Complete invariants for a quasi transversal Hopf bifurcation

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Introduction

Generic one parameter families of vector fields with simple recurrence bifurcate in two ways: one is the loss of hyperbolicity of a critical element (singularity or periodic orbit) and the other is loss of transversality between the stable and unstable manifolds of two hyperbolic critical elements. The saddle-node and the Andronov-Hopf bifurcation are the ones that persist under perturbations of the one-parameter family, in case of non-hyperbolicity of a critical element [1,4,11,12,13]. When bifurcating by loss of transversality, persistence is achieved by imposing quasi-transversality, i.e., least degenerate kind of non transversality for the invariant (stable and unstable) manifolds [7,8,9].

Here we shall deal with two-parameter families X_μ of vector fields with simple recurrence (i.e., the limit set has only a finite number of critical elements) in M^3. The class we are interested in is persistent, and appears in a natural way: X_μ bifurcates by simultaneously losing hyperbolicity and transversality. In [3], the author gives necessary and sufficient conditions for the existence of a mild equivalence between these families, i.e., equivalences that do not depend continuously on the parameters.

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This is done in terms of modulus of stability, a notion introced in Dynamical Systems by Palis in [10]. The bifurcation diagram and some moduli for strong equivalence (i.e., equivalences that depend continuously on the parameter) are also exhibited in terms of certain germs of functions. The main purpose of the present paper is to show that these moduli correspond to the complete list of invariants for strong topological equivalence between two of these two-parameter families of vector fields.

Let us recall some basic definitions and facts in order to state our results in a precise way.

Let $\mathfrak{X}_1(M^3)$ be the space of C^∞ vector fields endowed with the C^∞ Whitney topology and let $\mathfrak{X}_2(M)$ be the space of the mappings $X: I\times I \to \mathfrak{X}_1(M)$ with the usual C^∞ topology ($I = [-1,1]$). Suppose X_μ, is a 2-parameter family of vector fields such that X_μ at $\mu = (0,0)$, or shortly X_o is a field which exhibits:

a) a saddle type periodic orbit $\sigma_1 = \sigma_1(0)$ of period 1 hyperbolic, with associated eigenvalues $0 < |\beta_1| < 1 < \beta_2$, C^2 linearizable;

b) a quasi-hyperbolic singularity $p = p(0)$ of saddle type such that $\dim W^u(p(0)) = 1$, associated to the eigenvalue $\alpha_3 > 0$ and $\dim W^s(p(0)) = 2$, associated to eigenvalues $\alpha_1 \pm i\alpha_2$ $(\alpha_1=0)$. Consider $X/W^s(p(0))$ as a "vague attractor";

c) a unique orbit γ in $W^u(p(0)) \cap W^s(\sigma_1(0))$ that is, a quasi-transversal intersection.

Call $E \subset \mathfrak{X}_1(M^3)$ the set of fields like X_o and E' the set of families as the one described above.

There are different ways of defining equivalence between parametrized familis X_μ and $\tilde{X}_\mu$. In the first place, for fields we have that a topological equivalence between two vector fields X, $\tilde{X}$ on M is a homeomorphism $h: M \to M$ such that h sends orbits of X onto orbits of $\tilde{X}$, preserving time orientation. If in addition h preserves time, that is if $h X_t = \tilde{X}_t h$ holds, then h is called a conjugacy. A vector field X is called structurally stable if it is equivalent to any nearby vector field.

We say that a family X_μ at μ_o is mildly equivalent to $\tilde{X}_\mu$ at $\tilde{\mu}_o$, if there exists a reparametrization (homeomorphism) $\lambda: (U \subset \mathbb{R}^n, \mu_o) \to (U, \tilde{\mu}_o)$ and a family of homeomorphisms $h_\mu: M \to M$ so that h_μ maps orbits of X_μ onto orbits of $\tilde{X}_{\lambda(\mu)}$ for μ near μ_o. If we can choose the homeomorphisms h_μ to depend continuously on μ, then we say that X_μ is continuously (or strongly) equivalent to $\tilde{X}_\mu$ (at μ_o and $\tilde{\mu}_o$).

