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Some results about minimal immersions having flat normal bundle.

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Maio/1982

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Introduction

In the study of isometric immersions $M^n \hookrightarrow \mathbb{R}^{n+1}$, the local existence of functions w, defined in M and having the properties: $w \geq 0$, $\Delta w = \|A\|^2 + |\mathrm{grad}_M w|^2$, where A is the second fundamental tensor of the immersion, has led to an integral bound for minimal hypersurface curvatures [5], and to an "a priori" bound to the norm of the gradient of solutions of the minimal hypersurface equation [1] [6].

In what follows we found the existence of such functions to isometric minimal immersions in $\mathbb{R}^{n+2}$ having flat normal bundle. Indeed we proved the following results:

<u>Theorem 1</u> - Let $M^n \hookrightarrow \mathbb{R}^{n+2}$ be an isometric minimal immersion having flat normal bundle. Then in a neighborhood of each $p \in M$ there is a function w, of class C^2, such that $w \geq 0$ and $\Delta w = C^2 + |\delta w|^2$ where $C^2 = \|A\|^2$ and $\delta = \mathrm{grad}_M$.

<u>Theorem 2</u> - Let $M^n \hookrightarrow \mathbb{R}^{n+2}$ be an isometric minimal immersion having flat normal bundle. Then for each $p \in M$, there is $d > 0$ such that if $0 < \rho < \sigma < d$, we have

$$\int_{B_\rho(p)\cap M} C^2 \, dM \leq \frac{4}{(\sigma-\rho)^2} \int_{B_\sigma(p)\cap M} dM$$

where $B_\rho(p)$ is the ball in $\mathbb{R}^{n+2}$, centered in p and having radius ρ .

Theorem 3 - Let $M^n \hookrightarrow \mathbb{R}^{n+m}$ be an isometric minimal immersion having flat normal bundle. Then

$$\frac{1}{2}\,\Delta C^2 + C^4 \geq \left(1 + \frac{2}{n}\right) |\delta C|^2$$

If $\{V^{(k)}\}_{1 \leq k \leq m}$ is a local orthonormal family of parallel fields in the normal bundle, then

$$\frac{1}{2}\,\Delta C_k^2 + C_k^4 + \sum_{\substack{r=1 \\ r \neq k}}^{m} C_{kr}^2 \geq \left(1 + \frac{2}{n}\right) \left|\delta C_k^2\right| \; ; \; k = 1,2,\ldots,m$$

where $$C_{kr} = \langle A^{V^{(k)}}, A^{V^{(r)}} \rangle$$

and $$C_k^2 = C_{kk}$$

This result generalizes a known result to the hypersurface case [2] and implies the following Bernstein type theorem:

Theorem 4 - Let $M = \{(x, f(x), g(x))/x \in \mathbb{R}^n\}$ be a minimal graph in $\mathbb{R}^{n+2}$, having flat normal bundle and such that

$$\lim_{R \to \infty} \frac{1}{R^{n+\varepsilon}} \int_{B_R(p) \cap M} dM = 0 \; ; \; \forall p \in M \, , \; \forall \varepsilon > 0$$

a) If $n \leq 5$ and M is not totally geodesic and there is an orthonormal family $\{V, W\}$ of parallel fields in the normal bundle such that $< A^V, A^W > \equiv 0$, then M cannot be stable.

b) If $n \leq 3$ there is a family of vector fields $\{V,W\}$ satisfying the conditions of (a). Therefore, if M is stable then M is totally geodesic.

This theorem remains true for all complete manifolds that have an orthonormal family of parallel fields in the normal bundle, globally defined on M, if the above limit is valid for geodesic balls.

This result generalizes a theorem of do Carmo and Peng [2].

§ 1 - Basic facts

1.1 - Let $M^n \hookrightarrow \mathbb{R}^{n+m}$ be an isometric immersion and $f \in C^1(U)$, where U is an open set in $\mathbb{R}^{n+m}$ such that $U \cap M \neq \emptyset$. The gradient of the restriction of f to M will be denoted by δf. We observe that δf depends only on the values of f over M.

If $p \in U \cap M$, $\{u_i\}_{1 \le i \le n}$ is an orthonormal basis of $T_p(M)$, Df is the gradient of f in $\mathbb{R}^{n+m}$ and $\{V^{(k)}\}_{1 \le k \le m}$ is an orthonormal frame of normal fields defined on $U \cap M$, then we have:

$$\delta f(p) = \sum_{i=1}^{n} \langle Df(p) , u_i \rangle u_i \tag{1.1.1}$$

$$\delta f = Df - \sum_{k=1}^{m} \langle Df , V^{(k)} \rangle V^{(k)} \tag{1.1.2}$$

If $\{e_1, \ldots, e_{n+m}\}$ is a coordinate system to $\mathbb{R}^{n+m}$, that is each e_i is a constant field and $\langle e_i, e_j \rangle = \delta_{ij}$, then we have

$$\delta_i f = D_i f - \sum_{k=1}^{m} \langle Df , V^{(k)} \rangle V_i^{(k)} \quad ; \; i = 1,2,\ldots,n+m \tag{1.1.3}$$

where $D_i f = \langle Df, e_i \rangle$, $V_i^{(k)} = \langle V^{(k)}, e_i \rangle$ and $\delta_i f = \langle \delta f, e_i \rangle$

1.2 - If H is the mean curvature field of an isometric immersion $M^n \hookrightarrow \mathbb{R}^{n+m}$, $\bar{\nabla}$ is the riemannian connection of $\mathbb{R}^{n+m}$, $\{V^{(k)}\}_{1 \le k \le m}$ is a local orthonormal frame of normal vector fields defined in a neighborhood of $p \in M$ and $\{e_i\}_{1 \le i \le n+m}$ is a coordinate system of $\mathbb{R}^{n+m}$ such that

$$e_{n+k}(p) = V^{(k)}(p) \quad , \quad \text{we have at } p,$$

$$\delta_i = D_i \quad \text{for} \quad i \le n$$

$$\delta_{n+k} = 0 \quad \text{for} \quad k = 1,2,\ldots,m \,,$$

$$< H, V^{(k)} >_p = - \sum_{i=1}^{n+m} \delta_i \, V_i^{(k)} \, (p)$$

Since $\sum_{i=1}^{n+m} \delta_i \, V_i^{(k)}$ does not depend on the chosen coordinate system, we have

(1.2.1) $$< H, V^{(k)} = - \sum_{i=1}^{n+m} \delta_i \, V_i^{(k)}$$

at any point of M and in relation to any coordinate system of $\mathbb{R}^{n+m}$. So

(1.2.2) $$H = - \sum_{k=1}^{m} \left(\sum_{i=1}^{n+m} \delta_i \, V_i^{(k)} \right) V^{(k)}$$

The same way we have

(1.2.3) $$< A^{V^{(k)}} , A^{V^{(s)}} > = \sum_{ij=1}^{n+m} \left(\delta_i \, V_j^{(k)} \right) \left(\delta_j \, V_i^{(s)} \right)$$

at any point of M and with respect to any coordinate system of $\mathbb{R}^{n+m}$.

From now on

$$C_{ks} = < A^{V^{(k)}} , A^{V^{(s)}} >$$

(1.2.4) $$C_k^2 = C_{kK}$$

$$C^2 = \sum_{k=1}^{m} C_k^2 = \| A \|^2$$

1.3 - Now let $X \in \chi(M)$ and $\{e_i\}_{1\le i\le n+m}$ be a coordinate system of $\mathbb{R}^{n+m}$ such that

$$\nu^{(k)}(p) = e_{n+k} \; ; \; k = 1,2,\ldots m$$

Since

$$\operatorname{div}_M X(p) = \operatorname{trace}\,(w \longmapsto \nabla_w X) \; ; \; w \in T_pM,$$

where ∇ is the connection of M, it follows

$$\operatorname{div}_M X(p) = \sum_{i=1}^{n+m} \delta_i X_i(p)$$

and as $\sum_{i=1}^{n+m} \delta_i X_i$ does not depend on the coordinate system, we have

$$\operatorname{div}_M X = \sum_{i=1}^{n+m} \delta_i X_i \tag{1.3.1}$$

at any point of M and with respect to any coordinate system.

