $\mathbb{K}$ contains, as a full subcategory a category of type $\hat{\text{E}}1$, Then $\mathbb{K}$ contains as a full subcategory one of the categories of the list A^*.

Proof: The proof is by induction on the sum of the lengths of the paths $\varphi_1, \varphi_2, \varphi_3$ (called $n(\varphi)$). By 5.3, the result follows for $n(\varphi) = 3$. By the same lemma, we assume that $\varphi_1, \varphi_2, \varphi_3$ factor through the same direct predecessor of X, E (which, by 3.7, has dimension 1). Let us suppose it true for $n(\varphi) < n$ and let $\{|X|, |Z_1|, |Z_2|, |Z_3|\}$ be a category with $n(\varphi) = n$. Let us assume also that the length of φ_1 is greater than or equal to the length of φ_2 and φ_3. We claim that $Z_i \not\cong \tau X$ because, if $|\tau X| \neq 0$, then $dim\ Hom(|\tau X|, |X|) > 1$. Let T be a direct successor of Z_1 with $|T| \neq 0$. Dividing em cases:
a) $\| T \| = 2$
b) $\| T \| = 1$

Let us see a). If for some i, $dim\ Hom(|T|, |Z_i|) + dim\ Hom(|Z_i|, |T|) = 0$, the result follows from 3.7. So, there are three possibilities:
a$_1$) $f_i : Z_i \to T$ are irreducible morphisms for $i = 1, 2, 3$
a$_2$) $f_i : Z_i \to T$ are irreducible morphisms for $i = 1, 2$ and $g : T \to Z_3$
a$_3$) $f_1 : Z_1 \to T$ is irreducible and $g_i : T \to Z_i$ for $i = 2, 3$.

Let us see a$_1$. Then $\forall \psi : T \longrightarrow X$ with $|\psi| \neq 0$ and $\forall f_i : Z_i \longrightarrow T$, $Im|\psi f_i| = U$. If for all ψ, $Im(\psi) = U$ then $\{|T|, |X|\}$ is of type D. Let us suppose that there is $\overline{\psi} : T \longrightarrow X$ with $Im|\overline{\psi}| \neq 0$, such that $|\overline{\psi} f_i|$ is injective (and has image U) or is zero for all i. Then, $Im|f_i| \subset ker|\overline{\psi}|$, $\forall i$. Let us consider $V = ker|\overline{\psi}|$, then $Im|f_i| = V$ for all i, and the result follows from 5.3. The same happens for the case a$_2$. Let us see a$_3$. Let us consider $g_i : T \longrightarrow Z_i$ for $i = 2, 3$ $\psi_i : Z_i \longrightarrow X$. For any $\psi : T \longrightarrow X$, $|\psi f_1|$ has image U. If $Im|f_1| = V$ then $|\psi|(V) \subset U$. Moreover, $\varphi_2 : Z_2 \longrightarrow X$ is such that $Im|\varphi_2| = U$. Since $Hom(|Z_1|, |Z_2|) = 0$, $|g_2 f_1| = 0$ $\forall g_2 : T \longrightarrow Z_2$ so $Im|f_1| \subset ker|g_2|$ and the category $\{|T|, |Z_2|, |X|\}$ is a full subcategory of type $\hat{\text{D}}$.

Let us see b, that is, $\| T \| = 1$. In this case, $\forall \psi : T \longrightarrow X$, as $|f_1|$ is an isomorphism, then $Im|\psi| = Im|\psi f_1| = U$. If $Hom(|T|, |Z_i|) \neq 0$ for some $i = 2, 3$ then: $Z_1 \xrightarrow{f_1} T \xrightarrow{g_i} Z_i$, $|g_i f_1|$ is an isomorphism, a contradiction. If $Hom(|Z_2|, |T|) = 0$ and $Hom(|Z_3, |T|) = 0$, the full subcategory $\{|X|, |T|, |Z_2|, |Z_3|\}$ satisfies the induction hypothesis. If $Hom(|Z_i|, |T|) \neq 0$, there are two possibilities:
b$_1$) $f_i : Z_i \to T$ for $i = 1, 2, 3$ are irreducible morphisms or
b$_2$) $f_i : Z_i \to T$ for $i = 1, 2$ are irreducible and $Hom(|Z_3|, |T|) = 0$.

