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Top-Degree Global Solvability in CR and Locally Integrable Hypocomplex Structures

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Abstract
We study the top-degree cohomology for the $\bar{\partial}_b$ operator defined on a generic submanifold of the complex space as well as for the differential complex associated with a locally integrable structure $\mathcal{V}$ over a smooth manifold. The main assumptions are that $\mathcal{V}$ is hypocomplex and that the differential complex is locally solvable in degree one. One of the main tools is an adaptation of a sheaf theoretical argument due to Ramis–Ruget–Verdier.

Keywords Tangential Cauchy-Riemann complexes · Hypocomplex manifolds · Top-degree cohomology

Mathematics Subject Classification 35A01 · 35G05 · 32V05 and 32V40

1 Introduction

In this work, we recall classical sheaf theoretical arguments due to Cartan–Serre [10], Andreotti–Grauert [2], and Ramis–Ruget–Verdier [17] in order to study the top-degree cohomology of the $\bar{\partial}_b$ differential complex on a generic submanifold of the complex space.

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Let $\mathcal{M}$ be a smooth, oriented, connected, and generic submanifold of the complex space $\mathbb{C}^{n+d}$ of dimension $2n+d$ and CR dimension n. If U is an open subset of $\mathcal{M}$, we consider the $\bar{\partial}_b$ complex on U:

$$\bar{\partial}_b : \mathcal{C}^\infty(U, \Lambda_b^{p,q}) \to \mathcal{C}^\infty(U, \Lambda_b^{p,q+1}),$$

$0 \le p \le n+d$, $0 \le q \le n-1$, the associated cohomology spaces $H^{p,q}_{\mathcal{C}^\infty}(U; \bar{\partial}_b)$ as well as the sheaf cohomologies with stalks $\mathcal{H}^{p,q}_{\mathcal{C}^\infty}(\{z_0\}; \bar{\partial}_b)$, $z_0 \in \mathcal{M}$. Of course in these definitions, we can replace $\mathcal{C}^\infty$ by any distributional space. Our main goal in the present work is to derive sufficient conditions to ensure that, for any $0 \le p \le n+d$ the Fréchet spaces homomorphisms

$$\bar{\partial}_b : \mathcal{C}^\infty(\mathcal{M}, \Lambda_b^{p,n-1}) \to \mathcal{C}^\infty(\mathcal{M}, \Lambda_b^{p,n}) \tag{1}$$

have closed image. By the homomorphism theorem for Fréchet spaces, this property is equivalent to the (strong sequentially) closedness of the image of the maps

$$\bar{\partial}_b : \mathcal{E}'(\mathcal{M}, \Lambda_b^{p,0}) \to \mathcal{E}'(\mathcal{M}, \Lambda_b^{p,1}). \tag{2}$$

Now, to motivate our results, denote by $\mathcal{S}^{(p)}$ the sheaf of germs of $(p, 0)$-forms with distribution coefficients that are annihilated by the $\bar{\partial}_b$-operator. From Godement [14, Theorem 4.6.1], we have the existence of a spectral sequence for compactly supported cohomologies

$$E_2^{r,s} = H_c^r(\mathcal{M}, \mathcal{H}^{p,s}_{\mathcal{D}'}(\bar{\partial}_b)) \Rightarrow H^{p,q}_{\mathcal{E}'}(\mathcal{M}, \bar{\partial}_b)$$

which leads to an exact sequence[1]

$$0 \longrightarrow H_c^1(\Omega, \mathcal{S}^{(p)}) \longrightarrow H^{p,1}_{\mathcal{E}'}(\mathcal{M}, \bar{\partial}_b) \longrightarrow H_c^0(\Omega, \mathcal{H}^{p,1}_{\mathcal{D}'}(\bar{\partial}_b)).$$

Since our goal is to show $H^{p,1}_{\mathcal{E}'}(\mathcal{M}, \bar{\partial}_b)$ is a Hausdorff space, we can achieve it if we assume $\mathcal{H}^{p,1}_{\mathcal{D}'}(\bar{\partial}_b) = 0$ and show that $H_c^1(\Omega, \mathcal{S}^{(p)})$ is endowed with some "natural" Hausdorff topology.

This simple remark is the starting point of our approach. We first make the assumption that $\mathcal{H}^{p,1}_{\mathcal{D}'}(\bar{\partial}_b) = 0$ which, according to Proposition 4.1, is equivalent to $\mathcal{H}^{p,1}_{\mathcal{C}^\infty}(\bar{\partial}_b) = 0$. Next, in order to achieve the second requirement, we assume that every CR-function in $\mathcal{M}$ is the restriction to $\mathcal{M}$ of a holomorphic function defined in some open subset of $\mathbb{C}^{n+d}$. When this property holds, we have $\mathcal{S}^{(p)} \subset \mathcal{C}^\infty(\Lambda_b^{p,0})$ and we can prove the following crucial property: if $V \subset\subset U$ are open subsets of $\mathcal{M}$ then the homomorphism $\mathcal{S}^{(p)}(U) \to \mathcal{S}^{(p)}(V)$ induced by restriction is *compact* for the natural Fréchet topologies. Thanks to this property and the techniques we learned from

[1] This is a consequence of the well known "five-term exact sequence" associated to a first quadrant spectral sequence.

Ramis–Ruget–Verdier we are able to complete this program and show the following result:

Theorem 1.1 *Let $\mathcal{M}$ be a smooth, oriented, connected, and generic submanifold of $\mathbb{C}^{n+d}$ of CR dimension n. Introduce the following conditions:*

(a) $\mathcal{H}^{0,1}_{\mathcal{C}^\infty}(\bar\partial_b) = 0$*;*
(b) Every CR-function in $\mathcal{M}$ is the restriction to $\mathcal{M}$ of a holomorphic function defined in some open subset of $\mathbb{C}^{n+d}$.

Then (a) and (b) imply that (1) *has closed image. If moreover $\mathcal{M}$ is compact and real analytic then (b) implies that*

$$\bar\partial_b : \mathrm{B}(\mathcal{M}, \Lambda_b^{p,n-1}) \to \mathrm{B}(\mathcal{M}, \Lambda_b^{p,n})$$

has closed image. Here B *denotes the sheaf of hyperfunctions on $\mathcal{M}$.*

Notice that the last statement is not surprising since $\bar\partial_b$ is always locally solvable in the spaces of real-analytic forms.

A particular consequence of Theorem 1.1 is the following:

Corollary 1.2 *If $\mathcal{M}$ is a hypersurface (that is $d = 1$) whose Levi form is nondegenerate at each point and has at least two positive and two negative eigenvalues then* (1) *has closed image.*

Indeed in this case conditions (a) and (b) above hold (cf. [3,4] and Theorem 2.3).

For the exposition and proofs we prefer though to work in a more general set up: that of locally integrable structures, for which a natural differential complex is always associated. Condition (b) in Theorem 1.1 is then replaced by the structure being *hypocomplex* (cf. Sect. 4). We observe that this more general set up allows us to derive results for more general systems of complex vector fields as, for instance, for smooth hypoelliptic vector fields in the plane.

