All-creation and all-annihilation time-dependent $\mathcal{PT}$-symmetric bosonic Hamiltonians: An infinite squeezing degree at a finite time

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Here we introduce the all-creation and all-annihilation time-dependent (TD) $\mathcal{PT}$-symmetric bosonic Hamiltonians, which in the interaction picture are described only by creation or annihilation operators. These Hamiltonians are defined from the most general TD $\mathcal{PT}$-symmetric quadratic bosonic Hamiltonian, describing a cavity mode under linear and parametric amplifications. After presenting a general ansatz for the derivation of the TD Dyson map and metric operators, we solve the Schrödinger equations for both the $\mathcal{PT}$-symmetric Hamiltonian and its Hermitian counterpart. We then compute analytically the squeezing degree coming from the all-annihilation Hamiltonian and compare the result with that coming from an ordinary Hermitian Hamiltonian, showing a crucial result for interferometric procedures: instead of the asymptotic divergence of the squeezing degree that takes place for a Hermitian parametric pumping, the all-annihilation amplification leads to a divergence of that quantity at a finite controllable time.

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I. INTRODUCTION

Non-Hermitian quantum mechanics has been acquiring increasing visibility since the work by Bender and Boettcher [1], advancing that an autonomous Hamiltonian which is invariant under parity-time ($\mathcal{PT}$) transformation, i.e., $[H, \mathcal{PT}] = 0$, can exhibit an entirely real spectrum. When the Hamiltonian, in addition to being invariant, shares its eigenstates with the $\mathcal{PT}$ operator, it is said that it preserves the $\mathcal{PT}$ symmetry, which is broken when H and $\mathcal{PT}$ stop sharing the same eigenstates [2]. Shortly thereafter Bender and Boettcher's work, Mostafazadeh [3], starting from the reality of the spectrum, turned to the one remaining problem associated with a non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian: a probabilistic interpretation of the associated quantum theory. Such an indefiniteness of the unitarity of time evolution was overcome by constructing a Hilbert space based on a metric operator where the non-Hermitian Hamiltonian enjoys self-adjointness.

Attention has also been focused on the approach of nonautonomous non-Hermitian Hamiltonians. It has been shown in Ref. [4] that a pseudo-Hermiticity relation for a time-dependent (TD) non-Hermitian Hamiltonian demands a time-independent metric operator. As concluded in Ref. [4], for TD metric operators we cannot simultaneously have a unitary time evolution and an observable Hamiltonian; by ensuring the unitarity of time evolution we lose the observability of the Hamiltonian, which can nevertheless be forced at the expense of losing unitarity. By giving up the observability of the TD non-Hermitian Hamiltonian, in Ref. [5] it is shown that a pseudo-Hermiticity relation can be defined to all other observables of the system, thus rescuing the applicability of nonautonomous non-Hermitian systems. Going even further, in Ref. [6] a Schrödinger-like equation, governed by the non-Hermitian Hamiltonian itself, is derived to steer the evolution of a particular TD Dyson map, ensuring both the observability of the Hamiltonian and the unitarity of the evolution. The TD Dyson map derived in [6] is such that it still ensures the time independence of the metric operator, as required in Ref. [4].

Many interesting results have been recently derived from the study of $\mathcal{PT}$-symmetric systems. Among others, we mention that synthetic $\mathcal{PT}$-symmetric photonic lattices have been designed for light transport [7] and the realization of Bloch oscillations [8]; $\mathcal{PT}$-symmetry breaking has been observed in complex optical potentials [9] and coupled optical resonators [10], and it has been verified that this phase transition allows for $\mathcal{PT}$-symmetric lasers [11]. In this work we introduce the all-creation and all-annihilation nonautonomous $\mathcal{PT}$-symmetric Hamiltonians, which are composed, in the interaction picture, only by creation or annihilation bosonic operators. We are here taking a step further than that of Swanson [12] when proposing his Hamiltonian whose extensive study [13] was crucial for consolidating the method proposed in Ref. [3]. Whereas Swanson's Hamiltonian, given by $H = \omega(a^\dagger a + 1/2) + \alpha a^2 + \beta (a^\dagger)^2$, with $\omega, \alpha, \beta \in \mathbb{R}$ and $\alpha \neq \beta$ ($\hbar = 1$ from here on), describes a harmonic oscillator subjected to an unbalanced (non-Hermitian) parametric amplification (i.e., an unbalanced interplay between creation and annihilation bosonic operators), we are here addressing Hamiltonians where such imbalance is made extreme. These Hamiltonians are defined from the most general TD $\mathcal{PT}$-symmetric quadratic bosonic Hamiltonian, describing a cavity mode under linear and parametric amplifications.

Great attention was devoted to quadratic bosonic Hamiltonians during the 1990s, focusing on the generation of the squeezed states of the radiation field [14]. First introduced by Kennard [15], back in 1927, when studying the evolution of a generic Gaussian wave packet, the squeezed states experienced a remarkable resurgence in the late 1980s partly due to the development of techniques that allowed both the proposition of many distinct schemes for generating squeezed states [16] and the first experimental realizations of such propositions [17]. Among many interesting realizations for improving interferometry sensitivity via squeezed states, we mention the pioneering work by Caves [18] of recent great interest regarding the LIGO and Virgo [19] collaborations on the detection of gravitational waves.

Considering that the TD treatment of the Swanson Hamiltonian under a TD metric operator has been anticipated in Ref. [20], here we deal with a more general version of the nonautonomous Hamiltonian addressed in that reference, considering also the linear amplification process apart from the parametric one. However, for many reasons the present work is far from being a direct extension of the treatment presented in Ref. [20]. Apart from the fact that the introduction of linear amplification allows a more general definition of what we call all-creation and all-annihilation Hamiltonians, *(i)* we first mention that the addition of the linear process requires a more general Dyson map than the one used in Ref. [20]. And here we anticipate a method for the general construction of Dyson maps and thus metric operators. After presenting the general treatment for the more general TD $\mathcal{PT}$-symmetric quadratic Hamiltonian, we however disregard this linear process which has no effect on our next investigation: *(ii)* a comparison of the degrees of squeezing for the solutions of the Schrödinger equations governed by non-Hermitian and Hermitian Hamiltonians. Moreover, *(iii)* in Ref. [20] we simply present the mathematical expressions from which we can derive the solutions of the Schrödinger equation and then other relevant physical quantities associated with the non-Hermitian Hamiltonian. Here we go further, presenting analytical manipulation of the mathematical expressions to effectively compute the required physical quantities. *(iv)* We then introduce the all-creation and all-annihilation Hamiltonians to compare the degree of squeezing they generate with that obtained from the ordinary Hermitian Hamiltonian. We find that while the squeezing degree coming from the ordinary Hermitian Hamiltonians presents the well-known time asymptotic divergence, for the all-annihilation interaction (which presents easier mathematical treatment than the all-creation case), this divergence may occur in a controllable finite time. This result has important implications for the high performance interferometry, a crucial requirement for the recently inaugurated gravitational wave interferometers.

II. ALL-CREATION AND ALL-ANNIHILATION HAMILTONIANS

We start by defining the general quadratic TD non-Hermitian Hamiltonian ($\hbar = 1$)

$$H(t) = \omega(t)(a^\dagger a + 1/2) + \alpha(t)a^2 + \beta(t)a^{\dagger 2} + \gamma(t)a + \delta(t)a^\dagger, \tag{1}$$

where a and $a^\dagger$ are bosonic annihilation and creation operators with all TD parameters being, in principle, complex functions with $\beta(t) \neq \alpha^*(t)$ and $\delta(t) \neq \gamma^*(t)$. This Hamiltonian, which generalizes that treated in [20], becomes $\mathcal{PT}$ symmetric when demanding $\mathcal{PT}H(t)(\mathcal{PT})^{-1} = H(t) = H(-t)$, such that $\omega^*(t) = \omega(-t)$, $\alpha^*(t) = \alpha(-t)$, $\beta^*(t) = \beta(-t)$, $\gamma^*(t) = -\gamma(-t)$, and $\delta^*(t) = -\delta(-t)$. It enables us to define the all-creation

$$H_{ac}(t) = \omega(t)(a^\dagger a + 1/2) + \beta(t)a^{\dagger 2} + \delta(t)a^\dagger \tag{2}$$

and the all-annihilation

$$H_{aa}(t) = \omega(t)(a^\dagger a + 1/2) + \alpha(t)a^2 + \gamma(t)a \tag{3}$$

Hamiltonians which, in the interaction picture, are described only by creation and annihilation operators, respectively, making extreme the unbalanced interplay between creation or annihilation operators introduced by Swanson.