Let us see b$_1$. Since $\| Z_i \| = 1$, then $\| \tau T \| = 2$. But any morphism from τT to X factors through $Z_1 \oplus Z_2 \oplus Z_3$ so, $\{|\tau T|, |X|\}$ is a category of type D.

Let us see b$_2$. Let W be a direct successor of Z_3 with $|W| \neq 0$. If $\| W \| = 2$, and there is a path $W \longrightarrow Z_i \longrightarrow T \longrightarrow X$, then there exists a path $Z_3 \longrightarrow W \longrightarrow Z_i \longrightarrow T$ wich contradicts the maximality of the length of φ_1. By 3.7 we can assume then that $Hom(|Z_i|, |W|) \neq 0$. Hence, for all $\psi : W \longrightarrow X$, we have $Im|\psi f_i| = U$. If there exists $V \in P^1(W)$ such that $Im|f_i| = V$ for all i, the category $\{|W|, |Z_1|, |Z_2|, |Z_3|\}$ satisfies the induction hypothesis. If for some i and j with $i \neq j$, $Im|f_i| \neq Im|f_j|$, with $Im|\psi f_i| = Im|\psi f_j| = U$ for all ψ, then for all $\psi : W \longrightarrow X$, $Im|\psi| = U$, that is, $\{|W|, |X|\}$ is a category of type D. Let us suppose $\| W \|= 1$. Again if $Hom(|W|, |Z_i|) \neq 0$, then $Z_3 \longrightarrow W \longrightarrow Z_i$, a contradiction. On the other hand if $Hom(|Z_1|, |W|) = Hom(|Z_2|, |W|) = 0$, since, for any $\psi' : W \longrightarrow X$ with $|\psi'| \neq 0$, $Im|\psi'| = U$ the category $\{|X|, |Z_1|, |Z_2|, |W|\}$ satisfies the induction hypothesis. We show that there are two possibilities (*) and (**)

(*) If $Hom(|Z_1|, |W|) = 0$ and $Hom(|Z_2|, |W|) \neq 0$, then there exists a path: $Z_2 \longrightarrow A \longrightarrow W$ with $A \neq Z_3$. Let us observe that A can be equal to Z_2. If $\| A \|= 2$, $\{|A|, |Z_3|\}$ is path incomparable. If $\| A \|= 1$ and $Hom(|Z_1|, |A|) \neq 0$, this contradicts the assumption (*). Moreover, if there is $h : A \longrightarrow X$ such that $Im|h| \neq U$, since $Z_2 \xrightarrow{h_2} A \xrightarrow{h} X$, $|h_2|$ is an isomorphism and then $|hh_2| : |Z_2| \longrightarrow |X|$ is a morphism with image different from U, a contradiction. So if $A \neq Z_2$, the full category $\{|X|, |Z_1|, |A|, |Z_3|\}$ satisfies the induction hypothesis. Let us suppose that $A = Z_2$. If T or W have another direct predecessor with the same property then $\| \tau W \|= 2$ and all morphisms from τW to X have the same image, then the full subcategory $\{|\tau W|, |X|\}$ is of type D. Let us look at the other end of the path. Since $\| E \|= 1$, there exist other direct predecessors of X, and in this case $|B| \neq 0$ and X is not projective by 3.5. Dividing in cases:

1) $\| \tau X \| \neq 0$

2) $\| \tau X \| = 0$

Let us see 1)

As $\| \tau X \| \neq 0$, by 3.11 we can assume that $\| \tau X \|= 1$ and there exist B, C direct predecessors of X satisfying $Hom(|Z_i|, |B|) = Hom(|Z_i|, |C|) = 0$ (see 5.3). If $\{|Z_1|, |Z_2|, |Z_3|, |B|, |C|\}$ are path incomparable, the result follows from 3.3. Let us suppose then that for some i and for B or C there exists a zero path $Z_i \longrightarrow B$. This will lead us to a contradiction. Indeed, we have two paths from Z_i to X. So, none of this paths can be sectional and moreover, the non zero path cannot factor through τX because $dim\ Hom(|\tau X|, |X|) > 1$. So, there exists another predecessor of E and we have the following diagram:

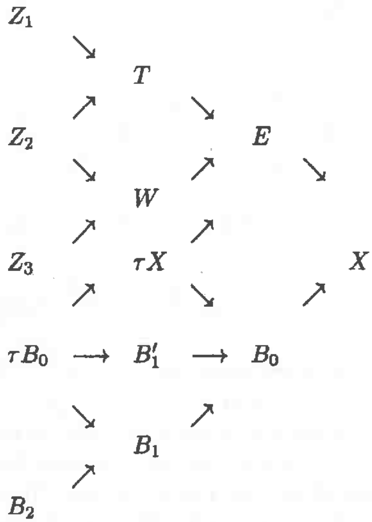
$$
\begin{array}{ccccccc}
Z_1 & & & & & & \\
 & \searrow & & & & & \\
 & & E_1 & & & & \\
 & \nearrow & & \searrow & & & \\
\tau E & & & & E & & \\
 & \searrow & & \nearrow & & \searrow & \\
 & & \tau X & \longrightarrow & B & \longrightarrow & X \\
 & & & \searrow & & \nearrow & \\
 & & & & C & &
\end{array}
$$

If $\| E_1 \|= 2$ and E_1 is indecomposable, then $\{|E_1|, |B|\}$ is path incomparable and the result follows from 3.3. If $\| E_1 \|= 1$, then $\| \tau E \|= 1$. If $Hom(|Z_i|, |\tau E|) \neq 0$, either one of the irreducible morphisms f between τE and B is such that $|f| = 0$ and the result follows from 3.13 or there is a non zero path between Z_i and B, a contradiction. Then, this path does not factor through τE and there exists E_2 direct predecessor of E_1 and a path $Z_1 \to E_2 \to E_1 \to E \to X$, this path is sectional and we begin again.

If $E_1 = E_1' \oplus E"_1$ then $\| \tau E \|= 2$ and by 3.7, $Hom(|Z_i|, |\tau E|) \neq 0$, so we have $Z_i \xrightarrow{f_1} \tau E \xrightarrow{\psi} X$. For all $\psi : \tau E \longrightarrow X$, $|\psi f_i|$ has image U, then either $Im|\psi| = U \ \forall \psi$, which cannot occur or there exists some $\overline{\psi} : \tau E \longrightarrow X$ such that $Im|\overline{\psi}| \neq U$. But since $|\overline{\psi} f_i|$ has image U, $Im|f_i|$ is the same for all i, that is, there exists $V \in P^1(\tau E)$ such that $Im|f_i| = V \ \forall i$ and we can use the induction hypothesis. Hence, either we have the result or we obtain one more step of a sectional path and that cannot continue indefinitely. If T and W are direct predecessors of E then $\| \tau E \|= 2$ and $\{|\tau E|, |Z_i|\}$ are path incomparable. Similarly if T and W are direct predecessors of E_i for some E_i in the sectional path, we see that $\| \tau E_{i-1} \|\neq 0$, and if $\tau E_{i-1} = W$, there is a non zero path from Z_2 to B, a contradiction. If $\tau E_{i-1} \neq W$, then $\| \tau E_i \|= 2$ and $\{|\tau E_i|, |Z_i|\}$ is path incomparable.

Let us see 2), that is $|\tau X| = 0$. In this case there exists a direct predecessor of B_0, say B_1, with $\| B_1 \|\neq 0$. If $\{|Z_1|, |Z_2|, |Z_3|, |B_1| \longrightarrow |B_0|\}$ is path incomparable, the result follows from 3.3. Let us suppose that there exists a zero path from Z_i to B_j.