Once the laplacean of $f \in C^2(M)$ is given by

$$\Delta f = \operatorname{div}_M (\operatorname{grad}_M f) \tag{1.3.2}$$

it follows that

$$\Delta f = \sum_{i=1}^{n+m} \delta_i \delta_i f \tag{1.3.3}$$

with respect to any coordinate system of $\mathbb{R}^{n+m}$.

1.4 - We will use the following well known results:

$$\int_M \delta\phi \, dM = -\int_M H\phi \, dM \; ; \; \forall \phi \in C^1(\mathbb{R}^{n+m}), \tag{1.4.1}$$

such that $\phi|_M$ has compact support.

(1.4.2) $$\int_M \phi\, \Delta\psi\, dM = \int_M \psi\, \Delta\phi\, dM = -\int_M < \delta\phi,\ \delta\psi > dM\ ;$$

$\forall\, \phi \in C^2\,(\mathbb{R}^{n+m})$ such that $\phi|_M$ has compact support and $\forall\, \psi \in C^2\,(\mathbb{R}^{n+m})$.

1.5 - Let $M^n \hookrightarrow N^{n+m}$ be an isometric immersion, $\nabla^\perp$ be the induced connection of the normal bundle and let $R^\perp$ be its curvature tensor. We say that the immersion has flat normal bundle if $R^\perp \equiv 0$.

We will consider the following well known results:

a) $R^\perp \equiv 0$ if, and only if, for each point $p \in M$ there is an orthonormal family $\{V^{(k)}\}_{1\le k\le m}$ of parallel fields in the normal bundle, defined in a neighborhood of p, that is,

(1.5.1) $$\nabla^\perp_X\, V^{(k)} \equiv 0\ ;\ \forall X\in \chi(M)\ ,\ k = 1,2,\dots,m$$

b) If N has constant curvature and $\{e_k\}_{1\le k\le m}$ is an ortho - normal frame of normal fields, then $R^\perp \equiv 0$ if, and only if, the tensors A^{e_k} are simultaneously diagonalized.

1.6 - <u>Proposition 1</u> - Let $M^n \hookrightarrow \mathbb{R}^{n+m}$ be an isometric immersion and let $\{V^{(k)}\}_{1\le k\le m}$ be an orthonormal family of parallel fields in the normal bundle. Thus, relatively to any coordinate system of $\mathbb{R}^{n+m}$, we have, at each point of M,

(1.6.1) $$\delta_i\, V_j^{(k)} = \delta_j\, V_i^{(k)}\ \ ;\ \ \begin{matrix}1\le i,j\le n+m\\ 1\le k\le m\end{matrix}$$

<u>Proof</u> - Let $p \in M$ and $\{e_i\}_{1 \leq i \leq n+m}$ be a coordinate system of $\mathbb{R}^{n+m}$ such that

$$V^{(k)}(p) = e_{n+k} \quad ; \quad k = 1,2,\ldots,m$$

Then,

$$[e_i, e_j](p) \in T_pM \quad \text{if} \quad 1 \leq i,j \leq n \implies$$

$$\implies \delta_i V_j^{(k)}(p) = \delta_j V_i^{(k)}(p) \quad \text{if} \quad 1 \leq i,j \leq n, \; 1 \leq k \leq m$$

and

$$\delta_{n+r} \equiv 0 \quad \text{at} \quad p \implies$$

$$\implies \delta_{n+r} V_i^{(k)}(p) = 0 \quad ; \quad 1 \leq i \leq n+p, \; 1 \leq k,r \leq m$$

On the other hand,

$$0 = \langle \bar{\nabla}_{ei} V^{(k)}, V^{(r)} \rangle_p = D_i V_{n+r}^{(k)}(p) = \delta_i V_{n+r}^{(k)}(p)$$

if $1 \leq i \leq n$ and $1 \leq k,r \leq m$

Therefore,

$$\delta_i V_{n+r}^{(k)}(p) = \delta_{n+r} V_i^{(k)}(p) \quad ; \quad 1 \leq i \leq n+m, \; 1 \leq k,r \leq m$$

Thus, relatively to $\{e_i\}$ we have

$$\delta_i V_j^{(k)}(p) = \delta_j V_i^{(k)}(p) \quad ; \quad 1 \leq i,j \leq n+m, \; 1 \leq k \leq m$$

Now let $\{\bar{e}_i\}_{1 \leq i \leq n+m}$ be another coordinate system such that

$$e_j = \sum_{i=1}^{n+m} \alpha_j^i \, \bar{e}_i \ , \ \bar{v}_i^{(k)} = < v^{(k)}, \bar{e}_i > \quad \text{and}$$

$\bar{\delta}_i = < \delta, \bar{e}_i >$. Then,

$$\left(\bar{\delta}_i \, \bar{v}_j^{(k)} - \bar{\delta}_j \, \bar{v}_i^{(k)}\right) (p) = \sum_{h,\ell=1}^{n+m} \alpha_h^j \, \alpha_\ell^i \left(\delta_i \, v_h^{(k)} - \delta_h \, v_i^{(k)}\right) (p) = 0$$

Then, $\delta_i \, v_j^{(k)} (p) = \delta_j \, v_i^{(k)} (p)$ with respect to any coordinate system of $\mathbb{R}^{n+m}$.

1.7 - Proposition 2 - Let $M^n \hookrightarrow \mathbb{R}^{n+m}$ be an isometric immersion and $\{v^{(k)}\}_{1 \le k \le m}$ be an orthonormal family of parallel fields in the normal bundle. Then at any point of M and relatively to any coordinate system of $\mathbb{R}^{n+m}$ we have

$$(1.7.1) \quad [\delta_i, \delta_j] = \sum_{k=1}^{m} \sum_{h=1}^{n+m} (v_i^{(k)} \, \delta_j \, v_h^{(k)} - v_j^{(k)} \, \delta_i \, v_h^{(k)}) \, \delta_h$$

where $[\delta_i, \delta_j] = \delta_i \delta_j - \delta_j \delta_i \quad ; \quad 1 \le i,j \le n+m$

Proof - It is made by a similar argument to that one in § 1.6 and by using (1.6.1) and the fact that

$$(1.7.2) \quad \sum_{h=1}^{n+m} v_h^{(r)} \, \delta_j \, v_h^{(k)} = \sum_{h=1}^{n+m} v_h^{(r)} \, \delta_h \, v_j^{(k)} = 0$$

if $1 \le j \le n+m$ and $1 \le k,r \le m$.

1.8 - Proposition 3 - Let $M^n \hookrightarrow \mathbb{R}^{n+m}$ be an isometric miniral immersion having an orthonormal family $\{v^{(k)}\}_{1 \le k \le m}$ of parallel fields in the normal bundle. If

$$\Delta v^{(k)} = (\Delta v_1^{(k)}, \ldots, \Delta v_{n+m}^{(k)}) \ , \quad \text{we have}$$

$$\Delta V^{(k)} = - C_k^2 V^{(k)} - \sum_{\substack{k,r=1 \\ k \neq r}}^{m} C_{kr} V^{(r)} \qquad (1.8.1)$$

Proof - From (1.6.1) and (1.7.1)

$$\Delta V_j^{(k)} = \sum_{i=1}^{n+m} \delta_i \delta_j V_i^{(k)} =$$

$$= \sum_{i=1}^{n+m} \delta_j \delta_i V_i^{(k)} + \sum_{r=1}^{m} \sum_{ih=1}^{n+m} (V_i^{(r)} \delta_j V_h^{(r)} - V_j^{(r)} \delta_i V_h^{(r)}) \delta_h V_i^{(k)}$$

Since $\sum_{i=1}^{n+m} \delta_i V_i^{(k)} \equiv 0$ and $\sum_{i=1}^{n+m} V_i^{(r)} \delta_h V_i^{(k)} = 0$, it follows that

$$\Delta V_j^{(k)} = - \sum_{r=1}^{m} V_j^{(r)} \left[\sum_{ih=1}^{n+m} (\delta_i V_h^{(r)}) (\delta_h V_i^{(k)}) \right]$$

Thus (1.2.3) and (1.24) imply (1.8.1).