The conditions for equivalence between two nearby one-parameter families are known, in the generic case. For example if a singularity loses hyperbolicity, the corresponding saddle-node and the Andronov-Hopf bifurcation are locally strongly equivalent to nearby families, or strongly stable. But some classes of families going through quasi-transversal bifurcations are not strongly or even mildly stable. Van Strien [14] characterized the conditions under which two one-parameter families are either mildly or strongly semilocally equivalent, that is, in a neighborhood of the orbit of tangency. Again, the concept of modulus of stability is essential for this purpose. For example, let f be a diffeomorphism of a surface which exhibits two hyperbolic fixed points p_1, p_2, such that $W^u(p_1)$ meets $W^s(p_2)$ quasi transversally, that is, with parabolic contact. Take g near f. Then it shall be topologically equivalent to f if and only if it "looks like" f and $\lambda = \dfrac{\ln \beta^s(p_1)}{\ln \beta^u(p_2)} = \dfrac{\ln \beta^s(\tilde{p}_1)}{\ln \beta^u(\tilde{p}_2)} = \tilde{\lambda}$ where $\beta^\sigma(p_i)$ $(\beta^\sigma(\tilde{p}_i))$ is the eigenvalue associated to $W^\sigma(p_i)$ $(W^\sigma(\tilde{p}_i))$ $(\sigma = s,u;$ $i = 1,2)$ [10], [6]. Thus, there exists a differentiable real invariant which is preserved under topological equivalence. In this context we call λ a modulus for the equivalence.

In [14], it is proved that for a generic one-parameter family f_μ of diffeomorphisms, going through a quasi transversal bifurcation, the moduli for strong equivalence are two: $\beta^s(p_1)$ and $\beta^u(p_2)$. Obviously, for mild equivalence the modulus is only λ.

Let us now give the semilocal versions of the notions of equivalence.

A semilocal equivalence between X and $\tilde{X}$, $\tilde{X} \in E$ shall be an equivalence defined from a neighborhood of $\bar{Y}$ onto a neighborhood of $\tilde{\bar{Y}}$.

In [3], it is proved:

Theorem A. Let X, $\tilde{X}$ be two C^r nearby vector fields in E. Then they are semilocally topologically equivalent iff $\beta_2 = \tilde{\beta}_2$.

In this case we say that X has modulus of stability one, and that β_2 is a modulus for X. In some sense, we are parametrizing the equivalence classes in a neighborhood of X by the parameter β_2.

Theorem B in [3] is concerned with families of vector fields in E', or more precisely, with families in E'', an open and dense subset of E' (see §2). A rough description of the type of vector fields that appear in families of E'' may motivate Theorem B. Let us give it before stating that theorem.

Due to the hyperbolicity of $\sigma_1(0)$ and non singularity of $dX(p(0))$, both σ_1 and p persist for nearby vector fields. We denote them by $\sigma_1(\mu_1,\mu_2)$ and $p(\mu_1,\mu_2)$ respectively.

As it will be shown later, we may suppose $\mu_1 = \alpha_1/\alpha_2$. This change of parameters is possible in E''.

When $\mu_1 > 0$ there shall appear new closed orbits $\sigma_2(\mu_1,\mu_2)$, unique for each value of (μ_1,μ_2), $\mu_1 > 0$, whose unstable manifold $W^u(\sigma_2)$ meets $W^s(\sigma_1)$ for some values of μ_2. These orbits σ_2 lie very near p, according to Hopf Theorem, and are hyperbolic with attractive eigenvalue β_3.

The intersection of those manifolds contains alternatively 0, 1 or 2 orbits. In case there is only 1 orbit, it shall be a quadratic tangency between $W^u(\sigma_2)$ and $W^s(\sigma_1)$ and this gives rise to a modulus of stability, as stated before.