III. DYSON MAP

We solve the Schrödinger equation for $H(t)$ by considering for the Dyson map

$$\eta(t) = \eta_2(t)\eta_1(t) \neq \eta^\dagger(t) \neq \eta^{-1}(t), \tag{4}$$

where

$$\eta_1(t) = \exp[\zeta(t)a + \zeta^*(t)a^\dagger + \kappa(t)], \tag{5a}$$

$$\eta_2(t) = \exp[\epsilon(t)(a^\dagger a + 1/2) + \mu(t)a^2 + \mu^*(t)(a^\dagger)^2] \tag{5b}$$

$$= \exp[\lambda(t)K_+]\exp\{\ln[\lambda_0(t)]K_0\}\exp[\lambda^*(t)K_-], \tag{5c}$$

the last equality following from the Gauss decomposition for the SU(1, 1) algebra, where $K_0 = (a^\dagger a + 1/2)/2$, $K_+ = (a^\dagger)^2/2$, and $K_- = a^2/2$, with $[K_-, K_+] = 2K_0$ and $[K_0, K_\pm] = \pm K_\pm$. The TD parameters introduced by the Dyson map, with $\kappa(t)$, $\epsilon(t)$ [$\zeta(t)$, $\mu(t)$] being real [complex] functions, are evidently derived from the construction of the Hermitian counterpart of $H(t)$. Finally, omitting, from now on, the time dependence of the variables, in Eq. (5c) we have defined

$$\lambda = \mu^* \frac{2\sinh\Xi}{\Xi\cosh\Xi - \epsilon\sinh\Xi} \equiv \Phi\, e^{-i\varphi}, \tag{6a}$$

$$\lambda_0 = \frac{\Xi^2}{[\Xi\cosh\Xi - \epsilon\sinh\Xi]^2} \tag{6b}$$

apart from $\Xi = \sqrt{\epsilon^2 - 4|\mu|^2} = \epsilon\sqrt{1 - |z|^2}$, where we have also introduced, as in [21] and [20], the parameter $z = 2\mu/\epsilon = |z|e^{i\varphi}$, where $|z| \in [0, 1]$ with the amplitude Φ and the phase φ being real parameters. In Ref. [21] the original time-independent Swanson model is treated using a time-independent η_2.

The map operator defined in Eq. (4) does not ensure the observability of the non-Hermitian Hamiltonian while establishing, however, the unitarity of the evolution. Since what matters here is the computation of the squeezing factor and eventually the variances of the radiation field, we consider here the framework established in Ref. [5] instead of that in Ref. [6]. The use of the framework in Ref. [6] requires a more elaborate calculation of the Dyson map using the above-mentioned Schrödinger-like equation, which is unnecessary for our purposes. As already anticipated in the Introduction, although the framework established in Ref. [5] does not assure the observability of a time-dependent non-Hermitian Hamiltonian, it assures that a pseudo-Hermiticity relation can be defined to all other observables of the system, as for example the variances of the radiation field.

However, we anticipate here a rule for deriving the Dyson map for complex Hamiltonians encompassing different algebras: it thus follows that the construction of the Dyson map must take into account a product of exponentials each related to a given algebra; in the present case, the two algebras related to the amplification processes: the Heisenberg and the SU(1,1) algebras for the linear and parametric amplifications, respectively.

IV. FROM THE HERMITIAN COUNTERPART OF H TO THE TD PARAMETERS OF THE DYSON MAP

Next, by defining the real parameter

$$\chi = |\lambda|^2 - \lambda_0 = \lambda_0 \frac{4|\mu|^2 \sinh^2 \Xi - \Xi^2}{\Xi^2} = \frac{2\Phi}{|z|} - 1, \tag{7}$$

we derive the relation

$$\eta \begin{pmatrix} a \\ a^\dagger \end{pmatrix} \eta^{-1} = \frac{1}{\sqrt{\lambda_0}} \begin{pmatrix} 1 & -\lambda \\ \lambda^* & -\chi \end{pmatrix} \begin{pmatrix} a \\ a^\dagger \end{pmatrix} + \begin{pmatrix} -\zeta^* \\ \zeta \end{pmatrix}, \tag{8}$$

and consequently the Hermitian counterpart of $H(t)$, given by [5,20]

$$\begin{aligned} h &= \eta H \eta^{-1} + i\dot{\eta}\eta^{-1} \\ &= W\left(a^\dagger a + \frac{1}{2}\right) + U a^2 + V\left(a^\dagger\right)^2 + Ra + Ta^\dagger + F, \end{aligned} \tag{9}$$

where

$$W(z, \Phi, \zeta, \kappa) = -\frac{1}{\lambda_0}\left[\omega(\chi + |\lambda|^2) + 2(\alpha\lambda + \beta\chi\lambda^*) + i\left(\lambda\dot{\lambda}^* - \frac{\dot{\lambda}_0}{2}\right)\right], \tag{10a}$$

$$U(z, \Phi, \zeta, \kappa) = \frac{1}{\lambda_0}\left[\omega\lambda^* + \alpha + \beta(\lambda^*)^2 + \frac{i}{2}\dot{\lambda}^*\right], \tag{10b}$$

$$V(z, \Phi, \zeta, \kappa) = \frac{1}{\lambda_0}\left[\omega\chi\lambda + \alpha\lambda^2 + \beta\chi^2 + \frac{i}{2}(\lambda^2\dot{\lambda}^* + \lambda_0\dot{\lambda} - \lambda\dot{\lambda}_0)\right], \tag{10c}$$

$$R(z, \Phi, \zeta, \kappa) = \frac{1}{\sqrt{\lambda_0}}[\omega(\zeta - \zeta^*\lambda^*) - 2(\alpha\zeta^* - \beta\lambda^*\zeta) + \gamma + \delta\lambda^* + i(\lambda^*\dot{\zeta}^* + \dot{\zeta})], \tag{10d}$$

$$T(z, \Phi, \zeta, \kappa) = \frac{1}{\sqrt{\lambda_0}}[\omega(\zeta^*\chi - \zeta\lambda) + 2(\alpha\zeta^*\lambda - \beta\chi\zeta) - \gamma\lambda - \delta\chi - i(\chi\dot{\zeta}^* + \lambda\dot{\zeta})], \tag{10e}$$

$$F(z, \Phi, \zeta, \kappa) = -\omega|\zeta|^2 + \alpha(\zeta^*)^2 + \beta\zeta^2 - \gamma\zeta^* + \delta\zeta + \frac{i}{2}(\dot{\zeta}^*\zeta - \zeta^*\dot{\zeta}) + i\dot{\kappa}. \tag{10f}$$