Note - If $m = 2$ and $V^{(1)} = V$, $V^{(2)} = W$, them the formulas (1.8.1) are

$$\begin{cases} \Delta V = - C_V^2 V - C_{VW} W \\ \Delta W = - C_W^2 W - C_{VW} V \end{cases} \qquad (1.8.2)$$

where $C_V^2 = \|A^V\|^2$; $C_W^2 = \|A^W\|^2$ and $C_{VW} = \langle A^V, A^W \rangle$

§ 2 - 2.1 Proof of theorem 1

Let $p \in M$ and U be a neighborhood of p such that

$$U = \{(x, f(x), g(x)) = F(x) \;/\; x \in \Omega \subset \mathbb{R}^n\}$$

where Ω is an open subset of $\mathbb{R}^n$. If

$$h_{ij} = \langle D_iF , D_jF \rangle \quad \text{and} \quad h = \det(h_{ij}), \quad \text{then}$$

$$(2.1.1) \qquad h = (1+ |Df|^2)(1+ |Dg|^2) - \langle Df,Dg \rangle^2$$

Now let

$$\tilde{v}^{(1)} = \frac{1}{\sqrt{1+ |Df|^2}} (- D_1f ,\ldots, - D_nf, 1, 0)$$

$$\tilde{v}^{(2)} = \frac{1}{\sqrt{1+ |Dg|^2}} (- D_1g ,\ldots, - D_ng, 0, 1)$$

and $\{v^{(1)}, v^{(2)}\}$ obtained from orthonormalization of $\{\tilde{v}^{(1)},\tilde{v}^{(2)}\}$ in such way that $v^{(2)} = \tilde{v}^{(2)}$.

Thus

$$\begin{cases} v_i^{(1)} = \dfrac{-(1+ |Dg|^2) D_if + \langle Df,Dg \rangle D_ig}{\sqrt{h}\sqrt{1+ |Dg|^2}}, \quad \text{if } i \le n \\[2ex] v_{n+1}^{(1)} = \dfrac{\sqrt{1+ |Dg|^2}}{\sqrt{h}} \;; \qquad v_{n+2}^{(1)} = - \dfrac{\langle Df,Dg \rangle}{\sqrt{h}\sqrt{1+ |Dg|^2}} \end{cases}$$

$$\begin{cases} V_i^{(2)} = - \dfrac{D_i f}{\sqrt{1+ |Dg|^2}} , \quad \text{if } \ i \le n \\ \\ V_{n+1}^{(2)} = 0 \quad ; \quad V_{n+2}^{(2)} = \dfrac{1}{\sqrt{1+ |Dg|^2}} \end{cases}$$

and

$$V_{n+1}^{(1)} \ V_{n+2}^{(2)} - V_{n+1}^{(2)} \ V_{n+2}^{(1)} = \frac{1}{\sqrt{h}} \tag{2.1.2}$$

But for any two normal fields V and W that constitute an orthonormal frame having the same orientation of $\{V^{(1)}, V^{(2)}\}$, we have

$$V_{n+1} \ W_{n+2} - V_{n+2} \ W_{n+1} = V_{n+1}^{(1)} \ V_{n+2}^{(2)} - V_{n+1}^{(2)} \ V_{n+2}^{(1)}$$

Then in a neighborhood of each $p \in U$ we have

$$\frac{1}{\sqrt{h}} = V_{n+1} \ W_{n+2} - V_{n+2} \ W_{n+1}$$

where $\{V,W\}$ is a frame of parallel fields in the normal bundle of M.

Thus

$$\Delta \left(\frac{1}{\sqrt{h}} \right) = \Delta \left(V_{n+1} \ W_{n+2} - V_{n+2} \ W_{n+1} \right) =$$

$$= V_{n+1} \ \Delta W_{n+2} + W_{n+2} \ \Delta V_{n+1} - V_{n+2} \ \Delta W_{n+1} - W_{n+1} \ \Delta V_{n+2}$$

$$+ 2 \langle \delta V_{n+1}, \delta W_{n+2} \rangle - 2 \langle \delta V_{n+2}, \delta W_{n+1} \rangle =$$

$$= - \frac{1}{\sqrt{h}} \ C^2 + 2 \langle \delta V_{n+1}, \delta W_{n+2} \rangle - 2 \langle \delta V_{n+2}, \delta W_{n+1} \rangle$$

(see 1.8.2).

<u>Claim</u>: $\langle \delta V_{n+1}, \delta W_{n+2} \rangle - \langle \delta V_{n+2}, \delta W_{n+1} \rangle \equiv 0$

Indeed, $\delta_i V_j = \delta_j V_i$ and $\delta_i W_j = \delta_j W_i$ imply

$$\langle \delta V_{n+1}, \delta W_{n+2} \rangle - \langle \delta V_{n+2}, \delta W_{n+1} \rangle =$$

$$= \langle \delta_{n+1} V , \delta_{n+2} W \rangle - \langle \delta_{n+2} V, \delta_{n+1} W \rangle =$$

$$= \sum_{i=1}^{n+2} D_{n+1} \tilde{V}_i \, (D_{n+2} \tilde{W}_i - \sum_j W_{n+2} W_j D_j \tilde{W}_i - \sum_j V_{n+2} V_j D_j \tilde{W}_i)$$

$$- \sum_{i=1}^{n+2} D_{n+2} \tilde{W}_i \, (\sum_j V_{n+1} V_j D_j \tilde{V}_i + \sum_j W_{n+1} W_j D_j \tilde{V}_i)$$

$$- \sum_{i=1}^{n+2} D_{n+2} \tilde{V}_i \, (D_{n+1} \tilde{W}_i - \sum_j W_{n+1} W_j D_j \tilde{W}_i - \sum_j W_{n+1} W_j D_j \tilde{W}_i)$$

$$+ \sum_{i=1}^{n+2} D_{n+1} \tilde{W}_i \, (\sum_j V_{n+2} V_j D_j \tilde{V}_i + \sum_j W_{n+2} W_j D_j \tilde{V}_i) +$$

$$+ \frac{1}{\sqrt{h}} \sum_{ijk=1}^{n+2} \left[(V_j D_j \tilde{V}_i)(W_k D_k \tilde{W}_i) - (W_j D_j \tilde{V}_i)(V_k D_k \tilde{W}_i) \right]$$

where $\tilde{V}$ and $\tilde{W}$ are orthonormal extensions of V and W to a neighborhood of M in $\mathbb{R}^{n+2}$. It is easy to show that the last term of the last member does not depend on the extensions of V and W. Thus the expression formed by the remaining four terms of the last member does not depend on the extensions too.

But if the extensions are given by

$$\tilde{V}(x_1, \ldots, x_{n+2}) = V(x_1, \ldots, x_n, f(x), g(x))$$

$$\tilde{W}(x_1, \ldots, x_{n+2}) = W(x_1, \ldots, x_n, f(x), g(x))$$

we have $D_{n+1} \tilde{V}_i = D_{n+2} \tilde{V}_i = D_{n+2} \tilde{W}_i = D_{n+1} \tilde{W}_i = 0, \ \forall_i$ which implies that

(2.1.3) $\quad <\delta V_{n+1}, \delta W_{n+2}> - <\delta V_{n+2}, \delta W_{n+1}> =$

$$= \frac{1}{\sqrt{h}} \sum_{ijk=1}^{n+2} \left[(V_j D_j \tilde{V}_i)(W_k D_k \tilde{W}_i) - (W_j D_j \tilde{V}_i)(V_k D_k \tilde{W}_i) \right]$$

holds for any extensions $\tilde{V}$ and $\tilde{W}$ of V and W, respectively.

Now let $\tilde{V}$ and $\tilde{W}$ be extensions of V and W that are constant on each plane spanned by $V(y)$ and $W(y)$, $y \in M$.

