DYNAMIC ASPECTS OF SPROTT BC CHAOTIC SYSTEM

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Abstract. In this paper we study global dynamic aspects of the quadratic system
$$\dot{x} = yz, \quad \dot{y} = x - y, \quad \dot{z} = 1 - x(\alpha y + \beta x),$$
where $(x, y, z) \in \mathbb{R}^3$ and $\alpha, \beta \in [0, 1]$ are two parameters. It contains the Sprott B and the Sprott C systems at the two extremes of its parameter spectrum and we call it Sprott BC system. Here we present the complete description of its singularities and we show that this system passes through a Hopf bifurcation at $\alpha = 0$. Using the Poincaré compactification of a polynomial vector field in $\mathbb{R}^3$ we give a complete description of its dynamic on the Poincaré sphere at infinity. We also show that such a system does not admit a polynomial first integral, nor algebraic invariant surfaces, neither Darboux first integral.

1. **Introduction and statement of the main results.** Chaos is an interesting phenomenon in nonlinear dynamical systems that has been intensively studied in the last decades. A chaotic system can be defined as a nonlinear system that displays a complex and unpredictable behavior.

In [16], J. C. Sprott introduced 19 three–dimensional quadratic chaotic systems, which were denoted by *Sprott A*, *Sprott B*, ..., *Sprott S*. Recently the dynamic aspects of such systems and also some generalized systems coming from them have been investigated. For instance, generalized Sprott B [5], Sprott C [17], and Sprott E [15], among others.

Sprott B system is given by

$$\begin{aligned}
\dot{x} &= yz, \\
\dot{y} &= x - y, \\
\dot{z} &= 1 - xy,
\end{aligned} \tag{1}$$

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and Sprott C system is described by

$$\begin{aligned}
\dot{x} &= yz, \\
\dot{y} &= x - y, \\
\dot{z} &= 1 - x^2.
\end{aligned} \tag{2}$$

In this paper we study some global dynamic aspects of the system

$$\begin{aligned}
\dot{x} &= yz, \\
\dot{y} &= x - y, \\
\dot{z} &= 1 - x(\alpha y + \beta x),
\end{aligned} \tag{3}$$

where $\alpha, \beta \in [0, 1]$ are two parameters. Note that system (3) is a quadratic system which contains systems Sprott B and Sprott C in the two extremes of its parameter spectrum, i.e. when $\alpha = 1$ and $\beta = 0$, system (3) represents Sprott B chaotic system and when $\alpha = 0$ and $\beta = 1$ we have Sprott C chaotic system. In this paper, system (3) will be called *Sprott BC system*. This quadratic system can be considered as a *bridge system*[1] between Sprott B and Sprott C systems.

Regarding system (3), we present the complete description of its singularities and we show that it passes through a Hopf bifurcation when $\alpha = 0$. The dynamic at infinity of system (3) is studied by using the Poincaré compactification of a vector field in $\mathbb{R}^3$. In this study we also show that such a system does not admit a polynomial first integral, nor algebraic invariant surfaces, neither Darboux first integral, for all the possible values of the parameters $\alpha, \beta \in [0, 1]$.

In order to start our study of the Sprott BC system we observe that if $\alpha = \beta = 0$ such a system does not possess finite singularities. If $(\alpha, \beta) \neq (0, 0)$ we can assume $\alpha > -\beta$ and in this case we have the following two finite singularities

$$\mathrm{E}_1 = \left(\frac{1}{\sqrt{\alpha + \beta}}, \frac{1}{\sqrt{\alpha + \beta}}, 0 \right), \qquad \mathrm{E}_2 = \left(-\frac{1}{\sqrt{\alpha + \beta}}, -\frac{1}{\sqrt{\alpha + \beta}}, 0 \right). \tag{4}$$

Observe that system (3) is symmetric with respect to the z–axes because it is invariant under the change of coordinates $(x, y, z) \longmapsto (-x, -y, z)$. Therefore, the local stability of singularity E_2 can be obtained by the study of E_1.

The linear part of system (3) evaluated in E_1 is

$$DX(\mathrm{E}_1) = \begin{pmatrix} 0 & 0 & \dfrac{1}{\sqrt{\alpha + \beta}} \\ 1 & -1 & 0 \\ \dfrac{-\alpha - 2\beta}{\sqrt{\alpha + \beta}} & -\dfrac{\alpha}{\sqrt{\alpha + \beta}} & 0 \end{pmatrix}, \tag{5}$$

which has two complex conjugate eigenvalues and one real eigenvalue (since we have $\det(DX(\mathrm{E}_1)) = -2$), so E_1 is locally a saddle–focus singularity (see page 47 from [7] for more details). Due to the mentioned z–symmetry, singularity E_2 has the same local behavior.

In Figures 1(A) and 1(B) we present a local behavior of the orbits around the finite singularities of systems Sprott B and Sprott C, respectively. For some specific values of the parameters $\alpha, \beta \in [0, 1]$ one can present a similar behavior of the orbits around the finite singularities of Sprott BC system.

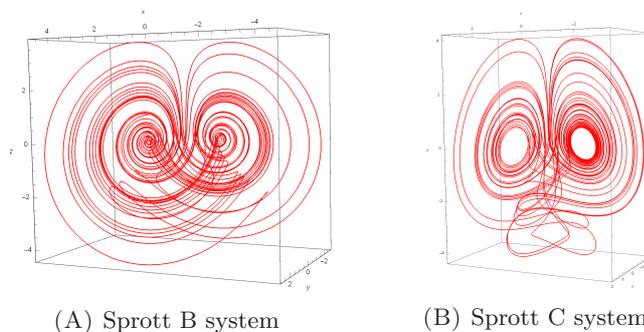[1] See the motivation for this notation in [4, 12, 13].

(A) Sprott B system   (B) Sprott C system

FIGURE 1. Local behavior of orbits around the finite singularities of Sprott B (in 1(A)) and Sprott C (in 1(B)) systems

Now our aim is to show that system (3) passes through a Hopf bifurcation and we compute the respective first Lyapunov coefficient. More precisely, our first result is the following one.

Theorem 1.1. *When $\alpha = 0$ system* (3) *passes through a Hopf bifurcation and in this case the first Lyapunov coefficient of system* (3) *in* E_1 *is* $l_1(0) = \sqrt{2}/144$.

Theorem 1.1 is proved in Sec. 3. Following [7] in Sec. 2.1 we present the definition of a Hopf point and we give a brief review of the method used to compute the first Lyapunov coefficient.

In order to analyze the global dynamic of system (3) studying its behavior at infinity, we use the Poincaré compactification for a polynomial vector field in $\mathbb{R}^3$ (see Sec. 2.2 for a summary of this technique and for the definition of Poincaré sphere). We formulate the following theorem.

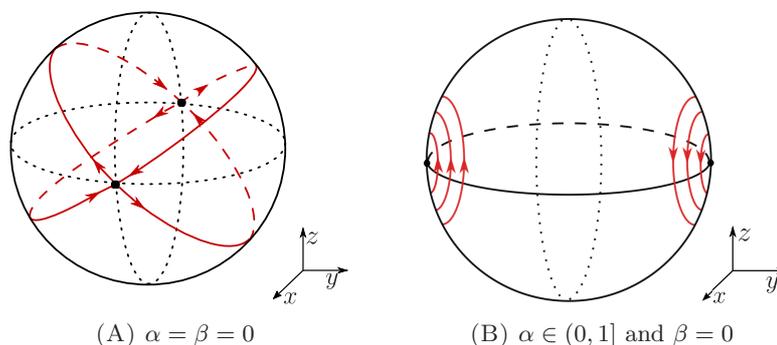**Theorem 1.2.** *The behavior of the flow of system* (3) *at infinity is described by Figures 2 and 3.*

(A) $\alpha = \beta = 0$ (B) $\alpha \in (0, 1]$ and $\beta = 0$

FIGURE 2. Phase portrait of system (3) on the Poincaré sphere. In Figure 2(A) there exist two closed curves filled up with singularities and one pair of distinguished singularities. These distinguished singularities possess two parabolic attractor sectors and two parabolic repelling sectors. In Figure 2(B) there exist one closed curve filled up with singularities and one pair of center type singularities