Finally we mention that for the Ramis–Ruget–Verdier argument we dedicate a full section (Sect. 5) where we work in a still more general set up. We believe that doing so such techniques could be applied to other situations.

2 Involutive Structures and Their Associated Differential Complexes

In this work, we shall let Ω denote a smooth, paracompact, oriented, and connected differentiable manifold of dimension $N \geq 2$. An involutive (or formally integrable structure) over Ω is the datum of a smooth subbundle $\mathcal{V}$ (of rank $n \geq 1$) of $\mathbb{C}T\Omega$ satisfying the Frobenius condition: $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$.

We denote by Λ^k the bundle $\Lambda^k\mathbb{C}T^*\Omega$ and by $\mathcal{C}^\infty(\Omega; \Lambda^k)$ the space of sections of Λ^k with smooth coefficients. There is a natural differential complex associated to each involutive structure $\mathcal{V}$. We briefly recall the construction of such complexes. For each $x \in \Omega$, we define

$$T'_x = \{u \in \mathbb{C}T^*_x\Omega : u(X) = 0,\ \forall X \in \mathcal{V}_x\}$$

and we have that $T' = \bigcup_{x\in\Omega} T'_x$ is a vector subbundle of $\mathbb{C}T^*\Omega$. For each $p, q \in \mathbb{Z}_+$, we denote by $T_x^{p,q}$ the subspace of Λ_x^{p+q} consisting of linear combinations of exterior products $u_1 \wedge \ldots \wedge u_{p+q}$ with $u_j \in \mathbb{C}T_x^*\Omega$ for $j = 1, \ldots, p+q$ and at least p of these factors belonging to T'_x. Notice that $T_x^{p+1,q-1} \subset T_x^{p,q}$ and so we can define $\Lambda_x^{p,q} \doteq T_x^{p,q}/T_x^{p+1,q-1}$ and $\Lambda^{p,q} = \bigcup_{x\in\Omega} \Lambda_x^{p,q}$ is a smooth vector bundle over Ω. Since $\mathcal{V}$ is involutive, we can easily verify that the exterior derivative take smooth sections of $T^{p,q}$ into smooth sections of $T^{p,q+1}$ and hence there exists a unique operator $\mathrm{d}'_{p,q}$ such that the diagram

$$\begin{array}{ccc} T^{p,q} & \xrightarrow{\ \mathrm{d}\ } & T^{p,q+1} \\ {\scriptstyle \pi_{p,q}}\downarrow & & \downarrow{\scriptstyle \pi_{p,q+1}} \\ \Lambda^{p,q} & \xrightarrow[\ \mathrm{d}'_{p,q}\ ]{} & \Lambda^{p,q+1} \end{array}$$

is commutative, with $\pi_{p,q} : T^{p,q} \to \Lambda^{p,q}$ being the quotient map.

When there is no risk of confusion, we will simplify the notation and write only d' for the operator $\mathrm{d}'_{p,q}$, or $\mathrm{d}'_{\mathcal{V}}$ when it is necessary to emphasize the associated involutive structure.

We denote by $\mathcal{C}^\infty(\Omega; \Lambda^{p,q})$ the space of sections of $\Lambda^{p,q}$ with smooth coefficients. Notice that the operator d' maps smooth sections of $\Lambda^{p,q}$ into smooth sections of $\Lambda^{p,q+1}$ and it also holds that $\mathrm{d}' \circ \mathrm{d}' = 0$. Therefore, for each $p \geq 0$, the operator d' defines a complex of $\mathbb{C}$-linear mappings

$$\mathcal{C}^\infty(\Omega; \Lambda^{p,0}) \xrightarrow{\mathrm{d}'} \mathcal{C}^\infty(\Omega; \Lambda^{p,1}) \xrightarrow{\mathrm{d}'} \ldots \xrightarrow{\mathrm{d}'} \mathcal{C}^\infty(\Omega; \Lambda^{p,q}) \xrightarrow{\mathrm{d}'} \mathcal{C}^\infty(\Omega; \Lambda^{p,q+1}) \xrightarrow{\mathrm{d}'} \ldots \tag{3}$$

Of course, we obtain another differential complex after replacing $\mathcal{C}^\infty$ by $\mathcal{C}_c^\infty$, where c stands for compact supports.

For each $p \geq 0$, we denote the set of the (p, q)-cocycles elements by

$$Z^{p,q}(\Omega; \mathcal{V}) = \ker\left(\mathrm{d}' : \mathcal{C}^\infty(\Omega; \Lambda^{p,q}) \to \mathcal{C}^\infty(\Omega; \Lambda^{p,q+1})\right),$$

the set of (p, q)-coboundaries by

$$B^{p,q}(\Omega; \mathcal{V}) = \mathrm{range}\left(\mathrm{d}' : \mathcal{C}^\infty(\Omega; \Lambda^{p,q-1}) \to \mathcal{C}^\infty(\Omega; \Lambda^{p,q})\right),$$

and the (p, q)-cohomology classes by

$$H^{p,q}(\Omega; \mathcal{V}) = \frac{Z^{p,q}(\Omega; \mathcal{V})}{B^{p,q}(\Omega; \mathcal{V})}$$

(we set $B^{p,0}(\Omega) = \{0\}$).

Given any open set $V \subset \Omega$, we can restrict the involutive structure $\mathcal{V}$ to V, which we still denote by $\mathcal{V}$, and consequently obtain the complexes $\big(\mathcal{C}^\infty(V; \Lambda^{p,q}), \mathrm{d}'\big)$ as well as homomorphisms of differential complexes

$$\big(\mathcal{C}^\infty(\Omega; \Lambda^{p,q}), \mathrm{d}'\big) \to \big(\mathcal{C}^\infty(V; \Lambda^{p,q}), \mathrm{d}'\big).$$

This then allows us to define, for a given $x \in \Omega$,

$$\mathcal{H}^{p,q}(\{x\}; \mathcal{V}) = \varinjlim_{V \ni x} H^{p,q}(V; \mathcal{V})$$

the inductive limit of $H^{p,q}(V; \mathcal{V})$ taken when V varies in the set of all open neighborhoods of x.

Example 2.1 By taking $\mathcal{V} = \mathbb{C}T\Omega$, we have the simplest involutive structure. In this case, the operator d' is just the usual exterior derivative and the space $H^p_{\mathcal{C}^\infty}(\Omega; \mathcal{V})$ is just the de Rham cohomology space.

Example 2.2 If $\mathcal{V} \oplus \overline{\mathcal{V}} = \mathbb{C}T\Omega$ then $\mathcal{V}$ defines a complex structure on Ω and the operator d' is nothing else than the Doubeault operator $\overline{\partial}$.

Example 2.3 If $\mathcal{V} \cap \overline{\mathcal{V}} = 0$ then $\mathcal{V}$ defines an abstract CR structure on Ω and in this case the operator d' is the operator $\overline{\partial}_b$.