If T and W are direct predecessors of E, we have:



and if $Z_2 = \tau E$, there exists the zero path $Z_2 \longrightarrow \tau X \longrightarrow B_0$. But B_0 is not projective, (there exists a path from M to τX). If $|\tau B_0| = 0$ there exists B_2 a direct predecessor of B_1, with $\| B_2 \| \neq 0$ and in this case, $\{|Z_1|, |Z_2|, |Z_2|, |B_2| \longrightarrow |B_1|\}$ is path incomparable. If $|\tau B_0| \neq 0$, we can assume that there exists B_1', with $\| B_1' \| = 1$ and again, $\{|Z_1|, |Z_2|, |Z_3|, |B_1'|, |B_1|\}$ is path incomparable. Let us suppose then that there exists a path $Z_i \longrightarrow B_j$. We have, again, two paths from Z_i to X, none of them sectional. But the path $Z_i \longrightarrow B_j$ is zero, and, in this case, by 6.2, either $|Z_i|$ or $|B_j|$ have some predecessor of dimension 2. But, if we had $Z_i \longrightarrow Y \longrightarrow B_j$, with $\| Y \| = 2$, the result follows by the induction hypothesis. So we assume that Y is a predecessor of both. Moreover, by 3.7, $Hom(|Y|, |Z_i|) \neq 0, \forall i$, $Hom(|Y|, |B_j|) \neq 0\ \forall j$ and then there exist two paths from Y to X, none of them sectional and a path $Y \longrightarrow B_t \longrightarrow B_{t-1} \longrightarrow \cdots B_i \longrightarrow \cdots B_1 \longrightarrow B_0 \longrightarrow X$ with $|B_j| \neq 0$, $\forall j$, non zero and non sectional from Y to X. We claim that there exists a j such that there exists a path from Z_i to B_j, but does not exist a path from Z_i to B_{j+1} so that the category: $\{|Z_1|, |Z_2|, |Z_3|, |B_{j+2}| \longrightarrow |B_{j+1}|\}$ is path incomparable. Let us observe that we cannot have a path from B_{j+1} to Z_i because there is a sectional path $B_{j+1} \longrightarrow B_j$. On the other hand, let us suppose that there exists a path $B_{j+2} \longrightarrow Z_i$. By the same argument above, the path $B_{j+2} \longrightarrow B_{j+1} \longrightarrow B_j$ cannot be sectional so that $B_{j+2} = \tau B_j$ for some j. Since $|B_{j+2}| \neq 0$, there exists B'_{j+1} with $|B'_{j+1}| \neq 0$ and

$$\begin{array}{ccccc} & & B'_{j+1} & & \\ & \nearrow & & \searrow & \\ \tau B_j = B_{j+2} & & & & B_j \\ & \searrow & & \nearrow & \\ & & B_{j+1} & & \end{array}$$

In this case, B_{j+1} cannot be Z_i. Moreover, there do not exist paths $Z_i \longrightarrow B'_{j+1}$ or $B'_{j+1} \longrightarrow Z_i$. In this case, let us consider the path incomparable category $\{|Z_1|, |Z_2|, |Z_3|, |B'_{j+1}|, |B_{j+1}|\}$. Let us suppose that $B_{j+2} \cong Y$. In this case,

$$\begin{array}{ccccc} & & Z_i & & \\ & \nearrow & & \searrow & \\ Y = B_{j+2} = \tau B_j & & & & B_j \\ & \searrow & & \nearrow & \\ & & B_{j+1} & & \end{array}$$

Since the path $Z_i \longrightarrow B_j$ is zero, if it was an irreducible morphism, the result follows from 3.13, if it is not an arrow, since the component is standard, we get a contradiction. So, it is enough to consider the case
(**) $Hom(|Z_1|, |W|) \neq 0$ and $Hom(|Z_2|, |W|) \neq 0$.

Again, this morphisms do not factor through Z_3, so that there exists another direct predecessor of W, say A, through wich this morphism factors. If $A = A_1 \oplus A_2$ with $Hom(|Z_1|, |A_1|) \neq 0$ and $Hom(|Z_2|, |A_2|) \neq 0$, then for all $h_i : A_i \longrightarrow X$ $Im|h_i| = U$ and, in this case, W is not projective, $\| \tau W \| = 2$ and $\{|\tau W|, |X|\}$ is a full subcategory of type D. So, we can assume that this morphisms from Z_1 and Z_2 to W factor through the same A (of dimension 1).