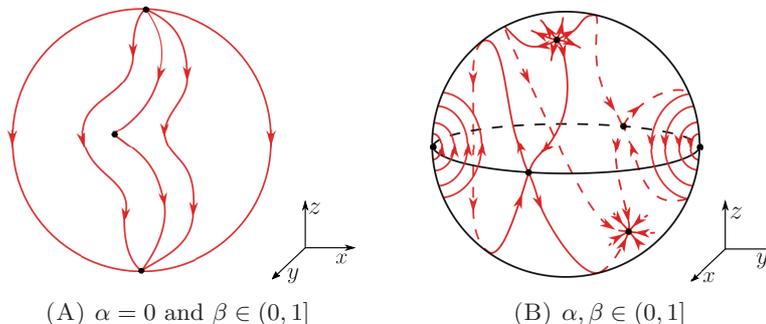(A) $\alpha = 0$ and $\beta \in (0,1]$ (B) $\alpha, \beta \in (0,1]$

FIGURE 3. Phase portrait of system (3) on the Poincaré sphere. In Figure 3(A) there exist a pair of cusp type singularities and a pair of node type singularities (being one attractor and other repelling). In Figure 3(B) there exist a pair of saddles, a pair of centers and a pair of nodes (being one attractor and other repelling)

Theorem 1.2 is proved in Sec. 4. Note that the dynamic at infinity depends explicitly on the values of the parameters α and β.

In an attempt to understand the global behavior of system (3), we investigate the existence of polynomial first integrals, invariant algebraic surfaces, and Darboux integrals. From this study we formulate the following theorem.

Theorem 1.3. *System* (3) *does not possess polynomial first integral, nor algebraic invariant surfaces, neither Darboux first integral.*

These properties show the evidences that the Sprott BC system is quite complex.

This paper is organized as follows. In Sec. 2 we present some preliminaries about Hopf bifurcation, first Lyapunov coefficient, Poincaré compactification, and integrability theory. In Sec. 3 we prove Theorem 1.1. In Sec. 4 we study the dynamic at infinity of system (3) and then we prove Theorem 1.2. Finally, by using several results, in Sec. 5 we prove Theorem 1.3.

2. **Preliminaries.** In this section we present some basic concepts and results in order to make this paper self–contained. We also indicate some references for more details.

2.1. **Hopf bifurcation and first Lyapunov coefficient.** In this subsection we present some basic definitions and a review of the method used to compute the first Lyapunov coefficient associated to Hopf bifurcations. We indicate [7] for more details.

A singularity (x_0, α_0) of a α–parameter family of vector fields $f(x, \alpha) \in \mathbb{R}^3$ is called a *Hopf point* if the Jacobian matrix $Df(x_0, \alpha_0)$ has a real eigenvalue $\lambda_1 \neq 0$ and a pair of purely imaginary eigenvalues $\lambda_{2,3} = \pm i\omega_0$, with $\omega_0 > 0$. If system $\dot{x} = f(x, \alpha)$ admits a Hopf point p then it has a two–dimensional center manifold at p which is invariant by the flow of the system (see [7], page 152).

Consider the differential equation

$$\dot{x} = Ax + F(x, \alpha), \tag{6}$$

where $x \in \mathbb{R}^3$ is a vector representing phase variables and $\alpha \in \mathbb{R}$ is a parameter of the system. Assume that $F \in \mathcal{C}^\infty(\mathbb{R}^3 \times \mathbb{R})$, whose components have Taylor

expansion in x starting with at least quadratic terms, i.e, $F = O(||x||^2)$. Suppose that $(0,0)$ is a singularity of (6) where the Jacobian matrix A has a pair of purely imaginary eigenvalues $\lambda_{2,3} = \pm i\omega_0$, with $\omega_0 > 0$, and admits no other eigenvalue with zero real part. Let T^c be the generalized eigenspace of A corresponding to $\lambda_{2,3}$, i.e. the largest subspace invariant by A on which the eigenvalues are $\lambda_{2,3}$. Let $p, q \in \mathbb{C}^3$ be vectors such that

$$Aq = i\omega_0 q, \quad A^T p = -i\omega_0 p, \quad \langle p, q \rangle = \sum_{i=1}^{3} \bar{p}_i q_i = 1,$$

where A^T is the transposed of the matrix A. Any vector $x \in T^c$ can be parametrized as $x = zq + \bar{z}\bar{q}$, where $z = \langle p, x \rangle$. Then, (6) with $\alpha = 0$ can be transformed into a single equation $\dot{z} = i\omega_0 z + g(z, \bar{z}, 0)$, where $g(z, \bar{z}, 0) = \langle p, F(zq + \bar{z}\bar{q}, 0) \rangle$ with a formal Taylor series expansion in two variables z and $\bar{z}$

$$g(z, \bar{z}, 0) = \sum_{k+l \geq 2} \frac{1}{k!l!} g_{kl}(0) z^k \bar{z}^l \quad \text{with} \quad g_{kl}(0) = \frac{\partial^{k+l}}{\partial z^k \partial \bar{z}^l} \langle p, F(zq + \bar{z}\bar{q}, 0) \rangle \bigg|_{z=0},$$

for $k + l \geq 2$ and $k, l \in \{0, 1, 2, \dots\}$. Suppose that for $\alpha = 0$ the function $F(x, \alpha)$ in (6) is represented as

$$F(x, 0) = \frac{1}{2} B(x, x) + \frac{1}{6} C(x, x, x) + O(||x||^4),$$

where $B(x, y)$ and $C(x, y, z)$ are symmetric multi–linear vector functions on the variables x, y, z. In coordinates one has

$$B_i(x, y) = \sum_{j,k=1}^{3} \frac{\partial F_i(\varepsilon, 0)}{\partial \varepsilon_j \partial \varepsilon_k} \bigg|_{\varepsilon=0} x_j y_k, \quad C_i(x, y, z) = \sum_{j,k,l=1}^{3} \frac{\partial F_i(\varepsilon, 0)}{\partial \varepsilon_j \partial \varepsilon_k \partial \varepsilon_l} \bigg|_{\varepsilon=0} x_j y_k z_l,$$

for $i = 1, 2, 3$. Then we have the following Taylor coefficients in $g(z, \bar{z}, 0)$

$$g_{20} = \langle p, B(q, q) \rangle, \quad g_{11} = \langle p, B(q, \bar{q}) \rangle, \quad \text{and} \quad g_{21} = \langle p, C(q, q, \bar{q}) \rangle.$$

The *first Lyapunov coefficient* is defined as

$$l_1(0) = \frac{1}{2\omega_0^2} \Re(i g_{20} g_{11} + \omega_0 g_{21}),$$

where, as usual, $\Re(z)$ stands for the real part of the complex number z.

A Hopf point is called *transversal* if the parameter–dependent complex eigenvalues cross the imaginary axis with nonzero derivative. When $l_1(0) < 0$ (respectively $l_1(0) > 0$) one family of stable (respectively unstable) periodic orbits can be found in the center manifold and its continuation shrinking to the Hopf point.

2.2. Poincaré compactification.

For this subsection we indicate [10] for more details. Consider in $\mathbb{R}^3$ the following polynomial differential system

$$\dot{x} = P_1(x, y, z), \quad \dot{y} = P_2(x, y, z), \quad \dot{z} = P_3(x, y, z), \tag{7}$$

or equivalently, its associated polynomial vector field $X = (P_1, P_2, P_3)$. The degree n of X is defined as $n = \max\{deg(P_i) : i = 1, 2, 3\}$. Let $\mathbb{S}^3 = \{y = (y_1, y_2, y_3, y_4) : ||y|| = 1\}$ be the unit sphere in $\mathbb{R}^4$ and $\mathbb{S}_+ = \{y \in \mathbb{S}^3 : y_4 > 0\}$ and $\mathbb{S}_- = \{y \in \mathbb{S}^3 : y_4 < 0\}$ be the northern and southern hemispheres of $\mathbb{S}^3$, respectively. The tangent space of $\mathbb{S}^3$ at the point y is denoted by $T_y\mathbb{S}^3$ and the tangent plane

$$T_{(0,0,0,1)}\mathbb{S}^3 = \{(x_1, x_2, x_3, 1) \in \mathbb{R}^4 : (x_1, x_2, x_3) \in \mathbb{R}^3\} \tag{8}$$

can be naturally identified with $\mathbb{R}^3$.