It is also important to consider the preceding complexes with more general coefficients and here the orientability of Ω will play an important role. Under this hypothesis, we can also construct a complex similar to (3) where now the coefficients are distributions over V; in this case, we denote the cohomology space by $H^{p,q}_{\mathcal{D}'}(V; \mathcal{V})$. In general, if $F(V)$ is a subspace of $\mathcal{D}'(V)$, we denote by $F(\Omega; \Lambda^{p,q})$ the space of sections of $\Lambda^{p,q}$ with coefficients in F and we denote the cohomology of d' with coefficients in $F(V)$ by $H^{p,q}_F(V; \mathcal{V})$. Furthermore by [20, Propositions VIII.1.2 and VIII.1.3], there is a natural bracket which turns the spaces

$$\mathcal{C}^\infty_c(V; \Lambda^{p,q}) \text{ and } \mathcal{D}'(V; \Lambda^{m-p,n-q}) \text{ (resp. } \mathcal{C}^\infty(V; \Lambda^{p,q}) \text{ and } \mathcal{E}'(V; \Lambda^{m-p,n-q}))$$

into the dual of one another and in such a way that the transpose of d' is also d' (up to a sign).

We can endow each space $\mathcal{C}^\infty(V; \Lambda^{p,q})$ with a locally convex structure of an Fréchet–Schwartz space ((FS)) space for short). Its dual $\mathcal{E}'(V; \Lambda^{m-p,n-q})$ is then dual of Fréchet–Schwartz space ((DFS)) space for short) and a sequence of definition for its topology can be taken by the sequence

$$\mathrm{G}_j(V; \Lambda^{m-p,m-q}) := \{u \in H^{-j}_{\mathrm{loc}}(V, \Lambda^{m-p,n-q}) : \operatorname{supp} u \subset K_j\}$$

where $\{K_j\}$ is an exhaustion of V by compact sets (the inclusions

$$\mathrm{G}_j(V; \Lambda^{m-p,m-q}) \hookrightarrow \mathrm{G}_{j+1}(V; \Lambda^{m-p,m-q})$$

being compact by the Relich lemma).

We introduce the following definition:

Definition 2.4 We shall say that $\mathcal{V}$ satisfies property $(\star)$ if given $V \subset \Omega$ open and $u \in \mathcal{D}'(V; \Lambda^{p,0})$ satisfying $\mathrm{d}' u = 0$ and vanishing in a nonempty connected open subset ω of V then u vanishes identically in the component of V that contains ω.

Proposition 2.5 *Assume that Ω is not compact and that $\mathcal{V}$ satisfies $(\star)$. Then*

1. *The kernel of the map $\mathrm{d}' : \mathcal{E}'(\Omega; \Lambda^{n-p,0}) \to \mathcal{E}'(\Omega; \Lambda^{n-p,1})$ is trivial;*
2. *If $K \subset \Omega$ is compact and if $u \in \mathcal{E}'(\Omega; \Lambda^{n-p,0})$ is such that supp $\mathrm{d}' u \subset K$ then supp $u \subset \tilde{K}$, where $\tilde{K}$ is the compact set obtained as the union of K with the relatively compact components of $\Omega \setminus K$.*

The proof is immediate.

3 Hypocomplex Structures

From now onwards, we will assume the stronger property that the structure $\mathcal{V}$ is *locally integrable*. This means that in a neighborhood of an arbitrary point $\mathfrak{p} \in \Omega$, there are defined $m = N - n$ smooth functions whose differentials span T' at each point in a neighborhood of $\mathfrak{p}$. Notice that each of these functions is annihilated by the operator d', that is, they are solutions for $\mathcal{V}$.

Let $\mathfrak{p} \in \Omega$. We denote by $\mathcal{S}(\mathfrak{p})$ the ring of germs of weak solutions of $\mathcal{V}$ at $\mathfrak{p}$, that is

$$\mathcal{S}(\mathfrak{p}) = \{f \in \mathcal{D}'(\mathfrak{p}) : \mathrm{d}' f = 0\}$$

and we denote by $\mathcal{O}_0^m$ the ring of germs holomorphic functions defined in a neighborhood of 0 in $\mathbb{C}^m$. Now consider $Z_1, \dots, Z_m$ solutions for $\mathcal{V}$ defined in an open neighborhood of $\mathfrak{p}$ with linearly independent differentials and denote by Z the map $Z = (Z_1, \dots, Z_m)$ defined in a neighborhood of $\mathfrak{p}$. Of course, we can assume that $Z(\mathfrak{p}) = 0$.

We introduce the ring homomorphism

$$\lambda : h \in \mathcal{O}_0^m \mapsto h \circ Z \in \mathcal{S}(\mathfrak{p}).$$

It is not difficult to prove that λ is injective. We shall say that $\mathcal{V}$ is *hypocomplex at* $\mathfrak{p}$ if λ is surjective, that is, if for every $f \in \mathcal{S}(\mathfrak{p})$, there exists a holomorphic function $h \in \mathcal{O}_0^m$ such that $f = h \circ Z$. Finally, we say that $\mathcal{V}$ defines a *hypocomplex struture in* Ω or that $\mathcal{V}$ *is hypocomplex* if $\mathcal{V}$ is hypocomplex at each point of Ω.

When $\mathcal{V}$ is hypocomplex then for each $U \subset \Omega$ open, for each $0 \leq p \leq m$ and each $s \in \mathbb{R}$, we have an inclusion

$$\{u \in H^s_{\mathrm{loc}}(U; \Lambda^{p,0}) : \mathrm{d}' u = 0\} \hookrightarrow \{u \in \mathcal{C}^\infty(U; \Lambda^{p,0}) : \mathrm{d}' u = 0\}. \tag{4}$$

An elementary application of the closed graph theorem gives:

Proposition 3.1 *If $\mathcal{V}$ is hypocomplex, then (4) is an isomorphism of (FS) spaces .*

The next result follows directly from the definitions:

Proposition 3.2 *If $\mathcal{V}$ is hypocomplex, then $\mathcal{V}$ satisfies the property* ($\star$).

Of course, an important class of hypocomplex structures is that of complex structures: indeed every complex structure is locally integrable thanks to the Newlander-Nirenberg theorem and hypocomplexity is immediate. The same applies to the class of elliptic structures. Recall that the *characteristic set of* $\mathcal{V}$ is the set $T^0 = T' \cap T^*\Omega$ and that the structure is *elliptic* if $T^0 = 0$. In this case, hypocomplexity follows from [20, Proposition III.5.1].

Another important class of hypocomplex structures is the one described in what follows. Assume that Ω is endowed with a locally integrable structure $\mathcal{V}$. Given $(\mathfrak{p}, \xi) \in T^0$, $\xi \neq 0$, the *Levi form* at $(\mathfrak{p}, \xi) \in T^0$ is the Hermitian form on $\mathcal{V}_x$ defined by

$$\mathcal{L}_{(\mathfrak{p},\xi)}(v, w) = (1/2i)\xi([L, \overline{M}]_{\mathfrak{p}}), \quad v, w \in \mathcal{V}_{\mathfrak{p}}$$

in which L and M are any smooth local sections of $\mathcal{V}$ in a neighborhood of $\mathfrak{p}$ such that $L_{\mathfrak{p}} = v$ and $M_{\mathfrak{p}} = w$. We shall say that $\mathcal{L}$ is *nondegenerate* if given any point $(\mathfrak{p}, \xi) \in T^0$, with $\xi \neq 0$, the Hermitian form $\mathcal{L}_{(\mathfrak{p},\xi)}$ is nondegenerate.