If $Hom(|T|, |A|) = 0$, since $Hom(|T|, |Z_3|) = 0$, the category $\{|X|, |T|, |A|, |Z_3|\}$ satisfies the induction hypothesis.

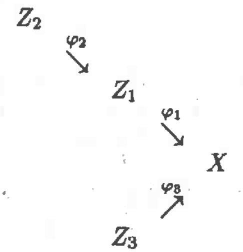
Suppose $Hom(|T|, |A|) \neq 0$.

If $T = A$, since W is not projective, there exists τW , $\tau W \neq Z_i$ $i = 1, 2$. Now, if $\| \tau W \| = 2$, then $\{|\tau W|, |Z_2|\}$ is a path incomparable category. If $\| \tau W \| = 1$, any morphism from τW to X factors through T or through Z_3, so any morphism from τW to X has image U. But, in this case, T is not projective, $\| \tau T \| = 2$ and again $\{|\tau T|, |X|\}$ is a full subcategory of type D.

Let us suppose $T \neq A$, then there exists a path $T \to A_1 \to \ldots \to A_n = A \to W$ with $Hom(|Z_1|, |\tau W|) = Hom(|Z_2|, |\tau W|) = Hom(|T|, |\tau W|) = 0$, so that $A_{n-1} \not\cong \tau W$. We can assume that $\| A_{n-1} \| = \| \tau W \| = 1$ so that A_n is not projective and again: $Hom(|Z_i|, |\tau A|) = Hom(|T|, |\tau A|) = 0$.

Hence $A_{n-2} \neq \tau A$, and again we assume that both have dimension 1. By iterating this we reach τA_1 and we have $Hom(|Z_i|, |\tau A_1|) = 0$ and so $Z_i \neq \tau A_1$, T is not projective and $\| \tau T \| = 2$. If there exists a morphism $\gamma : \tau A_1 \longrightarrow X$ with $Im|\gamma| \neq U$, since γ cannot factor through T, γ factors through τA_2,but this morphism factors through $A \oplus Z_3$, a contradiction. Then the category $\{|\tau T|, |X|\}$ is a full subcategory of type D.□

Proposition 6.4 Let $\mathbb{K}$ be as above and let us suppose that $\mathbb{K}$ contains, as a full subcategory, a category of the form:



with $\| X \| = 2$ and $\| Z_i \| = 1$ $\forall i$.
$Im\ |\varphi_1| = Im\ |\varphi_2| = U \in P^1(X)$
$Hom(|Z_2|,\ |Z_3|) = Hom(|Z_1|,\ |Z_3|) = Hom(|Z_3|,\ |Z_1|) =$
$Hom(|Z_3|,\ |Z_2|) = 0$ and $dim\ Hom(|Z_i|,\ |X|) = 1$. Then $\mathbb{K}$ contains, as a full subcategory, one of the categories (path incomparable) of the list A^*.

Proof: The proof is done by induction in the sum of the lengths of the paths φ_1 and φ_3. Let us see first that we can assume that φ_2 is an irreducible morphism. Let us suppose that the length of φ_2 is greater than 1 and let us consider $Z_2 \to T \to Z_1$ with T a direct predecessor of Z_1. Clearly, $|T| \neq 0$ and then, either

A) $\| T \| = 2$ or

B) $\| T \| = 1$. Let us consider this case B, that is, $\| T \| = 1$. If $Hom(|T|, |Z_3|) \neq 0$, then $Z_2 \to T \to Z_3$ is an isomorphism, a contradiction. If, on the other hand, $Hom(|Z_3|, |T|) \neq 0$, then $Z_3 \to T \to Z_1$ is an isomorphism, a contradiction. Now, let $h : T \to X$ be a morphism such that $|h| \neq 0$ and let us suppose that h is such that $Im|h| \neq U$ then $|h\ \varphi_2'|$ where $Z_2 \xrightarrow{\varphi_2'} T \xrightarrow{h} X$, has image different from U, a contradiction. So the category $\{|X|, |Z_3|, |T| \to |Z_1|\}$ satisfies the hypothesis.