Let $f_+ \colon \mathbb{R}^3 = T_{(0,0,0,1)}\mathbb{S}^3 \to \mathbb{S}_+$ and $f_- \colon \mathbb{R}^3 = T_{(0,0,0,1)}\mathbb{S}^3 \to \mathbb{S}_-$ be the central projections defined by $f_\pm(x) = \pm\frac{(x_1,x_2,x_3,1)}{\Delta(x)}$ where $\Delta(x) = \left(1 + \sum_{i=1}^3 x_i^2\right)^{1/2}$. Using these central projections, $\mathbb{R}^3$ is identified with the northern and southern hemispheres. The equator of $\mathbb{S}^3$ is $\mathbb{S}^2 = \{y \in \mathbb{S}^3 : y_4 = 0\}$.

The maps $f_\pm$ define two copies of X on $\mathbb{S}^3$, one $Df_+ \circ X$ in the northern hemisphere and the other $Df_- \circ X$ in the southern one. Denote by $\overline{X}$ the vector field on $\mathbb{S}^3 \setminus \mathbb{S}^2 = \mathbb{S}_+ \cup \mathbb{S}_-$, which restricted to $\mathbb{S}_+$ coincides with $Df_+ \circ X$ and restricted to $\mathbb{S}_-$ coincides with $Df_- \circ X$. Now we can extend analytically the vector field $\overline{X}(y)$ to the whole sphere $\mathbb{S}^3$ by $p(X) = y_4^{n-1}\overline{X}(y)$. This extended vector field $p(X)$ is called the *Poincaré compactification* of X on $\mathbb{S}^3$.

As $\mathbb{S}^3$ is a differentiable manifold, in order to compute the expression for $p(X)$, we can consider the eight local charts (U_i, F_i), (V_i, G_i), where

$$U_i = \{y \in \mathbb{S}^3 : y_i > 0\} \quad \text{and} \quad V_i = \{y \in \mathbb{S}^3 : y_i < 0\},$$

for $i = 1, 2, 3, 4$. The diffeomorphisms $F_i \colon U_i \to \mathbb{R}^3$ and $G_i \colon V_i \to \mathbb{R}^3$ for $i = 1, 2, 3, 4$ are the inverse of the central projections from the origin to the tangent hyperplane at the points $(\pm 1, 0, 0, 0)$, $(0, \pm 1, 0, 0)$, $(0, 0, \pm 1, 0)$, and $(0, 0, 0, \pm 1)$, respectively.

Now we do the computations on U_1. Suppose that the origin $(0, 0, 0, 0)$, the point $(y_1, y_2, y_3, y_4) \in \mathbb{S}^3$, and the point $(1, z_1, z_2, z_3)$ in the tangent hyperplane to $\mathbb{S}^3$ at $(1, 0, 0, 0)$ are collinear. Then we have

$$\frac{1}{y_1} = \frac{z_1}{y_2} = \frac{z_2}{y_3} = \frac{z_3}{y_4}$$

and consequently $F_1(y) = (y_2/y_1, y_3/y_1, y_4/y_1) = (z_1, z_2, z_3)$ defines the coordinates on U_1. As

$$DF_1(y) = \begin{pmatrix} -y_2/y_1^2 & 1/y_1 & 0 & 0 \\ -y_3/y_1^2 & 0 & 1/y_1 & 0 \\ -y_4/y_1^2 & 0 & 0 & 1/y_1 \end{pmatrix}$$

and $y_4^{n-1} = (z_3/\Delta(z)^{n-1})$, the analytical vector field $p(X)$ in the local chart U_1 becomes

$$\frac{z_3^n}{\Delta(z)^{n-1}}\big(-z_1 P_1 + P_2, -z_2 P_1 + P_3, -z_3 P_1\big),$$

where $P_i = P_i(1/z_3, z_1/z_3, z_2/z_3)$.

In a similar way, we can deduce the expressions of $p(X)$ in U_2 by

$$\frac{z_3^n}{\Delta(z)^{n-1}}\big(-z_1 P_2 + P_1, -z_2 P_2 + P_3, -z_3 P_2\big),$$

where $P_i = P_i(z_1/z_3, 1/z_3, z_2/z_3)$, and in U_3 by

$$\frac{z_3^n}{\Delta(z)^{n-1}}\big(-z_1 P_3 + P_1, -z_2 P_3 + P_2, -z_3 P_3\big),$$

where $P_i = P_i(z_1/z_3, z_2/z_3, 1/z_3)$.

The expression for $p(X)$ in U_4 is $z_3^{n+1}(P_1, P_2, P_3)$ and the expression for $p(X)$ in the local chart V_i is the same as in U_i multiplied by $(-1)^{n-1}$, where n is the degree of X, for $i = 1, 2, 3, 4$.

Doing a rescaling of the time variable we can omit the common factor $1/(\Delta(z))^{n-1}$ in the expression of the vector field $p(X)$.

From now on we will consider only the orthogonal projection of $p(X)$ from the northern hemisphere to $y_4 = 0$ which we will again denote by $p(X)$. Observe that the projection of the closed northern hemisphere is a closed ball of radius one denoted by B, whose interior is diffeomorphic to $\mathbb{R}^3$ and whose boundary $\mathbb{S}^2$ corresponds to the infinity of $\mathbb{R}^3$. Moreover, $p(X)$ is defined in the whole closed ball B in such a way that the flow on the boundary is invariant. The vector field induced by $p(X)$ on B is called the *Poincaré compactification* of X and B is called the *Poincaré sphere*.

All the points on the invariant sphere $\mathbb{S}^2$ at infinity in the coordinates of any local chart U_i and V_i have $z_3 = 0$.

In order to simplify the notation we write $(z_1, z_2, z_3) = (u, v, w)$.

2.3. **Integrability theory.** We start this subsection with the Darboux theory of integrability. We indicate [8, 14] for more details. As usual, $\mathbb{C}[x, y, z]$ denotes the ring of polynomial functions in the variables x, y and z. Given $f \in \mathbb{C}[x, y, z] \setminus \mathbb{C}$ we say that the surface $f(x, y, z) = 0$ is an *invariant algebraic surface* of system (3) if there exists $k \in \mathbb{C}[x, y, z]$ such that

$$\dot{x}\frac{\partial f}{\partial x} + \dot{y}\frac{\partial f}{\partial y} + \dot{z}\frac{\partial f}{\partial z} = kf. \tag{9}$$

The polynomial k is called *cofactor* of the invariant algebraic surface $f = 0$ and it has degree at most 1 because in (9) we have $\deg(k) + \deg(f)$ equal to

$$\max\{\deg(\dot{x}) + \deg(f) - 1, \deg(\dot{y}) + \deg(f) - 1, \deg(\dot{z}) + \deg(f) - 1\} \leq \deg(f) + 1,$$

since (3) is a quadratic system, the degree of the corresponding vector field is two, so $\deg(k) \leq 1$ and, therefore, without loss of generality we can assume that the cofactor is of the form $k(x, y, z) = k_0 + k_1 x + k_2 y + k_3 z$.

When $k = 0$, f is called a *polynomial first integral*.

Let $f, g \in \mathbb{C}[x, y, z]$ and assume that f and g are relatively primes in the ring $\mathbb{C}[x, y, z]$, or that $g = 1$. The function $\exp(f/g) \notin \mathbb{C}$ is called an *exponential factor* of system (3) if for some polynomial $L \in \mathbb{C}[x, y, z]$ of degree at most 1 we have

$$\dot{x}\frac{\partial \exp(f/g)}{\partial x} + \dot{y}\frac{\partial \exp(f/g)}{\partial y} + \dot{z}\frac{\partial \exp(f/g)}{\partial z} = L\exp(f/g). \tag{10}$$

As before, we say that L is the *cofactor* of the exponential factor $\exp(f/g)$.