The following result is due to Baouendi–Chang–Treves [6]:

Theorem 3.3 *Let Ω be endowed with a locally integrable structure for which the Levi form at each $(\mathfrak{p}, \xi) \in T^0$, $\xi \neq 0$, has one positive and one negative eigenvalue. Then $\mathcal{V}$ is hypocomplex.*

Assume now that the locally integrable structure $\mathcal{V}$ has rank $N - 1$, that is, T' is a complex line bundle. For such structures, hypocomplexity is completely characterized [20]:

Theorem 3.4 *If the locally integrable structure $\mathcal{V}$ has rank $N - 1$, then $\mathcal{V}$ is hypocomplex if and only if given any $\mathfrak{p} \in \Omega$ and given any local solution Z for $\mathcal{V}$ near $\mathfrak{p}$ with $\mathrm{d}\, Z_{\mathfrak{p}} \neq 0$ then Z is open at $\mathfrak{p}$.*

Example 3.5 Let Ω be an open subset of $\mathbb{R}^2$ and let X be a complex, nonvanishing smooth vector field in Ω. If X is $\mathcal{C}^\infty$-hypoelliptic then $\mathcal{V} := \mathrm{span}\{X\}$ is hypocomplex. Indeed since X satisfies the Nirenberg–Treves condition, (P) $\mathcal{V}$ is locally integrable and hypoellipticy implies that the first integrals of X are open (cf.[20, Section III.6]).

4 The Main Result

Let Ω be endowed with a locally integrable structure $\mathcal{V}$. We shall now make use of the local representation for $\mathcal{V}$ as described in [6,20]. Any point $\mathfrak{p} \in \Omega$ is the center of a coordinate system $(x, t) = (x_1, \ldots, x_m, t_1, \ldots, t_n)$ defined on an open neighborhood of p denoted by U such that $(x(\mathfrak{p}), t(\mathfrak{p})) = (0, 0) \in \mathbb{R}^m \times \mathbb{R}^n$. On U it is defined a smooth function $\phi(x,t) = (\phi_1(x,t), \dots, \phi_m(x,t))$ satisfying $\phi(0,0) = 0$ and $\phi_x(0,0) = 0$ such that the differentials of the functions

$$Z_k(x,t) = x_k + i\phi_k(x,t), \quad k = 1, \dots, m \tag{5}$$

are linearly independent and span T' over U. Contracting U if necessary, we can define the vector fields

$$M_k = \sum_{j=1}^{m} \mu_{kj}(x,t)\frac{\partial}{\partial x_j}, \quad k = 1, \dots, m,$$

characterized by the rule $M_k Z_j = \delta_{k,j}$ for $k, j = 1, \dots, m$, then the complex vector fields

$$L_j = \frac{\partial}{\partial t_j} - i\sum_{k=1}^{m} \frac{\partial \phi_k}{\partial t_j}(x,t)M_k, \quad j = 1, \dots, n,$$

span $\mathcal{V}$ over U. The vector fields L_j and M_k commute pairwise.

The differentials $\mathrm{d}t_k$, for $k = 1, \dots, n$, span a bundle over U which is supplementary to $T'|_U$. Therefore we can adjoin them with $\mathrm{d}Z_k$, for $k = 1, \dots, m$, to get a smooth basis for $\mathbb{C}T^*\Omega$. So if f is a smooth section of $\Lambda^{p,q}$ over U, we have the following unique representation for f

$$f = \sum_{|J|=p} \sum_{|K|=q} f_{J,K}(x,t)\mathrm{d}Z_J \wedge \mathrm{d}t_K.$$

The standard representation of $\mathrm{d}'f$ over U is then given by

$$\mathrm{d}'f = \sum_{|J|=p} \sum_{|K|=q} \sum_{l=1}^{n} \left(L_l f_{J,K}\right)(x,t)\mathrm{d}t_l \wedge \mathrm{d}Z_J \wedge \mathrm{d}t_K.$$

We now prove the following result:

Proposition 4.1 *Let Ω be endowed with a locally integrable structure $\mathcal{V}$ and let $\mathfrak{p} \in \Omega$. The following properties are equivalent, for a fixed $q \in \{1, \dots, n\}$:*

1. $\mathcal{H}^{0,q}_{\mathcal{C}^\infty}(\{\mathfrak{p}\}, \mathcal{V}) = 0$;
2. $\mathcal{H}^{0,q}_{\mathcal{D}'}(\{\mathfrak{p}\}, \mathcal{V}) = 0$;
3. $\mathcal{H}^{p,q}_{\mathcal{C}^\infty}(\{\mathfrak{p}\}, \mathcal{V}) = 0$ *for all* $p = 0, 1, 2, \dots, m$;
4. $\mathcal{H}^{p,q}_{\mathcal{D}'}(\{\mathfrak{p}\}, \mathcal{V}) = 0$ *for all* $p = 0, 1, 2, \dots, m$.

Proof From the discussion that precedes, it is clear that it suffices to prove the equivalence between (1) and (2). We will first prove that (1) implies (2) following in part an argument in [1]. We keep the notation just established and apply the well known Grothendieck argument as in [6], Theorem VII.6.1: given $U_0 \subset U$ an open neighborhood of the origin, there will correspond another such neighborhood $V_0 \Subset U_0$ such that if we set

$$\mathrm{E} := \{(f, u) \in Z^{0,q}(U_0) \times \mathcal{C}^\infty(V_0, \Lambda^{0,q-1}) : \mathrm{d}' u = f|_{V_0}\}$$

then the projection map $\pi : \mathrm{E} \to Z^{0,q}(U_0)$ (a continuous linear map between Fréchet spaces) is surjective and hence open. If we follow [1], we can ensure:

- Given $K' \subset V_0$ compact and ν' a positive integer, there are $K \subset U_0$ compact, ν a positive integer and $C > 0$ such that the following is true: given $f \in Z^{0,q}(U_0)$, there is $u \in \mathcal{C}^\infty(V_0, \Lambda^{0,q-1})$ such that $\mathrm{d}' u = f$ and

$$\|u\|_{\nu', K'} \leq C \|f\|_{\nu, K}.$$

We have written

$$\|f\|_{\nu, K} \doteq \sup_{|\alpha| \leq \nu} \sup_K |D^\alpha f|.$$

In order to conclude (2), we make use of [20, Theorem II.5.2]. It suffices to establish the following solvability property:

- Given an open neighborhood U_1 of the origin there is μ such that if $f \in \mathcal{C}^\mu(U_1; \Lambda^{0,q})$ satisfies $\mathrm{d}' f = 0$ we can find $u \in \mathcal{D}'(\{0\}; \Lambda^{0,q-1})$ solving $\mathrm{d}' u = f$.