Let us suppose now the case $\| T \| = 2$.

If $Hom(|T|, |Z_3|) + Hom(|Z_3|, |T|) = 0$, the result follows by 3.7. Dividing again in cases

A1) $Hom(|T|, |Z_3|) \neq 0$

A2) $Hom(|Z_3|, |T|) \neq 0$.

Let us consider A2. Assuming that $Hom(|Z_3|, |T|) \neq 0$, let us consider $f_3 : Z_3 \to T$, $f_2 : Z_2 \to T$, $f_1 : T \to Z_1$, $\varphi_1 : Z_1 \to X$. Since $Hom(|Z_3|, |Z_1|) = 0$, then $Im\ f_3 = ker\ f_1$ but $Im\ f_2 \neq Im\ f_3$. Let us assume that $V_1 = Im\ f_3$, $V_2 = Im\ f_2$ and $|T| = V_1 \oplus V_2$, so that for all $\psi : T \to X$ we have that $Im\ \psi \subset U$, and $\{|T|, |X|\}$ is a category of type D. Let us consider now the case A1, that is, $Hom(|T|, |Z_3|) \neq 0$, with $g : T \to Z_3$ and $\varphi_3 : Z_3 \to X$. If $\varphi : T \to X$ has $|\varphi| \neq 0$ then $|\varphi\ f_2|$

has image U. Let V be the image of f_2, then any $\varphi : T \to X$ is such that $|\psi|(V) \subset U$, and for all $g : T \to Z_3$, $ker|g| = Im|f_2| = V$. Since $Im|\varphi_3| = U$ then $\{|T| |Z_3| |X|\}$ is a category of type $\hat{\mathrm{D}}$. So, we can take φ_2 to be an irreducible morphism. Let us denote by $n(\varphi)$ the sum of the lengths of the paths φ_1 and φ_3. For $n(\varphi) = 2$, the result is given by 5.3. Moreover, we can assume that there exists E, a direct predecessor of X, with $\| E \| = 1$ and that there are φ_1 and φ_3 factoring through E. Let us suppose the statement true for $n(\varphi) \leq n$ and let $\{|X|, |Z_2| \to |Z_1|, |Z_3|\}$ be a subcategory with $n(\varphi) = n + 1$. We know that $\tau X \not\leq Z_i$, $\forall_i$. Let T be a direct predecessor of Z_1. Dividing in cases:
a) $\| T \| = 2$
b) $\| T \| = 1$

Let us assume $\| T \| = 2$. If $Hom(|T|, |Z_3|) + Hom(|Z_3|, |T|) = 0$, the result follows from 3.7. In fact, we have one of the following
a1) $Hom(|T|, |Z_3|) \neq 0$
a2) $Hom(|Z_3|, |T|) \neq 0$