For $\mu_1 < 0$ we also get tangencies when $W^u(p)$ meets

$W^s(\sigma_1)$, again implying the existence of a modulus [2]. Then, every such family exhibits curves of vector fields with one modulus of stability each, and we have

Theorem B. Let $X_\mu \in E''$. Then

a) the modulus of stability for mild topological equivalence is one, namely β_2,

b) the modulus of stability for strong topological equivalence is infinite, moreover, it is modelled over a space of germs of functions.

The question left in Theorem B is whether the moduli there exhibited are sufficient for strong topological equivalence. These moduli are given by germs of functions β_2, β_2^+, β_2^- associated to each family X_μ, which are described as follows. The set of values (μ_1,μ_2) for which X_μ has tangencies is a curve $(\mu_1,\mu_2(\mu_1))$ for $\mu_1 \leq 0$ and two other curves $(\mu_1,\mu_2^+(\mu_1))$, $(\mu_1,\mu_2(\mu_1))$ for $\mu_1 \geq 0$ by Proposition 12 in [3]. The moduli are given by the following 5 functions defined along these curves:

$\beta_2(\mu_1,\mu_2(\mu_1))$, $\quad(\beta_2,\beta_3)(\mu_1,\mu_2^+(\mu_1))$, $\quad(\beta_2,\beta_3)(\mu_1,\mu_2^-(\mu_1))$.

As β_3 depends monotonically on μ_1 (Prop. 13 in [3]) we get the following reduced set of moduli (3 functions):

$$\beta_2(\mu_1,\mu_2(\mu_1)) \quad \text{for } \mu_1 \leq 0 \quad \text{and}$$

$$\beta_2(\beta_3,\mu_2^+(\beta_3)), \quad \beta_2(\beta_3,\mu_2^-(\beta_3)) \quad \text{for } \mu_1 > 0.$$

We now state our main result in this paper.

<u>Theorem</u>. Two generic families X_μ and $\tilde{X}_\mu$ in E' are

strongly topologically equivalent when and only when

$$\beta_2(\mu_1,\mu_2(\mu_1)) = \tilde{\beta}_2(\mu_1,\tilde{\mu}_2(\mu_1)), \qquad \mu_1 \le 0$$

$$\beta_2(\beta_3,\mu_2^+(\beta_3)) = \tilde{\beta}_2(\beta_3,\mu_2^+(\beta_3)), \qquad \mu_1 > 0$$

$$\beta_2(\beta_3,\mu_2^-(\beta_3)) = \tilde{\beta}_2(\beta_3,\tilde{\mu}_2^-(\beta_3)), \qquad \mu_1 > 0.$$

Remark:

a) in this statement, $\mu_1 = \alpha_1/\alpha_2$ and $\tilde{\mu}_1 = \tilde{\alpha}_1/\tilde{\alpha}_2$.

b) We note that the functions depend only on μ_1 and β_3. So we denote them briefly by $\beta_2(\mu_1)$, $\beta_2^+(\beta_3)$, $\beta_2^-(\beta_3)$.

The families in E′

In order to show our bifurcation diagram, let us state Hopf Theorem, following [5].

Hopf Theorem in $\mathbb{R}^2$. Let X_λ be a C^k $(k \ge 4)$ vector field on $\mathbb{R}^2$ such that $X_\lambda(0) = 0$ for all λ and $X = (X_\lambda, 0)$ is also C^k. Let $dX_\lambda(0,0)$ have two distinct, complex conjugate eigenvalues $\alpha_1(\lambda) \pm i\alpha_2(\lambda)$, $\alpha_1 > 0$ for $\lambda > 0$. Also let $\frac{d}{d\lambda}\alpha_1(\lambda)\Big|_{\lambda=0} > 0$. Then

A: there is a C^{k-2} function $\lambda\colon (-\varepsilon,\varepsilon) \to \mathbb{R}$ such that $(x_1,0,\lambda(x_1))$ is on a closed orbit of period $\approx 2\pi/|\alpha_2(0)|$ and radius growing like $\sqrt{x_1}$, of X for $x_1 \ne 0$ and such that $\lambda(0) = 0$.