Indeed, we first apply the Approximate Poincaré Lemma [20, Section II.6]: there is an open neighborhood $U_0 \Subset U_1$ of the origin such that the conclusion of [20, Theorem II.6.1] holds. Now we refer to the discussion presented before: taking a sufficiently small closed ball K_0 centered at the origin there are μ and $C > 0$ such that for every $g \in Z^{0,q}(\overline{U_0}; \Lambda^{0,q})$ there is $v \in \mathcal{C}^\infty(K_0, \Lambda^{0,q-1})$ such that $\mathrm{d}' v = g$ in K_0 and

$$\|v\|_{0, K_0} \leq C \|g\|_{\mu, \overline{U_0}}.$$

This defines our sought μ. Given then $f \in \mathcal{C}^\mu(U_1, \Lambda^{0,q})$ we can find $f_j \in Z^{0,q}(\overline{U_0})$ such that $f_j \to f$ in $\mathcal{C}^\mu(\overline{U_0}; \Lambda^{0,q})$. Let $u_j \in \mathcal{C}^\infty(K_0, \Lambda^{0,q-1})$ be such that $\mathrm{d}' u_j = f_j$ in K_0 and

$$\|u_j\|_{0, K_0} \leq C \|f_j\|_{\mu, \overline{U_0}}.$$

This shows that u_j is bounded in $L^\infty(K_0; \Lambda^{0,q-1})$ and hence a subsequence of u_j will converge weakly to some u defined in the interior of K_0 which a fortiori satisfies $\mathrm{d}' u = f$.

For the proof of (2) implies (1), we again start by applying the Grothendieck argument, this time in the following way. Let U_0 be a neighborhood of the origin, let $\omega_j \Subset U_0$ be a fundamental system of neighborhoods of the origin and set

$$\mathrm{E}_j \doteq \{(f, u) \times Z^{0,q}(U_0) \times H^{-j}_{\mathrm{loc}}(\omega_j) : \mathrm{d}' u = f|_{\omega_j}\}.$$

By the standard argument, we can $V_0 \subset U_0$ an open neighborhood of the origin and $\sigma \in \mathbb{R}$ such that the following is true:

- Given $f \in Z^{0,q}(U_0)$, there is $u \in H^{\sigma}_{\text{loc}}(V_0, \Lambda^{0,q-1})$ such that $\mathrm{d}' u = f|_{V_0}$.

The proof can be completed after a close look at the argument presented in [20, Theorem VIII.9.1]

Remark It follows from the preceding proof that the equivalent conditions in Proposition 4.1 are further equivalent to the following property: given $s \in \mathbb{R}$, there is $t \in \mathbb{R}$ such that if $U \subset \Omega$ is an open neighborhood of $\mathfrak{p}$ there is another open neighborhood $V \subset U$ of $\mathfrak{p}$ such that the following holds: if $f \in H^s_{\text{loc}}(U, \Lambda^{p,q})$ satisfies $\mathrm{d}' f = 0$, there is $u \in H^t_{\text{loc}}(V, \Lambda^{p,q-1})$ with $\mathrm{d}' u = f$. Notice that if we introduce the Fréchet spaces

$$\begin{aligned} \mathrm{F} &= \{g \in H^s_{\text{loc}}(U, \Lambda^{p,q}) : \mathrm{d}' g = 0\}, \\ \mathrm{E} &= \{(g, v) \in \mathrm{F} \times H^t_{\text{loc}}(V, \Lambda^{p,q-1}) : \mathrm{d}' v = g \text{ in} V\} \end{aligned}$$

then the solvability property just stated is equivalent to the surjectivity of the homomorphism of Fréchet spaces $\lambda : \mathrm{E} \to \mathrm{F}$ induced by the projection on the first factor.

The following can be considered our main result:

Theorem 4.2 *Let Ω be endowed with a hypocomplex structure $\mathcal{V}$. Assume that the equivalent conditions in Proposition 4.1 hold for $q = 1$ and for every point of Ω. Then*

$$\mathrm{d}' : \mathcal{E}'(\Omega; \Lambda^{p,0}) \to \mathcal{E}'(\Omega; \Lambda^{p,1}) \tag{6}$$

has strongly sequentially closed range.

Corollary 4.3 *If Ω is endowed with a hypocomplex structure $\mathcal{V}$ and if furthermore Ω is not compact, then the validity of the equivalent conditions in Proposition 4.1 for $q = 1$ at each point of Ω imply that*

$$H^{p,n}_{\mathcal{C}^\infty}(\Omega) = 0, \qquad p = 0, 1, 2, \ldots$$

Proof This property is equivalent to the surjectivity of the map

$$\mathrm{d}' : \mathcal{C}^\infty(\Omega; \Lambda^{p,n-1}) \to \mathcal{C}^\infty(\Omega; \Lambda^{p,n}).$$

Its transpose is the map

$$\mathrm{d}' : \mathcal{E}'(\Omega; \Lambda^{m-p,0}) \to \mathcal{E}'(\Omega; \Lambda^{m-p,1})$$

which is injective because $\mathcal{V}$ is hypocomplex and strongly sequentially closed thanks to Theorem 4.2. The result is then a consequence of the homomorphism theorem for Fréchet-Montel spaces [15].

Corollary 4.4 *Let Ω be compact and endowed with a hypocomplex structure $\mathcal{V}$. If the equivalent conditions in Proposition 4.1 hold at each point of Ω, then*

$$\mathrm{d}' : \mathcal{C}^\infty(\Omega; \Lambda^{p,n-1}) \to \mathcal{C}^\infty(\Omega; \Lambda^{p,n})$$

has closed image of finite codimension.

Proof The result is again a consequence of the homomorphism theorem for Fréchet-Montel spaces.

In the next section, we shall develop some topological tools which may have some interest per se and which will be used in the proof of Theorem 4.2

5 Sheaf cohomology

In this section, we will fix a locally compact, second countable, Hausdorff, connected topological space X, and a sheaf $\mathcal{F}$ of complex vector spaces over X. Our main hypotheses are the following:

(i) For each open set $U \subset X$, the space $\mathcal{F}(U)$ is endowed with a (FS) topology for which the homomorphisms induced by restrictions are continuous;
(ii) If V is an open, relatively compact subset of the open set U then the homomorphism $\mathcal{F}(U) \to \mathcal{F}(V)$ is compact;
(iii) If $U \subset X$ is connected and if $u \in \mathcal{F}(U)$ vanishes in a nonempty subset of U, then u vanishes identically.

For each $K \subset X$ compact, it follows that $\mathcal{F}(K)$ has a natural (DFS) topology and moreover if L is compact and contained in the interior of K, then $\mathcal{F}(K) \to \mathcal{F}(L)$ is also compact.

We now follow Godement [14]. Let $\mathfrak{K} = \{K_j\}_{j\in\mathbb{N}}$ be a countable and locally finite covering of X by compact sets. We shall consider the cohomology spaces $H^*_c(\mathfrak{K}; \mathcal{F})$ where c stands for the family of supports consisting of compact sets of X. Recall that such cohomology spaces are the cohomology spaces of the complex of cochains

$$\delta_q : C^q_c(\mathfrak{K}, \mathcal{F}) \longrightarrow C^{q+1}_c(\mathfrak{K}, \mathcal{F})$$

where $C^q_c(\mathfrak{K}, \mathcal{F})$ stands for the set of all *finite* q-cochains with values in $\mathcal{F}$, that is, for a given $s \in C^q_c(\mathfrak{K}, \mathcal{F})$ there is only a finite number of q-simplexes σ of $\mathfrak{K}$ such that s_σ is non-zero. Observe that each $C^q_c(\mathfrak{K}, \mathcal{F})$ is a closed subspace of a direct sum of (DFS) spaces and then it also a (DFS) space.