Let us consider a1) $Hom(|T|, |Z_3|) \neq 0$ and let us take again $f_1 : Z_1 \to T$, $f_2 : T \to Z_3$. For any $\psi : T \to X$ with $|\psi| \neq 0$, $Im|\psi f_1| = U$, and $|f_2 f_1| = 0$, so $ker|f_2| = Im|f_1| = U_1 \in P^1(T)$ and $|\psi|(U_1) \subset U$ for any $\psi : T \to X$. Moreover, for all $\varphi : Z_3 \to X$, with $|\varphi| \neq 0$, then $Im|\varphi| = U$ for all $f : T \to Z_3$, with $|f| \neq 0$ and we have $ker|f| = U_1$ because $|f f_1| = 0$. Then the full subcategory $\{|T|, |Z_3|, |X|\}$ is a category of type $\hat{\mathrm{D}}$.
a2) $Hom(|Z_3|, |T|) \neq 0$. If $Im|f_1| = Im|f_3|$ then the category $\{|T|, |Z_2| \to |Z_1|, |Z_3|\}$ satisfies the induction hypothesis. Suppose $Im|f_1| \neq Im|f_3|$. In this case, for all $\psi : T \to X$, $Im|\psi\, f_i| = U$, so that for all $\psi : T \to X$, with $|\psi| \neq 0$ then $Im|\psi| = U$ and the full subcategory $\{|T|, |X|\}$ is a category of type D.
b) Let us suppose $\| T \| = 1$. For all $h : T \to X$ with $|h| \neq 0$, then $Im|h\, f_1| = U$ and since $|f_1|$ is an isomorphism, $Im|h| = U$. If $Hom(|T|, |Z_3|) \neq 0$, with $Z_1 \xrightarrow{f_1} T \xrightarrow{g} Z_3$ then $|g\ f_1|$ is an isomorphism, a contradiction. On the other hand, if $Hom(|Z_3|, |T|) = 0$, then the full subcategory $\{|X|,\ |Z_1| \to |T|,\ |Z_3|\}$ satisfies the induction hypothesis. Let us suppose that $Hom(|Z_3|, |T|) \neq 0$. Since $Hom(|Z_3|, |Z_1|) = 0$, there exists another direct predecessor of T, (say B) such that $|B| \neq 0$ and $Z_3 \to B \to T$. If $Z_1 \not\leq B$, and if $\| B \| = 2$ is indecomposable, then $\{|B|, |Z_1|\}$ is a category path incomparable. If $\| B \| = 1$, by the same arguments above, for any $h : B \to X$ with $|h| \neq 0$, $Im|h| = U$. Moreover, $Hom(|B|, |Z_1|) = Hom(|Z_1|, |B|) = 0$. So, the full subcategory $\{|X|, |Z_3| \longrightarrow |B|, |Z_1|\}$ satisfies the induction hypothesis. Let us see what happens when $Z_3 = B$. Then T is not projective and $|\tau T| \neq 0$. Let us suppose that there exists W, $|W| \neq 0$ with $W \neq Z_i$, a direct summand of the middle term of the ARS: $\tau T \longrightarrow Z_1 \oplus Z_3 \oplus W \longrightarrow T$. Then, $\| \tau T \| = 2$ and $\{|\tau T|, |Z_2|\}$ is a category path incomparable. Hence, we can assume that there does not exist such a W, $\| \tau T \| = 1$ and that there is at least another direct predecessor of X, say A, with $\| A \| = 1$, $Im(A \to X) \neq U$ and X is not

projective. There are two possibilities:
b1) $\| \tau X \| = 1$
b2) $\| \tau X \| = 0$

Let us consider b1.

If $\{|Z_2| \to |Z_1|, |Z_3|, |E_2|, |E_3|\}$ is a full subcategory path incomparable and the result follows (moreover if $T = E$, the above category is path incomparable). Let us suppose then that the category is not path incomparable.

Since there cannot exist a path $E_j \to Z_i \to X$, we can assume that there exists a zero path $Z_i \to E_j$, for $j \neq 1$. Then, there is a non zero path $Z_i \to E_1 \to X$ and a zero path $Z_i \to E_j \to X$ for $(j \neq 1)$, none of them sectional. In particular, the non zero path cannot factor through τX. Hence there exists a direct predecessor of E, (say A_1) such that $\| A_1 \| = 1$, and such that the non zero path $Z_i \to E_1$ factors through A_1: Again, E_1 is not projective and $|\tau E_1| \neq 0$. If $\| \tau E_1 \| = 2$, using the same arguments as in the case $\| T \| = 2$, we have the result.

If $\| \tau E_1 \| = 1$ and $Hom(|Z_i|, |\tau E_1|) \neq 0$, there exists an isomorphism $Z_i \longrightarrow \tau E_1 \longrightarrow E_2$ and so a morphism from $|Z_i|$ to $|X|$ with image different from U, a contradiction. Again, we have that there exists another direct predecessor of A_1, say A_2 with $\| A_2 \| = 1$, and $\| \tau A_1 \| = 1$. And if $Hom(|Z_i|, |\tau A_1|) \neq 0$, there is, again, a morphism from $|Z_i|$ to $|X|$, with image different from U and so on. Hence, the path $Z_i \to T \to E_1$ is sectional and the category $\{Z_2 \to Z_1, Z_3, E_2, E_3\}$ is path incomparable.