Let U be an open and dense subset of $\mathbb{R}^3$. We say that a nonconstant function $H : U \to \mathbb{R}$ is a *first integral* of system (3) in U if $H(x(t), y(t), z(t))$ is constant for all values of t for which $(x(t), y(t), z(t))$ is a solution of system (3) contained in U. Obviously, H is a first integral of system (3) if and only if

$$\dot{x}\frac{\partial H}{\partial x} + \dot{y}\frac{\partial H}{\partial y} + \dot{z}\frac{\partial H}{\partial z} = 0,$$

for all $(x, y, z) \in U$.

A first integral is called a *Darboux first integral* if it is a first integral of the form

$$f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q},$$

where $f_i = 0$ are the invariant algebraic surfaces of system (3), for $i = 1, \ldots, p$, and F_j are the exponential factors of system (3), for $j = 1, \ldots, q$, and $\lambda_i, \mu_j \in \mathbb{C}$.

The next result, proved in [3], explains how one can find Darboux first integrals.

Proposition 1. *Suppose that a polynomial system of degree m admits p invariant algebraic surfaces $f_i = 0$ with cofactors k_i, for $i = 1, \ldots, p$, and q exponential*

factors $\exp(g_j/h_j)$ *with cofactors* L_j, *for* $j = 1, \ldots, q$. *Then, there exists* λ_j *and* μ_j *complexes not all zero such that*

$$\sum_{i=1}^{p} \lambda_i k_i + \sum_{j=1}^{q} \mu_j L_j = 0, \tag{11}$$

if and only if the function

$$f_1^{\lambda_1} \ldots f_p^{\lambda_p} \left(\exp\left(\frac{g_1}{h_1} \right) \right)^{\mu_1} \ldots \left(\exp\left(\frac{g_q}{h_q} \right) \right)^{\mu_q}$$

is a Darboux first integral of such a system.

The next result whose proof is given in [10, 11] is useful to prove Theorem 1.3.

Lemma 2.1. *The following statements hold.*

(a) *If* $\exp(f/g)$ *is an exponential factor for the polynomial differential system* (3) *and* g *is not a constant polynomial, then* $g = 0$ *is an invariant algebraic surface.*

(b) *Eventually* $\exp(f)$ *can be an exponential factor, coming from the multiplicity of the infinity invariant plane.*

3. **Proof of Theorem 1.1.** We start this section with a lemma that provides a topological characterization parameter–dependent for singularity E_1.

Lemma 3.1. *Singularity* E_1 *is asymptotically stable if and only if* $\alpha \in (-\beta, 0)$. *In other words,* E_1 *and* E_2 *are unstable singularities of system* (3) *for* $\alpha, \beta \in [0, 1]$.

Proof. We have that the characteristic polynomial of the matrix (5) is given by

$$\lambda^3 + \lambda^2 + \frac{\alpha + 2\beta}{\alpha + \beta}\lambda + 2 = 0. \tag{12}$$

Note that the coefficients of (12) are non–negative since we have assumed $\alpha > -\beta$:

$$a_0 = 1, \quad a_1 = 1, \quad a_2 = \frac{\alpha + 2\beta}{\alpha + \beta}, \quad \text{and} \quad a_3 = 2.$$

Moreover,

$$\Delta_2 := a_1 a_2 - a_0 a_3 > 0 \Leftrightarrow \frac{\alpha + 2\beta}{\alpha + \beta} - 2 > 0 \Leftrightarrow \alpha < 0.$$

Applying Routh–Hurwitz criteria (see for instance [6]) we conclude that singularity E_1 is asymptotically stable if and only if $\alpha \in (-\beta, 0)$, i.e. in the $\beta\alpha$–parameter plane, if $\alpha > -\beta$ with $\alpha > 0$, singularity E_1 is unstable. Note also that by the z–symmetry of system (3), in some neighborhood of singularity E_2 we have the same topological behavior. □

From the previous lemma we conclude that $\alpha = 0$ is a bifurcation parameter. Note that for $\alpha = 0$, equation (12) takes the form $(\lambda^2 + 2)(\lambda + 1) = 0$, which has $\lambda = -1$ and $\lambda = \pm i\sqrt{2}$ as roots, so E_1 is non–elemental. By Center Manifold Theorem (see [7], page 152), near singularity E_1 is well–defined a local two–dimensional center manifold, which is invariant under the flow of system (3). According to the results described in Sec. 2.1 we now present the proof of Theorem 1.1.

Proof. First we show that when $\alpha = 0$ system (3) passes through a Hopf bifurcation. In fact, we saw that if $\alpha = 0$, the linear part of system (3) has a pair of purely imaginary eigenvalues $\pm i\sqrt{2}$. Taking the implicit derivative of equation (12) with respect to the parameter α, we obtain

$$\frac{d\lambda}{d\alpha} = \frac{\lambda\beta}{(\alpha + \beta)(3\lambda^2\alpha + 3\lambda^2\beta + 2\lambda\alpha + 2\lambda\beta + \alpha + 2\beta)}.$$

Replacing $\alpha = 0$ and taking $\lambda(0) = i\sqrt{2}$, we found

$$\left.\frac{d\lambda}{d\alpha}\right|_{\alpha=0} = \frac{-i\sqrt{2}+1}{6\beta} \quad \text{and so} \quad \left.\frac{d\Re(\lambda)}{d\alpha}\right|_{\alpha=0} = \frac{1}{6\beta} \neq 0,$$

where $\beta > 0$ because $\alpha > -\beta$ and $\alpha, \beta \in [0, 1]$. From the Hopf bifurcation theory we conclude that system (3) pass through a Hopf bifurcation in E_1 and E_2 when $\alpha = 0$. The Hopf theorem provides the birth of a closed orbit near singularities E_1 and E_2 when $\alpha = 0$, and such orbits have period $2\pi/w$, where $w = \Im(i\sqrt{2}) = \sqrt{2}$. On the other hand the study of the Hopf bifurcation gives us information about the stability of such closed orbits. In this direction we evaluate the first Lyapunov coefficient associated to this Hopf bifurcation. When $0 = \alpha > -\beta$, $E_1 = \left(\frac{\sqrt{\beta}}{\beta}, \frac{\sqrt{\beta}}{\beta}, 0\right)$. So the change of variables $(x_1, x_2, x_3) \longmapsto \left(x - \frac{\sqrt{\beta}}{\beta}, y - \frac{\sqrt{\beta}}{\beta}, z\right)$ carries singularity E_1 for the origin and system (3) becomes

$$\dot{x}_1 = x_2x_3 + \frac{\sqrt{\beta}}{\beta}x_3,$$
$$\dot{x}_2 = x_1 - x_2,$$
$$\dot{x}_3 = -\beta x_1^2 - 2\sqrt{\beta}x_1.$$

This last one can be written as $\dot{X} = AX + F(X)$, where

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & \frac{\sqrt{\beta}}{\beta} \\ 1 & -1 & 0 \\ -2\sqrt{\beta} & 0 & 0 \end{pmatrix}, \quad F(X) = \begin{pmatrix} F_1(X) \\ F_2(X) \\ F_3(X) \end{pmatrix} = \begin{pmatrix} x_2x_3 \\ 0 \\ -\beta x_1^2 \end{pmatrix}.$$

Note that matrix A has one real eigenvalue $\lambda_1 = -1$ and a pair of purely imaginary eigenvalues $\lambda_{2,3} = \pm i\sqrt{2}$. Besides, as in Sec. 2.1 we have

$$B_1(x, y) = x_2y_3 + x_3y_2, \quad B_2(x, y) = 0, \quad \text{and} \quad B_3(x, y) = -2\beta x_1y_1.$$