Notice also that we have $H^0_c(\mathfrak{K}, \mathcal{F}) = \mathcal{F}_c(X)$.

Let $A \subset \mathbb{N}$ be finite. We denote by $C^q_A(\mathfrak{K}; \mathcal{F})$ the set of all cochains $s \in C^q_c(\mathfrak{K}; \mathcal{F})$ such that $s_{j_0 j_1 \dots j_q} = 0$ if $\{j_0, j_1, \dots, j_q\} \cap A = \emptyset$.

Since $\delta_q(C^q_A(\mathfrak{K}; \mathcal{F})) \subset C^{q+1}_A(\mathfrak{K}; \mathcal{F})$, we have a well defined *complex of cochains with supports contained in* A. The corresponding cohomology spaces are denoted by $H^q_A(\mathfrak{K}; \mathcal{F})$ with $q = 0, 1, \dots$. We have natural homomorphisms

$$H^q_A(\mathfrak{K}; \mathcal{F}) \to H^q_B(\mathfrak{K}; \mathcal{F})$$

if $A \subset B$ and clearly we also have natural homomorphisms

$$H^q_A(\mathfrak{K}; \mathcal{F}) \to H^q_c(\mathfrak{K}; \mathcal{F}).$$

Proposition 5.1 *Under the above hypotheses for every finite subset A of $\mathbb{N}$, there is another finite subset B of $\mathbb{N}$ with the property that if $\sigma \in C^0_c(\mathfrak{K}, \mathcal{F})$ is such that $\delta_1\sigma \in C^1_A(\mathfrak{K}, \mathcal{F})$ then $\sigma \in C^0_B(\mathfrak{K}, \mathcal{F})$*

Proof The result is immediate if X is compact for our covering $\mathfrak{K}$ will a fortiori be finite. We then assume that X is not compact. Let $L \subset X$ be a compact such that $Y \doteq \Omega\backslash L$ is connected and $\bigcup_{j\in A} K_j \subset L$. We define $B = \{j \in \mathbb{N} : K_j \cap L \neq \emptyset\}$. Since $\mathfrak{K}$ is locally finite we have that B is finite. We also have, by the choice of L, that $A \subset B$.

Let $u \in C^0_c(\mathfrak{K}; \mathcal{F})$ be such that $s \doteq \delta_0 u \in C^1_A(\mathfrak{K}; \mathcal{F})$. We write $u = \{u_j\}$. Then have $u_j - u_k = s_{jk}$ as elements in $\mathcal{F}(K_j \cap K_k)$ if $K_j \cap K_k \neq \emptyset$. By definition

$$Y = \bigcup_{j\notin A}(Y \cap K_j).$$

This union defines a locally finite covering of Y by relatively closed subsets of Y. Moreover if $\{j, k\} \cap A = \emptyset$ we have $u_j = u_k$ in $Y \cap K_j \cap K_k$ and thus according to Godement (Theorem 5.2.2) there is $u \in \mathcal{F}(Y)$ such that $u = u_j$ in $Y \cap K_j$. Now since $u \in C^0_c(\mathfrak{K}; \mathcal{F})$ certainly there is $\ell \in \mathbb{N}$ such that $K_\ell \subset Y$ and $u_\ell = 0$ in K_ℓ. By property (iii) we conclude that $u = 0$ and hence $u_k = 0$ if $k \notin B$.

In the next result, we shall consider a countable and locally finite refinement $\mathfrak{L} = \{L_j\}$ of $\mathfrak{K}$ with each L_j compact. Recall (cf. Godement [14]) the existence of the refining homomorphisms

$$\rho_q : C^q_c(\mathfrak{K}, \mathcal{F}) \to C^q(\mathfrak{L}, \mathcal{F}), \quad q \geq 0.$$

Proposition 5.2 *The homomorphism ρ_1 induces an injective homomorphism*

$$\rho_1^* : H^1_c(\mathfrak{K}; \mathcal{F}) \to H^1_c(\mathfrak{L}; \mathcal{F}),$$

Proof Let $\gamma : \mathbb{N} \to \mathbb{N}$ be such that $L_j \subset K_{\gamma(j)}$ for every j. Let $s = \{s_{jk}\} \in C^1_c(\mathfrak{K}; \mathcal{F})$ satisfying $\delta_1 s = 0$ and suppose that $\rho_1 s = \delta^0 v$ for some $v = \{v_j\} \in C^0_c(\mathfrak{L}; \mathcal{F})$. Then $v_k - v_j = s_{\gamma(j)\gamma(k)}$ as elements of $\mathcal{F}(L_j \cap L_k)$ and since s is a cocycle we have that $s_{kr} - s_{jr} + s_{jk} = 0$ as elements of $\mathcal{F}(K_j \cap K_k \cap K_r)$ and consequently

$$v_k + s_{\gamma(k)r} = v_j + s_{\gamma(j)r}$$

as elements of $\mathcal{F}(L_j \cap L_k \cap K_r)$. Notice that the left-hand side does not depend on j and the right-hand side does not depend on k. By varying j and k, we see that there is $u_r \in \mathcal{F}(K_r)$ such that $u_r = v_j + s_{\gamma(j)r}$ as elements of $\mathcal{F}(L_j \cap K_r)$ whenever $K_j \cap K_r \neq \emptyset$.

Let $I, J \subset \mathbb{N}$ be such that $v_j = 0$ unless $j \in I$ and $s_{jk} = 0$ unless j and k belong to J. The set $I' \doteq J \cup \left\{r \in \mathbb{N} : \left(\bigcup_{j\in I} L_j\right) \cap K_r \neq \emptyset\right\}$ is finite and $u_r = 0$ if $r \notin I'$. Indeed, if $r \notin I'$ then $r \notin J$ and if j is such that $L_j \cap K_r \neq \emptyset$ then $j \notin I$. These two facts imply $u_r = 0$ in $K_r \cap L_j$. Thus $u \doteq \{u_j\}$ belongs to $C^0_c(\mathfrak{K}; \mathcal{F})$. Finally, as elements of $\mathcal{F}(L_j \cap K_r \cap K_t)$, we have

$$u_r - u_t = (v_j + s_{\gamma(j)r}) - (v_j + s_{\gamma(j)t}) = s_{\gamma(j)r} - s_{\gamma(j)t} = -s_{tr}.$$

And the proof is completed.

From now onwards, we assume that $\mathfrak{K}$ and $\mathfrak{L}$ satisfy the following property:

$$L_j \subset \text{int}(K_j) \text{ for all } j \in \mathbb{N}. \tag{7}$$

The construction of such $\mathfrak{L}$ and $\mathfrak{K}$ is very easy to attain: we introduce a metric in X and define

$$K_j = \{p \in \Omega : d(p, L_j) \leq \epsilon_j\}.$$

where $\epsilon_j > 0$ are chosen suitable small and d is any distance on X which defines its topology. Notice then that

(iv) If (j_0, j_1) (resp. (j_0, j_1, j_2)) is a 1-simplex for $\mathfrak{K}$ (resp. a 2-simplex for $\mathfrak{K}$) then it is also a 1-simplex for $\mathfrak{L}$ (resp.a 2-simplex for $\mathfrak{L}$).