Evidently, besides being a function of the metric TD functions (z, Φ, ζ, κ), the above parameters depend as well on the TD functions defining $H(t)$. To ensure the Hermiticity of $h(t)$, we demand W and F to be real, $V = U^*$, and $T = R^*$. Using the polar forms $\lambda = \Phi\, e^{-i\varphi}$, $\omega = |\omega| e^{i\varphi_\omega}$, $\alpha = |\alpha| e^{i\varphi_\alpha}$, and $\beta = |\beta| e^{i\varphi_\beta}$, we obtain from the conditions $W = W^*$ and $V = U^*$ the nonlinear system of three coupled differential equations

$$|\dot{z}| = |z|^2\left\{|\omega|\frac{\chi + \Phi^2}{\Phi}\sin\varphi_\omega - 2[|\alpha|\sin(\varphi - \varphi_\alpha) - |\beta|\chi\ \sin(\varphi + \varphi_\beta)]\right\} + \frac{|z|}{\Phi}\dot{\Phi}, \tag{11a}$$

$$\dot{\Phi} = \frac{2}{\chi - 1}\{[|\omega|\Phi\ \sin\varphi_\omega - |\alpha|\sin(\varphi - \varphi_\alpha)](1 - \Phi^2)|\beta|[(2\chi - 1)\Phi^2 - \chi^2]\sin(\varphi + \varphi_\beta)\}, \tag{11b}$$

$$\dot{\varphi} = \frac{2}{(1 - \chi)\Phi}[|\alpha|(1 - \Phi^2)\cos(\varphi - \varphi_\alpha)|\beta|(\Phi^2 - \chi^2)\cos(\varphi + \varphi_\beta)] + 2|\omega|\cos\varphi_\omega, \tag{11c}$$

noting from Eq. (7) that χ depends on $|z|$. It may be sometimes convenient to replace Eq. (11a), for $|z|$, by the equivalent equation for λ_0:

$$\dot{\lambda}_0 = -2\lambda_0\left\{|\omega|\left[1 + \frac{2\Phi^2}{\chi - 1}\right]\sin\varphi_\omega - 2\Phi\left[\frac{1}{\chi - 1}|\alpha|\sin(\varphi - \varphi_\alpha) - |\beta|\frac{2\chi - 1}{\chi - 1}\sin(\varphi + \varphi_\beta)\right]\right\}. \tag{12}$$

Therefore, as in Refs. [21] and [20], $|z|$ is the only free parameter determining η_2, with Φ and φ coming from Eqs. (11b) and (11c), with ϵ coming from the relation

$$\epsilon = \frac{1}{2\sqrt{1 - |z|^2}}\ln\frac{(1 + \sqrt{1 - |z|^2})\Phi - |z|}{(1 - \sqrt{1 - |z|^2})\Phi - |z|}, \tag{13}$$

directly derived from Eq. (6a). While Eq. (13) implies that a given pair $(|z|, \Phi)$ must be further corroborated by a real ϵ, such that $|z| \geqslant 2\Phi/(1 + \Phi^2)$, a numerical plot immediately shows that any value of Φ satisfies this inequality.

Having made clear the way in which we determine η_2, we then return to the Hermitization of $h(t)$ for the derivation of the parameters involved in η_1. Using the polar forms $\gamma = |\gamma|e^{i\varphi_\gamma}$ and $\delta = |\delta|e^{i\varphi_\delta}$, we obtain from the condition $T = R^*$ the coupled system for the amplitude and phase and of the parameter ζ

$$\begin{aligned}\frac{d|\zeta|}{dt} = {} & \frac{1}{\chi-1}\{|\zeta||\omega|[\chi+1-2\Phi\cos(\varphi-2\varphi_\zeta)]\sin\varphi_\omega - 2|\zeta||\alpha|[\Phi\sin(\varphi-\varphi_\alpha)-\sin(2\varphi_\zeta-\varphi_\alpha)]\\ & +2|\zeta||\beta|[\Phi\sin(\varphi+\varphi_\beta)-\chi\sin(2\varphi_\zeta+\varphi_\beta)] - |\gamma|[\sin(\varphi_\zeta-\varphi_\gamma)-\Phi\sin(\varphi-\varphi_\zeta-\varphi_\gamma)]\\ & -|\delta|[\chi\sin(\varphi_\zeta+\varphi_\delta)-\Phi\sin(\varphi-\varphi_\zeta+\varphi_\delta)]\},\end{aligned} \tag{14a}$$

$$\begin{aligned}\dot{\varphi}_\zeta = {} & \frac{1}{\chi-1}\bigg\{|\omega|[(\chi-1)\cos\varphi_\omega - 2\Phi\sin\varphi_\omega\sin(\varphi-2\varphi_\zeta)] + 2|\alpha|[\cos(2\varphi_\zeta-\varphi_\alpha)+\Phi\cos(\varphi-\varphi_\alpha)]\\ & -2|\beta|[\chi\cos(2\varphi_\zeta+\varphi_\beta)+\Phi\cos(\varphi+\varphi_\beta)] - \frac{|\gamma|}{|\zeta|}[\cos(\varphi_\zeta-\varphi_\gamma)+\Phi\cos(\varphi-\varphi_\zeta-\varphi_\gamma)]\\ & -\frac{|\delta|}{|\zeta|}[\chi\cos(\varphi_\zeta+\varphi_\delta)+\Phi\cos(\varphi-\varphi_\zeta+\varphi_\delta)]\bigg\},\end{aligned} \tag{14b}$$

and from $F = F^*$, we derive the equation for κ:

$$\dot{\kappa} = |\zeta|^2[|\omega|\sin\varphi_\omega - |\alpha|\sin(\varphi_\alpha-2\varphi_\zeta) - |\beta|\sin(\varphi_\beta+2\varphi_\zeta)] - |\zeta|[|\gamma|\sin(\varphi_\zeta-\varphi_\gamma)+|\delta|\sin(\varphi_\zeta+\varphi_\delta)], \tag{15}$$

noting that we have introduced the real κ to avoid constraints, coming from Eq. (15), on the TD functions γ and δ associated with linear pumping.

Next, by rewriting the parameters defining the Hermitian $h(t)$, we first verify that Eq. (11) leads to the real frequency

$$W = |\omega|\cos\varphi_\omega + \frac{2\Phi}{\chi-1}[|\alpha|\cos(\varphi-\varphi_\alpha) - |\beta|\cos(\varphi+\varphi_\beta)]. \tag{16}$$

We also have from Eq. (11) $U = V^* \equiv |U|e^{i\varphi_U}$, with

$$\begin{aligned}|U| = {} & \frac{1}{|\chi-1|}\{|\alpha|^2+|\beta|^2\chi^2 - 2|\alpha\beta|\chi\cos(\varphi_\alpha+\varphi_\beta)\\ & +|\omega|\Phi\sin\varphi_\omega[|\omega|\Phi\sin\varphi_\omega - 2|\alpha|\sin(\varphi-\varphi_\alpha)+2|\beta|\chi\sin(\varphi+\varphi_\beta)]\}^{1/2},\end{aligned} \tag{17a}$$

$$\tan\varphi_U = -\frac{|\omega|\Phi\cos\varphi\sin\varphi_\omega + |\alpha|\sin\varphi_\alpha + |\beta|\chi\sin\varphi_\beta}{|\omega|\Phi\sin\varphi\sin\varphi_\omega - |\alpha|\cos\varphi_\alpha + |\beta|\chi\cos\varphi_\beta}, \tag{17b}$$

and $R = T^* \equiv |R|e^{i\varphi_R}$, with

$$\begin{aligned}|R| = {} & \frac{2|\zeta|\sqrt{|\Phi^2-\chi|}}{|\chi-1|}\bigg\{|\omega|\sin\varphi_\omega\bigg[|\omega|\sin\varphi_\omega + 2|\alpha|\sin(2\varphi_\zeta-\varphi_\alpha)\\ & -2|\beta|\sin(2\varphi_\zeta+\varphi_\beta) - \frac{|\gamma|\sin(\varphi_\zeta-\varphi_\gamma)+|\delta|\sin(\varphi_\zeta+\varphi_\delta)}{|\zeta|}\bigg]\\ & +|\alpha|\bigg[|\alpha| - \frac{|\gamma|\cos(\varphi_\zeta-\varphi_\alpha+\varphi_\gamma)+|\delta|\cos(\varphi_\zeta-\varphi_\alpha-\varphi_\delta)}{|\zeta|}\bigg]\\ & +|\beta|\bigg[|\beta| + \frac{|\gamma|\cos(\varphi_\zeta+\varphi_\beta+\varphi_\gamma)+|\delta|\cos(\varphi_\zeta+\varphi_\beta-\varphi_\delta)}{|\zeta|}\bigg]\\ & -2|\alpha\beta|\cos(\varphi_\alpha+\varphi_\beta) + \frac{|\gamma|^2+|\delta|^2+2|\gamma\delta|\cos(\varphi_\gamma+\varphi_\delta)}{4|\zeta|^2}\bigg\}^{1/2},\end{aligned} \tag{18a}$$