Then,

$$B(x, y) = \begin{pmatrix} B_1(x, y) \\ B_2(x, y) \\ B_3(x, y) \end{pmatrix} = \begin{pmatrix} x_2y_3 + x_3y_2 \\ 0 \\ -2\beta x_1y_1 \end{pmatrix},$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. Proceeding, it is clear that $C(x, y, z) = (C_1(x, y, z), C_2(x, y, z), C_3(x, y, z)) \equiv 0$. Consider $\tilde{p}, \tilde{q} \in \mathbb{C}^3$ be such that

$$A\tilde{q} = i\sqrt{2}\tilde{q} \quad \text{and} \quad A^T\tilde{p} = -i\sqrt{2}\tilde{p}. \tag{13}$$

With some calculations we get that

$$\tilde{q} = \begin{pmatrix} \dfrac{-i}{\sqrt{2\beta}} \\ \dfrac{-1}{\sqrt{2\beta}(\sqrt{2}-1)} \\ 1 \end{pmatrix} \quad \text{and} \quad \tilde{p} = \begin{pmatrix} -i\sqrt{2\beta} \\ 0 \\ 1 \end{pmatrix}.$$

As $\langle \tilde{p}, \tilde{q} \rangle = \sum_{i=1}^{3} \bar{\tilde{p}}_i \tilde{q}_i = 2$, in order to have $\langle \tilde{p}, \tilde{q} \rangle = 1$ we take $\tilde{p} = p$ and $\tilde{q} = 2q$. Thus, $\langle p, q \rangle = 1$ and (13) holds. Therefore we can evaluate $B(q,q)$ and $B(q,\bar{q})$ and using these results we obtain

$$g_{20} = \frac{1}{4} + \frac{i}{2(\sqrt{2}-i)}, \quad g_{11} = -\frac{1}{4} + \frac{\sqrt{2}}{6}i, \quad \text{and} \quad g_{21} = 0.$$

In this way, the first Lyapunov coefficient takes the form

$$l_1(0) = \frac{1}{2w^2} \Re(ig_{20}g_{11} + wg_{21}) = \frac{\sqrt{2}}{144}.$$

This completes the proof of Theorem 1.1. $\qquad\square$

The previous lemma shows that the non–degeneracy condition of the Hopf theorem is also valid. Thus, singularity E_1 is a weak repelling focus (for the flow of system (3) restrict to the central manifold) and there exists an unstable limit cycle near to singularity E_1 for suitable parameter values, i.e., we have a *subcritical Hopf bifurcation*. Analogously we obtain the same results for singularity E_2.

4. **Proof of Theorem 1.2.** In this section we study the flow of system (3) at infinity by analyzing the Poincaré compactification of this system in local charts U_i and V_i, for $i = 1, 2, 3$, using the techniques presented in Sec. 2.2. We separate it in four subsections, depending on the values assumed by the parameters $\alpha, \beta \in [0, 1]$. Consequently, we have proved Theorem 1.2.

Note that the flow in the local chart V_i is the same as the flow in the respective local chart U_i, $i = 1, 2, 3$, because the compacted vector field $p(X)$ in V_i coincides with the vector field $p(X)$ in U_i multiplied by -1. Hence the flow in the local chart V_i is the same as the flow in the local chart U_i, for $i = 1, 2, 3$, reversing the time in a convenient manner. So for this reason here we only study the flow on the local charts U_i, for $i = 1, 2, 3$.

4.1. **Case 1:** $\alpha = \beta = 0$.

4.1.1. *Compactification in the local chart U_1.* As described in Sec. 2.2 system (3) in the local chart U_1 is given by

$$\begin{aligned} \dot{u} &= w - uw - u^2v, \\ \dot{v} &= w^2 - uv^2, \\ \dot{w} &= -uvw. \end{aligned} \tag{14}$$

For $w = 0$ (which correspond to the points on the sphere $\mathbb{S}^2$ at infinity) system (14) becomes

$$\dot{u} = -u^2v, \qquad \dot{v} = -uv^2. \tag{15}$$

This system has the origin as the unique singularity. It is easy to conclude that this system has uv as a common factor. So, by the Time Reparametrization Theorem, one can study the system

$$\dot{u} = -u, \qquad \dot{v} = -v,$$

which has the origin as a stable node. So, due to the common factor uv, the local behavior of the solutions near to the origin of system (15) is constituted by two parabolic attractor sectors (in $\{u > 0\} \cap \{v > 0\}$ and $\{u < 0\} \cap \{v < 0\}$) and two parabolic repelling sectors (in $\{u > 0\} \cap \{v < 0\}$ and $\{u < 0\} \cap \{v > 0\}$).

4.1.2. *Compactification in the local chart U_2.* Again from Sec. 2.2 system (3) in the local chart U_2 is given by

$$
\begin{aligned}
\dot{u} &= v + uw - u^2 w, \\
\dot{v} &= w^2 + vw - uvw, \\
\dot{w} &= w^2 - uw^2.
\end{aligned}
\tag{16}
$$

System (16) restricted to $w = 0$ becomes

$$\dot{u} = v, \qquad \dot{v} = 0.$$

Such a system has the curve $v = 0$ filled up with singularities.

4.1.3. *Compactification in the local chart U_3.* System (3) in the local chart U_3 is described as

$$
\begin{aligned}
\dot{u} &= v - uw^2, \\
\dot{v} &= uw - vw - vw^2, \\
\dot{w} &= -w^3.
\end{aligned}
\tag{17}
$$

Observe that system (17) restricted to the invariant uv-plane reduces to

$$\dot{u} = v, \qquad \dot{v} = 0,$$

which has the curve $v = 0$ filled up with singularities.

Therefore putting all these informations together we have the phase portrait presented in Figure 2(A).

4.2. **Case 2: $\alpha \in (0, 1]$ and $\beta = 0$.** Note that, in this case, when $\alpha = 1$ we have the Sprott B chaotic system.

4.2.1. *Compactification in the local chart U_1.* Under these conditions system (3) in the local chart U_1 is given by

$$
\begin{aligned}
\dot{u} &= w - uw - u^2 v, \\
\dot{v} &= -\alpha u + w^2 - uv^2, \\
\dot{w} &= -uvw.
\end{aligned}
\tag{18}
$$

For $w = 0$ system (18) becomes

$$\dot{u} = -u^2 v, \qquad \dot{v} = -\alpha u - uv^2.$$

This system has the curve $u = 0$ filled up with singularities.

4.2.2. *Compactification in the local chart U_2.* Under these conditions system (3) in the local chart U_2 is described by the differential equations

$$\dot{u} = v + uw - u^2 w,$$
$$\dot{v} = -\alpha u + vw + w^2 - uvw, \tag{19}$$
$$\dot{w} = w^2 - uw^2.$$

System (19) restricted to $w = 0$ becomes

$$\dot{u} = v, \qquad \dot{v} = -\alpha u.$$

This system has the origin as the unique singularity which is of center type because the respective Jacobian matrix has eigenvalues $\pm i\sqrt{\alpha}$ and $h(u,v) = \alpha u^2 + v^2$ is a first integral (restrict to the local chart U_2).

4.2.3. *Compactification in the local chart U_3.* System (3) in the local chart U_3 is given by

$$\dot{u} = v - uw^2 + \alpha u^2 v,$$
$$\dot{v} = uw - vw - vw^2 + \alpha uv^2, \tag{20}$$
$$\dot{w} = -w^3 + \alpha uvw.$$

Observe that system (20) restricted to the invariant uv–plane reduces to

$$\dot{u} = v + \alpha u^2 v, \qquad \dot{v} = \alpha uv^2,$$

and such a system has the curve $v = 0$ filled up with singularities.

As a result we arrive at the phase portrait presented in Figure 2(B).

4.3. **Case 3: $\alpha = 0$ and $\beta \in (0,1]$.** Note that, in this case, when $\beta = 1$ we have the Sprott C chaotic system.

4.3.1. *Compactification in the local chart U_1.* In this case system (3) in the local chart U_1 is given by

$$\dot{u} = w - uw - u^2 v,$$
$$\dot{v} = -\beta + w^2 - uv^2, \tag{21}$$
$$\dot{w} = -uvw.$$

For $w = 0$ system (21) becomes

$$\dot{u} = -u^2 v, \qquad \dot{v} = -\beta - uv^2.$$

This system does not have singularities.