Notice also that property (iv) above guarantees that the maps ρ_p are injective for $p = 0, 1, 2$. Moreover, if $A \subset \mathbb{N}$ is finite, then the homomorphisms

$$\rho_{q,A} : C^q_A(\mathfrak{K}; \mathcal{F}) \to C^q_A(\mathfrak{L}; \mathcal{F})$$

are compact operators, thanks to property (ii).

Theorem 5.3 *Assume that X is a locally compact, second countable, Hausdorff, connected topological space and that $\mathcal{F}$ is a sheaf of vector spaces over X satisfying properties (i), (ii) and (iii). Then the map $\delta_0 : C^0_c(\mathfrak{K}; \mathcal{F}) \to C^1_c(\mathfrak{K}; \mathcal{F})$ has closed range.*

Proof It suffices to show that range$(\delta^0) \cap C^1_A(\mathfrak{K}; \mathcal{F})$ is closed in $C^1_A(\mathfrak{K}; \mathcal{S}^p)$ for all finite subsets $A \subset \mathbb{N}$ (Banach–Dieudonné theorem).

Let $A \subset \mathbb{N}$ be finite and let B be as in Proposition 5.1. We define the following (DFS) spaces:

$$V \doteq \{(u, s) \in C^0_B(\mathfrak{K}; \mathcal{F}) \times C^1_A(\mathfrak{K}; \mathcal{F}) : \delta_0 u = s\},$$
$$W \doteq \{(u, s) \in C^0_B(\mathfrak{L}; \mathcal{F}) \times C^1_A(\mathfrak{K}; \mathcal{F}) : \delta_0 u = \rho_1 s\},$$

and let $\sigma : V \to W$ be the linear map given by $\sigma(u, s) = (\rho_{0,B} u, s)$.

We first prove that σ is a bijection. Indeed since δ^0 is injective, we have that σ is injective if $\sigma(u, s) = (0, 0)$ then $s = 0$ and since $(u, s) \in V$ we have $\delta^0 u = 0$. Now we are going to prove that σ is surjective. Let $(v, s) \in W$. We have that $\delta_1(\rho_1 s) = \delta_1(\delta_0 v) = 0$ and since $\rho_2(\delta_1 s) = \delta_1(\rho_1 s) = 0$ and ρ_2 is injective it follows that $\delta_1 s = 0$. Hence the class of s defines an element in $H^1_A(\mathfrak{K}; \mathcal{F})$ which belongs to the kernel of ρ_1^*. By Proposition 5.2, the class of s vanishes in $H^1_c(\mathfrak{K}; \mathcal{F})$ and so Proposition 5.1 shows that $s = \delta_0 u$ with $u \in C^0_B(\mathfrak{K}; \mathcal{F})$. In particular $(u, s) \in V$. Finally, we see that $\sigma(u, s) = (v, s)$ for $\delta_0(v - \rho_{0,B} u) = 0$ and δ^0 is injective.

We denote by E the (DFS) space $C^0_B(\mathfrak{L}; \mathcal{F}) \times C^1_A(\mathfrak{K}; \mathcal{F})$; we remark that W is a closed subspace of E and thus also a (DFS) space. Let ι be the inclusion map $W \subset E$. It follows that $\iota \circ \sigma : V \to E$ is a topological embedding.

We introduce the following continuous linear maps

$$\tau_1(u, s) = (\rho_{0,B}, 0), \qquad \tau_2(u, s) = (0, s).$$

Since τ_1 is compact and $\tau_2 = \iota \circ \sigma - \tau_1$, it follows from a classical result due to L. Schwartz [18] that τ_2 has closed range in E. This is the same as saying that the map $p_2 : V \to C^1_A(\mathfrak{K}; \mathcal{F})$ given by the projection on the second factor has closed range, but according to Proposition 5.1, we have

$$\text{range } p_2 = \text{range } (\delta_0) \cap C^1_A(\mathfrak{K}; \mathcal{F}),$$

which completes the proof.

6 Proof of Theorem 3.2

For $U \subset \Omega$ open, we set

$$\mathcal{S}^p_\infty(U) = \{u \in \mathcal{C}^\infty(U; \Lambda^{p,0}) : \mathrm{d}' u = 0\}.$$

Notice that the sheaf $\mathcal{F}(U) := \mathcal{S}^p_\infty(U)$ satisfies hypotheses (i), (ii), and (iii) of the previous section. Indeed, (i) is immediate, (iii) is a consequence of Proposition 3.2 whereas (ii) follows from [20, Corollary III.5.5]. We shall apply these properties in the argument that follows.

Let $\{u_\nu\} \subset \mathcal{E}'(\Omega; \Lambda^{p,0})$ be such that $\mathrm{d}' u_\nu \to f$ strongly in $\mathcal{E}'(\Omega; \Lambda^{p,1})$. It follows that there are $K \subset \Omega$ compact and $s \in \mathbb{R}$ such that

$$\text{supp } \mathrm{d}' u_\nu \subset K, \ \mathrm{d}' u_\nu \to f \text{ in} H^s(K; \Lambda^{p,0}).$$

By the remark that follows Proposition 3.1 and noticing that the homomorphism λ is sequentially invertible (cf. [15, p. 18]), we can assert the following: given $\mathfrak{p} \in K$ there are $t \in \mathbb{R}$, $V_\mathfrak{p} \subset \Omega$ an open neighborhood of $\mathfrak{p}$ and a Cauchy sequence $\{v_\nu\} \subset H^t_{\text{loc}}(V_\mathfrak{p}; \Lambda^{p,0})$ such that $\mathrm{d}' v_\nu = \mathrm{d}' u_\nu$ for every $\nu \geq 1$.

As a consequence, the compactness of K implies that we can construct a locally finite covering $\mathcal{K} = \{K_j\}_{j \in \mathbb{N}}$ by compact sets such that:

- $K \cap K_j = \emptyset$ if $j > \ell$;
- There are $U_j \supset K_j$ open, $t \in \mathbb{R}$ and $v_\nu^{(j)} \in H^t_{\text{loc}}(U_j, \Lambda^{p,0})$ such that $\mathrm{d}' v_\nu^{(j)} = \mathrm{d}' u_\nu$ in U_j and $v_\nu^{(j)} \to V_j$ in $H^t_{\text{loc}}(U_j, \Lambda^{p,0})$ (we set $U_j^{(\nu)} = 0$ if $j > \ell$).

Since $\mathrm{d}' V_j = f$ we have

$$\mathfrak{s} := \{(V_j - V_k)|_{K_j \cap K_k}\} \in C^1_c(\mathcal{K}, \mathcal{S}^p_\infty).$$

Furthermore, we have, by Proposition 5.2,

$$\delta_0 \left(\{(v_\nu^{(j)} - u_\nu)|_{K_j}\} \right) \to \mathfrak{s}.$$

By Theorem 5.3, the map δ_0 has closed image and hence there is $\{h_j\} \in C^0_c(\mathcal{K}, \mathcal{S}^p_\infty)$ such that $h_j - h_k = V_j - V_k$ in $K_j \cap K_k$. Hence if define $u \in \mathcal{E}'(\Omega, \Lambda^{p,0})$ as being equal to $V_j - h_j$ near K_j then $\mathrm{d}' u = v$ and the proof is complete.