For these extensions,

$$\bar{\nabla}_{v(y)} \tilde{V} = \bar{\nabla}_{w(y)} \tilde{W} \equiv 0$$

Since the second member of (2.1.3) does not depend on the coordinate system of $\mathbb{R}^{n+2}$, if $\{e_i\}_{1\le i\le n+2}$ is one such that

$$V(y) = e_{n+1} \quad \text{and}$$

$$W(y) = e_{n+2} \quad \text{we have}$$

$$D_{n+1} \tilde{V}_i \,(y) = < \bar{\nabla}_{v(y)} \tilde{V}, e_i > = 0 \quad , \quad \forall i$$

$$D_{n+2} \tilde{V}_i \,(y) = < \bar{\nabla}_{w(y)} \tilde{V}, e_i > = 0 \quad , \quad \forall i$$

This reduces the second member of (2.1.3) to

$$\frac{1}{\sqrt{h}}(y) \sum_{i=1}^{n+2} \left[(D_{n+1} \tilde{V}_i)(D_{n+2} \tilde{W}_i) - (D_{n+2} \tilde{V}_i)(D_{n+1} \tilde{W}_i) \right] (y) \equiv 0$$

and then

$$< \delta V_{n+1}, \delta W_{n+2} > - < \delta V_{n+2}, \delta W_{n+1} > = 0$$

Thus

$$\Delta\left(\frac{1}{\sqrt{h}}\right) = -\frac{1}{\sqrt{h}}\, c^2 \tag{2.1.4}$$

The function $w = \log\sqrt{h}$ is the one looked for, since

$$h \geq 1 + |Df|^2 + |Dg|^2 \geq 1 \implies w \geq 0$$

and

$$\Delta W = -\Delta \log\left(\frac{1}{\sqrt{h}}\right) \implies$$

$$\Delta W = -\sqrt{h}\sum_{i=1}^{n+2} \delta_i\delta_i\left(\frac{1}{\sqrt{h}}\right) + \sum_{i=1}^{n+2}\left(\frac{\delta_i(\sqrt{h})}{\sqrt{h}}\right)^2 \implies$$

$$\Delta W = c^2 + |\delta w|^2 \tag{2.1.5}$$

2.2 - Proof of theorem 2

Theorem 1 implies that for each $p \in M$ there is a neighbohood U of p and a function $w \geq 0$ on it such that

$$\Delta w = c^2 + |\delta w|^2$$

If $d = \text{dist}(p, \partial U)$ and $\phi^2 \in C_0^2(B_d(p))$, we have

$$\Delta w \geq |\delta w|^2 \implies \int_M \phi^2\,\Delta w\, dM \geq \int_M \phi^2\,|\delta w|^2\, dM \tag{2.2.1}$$

Since

$$\int_M \phi^2\,\Delta w\, dM = -\int_M \langle\delta\phi^2, \delta w\rangle\, dM = -2\int_M \phi\langle\delta\phi, \delta w\rangle\, dM$$

and

$$-2\phi\langle\delta\phi, \delta w\rangle \leq \tfrac{1}{2}\phi^2\,|\delta w|^2 + 2\,|\delta\phi|^2,$$

we have

$$(2.2.2) \qquad \int_M \phi^2 \, \Delta w \, dM \le \frac{1}{2} \int_M \phi^2 \, |\delta w|^2 \, dM + 2 \int_M |\delta \phi|^2 \, dM$$

From (2.2.1) and (2.2.2) it follows

$$(2.2.3) \qquad \int_M \phi^2 \, |\delta w|^2 \, dM \le 4 \int_M |\delta \phi|^2 \, dM \ , \ \forall \ \phi \in C_o^1(B_d(p))$$

On the other hand,

$$\Delta w \ge C^2 \implies \int_M \phi^2 \, \Delta w \, dM \ge \int_M \phi^2 \, C^2 \, dM \ ,$$

which, together with (2.2.2) and (2.2.3), implies

$$(2.2.4) \qquad \int_M \phi^2 \, C^2 \, dM \le 4 \int_M |\delta \phi|^2 \, dM \ , \ \forall \ \phi \in C_o^1(B_d(p))$$

Now given ρ and σ such that $0 < \rho < \sigma < d$, let $\{\phi_j\}$ be a sequence of functions of $C_o^1 \, (B_d(p))$ such that

$$0 \le \phi_j \, (x) \le 1 \ , \ \forall \ x \in B_d(p)$$

$$\phi_j \, (x) = 1 \ \text{ if } \ x \in B_\rho(p)$$

$$\phi_j \, (x) = 0 \ \text{ if } \ x \notin B_\sigma(p)$$

and

$$\limsup_j \ |D\phi_j(x)| = \frac{1}{\sigma - \rho}$$

For each j, (2.2.1) implies that

$$(2.2.5) \qquad \int_M \phi_j^2 \, C^2 \, dM \le 4 \int_M |\delta\phi_j|^2 \, dM \le$$

$$\le 4 \limsup_j |D\phi_j(x)|^2 \int_{M\cap B_\sigma(p)} dM \le$$

$$\le \frac{4}{(\sigma-\rho)^2} \int_{M\cap B_\sigma(p)} dM$$

Then

$$(2.2.6) \qquad \int_{M\cap B_\rho(p)} C^2 \, dM \le \frac{4}{(\sigma-\rho)^2} \int_{M\cap B_\sigma(p)} dM$$

Note - If $d = +\infty$ and $\sigma = 2_\rho$, (2.2.6) implies

$$(2.2.7) \qquad \int_{M\cap B_\rho(p)} C^2 \, dM \le \frac{4}{\rho^2} \int_{M\cap B_{2\rho}(p)} dM$$

Besides if $\frac{1}{4\rho^n} \int_{M\cap B_\rho(p)} dM$ is a bounded function of ρ, that is if the area of $M\cap B_\rho(p)$ is not much bigger then the area of sphere in $\mathbb{R}^n$, we have

$$(2.2.8) \qquad \lim_{\rho\to\infty} \frac{1}{\rho^{n-2+\varepsilon}} \int_{M\cap B_\rho(p)} C^2 \, dM = 0 \, , \ \forall \, \varepsilon > 0$$

§ 3 - 3.1 - Proposition 4 - Let $M^n \hookrightarrow \mathbb{R}^{n+m}$ be an isometric minimal immersion having flat normal bundle and let $\{V^{(k)}\}_{1\le k\le m}$ be an orthonormal family of parallel fields defined in an open set $U \subset M$. If $\{e_i\}_{1\le i\le n+m}$ is a coordinate system of $\mathbb{R}^{n+m}$ such that $V^{(k)}(p) = e_{n+k}$, $k = 1,2,\ldots,m$, at some $p \in U$, we have

$$(3.1.1) \qquad \left(\tfrac{1}{2}\,\Delta C_k^2 + C_k^4 + \sum_{\substack{k,r=1\\k\neq r}}^{m} C_{kr}\right)(p) = \sum_{h,i,j=1}^{m} \left(\delta_i\delta_j V_h^{(k)}\right) \quad (p)$$

Proof - $R^{\perp} \equiv 0$ implies

$$C_k^2 = \sum_{ij=1}^{n+m} \left(\delta_i V_j^{(k)}\right)\left(\delta_j V_i^{(k)}\right) = \sum_{ij=1}^{n+m} \left(\delta_i V_j^{(k)}\right)^2$$

Thus

$$\tfrac{1}{2}\,\Delta C_k^2 = \tfrac{1}{2}\sum_{i=1}^{n+m} \delta_i\delta_i\left(\sum_{hj=1}^{n+m}\left(\delta_h V_j^{(k)}\right)^2\right) =$$

$$\sum_{hij=1}^{n+m} \left(\delta_i\delta_h V_j^{(k)}\right)^2 + \sum_{hij=1}^{n+m} \left(\delta_h V_j^{(k)}\right)\left(\delta_i\delta_i\delta_h V_j^{(k)}\right)$$

Using the comutator $[\delta_i\delta_h]$ twice, we get

$$(3.1.2) \qquad \tfrac{1}{2}\,\Delta C_k^2 = \sum_{hij=1}^{n+m} \left(\delta_i\delta_h V_j^{(k)}\right)^2 + \sum_{hij=1}^{n+m} \left(\delta_h V_j^{(k)}\right)\delta_h\left(\Delta V_j^{(k)}\right) +$$

$$+ \sum_{\ell=1}^{m}\sum_{hijs=1}^{n+m} V_i^{(\ell)}\left(\delta_h V_j^{(k)}\right)\left(\delta_h V_s^{(\ell)}\right)\left(\delta_s\delta_i V_j^{(k)}\right) -$$

$$- \sum_{\ell=1}^{m}\sum_{hijs=1}^{n+m} \left(\delta_h V_j^{(k)}\right)\left(\delta_i V_h^{(\ell)}\right)\left(\delta_i V_s^{(\ell)}\right)\left(\delta_s V_j^{(k)}\right)$$