4.3.2. *Compactification in the local chart U_2.* System (3) in the local chart U_2 is given by

$$\dot{u} = v + uw - u^2 w,$$
$$\dot{v} = vw + w^2 - \beta u^2 - uvw, \tag{22}$$
$$\dot{w} = w^2 - uw^2,$$

which restricted to $w = 0$ becomes

$$\dot{u} = v, \qquad \dot{v} = -\beta u^2.$$

This last one has the origin as the unique singularity, which is clearly a nilpotent singularity. Applying the Nilpotent Singular Points Theorem (Theorem 3.5 from

[3]) we conclude by item $(3) - (ii)$ of such a theorem that $(0,0)$ is a cusp type singularity, since we have $F(u) = -\beta u^2$ and $G(u) \equiv 0$.

4.3.3. *Compactification in the local chart U_3.* System (3) in the local chart U_3 is

$$
\begin{aligned}
\dot{u} &= v - uw^2 + \beta u^3, \\
\dot{v} &= uw - vw - vw^2 + \beta u^2 v, \\
\dot{w} &= -w^3 + \beta u^2 w,
\end{aligned}
\tag{23}
$$

which restricted to the invariant uv–plane reduces to

$$
\dot{u} = v + \beta u^3, \qquad \dot{v} = \beta u^2 v.
$$

This system has the origin as the unique singularity. The linear part evaluated at this singularity is described by a nilpotent matrix. So, applying the Nilpotent Singular Points Theorem (Theorem 3.5 from [3]) we conclude by item $(4) - (iii3)$ of such a theorem that $(0,0)$ is a repelling node, since $F(u) = -\beta^2 u^5$ and $G(u) = 4\beta u^2$.

Therefore we have the phase portrait presented in Figure 3(A).

4.4. **Case 4:** $\alpha, \beta \in (0,1]$. This case presents a very rich behavior at infinity.

4.4.1. *Compactification in the local chart U_1.* Now system (3) in the local chart U_1 is given by

$$
\begin{aligned}
\dot{u} &= w - uw - u^2 v, \\
\dot{v} &= -\beta - \alpha u + w^2 - uv^2, \\
\dot{w} &= -uvw.
\end{aligned}
\tag{24}
$$

For $w = 0$ system (24) becomes

$$
\dot{u} = -u^2 v, \qquad \dot{v} = -\beta - \alpha u - uv^2.
$$

This system has the singularity $(-\beta/\alpha, 0)$. As the respective eigenvalues are $\pm\beta/\sqrt{\alpha}$, by Hartman–Grobman Theorem we conclude that this singularity if of saddle type.

4.4.2. *Compactification in the local chart U_2.* System (3) in the local chart U_2 is given by

$$
\begin{aligned}
\dot{u} &= v + uw - u^2 w, \\
\dot{v} &= -\alpha u + vw + w^2 - \beta u^2 - uvw, \\
\dot{w} &= w^2 - uw^2.
\end{aligned}
\tag{25}
$$

System (25) restricted to $w = 0$ becomes

$$
\dot{u} = v, \qquad \dot{v} = -\alpha u - \beta u^2.
\tag{26}
$$

This system has $(0,0)$ and $(-\alpha/\beta, 0)$ as singularities, but only the origin interests us. We point out that the origin is a center type singularity, because the linear part of system (26) has eigenvalues $\pm i\sqrt{\alpha}$ and $h(u,v) = v^2 + \alpha u^2 + 2\beta u^3/3$ is a first integral (restrict to the local chart U_2).

4.4.3. *Compactification in the local chart U_3.* System (3) under these conditions in the local chart U_3 is described by the following differential equations

$$
\begin{aligned}
\dot{u} &= v - uw^2 + \alpha u^2 v + \beta u^3, \\
\dot{v} &= uw - vw - vw^2 + \alpha uv^2 + \beta u^2 v, \\
\dot{w} &= -w^3 + \alpha uvw + \beta u^2 w.
\end{aligned}
\tag{27}
$$

Observe that system (27) restricted to the invariant uv–plane reduces to

$$
\dot{u} = v + \alpha u^2 v + \beta u^3, \qquad \dot{v} = \alpha uv^2 + \beta u^2 v.
$$

This system has the origin as a singularity. The linear part evaluated at $(0,0)$ is given by a nilpotent matrix. So applying the Nilpotent Singular Points Theorem (Theorem 3.5 from [3]) we conclude by item $(4)-(iii3)$ of such a theorem that $(0,0)$ is a repelling node, since $F(u) = -\beta^2 u^5 + \alpha\beta^2 u^7$ and $G(u) = 4\beta u^2 - 4\alpha\beta u^4$.

In conclusion, we have the phase portrait presented in Figure 3(B).

5. **Proof of Theorem 1.3.** The proof of this theorem follows from several results, which are described according to the values of parameters $\alpha, \beta \in [0, 1]$.

Proposition 2. *System (3) does not possess algebraic invariant surfaces for $\alpha = \beta = 0$.*

Proof. Suppose $f = 0$ be an algebraic invariant surface of degree $n \geq 1$ for system (3) with cofactor $k = k_0 + k_1 x + k_2 y + k_3 z$, being $k_0, k_1, k_2, k_3 \in \mathbb{R}$ not all zero. Let us write f as a sum of its homogeneous parts, there is, $f = \sum_{i=0}^{n} f_i$, where f_i is a homogeneous polynomial of degree i. As f satisfies the partial differential equation (from now on we will summarize writing PDE)

$$
yz\frac{\partial f}{\partial x} + (x-y)\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = (k_0 + k_1 x + k_2 y + k_3 z)f,
\tag{28}
$$

considering the terms of degree $n+1$ in (28) we have

$$
yz\frac{\partial f_n}{\partial x} = (k_1 x + k_2 y + k_3 z)f_n.
$$

Solving this equation we obtain

$$
f_n(x, y, z) = \exp\left(\frac{1}{yz}\left(\frac{k_1 x^2}{2} + k_2 xy + k_3 xz\right)\right)C(y, z),
$$

where C is a $\mathcal{C}^1$ function of variables y and z. Then for f_n be a homogeneous polynomial of degree n we should have $k_1 = k_2 = k_3 = 0$ and so $f_n = f_n(y, z)$. Computing the terms of degree n in (28) we get

$$
yz\frac{\partial f_{n-1}}{\partial x} + (x-y)\frac{\partial f_n}{\partial y} = k_0 f_n,
$$

which implies that

$$
\frac{\partial f_n}{\partial y} = \frac{k_0}{x-y}f_n - \frac{yz}{x-y}\frac{\partial f_{n-1}}{\partial x}.
$$

As f_n does not depend on x, we also should have $\dfrac{\partial f_n}{\partial y} = 0$ (therefore, $f_n = f_n(z)$) and

$$
\frac{\partial f_{n-1}}{\partial x} = \frac{k_0}{yz}f_n = \frac{k_0}{y}g_n(z),
$$

where $g_n(z) = f_n(z)/z$. So $k_0 = 0$, otherwise f_{n-1} would not be a homogeneous polynomial of degree $n - 1$. Thus $k \equiv 0$, which contradicts our hypotheses over k. □

Proposition 3. *System (3) does not possess algebraic invariant surfaces for $\beta = 0$ and $\alpha \in (0, 1]$.*

Proof. As in Proposition 2, suppose $f = 0$ be an algebraic invariant surface of degree $n \geq 1$ for system (3) with cofactor $k = k_0 + k_1 x + k_2 y + k_3 z$, being $k_0, k_1, k_2, k_3 \in \mathbb{R}$ not all zero. We write $f = \sum_{i=0}^{n} f_i$, where f_i is a homogeneous polynomial of degree i. As f satisfies the PDE

$$yz\frac{\partial f}{\partial x} + (x - y)\frac{\partial f}{\partial y} + (1 - \alpha xy)\frac{\partial f}{\partial z} = (k_0 + k_1 x + k_2 y + k_3 z)f, \qquad (29)$$

considering the terms of degree $n + 1$ in (29) we have

$$yz\frac{\partial f_n}{\partial x} - \alpha xy\frac{\partial f_n}{\partial z} = (k_1 x + k_2 y + k_3 z)f_n. \qquad (30)$$

Using the method of characteristical curves (see for instance chapter two of [1]), the solution of PDE (30) is

$$f_n(x, y, z) = \exp\left(\frac{k_3 x}{y} + \frac{k_1|z|}{\alpha y} + \frac{k_2 \arctan\left(\frac{\sqrt{\alpha}x}{|z|}\right)}{\sqrt{\alpha}}\right) C(y, z^2 + \alpha x^2),$$

where C is a $\mathcal{C}^1$ function of variables y and $z^2 + \alpha x^2$. So, for f_n be a homogeneous polynomial of degree n we should have $k_1 = k_2 = k_3 = 0$.