7 The real-analytic case

We assume now that Ω is a compact, real-analytic, orientable, and connected manifold. In such a situation, $\mathcal{C}^\omega(\Omega)$ can be endowed with a natural topology of (DFS) space. This topology $\mathfrak{T}$ can be described as follows: if we embed $\Omega \subset \mathcal{N}$ as a maximally real submanifold of a Stein manifold $\mathcal{N}$, then $(\mathcal{C}^\omega(\Omega), \mathfrak{T})$ is the inductive limit of a sequence $\mathcal{O}_\infty(\mathcal{U}_\nu)$, where $\mathcal{U}_\nu \searrow \Omega$ is a fundamental system of neighborhoods of Ω in $\mathcal{N}$ and the symbol ∞ stands for bounded holomorphic functions.

If we denote by B the sheaf of hyperfunctions on Ω then $\mathrm{B}(\Omega)$, the space of its global sections, can be identified with the topological dual of $(\mathcal{C}^\omega(\Omega), \mathfrak{T})$ or else with the space of all analytic functionals on $\mathcal{N}$ carried by Ω.

Let $\mathcal{V}$ be a real-analytic, locally integrable structure over Ω. We can assume that $\mathcal{V}$ and consequently all $\Lambda^{p,q}$ can be extended as holomorphic bundles on $\mathcal{M}$. Then (see, for instance [5]) the property below holds:

(†) Given an open neighborhood $\mathcal{U}$ of Ω in $\mathcal{N}$, the following holds: for every $\mathfrak{p} \in \Omega$, there is an open neighborhood $\mathcal{V}_\mathfrak{p} \subset \mathcal{U}$ of $\mathfrak{p}$ and a constant $C > 0$ such that if $G \in \mathcal{O}_\infty(\mathcal{U}, \Lambda^{p,1})$ satisfies $\mathrm{d}' G = 0$ there is $H \in \mathcal{O}_\infty(U_\mathfrak{p}, \Lambda^{p,0})$ such that $\mathrm{d}' H = G$ and $\|H\|_{U_\mathfrak{p}} \leq C \|G\|_{\mathcal{U}}$.

Here, the norms are defined by taking sup-norms of the coefficients of the forms.

Proposition 7.1 *Let Ω be as described above and assume that $\mathcal{V}$ is a real-analytic, hypocomplex structure over Ω. Then for any $0 \leq p \leq m$, the map*

$$\mathrm{d}' : \mathcal{O}(\Omega; \Lambda^{p,0}) \to \mathcal{O}(\Omega; \Lambda^{p,1})$$

has strongly sequentially closed image.

Proof The proof follows exactly the same lines of that of Theorem 3.2. Let $\{u_\nu\} \subset \mathcal{C}^\omega(\Omega; \Lambda^{p,0})$ be such that $\mathrm{d}' u_\nu \to f$ strongly in $\mathcal{C}^\omega(\Omega; \Lambda^{p,1})$. Then there is an open neighborhood $\mathcal{U}$ of Ω in $\mathcal{M}$ such that each $\mathrm{d}' u_\nu$ extends to an element in $\mathcal{O}_\infty(\mathcal{U}, \Lambda^{p,1})$ and $\mathrm{d}' u_\nu \to f$ in $\mathcal{O}_\infty(\mathcal{U}, \Lambda^{p,1})$. But then taking into account (†), we can assert:

- There is a finite covering $K_1, \ldots K_\ell$ of Ω by compact sets, open sets $K_j \subset U_j \subset \Omega$ and forms $v_\nu^{(j)} \in \mathcal{C}^\omega(U_j, \Lambda^{p,0})$ such that $\mathrm{d}' v_\nu^{(j)} = \mathrm{d}' u_\nu$ and $v_\nu^{(j)} \to V_j$ in $\mathcal{C}^\infty(U_j, \Lambda^{p,0})$.

Proceeding as in the proof of Theorem 4.2, we obtain $u \in \mathcal{C}^\infty(\Omega, \Lambda^{p,0})$ such that $\mathrm{d}' u = f$ in Ω. Since $\mathcal{V}$ is hypocomplex by [20, Proposition III.5.3], it follows that $u \in \mathcal{C}^\omega(\Omega, \Lambda^{p,0})$ and this completes the proof.

Corollary 7.2 *Let Ω be compact, real-analytic, and endowed with a real-analytic hypocomplex structure $\mathcal{V}$. Then*

$$\mathrm{d}' : \mathrm{B}(\Omega; \Lambda^{p,n-1}) \to \mathrm{B}(\Omega; \Lambda^{p,n})$$

has closed image of finite codimension.

Under the hypothesis of Corollary 7.2, we in particular obtain local solvability in top degree and in the hyperfunction sense in hypocomplex structure, a result obtained in [12].

8 Consequences of the main theorem

In this section, we present some consequences of Theorem 4.2. In all that follows, we have examples of an involutive structure $\mathcal{V}$ for which the operator $\mathrm{d}' : \mathcal{C}^\infty(\Omega; \Lambda^{p,n-1}) \to \mathcal{C}^\infty(\Omega; \Lambda^{p,n})$ has closed range and, when Ω is not compact, is surjective.

7.1. $\mathcal{V}$ is an elliptic structure. As pointed out right after the statement of Proposition 3.2, $\mathcal{V}$ is hypocomplex and local solvability in every degree follows from [7,20]. Notice that the class of complex structures is contained in the class of elliptic structures.

7.2. $\mathcal{V}$ is the CR structure over a generic submanifold of the complex space satisfying conditions (a) and (b) in Theorem 1.1. Indeed, it is clear that condition (b) is equivalent to the hypocomplexity of $\mathcal{V}$. Notice for instance that this property can be achieved under certain conditions imposed on the extrinsic Levi form of $\mathcal{M}$ (cf. [8, Theorem 2, p.202]).

7.3. Ω and $\mathcal{V}$ are real analytic, the Levi form form is nondegenerate at each characteristic point it has at least two positive and two negative eigenvalues. The hypocomplexity of $\mathcal{V}$ follows from Theorem 3.3 and the local solvability in degree one follows from the fact that the associated box operator has an analytic parametrix in degree one [19].

7.4. $\mathcal{V}$ has rank $N-1$, is hypocomplex (Theorem 3.4) and satisfies Treves condition $(\star)_0$ at each point (cf. [11] and the references therein). This latter condition is equivalent to the local solvability of the associated differential complex in degree one (cf. [13], [16]).

7.5. $\mathcal{V}$ is the line bundle spanned by a smooth, hypoelliptic vector field X on an open subset Ω of the plane (cf Example 3.5). In particular, we obtain the surjectivity of the map $X : \mathcal{C}^\infty(\Omega) \to \mathcal{C}^\infty(\Omega)$, a result previously obtained in [9, Teorema 19, página 126] by using the Riemann surface structure on Ω defined by X.

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