But from (1.8.1) ,

$$\delta_h(\Delta V_j^{(k)}) = -(\delta_h C_k^2)\, V_j^{(k)} - (\delta_h V_j^{(k)})\, C_k^2 - \sum_{\substack{r=1 \\ r\neq k}}^{m} \delta_h(C_{kr})\, V_j^{(r)} -$$

$$- \sum_{\substack{r=1 \\ r\neq k}}^{m} C_{kr}\, \delta_h V_j^{(r)}$$

Thus, (1.2.3) and (1.7.2), implies

$$\sum_{hj=1}^{n+m} (\delta_h V_j^{(k)})\, \delta_h(\Delta V_j^{(k)}) = - C_k^4 - \sum_{\substack{r=1 \\ r\neq k}}^{m} C_{kr}^2$$

So from (3.1.2) we have

$$\text{(3.1.3)} \qquad \tfrac{1}{2}\Delta C_k^2 + C_k^4 + \sum_{\substack{r=1 \\ r\neq k}}^{m} C_{kr}^2 =$$

$$= \sum_{hij=1}^{n+m} (\delta_i \delta_h V_j^{(k)})^2 - 2 \sum_{\ell=1}^{m} \sum_{hijs=1}^{n+m} (\delta_h V_j^{(k)})(\delta_h V_s^{(\ell)})(\delta_s V_i^{(\ell)})(\delta_i V_j^{(k)})$$

Now, using $[\delta_h, \delta_{n+r}]$,

$$\delta_n\, \delta_{n+r}\, V_j^{(k)}\ (p) = - \sum_{s=1}^{n+m} (\delta_h V_s^{(r)})\ (\delta_s V_j^{(k)})\ (p)$$

Then we have at p:

$$\text{(3.1.4)} \qquad (\tfrac{1}{2}\Delta C_k^2 + C_k^4 + \sum_{\substack{r=1 \\ r\neq k}}^{m} C_{kr}^2)\ (p) =$$

$$= \sum_{hij=1}^{n+m} (\delta_i \delta_h V_j^{(k)})^2\ (p) - 2 \sum_{\ell=1}^{m} \sum_{sj=1}^{n+m} (\delta_s \delta_{n+\ell} V_j^{(k)})^2\ (p)$$

But,

$$\delta_i \; \delta_{n+\ell} \; V_{n+j}^{(k)} \; (p) = - \sum_{s=1}^{n} (\delta_i V_s^{(\ell)}) \; (\delta_s V_{n+j}^{(k)}) \; (p)$$

and
$$\delta_s V_{n+j}^{(k)} \; (p) = < \nabla^{\perp}_{e_s} \; V^{(k)} , e_{n+j} >_p = 0$$

Thus

$$(\tfrac{1}{2} \Delta C_k^2 + C_k^4 + \sum_{\substack{r=1 \\ r\neq k}}^{m} C_{kr}^2) \; (p) =$$

$$= \sum_{hij=1}^{n} (\delta_i \delta_h V_j^{(k)})^2 \; (p) + \sum_{\ell=1}^{m} \sum_{ih=1}^{n} (\delta_i \delta_h V_{n+\ell}^{(k)})^2 \; (p) +$$

$$+ \sum_{\ell=1}^{m} \sum_{ij=1}^{n} (\delta_i \; \delta_{n+\ell} \; V_j^{(k)})^2 \; (p) - 2 \sum_{\ell=1}^{m} \sum_{sj=1}^{n} (\delta_i \delta_{n+\ell} \; V_j^{(k)})^2 \; (p) =$$

$$= \sum_{hij=1}^{n} (\delta_i \delta_h V_j^{(k)})^2 \; (p)$$

which gives (3.1.1)

Note - $\quad C^2 = \sum_{k=1}^{m} C_k^2 \Rightarrow \Delta C^2 = \sum_{k=1}^{m} \Delta C_k^2$

and
$$C_{kr} = < A^{V^{(k)}} , A^{V^{(r)}} > \leq C_k \; C_r$$

implies
$$\sum_{k=1}^{m} C_k^4 + \sum_{k\neq r}^{m} C_{kr}^2 \leq \sum_{k=1}^{m} C_k^4 + \sum_{k\neq r}^{m} C_k^2 \; C_r^2 =$$

$$= \sum_{k=1}^{m} C_k^4 + 2 \sum_{k<r}^{m} C_k^2 \; C_r^2 = (\sum_{k=1}^{m} C_k^2)^2 = C^4$$

Then, from (3.1.1) we have

$$(3.1.5)\qquad \left(\tfrac{1}{2}\,\Delta C^2 + C^4\right)(p) \geq \sum_{k=1}^{m}\ \sum_{hij=1}^{n} (\delta_i\delta_h V_j^{(k)})^2\,(p)$$

3.2 - Proof of theorem 3

Since $R^{\perp} \equiv 0$ we may choose a coordinate system $\{e_i\}_{1\leq i\leq n+m}$ of $\mathbb{R}^{n+m}$ such that $V^{(k)}(p) = e_{n+k}$, $k = 1,2,\ldots,m$ at some $p \in M$ and it diagonalizes simultaneously the tensors $A^{V^{(k)}}$ at p, which implies that it diagonalizes the matrix $(\delta_i V_j^{(k)}(p))$, $k = 1,2,\ldots,m$, for

$$\langle A^{V^{(k)}} e_i, e_j\rangle_p = -\langle\bar{\nabla}_{e_i} V^{(k)}, e_j\rangle_p = -D_i V_j^{(k)}(p) = -\delta_i V_j^{(k)}(p),$$

$1 \leq i,j \leq n$; $1 \leq k \leq m$.

Thus,

$$C^2 = \sum_{k=1}^{m}\ \sum_{hj=1}^{n+m} (\delta_h V_j^{(k)})^2$$

$$|\delta C|^2\,(p) = \frac{1}{C^2(p)} \sum_{i=1}^{n+m}\left[\sum_{k=1}^{m}\ \sum_{hj=1}^{n+m} (\delta_h V_j^{(k)})(\delta_i\delta_h V_j^{(k)})\right]^2 (p) =$$

$$= \frac{1}{2C^2(p)} \sum_{i=1}^{n}\left[\sum_{\ell k=1}^{m}\ \sum_{hr=1}^{n} 2(\delta_h V_h^{(k)})(\delta_i\delta_h V_h^{(k)})(\delta_r V_r^{(\ell)})(\delta_i\delta_r V_r^{(\ell)})\right](p) \leq$$

$$\leq \frac{1}{2C^2(p)} \sum_{i=1}^{n}\left[\sum_{k\ell=1}^{m}\ \sum_{hr=1}^{n} (\delta_h V_h^{(k)})^2 (\delta_i\delta_r V_r^{(\ell)})^2 + \sum_{k\ell=1}^{m}\ \sum_{hr=1}^{n} (\delta_r V_r^{(\ell)})^2 (\delta_i\delta_h V_h^{(k)})^2\right](p)$$

So,

$$(3.2.1) \qquad |\delta c|^2 (p) \leq \sum_{k=1}^{m} \sum_{ih=1}^{n} (\delta_i \delta_h V_n^{(k)})^2 (p) =$$

$$= \sum_{k=1}^{m} \left[\sum_{i=1}^{n} (\delta_i \delta_i V_i^{(k)})^2 + \sum_{i=1}^{n} \sum_{\substack{h=1 \\ h \neq i}}^{n} (\delta_i \delta_h V_h^{(k)})^2 \right] (P)$$

But,

a) $\sum_{h=1}^{n+m} \delta_i \delta_h V_h^{(k)} = 0 \; ; \; k = 1,2,\ldots,m \; , \; i = 1,2,\ldots,n+m$

for the immersion is minimal,

b) $\delta_{n+\ell} \equiv 0$ at p implies

$$\delta_i \delta_{n+\ell} V_{n+\ell}^{(k)} (p) = \delta_{n+\ell} \delta_i V_{n+\ell}^{(k)} (p) + \sum_{r=1}^{m} \sum_{s=1}^{n+m} (V_i^{(r)} \delta_{n+\ell} V_s^{(r)} - V_{n+\ell}^{(r)} \delta_i V_s^{(r)}) \delta_s V_{n+\ell}^{(k)} (p) =$$

$$= - \sum_{s=1}^{n} (\delta_i V_s^{(\ell)}) (\delta_s V_{n+\ell}^{(k)}) (p)$$

c) $R^{\perp} \equiv 0$ implies $\delta_s V_{n+\ell}^{(k)} (p) = 0$

Thus, $\sum_{\substack{h=1 \\ h \neq i}}^{n} \delta_i \delta_h V_h^{(k)} (p) = - \delta_i \delta_i V_i^{(k)} (p) \; ; \; i \leq n$