$$\tan\varphi_R = -\frac{|\omega|\cos\varphi_\zeta\sin\varphi_\omega + |\alpha|\sin(\varphi_\zeta-\varphi_\alpha) - |\beta|\sin(\varphi_\zeta+\varphi_\beta) + \frac{1}{2|\zeta|}(|\gamma|\sin\varphi_\gamma - |\delta|\sin\varphi_\delta)}{|\omega|\sin\varphi_\zeta\sin\varphi_\omega + |\alpha|\cos(\varphi_\zeta-\varphi_\alpha) - |\beta|\cos(\varphi_\zeta+\varphi_\beta) - \frac{1}{2|\zeta|}(|\gamma|\cos\varphi_\gamma + |\delta|\cos\varphi_\delta)}. \tag{18b}$$

Finally, from Eq. (15) we have the real parameter

$$\begin{aligned}F = {} & \frac{|\zeta|^2}{1-\chi}\bigg\{2|\omega|\Phi\sin\varphi_\omega\sin(\varphi-2\varphi_\zeta) - |\alpha|[(\chi+1)\cos(2\varphi_\zeta-\varphi_\alpha)+2\Phi\cos(\varphi-\varphi_\alpha)]\\ & +|\beta|[(\chi+1)\cos(2\varphi_\zeta+\varphi_\beta)+2\Phi\cos(\varphi+\varphi_\beta)]\end{aligned}$$

$$+\frac{|\gamma|}{|\zeta|}[\chi\cos(\varphi_\zeta-\varphi_\gamma)+\Phi\cos(\varphi-\varphi_\zeta-\varphi_\gamma)] +\frac{|\delta|}{|\zeta|}[\cos(\varphi_\zeta+\varphi_\delta)+\Phi\cos(\varphi-\varphi_\zeta+\varphi_\delta)]\bigg\}. \tag{19}$$

As a last remark on the Dyson map, we note that from Eqs. (10d) and (10e) it is immediate to verify the necessity of the product $\eta=\eta_1\eta_2$ instead of considering only $\eta=\eta_2$ as in Refs. [20,21]. Otherwise, with $\eta_1=\mathbf{1}$ and consequently $\zeta(t)=\kappa(t)=0$, Eqs. (10d) and (10e) reduce to

$$R(z,\epsilon,\zeta,\kappa)=\frac{1}{\sqrt{\lambda_0}}(\gamma+\delta\lambda^*), \tag{20a}$$

$$T(z,\epsilon,\zeta,\kappa)=-\frac{1}{\sqrt{\lambda_0}}(\gamma\lambda+\delta\chi), \tag{20b}$$

and the relation $T=R^*$ evidently imposes a constraint on the TD functions γ and δ, ruling the linear amplification process. As mentioned above, the choice $\kappa(t)=0$ also results in constraints between these TD functions, although not as strong as in Eq. (20).

V. SOLUTIONS OF THE SCHRÖDINGER EQUATIONS FOR $h(t)$ AND $H(t)$ FROM THE LEWIS AND RIESENFELD TD INVARIANTS

As in Ref. [20], we first solve the Schrödinger equation for the general quadratic bosonic Hermitian Hamiltonian $h(t)$, using the method presented in Refs. [22] based on the Lewis and Riesenfeld TD invariants [23]. In Refs. [22] the quadractic bosonic Hamiltonian is reduced to a linear one, through a unitary transformation defined by the squeeze operator $S[\xi(t)]=\exp\{[\xi(t)(a^\dagger)^2-\xi^*(t)a^2]/2\}$, with $\xi(t)=r(t)e^{i\phi(t)}$, $r(t)$ being the degree of squeezing and $\phi(t)$ the squeezing direction in phase space [24]. The Schrödinger equation for the linear Hamiltonian is then solved through an already known linear TD invariant [25]. Besides being used in Ref. [20], the method presented in Ref. [22] was also used to study a TD atom-field interaction in cavity quantum electrodynamics [26]. We mention that a method was recently presented in Ref. [27] for a step-by-step construction of TD Lewis and Riesenfeld invariants, as a product of operators, and for a direct derivation of their corresponding eigenvalue equations.

Following the steps outlined above the solutions of the Schrödinger equation for $h(t)$, from which we immediately obtain that for $H(t)$, constitute a complete basis given by the states

$$|\Pi_n(t)\rangle=\mathcal{T}(t)|n\rangle, \tag{21}$$

where $\{|n\rangle\}$ define the Fock-state basis and the unitary operator $\mathcal{T}(t)$ is given by

$$\mathcal{T}(t)=\Upsilon(t)S[\xi(t)]D[\theta(t)]\mathcal{R}[\Omega(t)], \tag{22}$$

where $D[\theta(t)]=\exp[\theta(t)a^\dagger-\theta^*(t)a]$ is the displacement operator, with

$$i\dot\theta(t)=\Omega(t)\theta(t)+G(t), \tag{23a}$$

$$\Omega(t)=W(t)+2|U(t)|\tanh[r(t)]\cos[\phi(t)+\varphi_U], \tag{23b}$$

$$G(t)=R^*(t)\cosh[r(t)]+R(t)e^{i\phi(t)}\sinh[r(t)], \tag{23c}$$

while $\mathcal{R}[\Omega(t)]=\exp[-i\varpi(t)a^\dagger a]$ is a rotation operator, with $\varpi(t)=\int_0^t\Omega(\tau)d\tau$, and, finally, $\Upsilon(t)=\exp[-i\varpi(t)/2]$ is a global phase factor.

For reducing the quadractic Hermitian Hamiltonian $h(t)$ into a linear one, the squeezing parameters must obey the coupled nonlinear differential equations

$$\dot r(t)=2|U(t)|\sin[\varphi_U(t)+\phi(t)], \tag{24a}$$

$$\dot\phi(t)=-2W(t)-4|U(t)|\coth[2r(t)]\cos[\phi(t)+\varphi_U]. \tag{24b}$$

With the wave vectors in Eq. (21), we directly obtain the solutions of the Schrödinger equation for $H(t)$, given by

$$|\psi_n(t)\rangle=\eta^{-1}(t)|\Pi_n(t)\rangle=\eta^{-1}(t)\mathcal{T}(t)|n\rangle. \tag{25}$$

For a generic superposition $|\psi(t)\rangle=\sum_n c_n|\psi_n(t)\rangle$ it follows that

$$|\psi(t)\rangle=\eta^{-1}(t)\mathcal{T}(t)\sum_n c_n|n\rangle =\eta^{-1}(t)\mathcal{U}(t)\eta(0)|\psi(0)\rangle, \tag{26}$$

with the evolution operator

$$\mathcal{U}(t)=\mathcal{T}(t)\mathcal{T}^\dagger(0) =\Upsilon(t)S[\xi(t)]D[\theta(t)]\mathcal{R}[\Omega(t)]D^\dagger[\theta(0)]S^\dagger[\xi(0)]. \tag{27}$$