B: there is a neighborhood U of $(0,0,0)$ in $\mathbb{R}^3$ such that any closed orbit in U is one of those above. Furthermore, if 0 is a "vague attractor" for X_o, then

C: $\lambda(x_1) > 0$ for all $x_1 \neq 0$ and the orbits are attracting.

In our case $X_\mu \in \mathfrak{X}_1(M^3)$, and it is hyperbolic (expansive) along the x_3-direction.

<u>Corollary</u>. For small (positive) values of α_1 the unique periodic orbit σ_2 which appears near p, is hyperbolic of saddle type.

We want to distinguish a subset of E'. Namely, those families that meet E only at one point, that is, only one member of the family is simultaneously non hyperbolic and non transversal.

To do so, consider a family $X_\mu \in E'$ and call $v(\mu_1,\mu_2) = \pi_2(W^{uu}(p(\mu_1,\mu_2))) \cap N$ where $W^{uu} = W^u$ when p is a saddle type singularity.

Identify $W^u(\sigma_1) \cap \Sigma$ with a neighborhood of $o \in R$, so that $v: U \to R$, $v \in C^1$.

As before, $\alpha_1(\mu_1,\mu_2)$ is the real part of the complex eigenvalue of $dX(\mu_1,\mu_2)$, and $\beta_2(\mu_1,\mu_2)$ is the expansive eigenvalue associated to $\sigma_1(\mu_1,\mu_2)$.

<u>Definition of E"</u>. Let $X_\mu \in E'$. We say that $X_\mu \in E''$ if

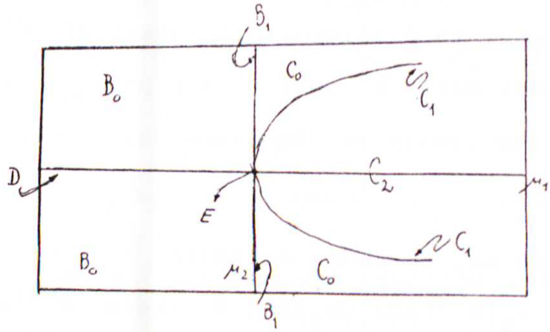
1) $J(\alpha_1(\mu_1,\mu_2), v(\mu_1,\mu_2))(0,0)$ is non singular, where J is the Jacobian matrix of (α_1, v).

2) $\partial\beta_2/\partial\mu_2\,(0,0) \neq 0$.

E'' is open and dense in E', so we shall call $X_\mu \in E''$ a generic family.

We want to establish the necessity of some moduli of stability for equivalence between members of E''.

In the first place, let us mention which kind of vector fields shall appear in a generic family (see Figure).



Let B_0, B_1, C_0, C_1, C_2, D and E be as in Figure 1. Then we have:

1) a unique vector field in E, the central bifurcation,

2) vector fields in D, which look like those in E, but are hyperbolic,

3) vector fields in C, which exhibit a singularity of the source type, hyperbolic and a periodic orbit σ_2 very near this singularity.

This set C we subdivide into:

a) C_1, the set of fields such that $W^u(\sigma_2) \cap W^s(\sigma_1)$ consista only of one orbit, giving rise to a quasi transversal intersection.

b) C_2, the set of vector fields such that $W^u(\sigma_2) \pitchfork W^s(\sigma_1)$ along 2 orbits.

c) C_o, the set of vector fields such that $W^u(\sigma_2) \cap W^s(\sigma_1) = \emptyset$.

4) Vector fields in B, with a singularity p of saddle type, such that $W^u(p) \cap W^s(\sigma_1) = \emptyset$, which we subdivide into: a) B_o the set of fields where p is hyperbolic; b) B_1 the set of fields where p is quasi-hyperbolic.

Observe that $W^u(\sigma_2) \cap W^s(\sigma_1)$ consists alternatively of none, one or two orbits, and that there might be tangencies only for those fields of C_1.

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