So,

$$(3.2.2) \qquad \sum_{i=1}^{n} (\delta_i \delta_i V_i^{(k)})^2 (p) = \sum_{i=1}^{n} \left(\sum_{\substack{h=1 \\ h \neq i}}^{n} \delta_i \delta_h V_h^{(k)} \right)^2 (p) \leq$$

$$\leq (n-1) \sum_{i=1}^{n} \sum_{\substack{h=1 \\ h \neq i}}^{n} (\delta_i \delta_h V_n^{(k)})^2 (p)$$

which together with (3.2.1) implies

$$(3.2.3) \qquad |\delta c|^2 (p) \le n \sum_{k=1}^{m} \sum_{\substack{ih=1 \\ i\neq h}}^{n} (\delta_i \delta_h V_h^{(k)})^2 (p)$$

Now, from (1.7.1) we have

$$\delta_i \delta_h V_h^{(k)} (p) = \delta_h \delta_i V_h^{(k)} (p) + \sum_{\ell=1}^{m} \sum_{s=1}^{n+m} (V_i^{(\ell)} \delta_h V_s^{(\ell)} - V_h^{(\ell)} \delta_i V_s^{(\ell)}) \delta_s V_h^{(k)} (p) =$$

$$= \delta_h \delta_i V_h^{(k)} (p) = \delta_h \delta_h V_i^{(k)} (p) \quad ; \quad \begin{matrix} 1 \le i,h \le n \\ 1 \le k \le m \end{matrix}$$

and so,

$$\sum_{hij=1}^{n} (\delta_i \delta_h V_j^{(k)})^2 (p) = \sum_{i=1}^{n} (\delta_i \delta_i V_i^{(k)})^2 (p) + 3 \sum_{\substack{ih=1 \\ i\neq h}}^{n} (\delta_i \delta_h V_h^{(k)})^2 (p) +$$

$$+ \sum_{\substack{hij=1 \\ i\neq h, h\neq j, j\neq i}}^{n} (\delta_i \delta_h V_j^{(k)})^2 (p) \ge$$

$$\ge 2 \sum_{\substack{ih=1 \\ i\neq h}}^{n} (\delta_i \delta_h V_h^{(k)})^2 (p) + \left[\sum_{i=1}^{n} (\delta_i \delta_i V_i^{(k)})^2 + \sum_{\substack{ih=1 \\ i\neq h}}^{n} (\delta_i \delta_h V_h^{(k)})^2 \right] (p)$$

Thus (3.2.2) and (3.2.3) imply

$$(3.2.4) \qquad \sum_{k=1}^{m} \sum_{hij=1}^{n} (\delta_i \delta_h V_j^{(k)})^2 (p) \ge (1 + \tfrac{2}{n}) |\delta c|^2 (p)$$

From (3.1.5) and (3.2.4) we have

$$(3.2.5) \qquad (\tfrac{1}{2} \Delta C^2 + C^4) (p) \ge (1 + \tfrac{2}{n}) |\delta c|^2 (p)$$

which is true at any $p \in M$, for the elements involved in the inequality are independent of $\{e_i\}$.

Analogous computations give the inequalities

$$|\delta C_k|^2 (p) \le n \sum_{\substack{ih=1 \\ i \neq h}}^{n} (\delta_i \delta_h V_h^{(k)}) (p) \; ; \; k = 1,2,\ldots,m$$

which, together with (3.1.1), gives

$$(3.2.6) \quad \tfrac{1}{2} \Delta C_k^2 + C_k^4 + \sum_{\substack{r=1 \\ r \neq k}}^{m} C_{kr}^2 \ge (1 + \tfrac{2}{n}) |\delta C_k|^2 \; ; \; k = 1,2,\ldots,m$$

§ 4 - 4.1 - The second variation of the area.

Let $f: M^r \hookrightarrow \bar{M}$ be an immersion of a compact Manifold M in $\bar{M}$ and let $F: M \times (-\varepsilon,\varepsilon) \longrightarrow \bar{M}$ be a variation of f such that $F_t(\partial M) = f(\partial M)$, $\forall t$.

If E is the variational field of F, we set

$$(4.1.1) \qquad A(t) = \int_M dM_t \quad ; \quad t \in (-\varepsilon,\varepsilon)$$

where dM_t is the volume form of $M_t = F(M \times \{t\})$.

Thus,

$$(4.1.2) \qquad A'(0) = -\int_M \langle E^N , H \rangle \, dM$$

where $dM = dM_0$, and if f is a minimal immersion,

$$(4.1.3) \qquad A''(0) = \int_M \langle -\nabla^2 E^N + \bar{R}(E^N) - \tilde{A}(E^N), E^N \rangle \, dM$$

where $\bar{R}$ is the curvature tensor of $\bar{M}$

$\tilde{A}$ is defined by $\langle \tilde{A}(v), w \rangle_p = \langle A^v, A^w \rangle_p$ where v and w are normal vectors of M at $p \in M$.

∇^2 is the laplacian in the normal bundle which is defined by

$$\nabla^2 v = \sum_{i=1}^{r} (\nabla^{\perp}_{e_i} \nabla^{\perp}_{e_i} v - \nabla^{\perp}_{\nabla_{e_i} e_i} v)$$

where $\{ei\}_{1\le i\le r}$ is an orthonormal tangent frame, ∇ is the connection of M and $\nabla^{\perp}$ is the connection of the normal bundle. (see [4])

Besides if $f: M^n \hookrightarrow \mathbb{R}^{n+m}$ is an isometric minimal immersion having flat normal bundle $\{V^{(k)}\}_{1\le k\le m}$ is an orthonormal family of parallel fields in the normal bundle defined in an open set $U\subset M$, and E is the variational field of a variation F of f such that

$$E^N = \sum_{k=1}^{m} h_k V^{(k)} \quad ; \; h_k \in C_0^1(U) \quad , \; k = 1,2,\ldots,m$$

we have

$$\int_M < -\nabla^2 E^N, E^N > dM = - \int_M < \nabla^{\perp} E^N, \nabla^{\perp} E^N > dM$$

and

$$(4.1.4) \qquad < \nabla^{\perp} E^N, \nabla^{\perp} E^N > = \sum_{i=1}^{n} < \nabla^{\perp}_{ei} E^N, \nabla^{\perp}_{ei} E^N > =$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{m} (ei\,[h_k])^2$$

where $\{e_i\}_{1\le i\le n}$ is an orthonormal tangent frame.

Now let $\{e_i\}_{1\le i\le n+m}$ be a coordinate system of $\mathbb{R}^{n+m}$ such that

$$V^{(k)}(p) = e_{n+k} \quad ; \quad k = 1,2,\ldots,m$$

at some $p \in U$.

Then, from (4.1.4),

(4.1.5) $$< \nabla^{\perp} E^N, \nabla^{\perp} E^N >_p = \sum_{i=1}^{n} \sum_{k=1}^{m} (D_i h_k)^2 (p) -$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{m} (\delta_i h_k)^2 (p) = \sum_{k=1}^{m} \sum_{i=1}^{n+m} (\delta_i h_k)^2 (p) =$$

$$= \sum_{k=1}^{m} |\delta h_k|^2 (p)$$

On the other hand, we have

(4.1.6) $$< \bar{R}(E^N), E^N > = 0$$ and

(4.1.7) $$- < \tilde{A}(E^N), E^N > = - < A^{E^N}, A^{E^N} > =$$

$$= - \sum_{k=1}^{m} h_k^2 C_k^2 - \sum_{\substack{kr=1 \\ k \neq r}}^{m} h_k h_r C_{kr}$$

Thus from (4.1.3), (4.1.5), (4.1.6) and (4.1.7) we have

(4.1.8) $$A''(0) = \int_M \left[\sum_{k=1}^{m} (|\delta h_k|^2 - h_k^2 C_k^2) - \sum_{\substack{kr=1 \\ k \neq r}}^{m} h_k h_r C_{kr} \right] dM$$

In particular if $m = 2$ and $V^{(1)} = V$ and $V^{(2)} = W$,

(4.1.9) $$A''(0) = \int_M (|\delta h_1|^2 + |\delta h_2|^2 - h_1^2 C_V^2 - h_2^2 C_W^2 - h_1 h_2 C_{VW}) \, dM$$

where $E^N = h_1 V + h_2 W$; $h_1, h_2 \in C_o^1(U)$

4.2 - Proof of theorem 4

Since M is simply connected, there is an orthonormal family of parallel fields $\{V,W\}$, globally defined in the normal bundle [3].