From Eqs. (25) and (26), we conclude that a generic superposition for $h(t)$, given by $|\Pi(t)\rangle=\sum_n c_n|\Pi_n(t)\rangle$, naturally evolves as

$$|\Pi(t)\rangle=\mathcal{U}(t)|\Pi(0)\rangle. \tag{28}$$

VI. VARIANCES OF THE FIELD QUADRATURES AND THE FIELD ENERGY

In the Heisenberg picture the annihilation operator $a_h(t)$ regarding the Hermitian Hamiltonian $h(t)$ follows from the Bogoliubov transformation

$$a_h(t)=\mathcal{U}^\dagger(t)a\mathcal{U}(t)=u(t)a+v(t)a^\dagger+w(t), \tag{29}$$

where we have defined

$$u(t)=e^{-i\varpi(t)}\cosh[r(t)]\cosh[r(0)] -e^{i[\varpi(t)+\phi(t)-\phi(0)]}\sinh[r(t)]\sinh[r(0)], \tag{30a}$$

$$v(t)=e^{i[\varpi(t)+\phi(t)]}\sinh[r(t)]\cosh[r(0)] -e^{-i[\varpi(t)-\phi(0)]}\cosh[r(t)]\sinh[r(0)], \tag{30b}$$

$$w(t)=[\theta(t)-e^{-i\varpi(t)}\theta(0)]\cosh[r(t)] +e^{i\varpi(t)}[\theta^*(t)-e^{i\varpi(t)}\theta^*(0)]\sinh[r(t)]. \tag{30c}$$

For the field quadratures, also in the Heisenberg picture, we have

$$x_h(t) = \frac{a_h(t) + a_h^\dagger(t)}{2}, \tag{31a}$$

$$p_h(t) = \frac{a_h(t) - a_h^\dagger(t)}{2i}, \tag{31b}$$

from which we derive the pseudo-Hermitian forms associated with $H(t)$:

$$\begin{aligned} X(t) &= \eta^{-1}(t)x_h(t)\eta(t) \\ &= -\frac{1}{|z|\sqrt{\Phi^2-\chi}}\left\{[(1+i|z|\sin\varphi)\Phi - |z|]x_h(t) + \frac{i}{2}(1+|z|\cos\varphi)\Phi p_h(t) + \frac{|z|}{2}[\zeta + \chi\zeta^* + 2\Phi\,\mathrm{Re}(e^{-i\varphi}\zeta)]\right\}, \end{aligned} \tag{32a}$$

$$\begin{aligned} P(t) &= \eta^{-1}(t)p_h(t)\eta(t) \\ &= -\frac{1}{|z|\sqrt{\Phi^2-\chi}}\left\{[(1-i|z|\sin\varphi)\Phi - |z|]p_h(t) - i\frac{1}{2}(1-|z|\cos\varphi)\Phi x_h(t) + i|z|\frac{1}{2}[\zeta - \chi\zeta^* - 2i\Phi\,\mathrm{Im}(e^{-i\varphi}\zeta)]\right\}. \end{aligned} \tag{32b}$$

Therefore, the pseudo-Hermitian observables $X(t)$ and $P(t)$ are composed of superpositions of the Hermitians $x_h(t)$ and $p_h(t)$, and their experimental checks must involve simultaneous measurements of canonically conjugated variables [28,29]. By considering that the radiation field is initially in the vacuum state $|\psi(0)\rangle = |0\rangle$ for $H(t)$, such that $|\Pi(0)\rangle = \mathcal{T}^\dagger(0)|0\rangle = |0\rangle$ for $h(t)$ [since $\Upsilon(0) = \mathcal{R}[\Omega(0)] = D[\theta(0)] = S[\xi(0)] = 1$ under the choice $\theta(0) = 0$], the variances of the field quadratures are given by

$$\begin{aligned} [\Delta X(t)]_\Theta^2 &= \langle X^2(t)\rangle_\Theta - \langle X(t)\rangle_\Theta^2 = \langle x_h^2(t)\rangle - \langle x_h(t)\rangle^2 = [\Delta x_h(t)]^2 \\ &= \tfrac{1}{4}\{[\Delta a_h(t)]^2 + [\Delta a_h^\dagger(t)]^2 + \Delta[a_h^\dagger(t)a_h(t)]\} \\ &= \tfrac{1}{4}|u(t) + v^*(t)|^2, \end{aligned} \tag{33a}$$

$$\begin{aligned} [\Delta P(t)]_\Theta^2 &= \langle P^2(t)\rangle_\Theta - \langle P(t)\rangle_\Theta^2 = \langle p_h^2(t)\rangle - \langle p_h(t)\rangle^2 = [\Delta p_h(t)]^2 \\ &= \tfrac{1}{4}\{1 - [\Delta a_h^\dagger(t)]^2 - [\Delta a_h(t)]^2 + 2\Delta[a_h^\dagger(t)a_h(t)]\} \\ &= \tfrac{1}{4}|u(t) - v^*(t)|^2, \end{aligned} \tag{33b}$$

where we have defined

$$\Delta[a_h^\dagger(t)a_h(t)] = \langle a_h^\dagger(t)a_h(t) + a_h(t)a_h^\dagger(t)\rangle - 2\langle a_h(t)\rangle\langle a_h^\dagger(t)\rangle. \tag{34}$$

Since $r(0) = 0$, such that $u(t) = e^{-i\varpi(t)}\cosh[r(t)]$ and $v(t) = e^{i[\varpi(t)+\phi(t)]}\sinh[r(t)]$, we finally obtain

$$[\Delta x_h(t)]^2 = \tfrac{1}{4}\{\cosh[2r(t)] + \sinh[2r(t)]\cos[\phi(t)]\}, \tag{35a}$$

$$[\Delta p_h(t)]^2 = \tfrac{1}{4}\{\cosh[2r(t)] - \sinh[2r(t)]\cos[\phi(t)]\}. \tag{35b}$$

Now, defining a rotated complex amplitude of the radiation field at an angle $\phi(t)/2$, given by

$$\tilde{x}_h(t) + i\tilde{p}_h(t) = [x_h(t) + ip_h(t)]e^{-i\phi(t)/2}, \tag{36}$$

and considering again the initial vacuum state $|\psi(0)\rangle = |\Pi(0)\rangle = |0\rangle$, we obtain the variances

$$[\Delta\tilde{x}_h(t)]^2 = \tfrac{1}{4}|u\,e^{-i\phi(t)/2} + v^*e^{i\phi(t)/2}| = \tfrac{1}{4}e^{2r(t)}, \tag{37a}$$

$$[\Delta\tilde{p}_h(t)]^2 = \tfrac{1}{4}|u\,e^{-i\phi(t)/2} - v^*e^{i\phi(t)/2}| = \tfrac{1}{4}e^{-2r(t)}. \tag{37b}$$

Since these variances depend only on the squeezing degree, which in turn follows only from the parametric amplification, we henceforth disregard the linear pumping by imposing $\gamma = \delta = 0$. Back to the system of coupled equations for $r(t)$ and $\phi(t)$, we then obtain

$$\dot{r}(t) = 2|U(t)|\sin[\varphi_U(t) + \phi(t)], \tag{38a}$$

$$\dot{\phi}(t) = -2W(t) - 4|U(t)|\coth[2r(t)]\cos[\varphi_U(t) + \phi(t)], \tag{38b}$$

leading to a maximized rate of squeeze degree under the choice $\varphi_U(t) = \pi/2 - \phi(t)$, which implies that

$$r(t) = 2\int_0^t |U(\tau)|d\tau, \tag{39a}$$

$$\dot{\varphi}_U(t) = -\dot{\phi}(t) = 2W(t). \tag{39b}$$

The relation (39b) leads to the useful constraint

$$\frac{d}{dt}\left\{\arctan\left[-\frac{|\omega|\Phi\cos\varphi\sin\varphi_\omega + |\alpha|\sin\varphi_\alpha + |\beta|\chi\sin\varphi_\beta}{|\omega|\Phi\sin\varphi\sin\varphi_\omega - |\alpha|\cos\varphi_\alpha + |\beta|\chi\cos\varphi_\beta}\right]\right\}$$
$$= 2|\omega|\cos\varphi_\omega + \frac{4\Phi}{\chi-1}[|\alpha|\cos(\varphi-\varphi_\alpha) - |\beta|\cos(\varphi+\varphi_\beta)], \tag{40}$$

linking together the parameters ω, α, and β of H with those defining the Dyson map η.