Now, let us consider b2, $|\tau X| = 0$. Since $\| X \| = 2$, there are at least two different paths from M to X, none of them sectional. Then the path $M \longrightarrow E_2 \longrightarrow X$ is non zero, non sectional and does not factor through τX. We can assume that there is a non zero and non sectional path

$$\begin{array}{ccccccccc} & & & A'_j & & & & & \\ & & \nearrow & & \searrow & & & & \\ M \longrightarrow & A_{j+1} & & & & A_{j-1} & \longrightarrow & E_2 & \to X \\ & & \searrow & & \nearrow & & & & \\ & & & A''_j & & & & & \end{array}$$

Let us consider j such that the paths $A'_j \longrightarrow E_2 \to X$ and $A''_j \longrightarrow E_2 \to X$ are sectional. If $\| A'_j \|$ or $\| A''_j \| = 2$, the result follows from 3.7. Let us suppose then that $\| A'_j \| = \| A''_j \| = 1$. If the category $\{|Z_2| \to |Z_1|, |Z_3|, |A'_j|, |A''_j|\}$ is path incomparable, we are finished.

Since the path $A'_j \longrightarrow E_2 \to X$ is sectional, we cannot have a path $A'_j \longrightarrow Z_i \to X$. Let us suppose then that there exists a zero path $Z_i \longrightarrow A'_j \to X$. By 6.2, there exists an object $|Y|$ such that $\| Y \| = 2$, and Y is a predecessor either of Z_i or of A'_j. But if there exists $Z_i \longrightarrow Y \longrightarrow A'_j$, again, repeating the arguments of the case $\| T \| = 2$, the result follows. So, we can assume that Y is a predecessor of both. Moreover, if $Hom(|Y|, |Z_i|) = 0$ or

$Hom(|Y|, |A_i|) = 0$, the result follows from 3.7. Let us consider the path

$$\begin{array}{ccccccc} & & & A'_j & & & \\ & & \nearrow & & \searrow & & \\ Y \longrightarrow & A_{j+1} & & & & A_j & \longrightarrow E_2 \\ & & \searrow & & \nearrow & & \\ & & & A''_j & & & \end{array}$$

Let t be a minimal index such that the path $Y \longrightarrow A_t$ is sectional (clearly $t \geq j$). Then we cannot have a path $Y \longrightarrow Z_i \longrightarrow A_t$. If $\{|Z_2| \to |Z_1|, |Z_3|, |A_t|, |A'_t|\}$ is path incomparable then the result follows. Let us suppose that there exists a path $A_t \longrightarrow Z_i$, for some i. Then we have two paths: $A_t \longrightarrow Z_i \longrightarrow A_j$ a zero path and a non zero path $A_t \longrightarrow A_j$. Let us consider now t^* an index such that there exists a path $A_{t^*} \longrightarrow Z_i$, but there does not exist a path from $A_{t^*+1} \longrightarrow Z_i$ and similarly, let j^* be such that there exists a path $Z_i \longrightarrow A_{j^*}$, but there does not exist a path from Z_i to A_{j^*+1}. Since the component is directed and of tree type, $t^* > j^*$ and we have two paths, none of them are sectional. Hence, in the non zero path there is at least one mesh:

$$\begin{array}{ccccccc} & & & B_1 & & & \\ & & \nearrow & & \searrow & & \\ A_{t^*} & \longrightarrow & \cdot & & & \longrightarrow & A_{j^*} \\ & & \searrow & & \nearrow & & \\ & & & B_2 & & & \end{array}$$

and in this case, the category $\{|Z_2| \to |Z_1|, |Z_3|, |B_1|, |B_2|\}$ is path incomparable.□

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