Thus in PDE (29), as $\alpha \neq 0$, taking $(x, y, z) = \left(\frac{1}{\sqrt{\alpha}}, \frac{1}{\sqrt{\alpha}}, 0\right)$ and using that $f \not\equiv 0$, we have $k_0 = 0$. Therefore $k \equiv 0$, a contradiction. This proves Proposition 3. □

The following lemma will be fundamental for the proof of Proposition 4. This result was motivated by Lemma 3.4 of [2].

Lemma 5.1. *Consider the differential system*

$$\dot{x} = P(x, y, z), \qquad \dot{y} = Q(x, y, z), \qquad \dot{z} = R(x, y, z), \qquad (31)$$

where $P, Q, R \in \mathbb{C}_m[x, y, z]$, the set of all complex polynomials of degree at most m in the variables x, y and z, with m the degree of the respective vector field. If system (31) possesses an invariant algebraic surface of degree n, say $f = \sum_{i=0}^{n} f_i$, then the linear factors (including the complex ones) of f_n are the linear factors of the term

$$y(P_m + R_m) - (x + z)Q_m,$$

which has degree $m + 1$.

Proof. Let $\gamma = rx + sy + tz$ be a linear factor (possibly complex) of multiplicity p in f_n and let $k \in \mathbb{C}_{n-p}[x, y, z]$ the quotient, such that

$$f_n(x, y, z) = \gamma^p k. \qquad (32)$$

As f is an invariant algebraic surface for system (3), there exists a cofactor $L \in \mathbb{C}_{m-1}[x, y, z]$ such that

$$\dot{f} = f_x\dot{x} + f_y\dot{y} + f_z\dot{z} = fL. \qquad (33)$$

Note that the greatest order terms in (33) provide, by (32),

$$pk\gamma^{p-1}(r\dot{x} + s\dot{y} + t\dot{z}) + \gamma^p(k_x\dot{x} + k_y\dot{y} + k_z\dot{z}) = \gamma^p kL,$$

there is

$$pk\gamma^{p-1}(rP_m + sQ_m + tR_m) + \gamma^p(k_xP_m + k_yQ_m + k_zR_m) = \gamma^p kL,$$

where P_m, Q_m and R_m are terms of degree m in P, Q and R, respectively. Thus, γ is a linear factor of $rP_m + sQ_m + tR_m$ and, consequently, γ is a linear factor of

$$y(rP_m + sQ_m + tR_m) = (rx + sy + tz)Q_m - rxQ_m - tzQ_m + ryP_m + tyR_m$$

and, as this last one is equal to $(rx + sy + tz)Q_m + r(yP_m - xQ_m) + t(yR_m - zQ_m)$, we have that γ is a linear factor of $yP_m - xQ_m$ and also of $yR_m - zQ_m$, thus γ is a linear factor of the $(m+1)$–degree terms

$$yP_m - xQ_m + yR_m - zQ_m = y(P_m + R_m) - (x + z)Qm,$$

as we wanted to show. $\qquad\square$

The next proposition is the last case to be considered about the non–existence of algebraic invariant surfaces for system (3).

Proposition 4. *System* (3) *does not have algebraic invariant surfaces for* $\alpha \in [0, 1]$ *and* $\beta \in (0, 1]$.

Proof. Let $f = \sum_{i=0}^n f_i = 0$ be an algebraic invariant surface of degree $n \geq 1$ for system (3) with cofactor $k = k_0 + k_1x + k_2y + k_3z$, being $k_0, k_1, k_2, k_3 \in \mathbb{R}$ not all zero. By definition we have that

$$yz \sum_{i=0}^n \frac{\partial f_i}{\partial x} + (x-y) \sum_{i=0}^n \frac{\partial f_i}{\partial y} + (1 - \alpha xy - \beta x^2) \sum_{i=0}^n \frac{\partial f_i}{\partial z} = (k_0 + k_1x + k_2y + k_3z) \sum_{i=0}^n f_i. \tag{34}$$

The terms of degree $n + 1$ in (34) are

$$yz \frac{\partial f_n}{\partial x} - (\alpha xy + \beta x^2) \frac{\partial f_n}{\partial z} = (k_1x + k_2y + k_3z)f_n. \tag{35}$$

Using the notation of Lemma 5.1, we get

$$y(P_2 + R_2) - (x + z)Q_2 = y(yz - \alpha xy - \beta x^2),$$

so we can write

$$f_n(x, y, z) = y^{n-2m}(yz - \alpha xy - \beta x^2)^m, \quad m \in \mathbb{N} \cup \{0\}.$$

In this way, taking the partials derivatives of f_n with respect to x and z and replacing in (35) we obtain

$$-my(\alpha xy + \alpha yz + \beta x^2 + 2\beta xz) = (k_1x + k_2y + k_3z)(yz - \alpha xy - \beta x^2). \tag{36}$$

As $\alpha, \beta \in [0, 1]$ from (36) we get $k_1 = k_2 = k_3 = m = 0$. Thus $f_n = y^n$ and $k = k_0$. Note that, taking $(x, y, z) = \left(\dfrac{1}{\sqrt{\alpha + \beta}}, \dfrac{1}{\sqrt{\alpha + \beta}}, 0 \right)$ we have that $k_0 = 0$, contradiction. This proves Proposition 4. $\qquad\square$

The following results discuss about the non–existence of polynomial first integral for system (3).

Proposition 5. *When* $\alpha = \beta = 0$ *system* (3) *does not possess polynomial first integral.*

Proof. Suppose that exists $f = \sum_{i=0}^{n} f_i$ be a polynomial first integral for system (3). Without loss of generality, assume that $f_n \not\equiv 0$. Thus f satisfies the PDE

$$yz\frac{\partial f}{\partial x} + (x - y)\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0. \tag{37}$$

Computing the terms of degree $n + 1$ in (37) we have

$$yz\frac{\partial f_n}{\partial x} = 0.$$

Solving this equation we conclude that $f_n = f_n(y, z)$. Now computing the terms of degree n in (37) we obtain

$$yz\frac{\partial f_{n-1}}{\partial x} + (x - y)\frac{\partial f_n}{\partial y} = 0,$$

which implies that

$$\frac{\partial f_n}{\partial y} = \frac{yz}{y - x}\frac{\partial f_{n-1}}{\partial x}.$$

Since f_n does not depend on the variable x (the same happens for its derivative) we must have $\dfrac{\partial f_n}{\partial y} = 0$ and $\dfrac{\partial f_{n-1}}{\partial x} = 0$, that is,

$$f_n = f_n(z) \quad \text{and} \quad f_{n-1} = f_{n-1}(y, z). \tag{38}$$

Once more, computing the terms of degree $n - 1$ in (37) we get

$$yz\frac{\partial f_{n-2}}{\partial x} + (x - y)\frac{\partial f_{n-1}}{\partial y} + \frac{\partial f_n}{\partial z} = 0,$$

and this implies that

$$\frac{\partial f_{n-2}}{\partial x} = \frac{y - x}{yz}\frac{\partial f_{n-1}}{\partial y} - \frac{1}{yz}\frac{\partial f_n}{\partial z}.$$

Integrating with respect to the variable x and using (38), we obtain

$$f_{n-2} = \left(\frac{x}{z} - \frac{x^2}{2yz}\right)\frac{\partial f_{n-1}}{\partial y} - \frac{x}{yz}\frac{\partial f_n}{\partial z} + C(x),$$

where $C(x)$ is a function of the variable x. Using (38) and the fact that f_{n-2} is a homogeneous polynomial, we must have $\partial f_n / \partial z = 0$, which yields f_n to be a constant, contradicting the fact that f_n is a homogeneous polynomial of degree n. As a result we conclude that does not exist a polynomial first integral for system (3). $\qquad \square$

The next proposition together to Lemma 5.2 are important results in the proof of Proposition 7.