Another quantity of interest, when considering the squeezing of the field quadratures, is the field mean energy given by

$$\langle E(t)\rangle_\Theta = |\omega(t)|\langle 0|a_h^\dagger(t)a_h(t)|0\rangle = \langle\mathcal{E}(t)\rangle = |\omega(t)|[|v(t)|^2 + |w(t)|^2]. \tag{41}$$

In what follows we compute $r(t)$ for two different cases: the Hermitian case, $H = H^\dagger$, where $\varphi_\omega = 0$ such that $\omega = |\omega|$ is real, $|\beta| = |\alpha|$ and $\varphi_\beta = -\varphi_\alpha$, and the all-annihilation case, where $|\beta| = 0$ and $\varphi_\beta = 0$, under the additional choice $\varphi_\omega = 0$. As anticipated in the Introduction, we compute the squeezing degree for the all-annihilation case because it presents a simpler mathematical treatment than the all-creation case.

VII. CIRCUMVENTING THE NEED FOR MEASURING CANONICALLY CONJUGATED VARIABLES

To circumvent the need for measuring canonically conjugated variables we must set the parameters of the Dyson map to obtain $X = x_h$ or $P = p_h$ in Eq. (32). Considering $\zeta = 0$, once the linear pumping is disregarded, in the former case we set $|z|\cos\varphi = -1$ such that $|z| = 1$ and $\varphi = \pi$, while in the latter case we set $|z|\cos\varphi = 1$, implying that $|z| = 1$ and $\varphi = 0$. Assuming a real frequency ω and $|z| = 1$, the coupled Eq. (11) reduces to

$$\mp\dot{\Phi} = -2\Phi|\alpha|\sin\varphi_\alpha - 2|\beta|\Phi\chi\sin\varphi_\beta, \tag{42a}$$

$$\sin\varphi_\alpha = \frac{|\beta|}{|\alpha|}\frac{(2\chi-1)\Phi^2 - \chi^2 - (\chi-1)\chi\Phi}{(1-\Phi^2)+(\chi-1)\Phi}\sin\varphi_\beta, \tag{42b}$$

$$\mp|\omega| = -|\alpha|\frac{1-\Phi^2}{(1-\chi)\Phi}\cos\varphi_\alpha - |\beta|\frac{\Phi^2-\chi^2}{(1-\chi)\Phi}\cos\varphi_\beta, \tag{42c}$$

where the upper and lower signs refer to $\varphi = \pi$ and $\varphi = 0$, respectively.

VIII. HERMITIAN H

For $H^\dagger = H = \omega(t)(a^\dagger a + 1/2) + \alpha(t)a^2 + \alpha^*(t)a^{\dagger 2}$, in which case $W(t) = |\omega(t)|$, $U(t) = |\alpha(t)|e^{i\varphi_\alpha(t)}$, the constraint (40) leads to $\dot{\varphi}_\alpha = 2|\omega|$, such that

$$\varphi_\alpha(t) = 2\int_0^t |\omega(\tau)|d\tau, \tag{43}$$

and consequently (with the subscript h standing for Hermitian)

$$r_h(t) = 2\int_0^t |\alpha(\tau)|d\tau, \tag{44a}$$

$$\phi(t) = \pi/2 - \varphi_\alpha(t). \tag{44b}$$

Noting that the TD parameters in Eq. (30) reduce to $|u(t)|^2 = \cosh^2[r(t)]$, $|v(t)|^2 = \sinh^2[r(t)]$, and $|w(t)|^2 = 0$, the energy of the radiation field, for the initial vacuum state $|0\rangle$, evolves as

$$\langle\mathcal{E}(t)\rangle = |\omega(t)||v(t)|^2 = |\omega(t)|\sinh^2[r(t)]. \tag{45}$$

IX. ALL-ANNIHILATION H

For the all-annihilation Hamiltonian $H_{aa}(t) = \omega(t)(a^\dagger a + 1/2) + \alpha(t)a^2$, derived from the Hermitian one with real $\omega(t)$, such that $|U| = |\alpha|/|\chi - 1|$ and $\varphi_U = \varphi_\alpha$, we obtain from the constraint (40) the relation

$$\dot{\varphi}_\alpha = 2|\omega| - 2|\alpha|\frac{2\Phi}{1-\Phi^2+\lambda_0}\cos(\varphi-\varphi_\alpha), \tag{46}$$

and from the system (11), the coupled equations

$$\frac{\dot{\lambda}_0}{\lambda_0} = -2|\alpha|\frac{2\Phi}{1-\Phi^2+\lambda_0}\sin(\varphi-\varphi_\alpha), \tag{47a}$$

$$\dot{\Phi} = -2|\alpha|\frac{\Phi^2-1}{1-\Phi^2+\lambda_0}\sin(\varphi-\varphi_\alpha), \tag{47b}$$

$$\dot{\varphi} = 2|\omega| - 2|\alpha|\frac{\Phi^2-1}{(1-\Phi^2+\lambda_0)\Phi}\cos(\varphi-\varphi_\alpha). \tag{47c}$$

By rewriting Eq. (47) under the change of variables

$$\varphi - \varphi_\alpha = \varphi_-, \tag{48a}$$

$$\varphi + \varphi_\alpha = \varphi_+, \tag{48b}$$

it follows that

$$\frac{\dot{\lambda}_0}{\lambda_0} = -2|\alpha|\frac{2\Phi}{1-\Phi^2+\lambda_0}\sin\varphi_-, \tag{49a}$$

$$\dot{\Phi} = -2|\alpha|\frac{\Phi^2-1}{1-\Phi^2+\lambda_0}\sin\varphi_-, \tag{49b}$$

$$\dot{\varphi}_- = 2|\alpha|\frac{1+\Phi^2}{\Phi(1-\Phi^2+\lambda_0)}\cos\varphi_-, \tag{49c}$$

$$\dot{\varphi}_+ = 4|\omega| - 2|\alpha|\frac{3\Phi^2-1}{\Phi(1-\Phi^2+\lambda_0)}\cos\varphi_-, \tag{49d}$$

and by imposing $\varphi_- = \pi/2$, we obtain

$$\frac{1}{2\Phi}\frac{\dot{\lambda}_0}{\lambda_0} = -\frac{2|\alpha|}{1-\Phi^2+\lambda_0}, \tag{50a}$$

$$\frac{\dot{\Phi}}{\Phi^2-1} = -\frac{2|\alpha|}{1-\Phi^2+\lambda_0}, \tag{50b}$$

$$\varphi_+ = \int_0^t 4|\omega(\tau)|d\tau, \tag{50c}$$

$$\varphi = \frac{\varphi_+ + \varphi_-}{2} = 2\int_0^t |\omega(\tau)|d\tau + \frac{\pi}{4}. \tag{50d}$$

From Eqs. (50a) and (50b) we obtain

$$\lambda_0(t) = \lambda_0(0)\frac{\Phi^2(t)-1}{\Phi^2(0)-1}. \tag{51}$$

By substituting $\lambda_0(t)$ back in Eq. (50b), we obtain

$$\Phi(t) = \Phi(0) - \Lambda\int_0^t |\alpha(\tau)|d\tau, \tag{52}$$

where

$$\Lambda = 2\frac{\Phi^2(0)-1}{1-\Phi^2(0)+\lambda_0(0)}$$

and, consequently, substituting Eq. (51) into (7) we obtain

$$|z(t)| = \frac{2\Phi(t)}{\Phi^2(t)+1-\lambda_0(t)}. \tag{53}$$

Finally, from Eqs. (39a), (17a), and (53), we obtain (with *aa* standing for all-annihilation)

$$\begin{aligned} r_{aa}(t) &= 2\int_0^t |U(\tau)|d\tau = 2\int_0^t \frac{|\alpha(\tau)|}{|1-\chi(\tau)|}d\tau \\ &= \int_0^t \frac{|\Lambda\alpha(\tau)|}{|\Phi^2(t)-1|}d\tau. \end{aligned} \tag{54}$$

For the all-annihilation Hamiltonian, the evolution of the mean energy of the radiation field for the initial vacuum state obeys the same expression, in Eq. (45), as that for the Hermitian Hamiltonian.