Let E be the variational field of a variation of M in $\mathbb{R}^{n+2}$ given by

$$E^N = h_1 V + h_2 W \quad ; \quad h_1, h_2 \in C_o^1(M)$$

As M is stable, $A''(0) \geq 0, \forall\ h_1 h_2 \in C_o^1(M)$.

In particular let

$$h_1 = \phi\, C_V^{1+q} \qquad \text{and} \qquad h_2 = 0$$

where

$$q \leq \sqrt{\frac{2}{n}} \qquad \text{and} \qquad \phi \in C_o^1(M)$$

Thus,

$$A''(0) = \int_M (|\delta h_1|^2 - h_1^2\, C_V^2)\, dM \geq 0 \qquad \text{implies}$$

$$\int_M |\delta h_1|^2\, dM \geq \int_M \phi^2\, C_V^{2q+4}\, dM \tag{4.2.1}$$

But

$$|\delta h_1|^2 = |\ \delta(\phi\, C_V^{1+q})\ |^2 = \tag{4.2.2}$$

$$= C_V^{2q+2}\, |\delta\phi|^2 + (1+q)^2\, \phi^2 C_V^{2q}\, |\delta C_V|^2 + \tfrac{1}{2} \langle \delta\phi^2, \delta C_V^{2q+2} \rangle$$

and

$$(4.2.3) \qquad \int_M |\delta h_1|^2 \, dM = \int_M C_v^{2q+2} |\delta\phi|^2 \, dM + (1+q)^2 \int_M \phi^2 C_v^{2q} |\delta C_v|^2 \, dM - $$

$$- \frac{1}{2} \int_M \phi^2 \, \Delta C_v^{2q+2} \, dM$$

Using (3.2.6)

$$\Delta C_v^{2q+2} = 4q\,(1+q)\, C_v^{2q} |\delta C_v|^2 + (1+q)\, C_v^{2q} \Delta C_v^2 \ \geq$$

$$\geq 4q\,(1+q)\, C_v^{2q} |\delta C_v|^2 + 2(1+q)\, C_v^{2q} \left(-C_v^4 - C_{vw}^2 + \left(1 + \frac{2}{n}\right) |\delta C_v|^2\right)$$

So, from (4.2.3) we have

$$(4.2.4) \qquad \int_M |\delta h_1|^2 \, dM \leq \int_M C_v^{2q+2} |\delta\phi|^2 \, dM - (1+q)\left(q + \frac{2}{n}\right) \int_M \phi^2 C_v^{2q} |\delta C_v|^2 \, dM +$$

$$+ (1+q) \int_M \phi^2 C_v^{2q+4} \, dM + (1+q) \int_M \phi^2 C_v^{2q} C_{vw}^2 \, dM$$

(4.2.1) and (4.2.4) imply

$$(4.2.5) \qquad (1+q)\left(q + \frac{2}{n}\right) \int_M \phi^2 C_v^{2q} |\delta C_v|^2 \, dM \leq$$

$$\leq \int_M C_v^{2q+2} |\delta\phi|^2 \, dM + q \int_M \phi^2 C_v^{2q+4} \, dM + (1+q) \int_M \phi^2 C_v^{2q} C_{vw}^2 \, dM$$

On the other hand, from (4.2.2) again, we have

$$|\delta h_1|^2 \le C_V^{2q+2}\,|\delta\phi|^2 + (1+q)^2\,\phi^2\,C_V^{2q}\,|\delta C_V|^2 +$$

$$+ (1+q)\left[\varepsilon\phi^2 C_V^{2q}\,|\delta C_V|^2 + \frac{1}{\varepsilon}\,C_V^{2q+2}\,|\delta\phi|^2\right],\ \forall\,\varepsilon > 0$$

where we have used that $2ab \le \varepsilon a^2 + \frac{1}{\varepsilon}\,b^2$ with $a = \phi\,C_V^q\,\delta_i C_V$ and $b = C_V^{1+q}\,\delta_i\phi$.

From (4.2.1) we have

$$(4.2.6)\qquad \int_M \phi^2\,C_V^{2q}\,|\delta C_V|^2\,dM \ge -\frac{1+\frac{1+q}{\varepsilon}}{(1+q)(1+q+\varepsilon)}\int_M C_V^{2q+2}\,|\delta\phi|^2\,dM + \frac{1}{(1+q)(1+q+\varepsilon)}\int_M \phi^2\,C_V^{2q+4}\,dM$$

Now from (4.2.5) and (4.2.6) it follows

$$(4.2.7)\qquad \left(\frac{q+\frac{2}{n}}{1+q+\varepsilon} - q\right)\int_M \phi^2\,C_V^{2q+4}\,dM - (1+q)\int_M \phi^2\,C_V^{2q}\,C_{VW}^2\,dM \le$$

$$\le \left(1 + \frac{\left(1+\frac{1+q}{\varepsilon}\right)\left(q+\frac{2}{n}\right)}{1+q+\varepsilon}\right)\int_M C_V^{2q+2}\,|\delta\phi|^2\,dM$$

We observe that

$$\frac{q+\frac{2}{n}}{1+q} - q > 0 \iff q^2 < \frac{2}{n}$$

Thus, for $\varepsilon > 0$ sufficiently small and $q^2 < \frac{2}{n}$,

$\dfrac{q+\frac{2}{n}}{1+q+\varepsilon} - q > 0$, which together with $C_{VW} = \langle A^V, A^W \rangle \equiv 0$ and

(4.2.7) imply

$$(4.2.8) \qquad \int_M \phi^2 \, C_v^{2q+4} \, dM \le \gamma(n,q,\varepsilon) \int_M C_v^{2q+2} \, |\delta\phi|^2 \, dM$$

On the same way taking a variation such that $h_1 = 0$ and $h_2 = \phi \, C_w^{1+q}$, where ϕ and q have the same meaning as above, we have

$$(4.2.9) \qquad \int_M \phi^2 \, C_w^{2q+4} \, dM \le \gamma(n,q,\varepsilon) \int_M C_w^{2q+2} \, |\delta\phi|^2 \, dM$$

Thus from (4.2.8) and (4.2.9) we have

$$(4.2.10) \qquad \int_M \phi^2 \, (C_v^{2q+4} + C_w^{2q+4}) \, dM \le \gamma(n,q,\varepsilon) \int_M (C_v^{2q+2} + C_w^{2q+2}) \, |\delta\phi|^2 \, dM$$

Recalling the Young's inequality $ab \le \frac{\alpha^s a^s}{s} + \frac{\alpha^{-t} b^t}{t}$,

that is true for all $a \ge 0$, $b \ge 0$, $\alpha > 0$,
$1 < s < +\infty$ and $1 < t < +\infty$ with $\frac{1}{s} + \frac{1}{t} = 1$

Now choose r such that $0 < r < 2q + 2$ and $2 > 0$, and set

$$C_v^{2q+2} \, |\delta\phi|^2 = \phi^2 \, (C_v^{2q+2-r} \, C_v^r \, \frac{|\delta\phi|^2}{\phi^2}) \le$$

$$\le \phi^2 \left[\frac{\alpha^s}{s} \, C_v^{s(2q+2-r)} + \frac{\alpha^{-t}}{t} \, (C_v^r \, \frac{|\delta\phi|^2}{\phi^2})^t \right]$$

Taking r,s and t such that

$$\begin{cases} s\ (2q + 2-r) = 2q + 4 \\ rt = 2 \\ \frac{1}{s} + \frac{1}{t} = 1 \end{cases}$$

we have that $t = 1+q$, $r = \frac{2}{1+q}$ and $s = \frac{1+q}{q}$ is a solution of this system of equations.