X. ALL-ANNIHILATION *H* FOR THE CASE WHERE $X = x_h$ AND $P = p_h$

For the all-annihilation Hamiltonian $H_{aa}(t) = \omega(t)(a^\dagger a + 1/2) + \alpha(t)a^2$, with real $\omega(t)$, such that $|U| = |\alpha|/|\chi - 1|$ and $\varphi_U = \varphi_\alpha$, we next consider the cases $X = x_h$ and $P = p_h$, with $|z| = 1$ for both cases, while $\varphi = \pi$ for the former and $\varphi = 0$ for the latter case. From the constraint (40) we obtain

$$\dot{\varphi}_\alpha = 2|\omega| \mp 2|\alpha|\frac{\Phi}{\Phi-1}\cos\varphi_\alpha, \tag{55}$$

where the upper and lower signs refer to $\varphi = \pi$ and $\varphi = 0$, respectively. From the system (42), we obtain the solutions $\Phi(t) = cte$, $|\omega| = |\alpha|(1+\Phi)/2\Phi$, and $\varphi_\alpha = \ell\pi$, ℓ being even for $\varphi = \pi$ and odd for $\varphi = 0$. Both cases, $\varphi = \pi$ and $\varphi = 0$, lead to the same squeezing degree

$$r_{aa}(t) = \frac{1}{|\Phi-1|}\int_0^t |\alpha(\tau)|d\tau = \frac{1}{2|\Phi-1|}r_h(t), \tag{56a}$$

which is the same as the Hermitian one apart from a scale factor $[2|\Phi-1|]^{-1}$. For the cases where $\Phi \in [0, 1/2)$ and $\Phi > 3/2$ it follows that $r_{aa}(t) < r_h(t)$, while for the cases $\Phi \in (1/2, 1)$ and $\Phi \in (1, 3/2)$ we obtain $r_{aa}(t) > r_h(t)$. In the case where $\Phi = 1/2$ and $\Phi = 3/2$ we have $r_{aa}(t) = r_h(t)$.

XI. COMPARING THE SQUEEZING DEGREES DERIVED FROM THE HERMITIAN AND ALL-ANNIHILATION HAMILTONIANS

We now compare the squeezing degrees found in Eqs. (54) and (44a), starting with a pumping amplitude which decreases exponentially over time as $\alpha(t) = \alpha_0 e^{-\gamma|t|}$. The degree of squeezing for the Hermitian case (*h*) reaches, asymptotically, the value

$$r_h(t) = 2\int_0^t |\alpha(\tau)|d\tau = 2(1-e^{-\gamma t}) \simeq 2, \tag{57}$$

while for the all-annihilation case (*aa*) it goes as

$$\begin{aligned} r_{aa}(t) &= |\Lambda|\int_0^t \frac{\gamma\, e^{-\gamma\tau}}{|1-[\Phi(0)-\Lambda(1-e^{-\gamma\tau})]^2|}d\tau \\ &= -\frac{|\Lambda|}{\Lambda}\int_{\Phi(0)}^{\Phi(0)-\Lambda(1-e^{-\gamma t})} \frac{1}{|1-x^2|}dx. \end{aligned} \tag{58}$$

Assuming first $\Phi(0) > 1$, in which case the above integral is nonsingular, we obtain

$$r_{aa}(t) = -\frac{|\Lambda|}{\Lambda}\int_{\Phi(0)}^{\Phi(0)-\Lambda(1-e^{-\gamma t})} \frac{1}{x^2-1}dx \tag{59}$$

$$= -\frac{|\Lambda|}{2\Lambda}\ln\left(\frac{\Phi(0)+1}{\Phi(0)-1}\frac{\Phi(0)-1-\Lambda(1-e^{-\gamma t})}{\Phi(0)+1-\Lambda(1-e^{-\gamma t})}\right), \tag{60}$$

where, from the choice $|z(0)| = 1$, we set $\lambda_0(0) = [1-\Phi(0)]^2$.

In Fig. 1 we show the squeezing factors r_h and r_{aa} against γt, the thick solid line standing for r_h, while the solid, dashed, dashed-dotted, and dotted stand for $\Phi(0) = 1.5$, 1.1, 1.01, and $1 + 10^{-10}$, respectively, as indicated in the inset. We note that the squeezing factor r_{aa} increases as $\Phi(0)$ approaches unity from above. For $\Phi(0) = 1.01$, r_{aa} becomes higher than r_h for any instant and with $\Phi(0) = 1 + 10^{-10}$, r_{aa} is around an order of magnitude higher than r_h in the time interval where $t < 3\gamma/2$. It is worth noting that, as $\Phi(0)$ approaches unity, the time derivative at the origin increases strongly, a most desirable feature for high performance interferometry.

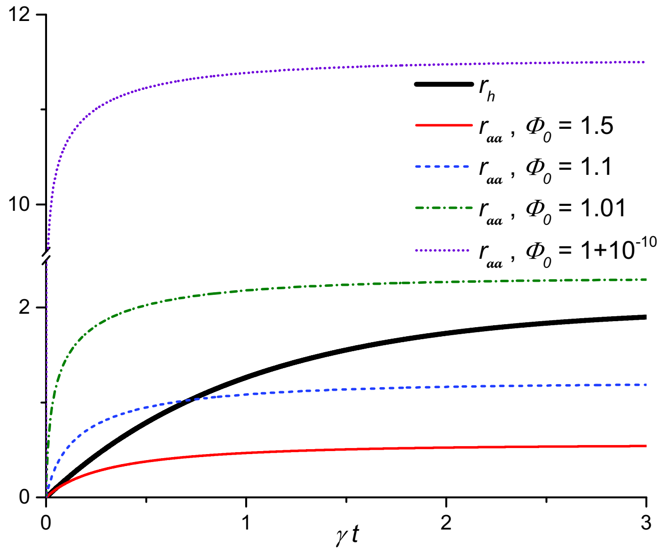

FIG. 1. Plot of the squeezing factors r_h and r_{aa} against γt, for $\Phi(0) > 1$, $|z(0)| = 1$, and $\lambda_0(0) = [1 - \Phi(0)]^2$. The thick solid line stands for r_h, while the solid, dashed, dashed-dotted, and dotted stand for $\Phi(0) = 1.5, 1.1, 1.01$, and $1 + 10^{-10}$, respectively, as indicated in the legend.

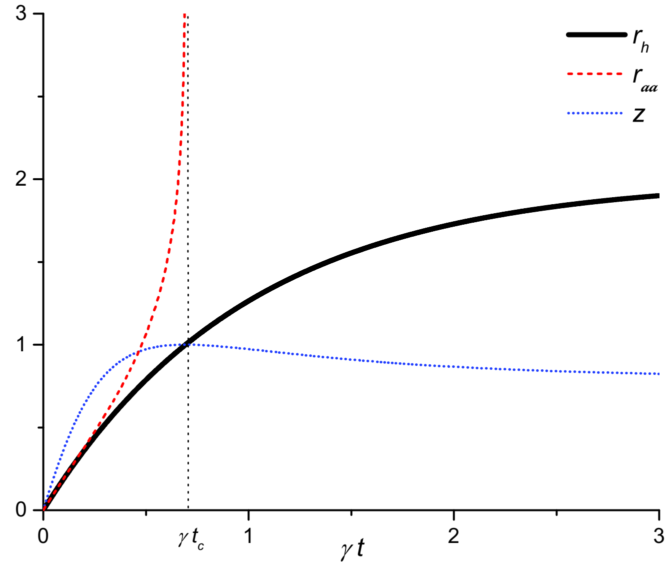

FIG. 2. Plot of the squeezing factors r_h and r_{aa} against γt, for $\Phi(0) = 0$, $|z(0)| = 0$, and $\lambda_0(0) = 0$. The thick solid line stands for r_h, while the dashed line indicates a divergence of the squeezing degree r_{aa} at a controllable finite time $t_c = \gamma^{-1} \ln[\Lambda/(\Lambda + 1)]$. Finally, the dotted line indicates the function $z(t)$.

With $\Phi(0) < 1$, in which case the integral in Eq. (58) exhibits a singularity at $x = 1$, we consider the upper limit $\Phi(t_c) = \Phi(0) - \Lambda \int_0^{t_c} |\alpha(\tau)| d\tau = 1 - \epsilon$, at a critical time t_c where $|z(t_c)| \to 1$ when $\epsilon \to 0$. In this way, when integrating until the upper limit $x = \Phi(0) - \Lambda(1 - e^{-\gamma t_c}) = 1$, we obtain from Eq. (58) the analytical result

$$r_{aa}(t) = \frac{|\Lambda|}{2\Lambda} \ln \left(\frac{1 + \Phi(0)}{1 - \Phi(0)} \frac{1 - \Phi(0) + \Lambda\left(1 - e^{-\gamma t}\right)}{1 + \Phi(0) - \Lambda(1 - e^{-\gamma t})} \right). \tag{61}$$

Considering $|z(0)| = 0$, such that $\Phi(0) = 0$ whatever $\lambda_0(0)$, in Fig. 2 we set $\lambda_0(0) = 0$ to show the squeezing factors r_h and r_{aa} against γt, the thick solid line standing for r_h. Here we observe, as indicated by the dashed line, a remarkable divergence of the squeezing degree r_{aa} at a controllable finite time $t_c = \gamma^{-1} \ln[\Lambda/(\Lambda + 1)]$. This divergence can also be observed for a constant pumping amplitude $\alpha(t) = \alpha_0$, in which case $r_h(t) = 2\alpha_0 t$, and

$$r_{aa}(t) = |\Lambda|\alpha_0 \int_0^t \frac{1}{|[\Phi(0) - \Lambda\alpha_0\tau]^2 - 1|} d\tau, \tag{62}$$

the divergence occurring for $t_c = (2\alpha_0)^{-1}$ when considering $|z(0)| = \Phi(0) = \lambda_0(0) = 0$. Back to Fig. 2, the dashed line parallel to the ordinate axis indicates the point γt_c where the divergence of r_{aa} occurs. Finally, we plot the function $z(t)$, as indicated by the dotted line, to ensure that this function is always smaller than unity regardless the divergence of r_{aa}. We have also verified that $z(t)$ is always smaller than unity for all the four values of $\Phi(0)$ considered in Fig. 1.

XII. PRICE TO PAY FOR THE HIGH SQUEEZING FACTORS

It is worth saying a few words about the cost-benefit aspects of the high compression factors achieved through non-Hermitian Hamiltonians. Fist of all, there is the energy cost. The higher the degree of squeezing, the greater the energy required for the process. Then there are other costs to be considered as the preparation of the state to be squeezed, associated with the non-Hermitian Hamiltonian H. As is well known [5], the state to be prepared is given by $\eta(t)|\Pi(0)\rangle$, with the Dyson map defined by Eq. (5), i.e., we must implement the displacement of the initial state $|\Pi(0)\rangle$ of the Hermitian h, followed by the squeezing of such a displaced state. This is not an easy operation to perform, whatever the experimental platform used. Finally, there is the cost associated with the measurement of the observables associated with the non-Hermitian H, which comprises a superposition of the Hermitian observables of h. In other words there is the cost associated with the measurement of canonically conjugated variables.

Therefore, to the remarkable gains achieved with the non-Hermitian Hamiltonians in increasing the squeezing factor, there is in contrast the difficulties of the experimental implementation of the squeezing process, starting with the engineering of the non-Hermitian Hamiltonians themselves. However, beyond the difficulties of the experimental implementation of the process, the core of the present work is to bring attention to the possibilities offered by non-Hermitian Hamiltonians, in particular the all-creation and all-annihilation interactions, for high performance interferometry.

Regarding the engineering of effective all-creation and all-annihilation Hamiltonians, a crucial step for the implementation of the ideas developed above, we anticipate that this procedure can be performed by the method proposed in Ref. [30] and extensively used in the literature, especially in the field of radiation-matter interaction [31].

XIII. CONCLUSIONS

We have here defined the all-creation and all-annihilation TD $\mathcal{PT}$-symmetric bosonic Hamiltonians. To this end, we started from the most general TD quadratic bosonic $\mathcal{PT}$-symmetric Hamiltonian, describing a cavity mode under linear and parametric amplifications. To solve this general $\mathcal{PT}$-symmetric Hamiltonian, and consequently the particular all-creation and all-annihilation Hamiltonians, we first present a strategy to define the TD Dyson map and metric operators. Then, using the Lewis and Riesenfeld TD invariants, under the steps outlined in Refs. [22], we get the solutions of the Schrödinger equations for the quadratic $\mathcal{PT}$-symmetric Hamiltonian and its Hermitian counterpart. From this general solution, and neglecting the TD linear amplification process, which does not affect the squeezed state mechanism in which we are interested, we directly obtain the solutions for the all-creation and all-annihilation Hamiltonians. It happens that the analytical treatment of the all-annihilation Hamiltonian becomes easier than for the all-creation one.

After solving these Schrödinger equations we turn to the observables associated with the non-Hermitian Hamiltonians, computing the variances of the field quadratures and the field energy. Since these observables, coming from the non-Hermitian Hamiltonians, are normally superpositions of L^2-Hermitian operators, special attention was devoted to circumvent the need for measuring canonically conjugated variables. On this regard, we have found the parameters of the Dyson map enabling each field quadrature to be Hermitian with respect to both the L^2 metric and that enabling a probabilistic interpretation of our non-Hermitian models. Our main concern, however, was the squeezing degree generated by such non-Hermitian Hamiltonians. Focusing on the all-annihilation system, we have derived a result that can contribute decisively for improving interferometry sensitivity: while the Hermitian parametric amplification leads to an asymptotic divergence of the squeezing degree, the all-annihilation amplification leads to a divergence of such quantity in a controllable finite time. We also present a discussion of the cost-benefit aspects for achieving a higher degree of squeezing from non-Hermitian Hamiltonians. Our belief is that, despite the difficulties in the experimental implementation of the process, the main contribution of this work is the demonstration that non-Hermitian Hamiltonians can decisively outperform conventional interferometry methods.

Recently, the LIGO and Virgo collaborations have reported the detection of gravitational waves through laser interferometry, a seminal result for both theoretical and experimental physics. The next step of such collaborations is to enhance the interferometric sensitivity using squeezed states of light. The successful engineering of all-annihilation or all-creation Hamiltonians is then an important step towards high performance interferometry, in special gravitational wave interferometers.

Beyond gravitational interferometry, the possibility of producing states of the radiation field with a high degree of squeezing is evidently of crucial relevance for quantum metrology, which by its turn can be applied for studying fundamental phenomena such as, among many others, quantum coherence and nonclassical statistics. A direct extension of the present work regards the all-creation and all-annihilation TD fermionic Hamiltonians, useful for nuclear magnetic resonance where the advances in quantum information processing have been decisive.

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