An analogous inequality is true for C_w. Thus (4.2.10) implies

$$\int_M \phi^2 \ (C_v^{2q+4} + C_w^{2q+4})\ dM \le$$

$$\le \gamma(n,q,\varepsilon) \left[\frac{\alpha^s}{s} \int_M \phi^2 (C_v^{2q+4} + C_w^{2q+4})\ dM + \frac{\alpha^{-t}}{t} \int_M \frac{(C_v^2+C_w^2)\ |\delta\phi|^{2q+2}}{\phi^{2q}}\ dM \right]$$

So, choosing α small enough to have

$$1 - \frac{\gamma(n,q,\varepsilon)\ \alpha^s}{s} > 0 , \quad \text{we have}$$

$$\int_M \phi^2 \ (C_v^{2q+4} + C_w^{2q+4})\ dM \le \beta(n,q,\varepsilon) \int_M \frac{C^2\ |\delta\phi|^{2q+2}}{\phi^{2q}}\ dM$$

Taking ϕ in the place of ϕ^{1+q} we get

$$\text{(4.2.11)} \quad \int_M \phi^{2q+2}\ (C_v^{2q+4} + C_w^{2q+4})\ dM \le$$

$$\le \beta(n,q,\varepsilon) \int_M C^2\ |\delta\phi|^{2q+2}\ dM\ ;\ \forall\ \phi \in C_0^1(M)$$

Now, let $p \in M$ and $B_R(p)$ be a family of balls in $\mathbb{R}^{n+2}$ such that

$$\bigcup_{R \in (0,+\infty)} (B_R(p) \cap M) = M$$

For a given R, if ϕ is a function such that

$$0 \le \phi(x) \le 1, \quad \forall x$$
$$\phi(x) = 1 \quad \text{if} \quad x \in B_{R/2}(p)$$
$$\phi(x) = 0 \quad \text{if} \quad x \in B_R(p)$$
$$|D\phi| \le \frac{2}{R},$$

we have from (4.2.11) ,

$$\int_{M \cap B_{R/2}(p)} (C_V^{2q+4} + C_W^{2q+4})\, dM \le \beta(n,q,\varepsilon) \frac{2^{2q+2}}{R^{2q+2}} \int_{M \cap B_R(p)} C^2\, dM$$

and so from (2.2.7) ,

$$\int_{M \cap B_{R/2}(p)} (C_V^{2q+4} + C_W^{2q+4})\, dM \le \beta(n,q,\varepsilon) \frac{2^{2q+2}}{R^{2q+4}} \int_{M \cap B_{2R}(p)} dM$$

Besides if $2q+4 > n$ and $R \longrightarrow +\infty$, the limit of the second side will be zero. On this case,

$$\int_M C_V^{2q+4} + C_W^{2q+4})\, dM = 0$$

which implies $C_V = C_W = 0$, that is, $C = 0$. But then M will be totally geodesic in $\mathbb{R}^{n+2}$, against the hypothesis.

This proves (a) because

$$2q + 4 > n \quad \text{and} \quad q^2 < \frac{2}{n} \quad \text{imply}$$

$$n < 2\sqrt{\frac{2}{n}} + 4 \text{ , i.e, } n \le 5.$$

Now let $\{\tilde{v}, \tilde{w}\}$ be an orthonormal family of parallel fields in the normal bundle. We have that

$$\frac{q + \frac{2}{n}}{1+q} - q > 1+q \iff 1 + 2q + 2q^2 < \frac{2}{n}$$

and so, for $\varepsilon > 0$ sufficiently small,

$$\frac{q + \frac{2}{n}}{1+q+\varepsilon} - q > 1+q \quad \text{if} \quad 1 + 2q + 2q^2 < \frac{2}{n}$$

Then from (4.2.7),

$$\int_M \phi^2 \, (C_{\tilde{v}}^{2q+4} - C_{\tilde{v}}^{2q} \, C_{\tilde{v}\tilde{w}}^2) \, dM \le$$

$$\le \beta(n,q,\varepsilon) \int_M C_{\tilde{v}}^{2q+2} \, |\delta\phi|^2 \, dM$$

and

$$\int_M \phi^2 \, (C_{\tilde{w}}^{2q+4} - C_{\tilde{w}}^{2q} \, C_{\tilde{v}\tilde{w}}^2) \, dM \le$$

$$\le \beta(n,q,\varepsilon) \int_M C_{\tilde{w}}^{2q+2} \, |\delta\phi|^2 \, dM$$

Then adding these two inequalities,

$$\int_M \phi^2\left[C_{\tilde{v}}^{2q}\ (C_{\tilde{v}}^4 - C_{\tilde{v}\tilde{w}}^2) + C_{\tilde{w}}^{2q}\ (C_{\tilde{w}}^4 - C_{\tilde{v}\tilde{w}}^2)\right] dM \le$$

$$\le \beta(n,q,\varepsilon) \int_M (C_{\tilde{v}}^{2q+2} + C_{\tilde{w}}^{2q+2})\ |\delta\phi|^2\ dM$$

Since $C_{\tilde{v}\tilde{w}}^2 \le C_{\tilde{v}}^2\, C_{\tilde{w}}^2$ we get

$$(4.2.12) \quad \int_M \phi^2\ (C_{\tilde{v}}^{2q+2} - C_{\tilde{w}}^{2q+2})(C_{\tilde{v}}^2 - C_{\tilde{w}}^2)\ dM \le$$

$$\le \beta(n,q,\varepsilon) \int_M (C_{\tilde{v}}^{2q+2} + C_{\tilde{w}}^{2q+2})\ dM$$

By a similar argument to that one made with (4.2.7) we get from (4.2.12)

$$\int_{M\cap B_{R/2}(p)} (C_{\tilde{v}}^{2q+2} - C_{\tilde{w}}^{2q+2})(C_{\tilde{v}}^2 - C_{\tilde{w}}^2)\ dM \le$$

$$\le \beta(n,q,\varepsilon)\ \frac{2^{2q+2}}{R^{2q+4}} \int_{M\cap B_{2R}(p)} dM$$

Now if we let $R \longrightarrow +\infty$ when

$2q + 4 > n$ and $1 + 2q + 2q^2 < \frac{2}{n}$, we have

$$\int_M (C_{\tilde{v}}^{2q+2} - C_{\tilde{w}}^{2q+2})(C_{\tilde{v}}^2 - C_{\tilde{w}}^2)\ dM = 0$$

which gives

$$(C_{\tilde{V}}^{2q+2} - C_{\tilde{W}}^{2q+2})(C_{\tilde{V}}^2 - C_{\tilde{W}}^2) = 0 \text{ , i e, } C_{\tilde{V}}^2 \equiv C_{\tilde{W}}^2$$

Then $\{V,W\}$, where $V = \dfrac{\tilde{V} + \tilde{W}}{\sqrt{2}}$ and $W = \dfrac{\tilde{V} - \tilde{W}}{\sqrt{2}}$, is an orthonormal family of parallel fields in the normal bundle such that $C_{VW} \equiv 0$. Thus (a) implies $n \leq 5$. And so, if M is not totally geodesic, it cannot be stable.

But there is q such that $2q + 4 > n$ and $1 + 2q + 2q^2 < \frac{2}{n}$ only if $n = 2$ or $n = 3$. This próves (b).

Note - The theorem 4 is true for manifolds that are not graphs in the following way:

"let M be simply connected and $M^n > \mathbb{R}^{n+2}$ be an isometric complete minimal immersion having flat normal bundle such that there exist

$$\lim_{R \to +\infty} \frac{1}{R^{n+2}} \int_{M_R(p)} C^2 \, dM \text{ , } \quad p \in M \text{ ,}$$

where $M_R(p)$ is a geodesic ball of radius R, centered in p. Then (a) and (b) are true".

Taking $\phi \in C_0^1(M)$ such that

$$\begin{aligned} 0 \leq \phi(x) \leq 1, &\quad \forall x \in M \\ \phi(x) = 1 &\quad \text{if} \quad x \in M_{R/2}(p) \\ \phi(x) = 0 &\quad \text{if} \quad x \notin M_R(p) \\ |\delta\phi| \leq \frac{2}{R}, & \end{aligned}$$

the proof is exactly the same.

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