Proposition 6. *The linear differential system*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

with $abc \in \mathbb{R} \setminus \{0\}$, has two first integrals of the form

$$F_1(x, y, z) = \frac{(x^2 + y^2)^c}{z^{2a}}, \quad F_2(x, y, z) = \frac{\exp\left(2a \arctan\left(\frac{y}{x}\right)\right)}{(x^2 + y^2)^b} \tag{39}$$

and, if $a = 0$, such a system has two first integrals of the form

$$F_1(x, y, z) = x^2 + y^2, \quad F_2(x, y, z) = \exp\left(-c \arctan\left(\frac{y}{x}\right)\right) z^b. \tag{40}$$

Both first integrals in (39) *and* (40) *are functionally independent, there is,*

$$rank \begin{pmatrix} \dfrac{\partial F_1}{\partial x} & \dfrac{\partial F_1}{\partial y} & \dfrac{\partial F_1}{\partial z} \\[2mm] \dfrac{\partial F_2}{\partial x} & \dfrac{\partial F_2}{\partial y} & \dfrac{\partial F_2}{\partial z} \end{pmatrix} = 2$$

at all points $(x, y, z) \in \mathbb{R}^3$ where they are defined, with the possible exception of a Lebesgue null set. The linear system in question is said completely integrable (see [8] *for more details).*

Proof. It is immediate. $\qquad \square$

Lemma 5.2. *The linear differential system*

$$\begin{aligned} \dot{x} &= \frac{-z}{\sqrt{\alpha + \beta}}, \\ \dot{y} &= x - y, \qquad\qquad \alpha + \beta > 0, \\ \dot{z} &= \frac{\alpha x + \alpha y + 2\beta x}{\sqrt{\alpha + \beta}}, \end{aligned} \tag{41}$$

does not have polynomial first integral.

Proof. As in [9], we rewrite (41) as $\dot{X} = AX$, there is,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dfrac{-1}{\sqrt{\alpha + \beta}} \\[2mm] 1 & -1 & 0 \\[2mm] \dfrac{\alpha + 2\beta}{\sqrt{\alpha + \beta}} & \dfrac{\alpha}{\sqrt{\alpha + \beta}} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

The characteristic polynomial of matrix A is

$$\lambda^3 + \lambda^2 + \frac{\alpha + 2\beta}{\alpha + \beta}\lambda + 2 = 0,$$

which has roots of the form a and $b \pm ic$, with $a < 0$ and $b, c \neq 0$ and the corresponding real canonical Jordan form of A has the form

$$\begin{pmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{pmatrix}.$$

It follows from Proposition 6 that system (41) does not have polynomial first integral. $\qquad \square$

Proposition 7. *When $\alpha \neq 0$ and $\beta \neq 0$, system* (3) *does not have polynomial first integral.*

Proof. As $\alpha, \beta \in (0, 1]$, so $\alpha + \beta > 0$. We know that system (3) has two singularities, which are described in (4). By translating E_1 to the origin, we get the differential

system

$$\dot{x} = yz - \frac{z}{\sqrt{\alpha + \beta}},$$
$$\dot{y} = x - y, \tag{42}$$
$$\dot{z} = \frac{\alpha x}{\sqrt{\alpha + \beta}} + \frac{\alpha y}{\sqrt{\alpha + \beta}} + \frac{2\beta x}{\sqrt{\alpha + \beta}} - \alpha xy - \beta x^2.$$

Suppose that exists $f = \sum_{i \geq 0} f_i$ be a polynomial first integral for system (42), where f_i is a homogeneous polynomial of degree i. We shall prove by induction over i that

$$f_i = 0, \quad \forall\, i \geq 1 \tag{43}$$

and so we get $f = f_0 = $ const. contradicting the hypotheses that f is a first integral. In fact, being f a polynomial first integral, from (42) it verifies

$$\dot{x}\frac{\partial f}{\partial x} + \dot{y}\frac{\partial f}{\partial y} + \dot{z}\frac{\partial f}{\partial z} = 0. \tag{44}$$

The terms of degree 1 on the variables x, y, and z of (44) verify

$$\frac{-z}{\sqrt{\alpha + \beta}}\frac{\partial f_1}{\partial x} + (x - y)\frac{\partial f_1}{\partial y} + \frac{1}{\sqrt{\alpha + \beta}}(\alpha x + \alpha y + 2\beta x)\frac{\partial f_1}{\partial z} = 0. \tag{45}$$

Thus, either $f_1 = 0$ or f_1 is a polynomial first integral of linear system (41). By Lemma 5.2, this last case is not possible. Then $f_1 = 0$, and this proves (43) for $i = 1$.

By induction hypotheses, we assume that (43) holds for $i = 2, \ldots, n-1$.

So, evaluating the terms of degree n in (44) and using the induction hypotheses we obtain the PDE

$$\frac{-z}{\sqrt{\alpha + \beta}}\frac{\partial f_n}{\partial x} + (x - y)\frac{\partial f_n}{\partial y} + \frac{1}{\sqrt{\alpha + \beta}}(\alpha x + \alpha y + 2\beta x)\frac{\partial f_n}{\partial z} = 0,$$

which is a PDE analogous to PDE (45). In this way, using the same argument as before, we conclude that $f_n = 0$ and, therefore, (43) holds. Thus, system (42) (and, consequently, system (3)) does not have polynomial first integral. □

Lemma 5.3. *The linear differential system*

$$\dot{x} = \frac{-z}{\sqrt{\beta}},$$
$$\dot{y} = x - y, \qquad \beta > 0, \tag{46}$$
$$\dot{z} = 2\sqrt{\beta},$$

does not have polynomial first integral.

Proof. Here we proceed as described in Lemma 5.2. As in [9], we rewrite (46) as $\dot{X} = AX$, where the characteristic polynomial of matrix A is $(\lambda^2 + 2)(\lambda + 1) = 0$, which has roots -1 and $\pm i\sqrt{2}$. Thus, using the real canonical Jordan form of A and Proposition 6 we conclude that system (46) does not have polynomial first integral. □

Proposition 8. *When $\alpha = 0$ and $\beta \neq 0$, system (3) does not have polynomial first integral.*

Proof. In this case, system (3) has two singularities

$$\mathrm{E}_1 = \left(\frac{1}{\sqrt{\beta}}, \frac{1}{\sqrt{\beta}}, 0\right), \quad \mathrm{E}_2 = \left(-\frac{1}{\sqrt{\beta}}, -\frac{1}{\sqrt{\beta}}, 0\right).$$

By translating E_1 to the origin, we get the differential system

$$\begin{aligned}
\dot{x} &= yz - \frac{z}{\sqrt{\beta}}, \\
\dot{y} &= x - y, \\
\dot{z} &= 2\sqrt{\beta}x - \beta x^2.
\end{aligned} \tag{47}$$

Proceeding as described in the proof of Proposition 7 and applying Lemma 5.3, we conclude this proof. $\qquad\square$

Lemma 5.4. *The linear differential system*

$$\begin{aligned}
\dot{x} &= yz - \frac{z}{\sqrt{\alpha}}, \\
\dot{y} &= x - y, \qquad\qquad \alpha > 0, \\
\dot{z} &= \sqrt{\alpha}x + \sqrt{\alpha}y,
\end{aligned} \tag{48}$$

does not have polynomial first integral.

Proof. It is enough to proceed exactly as in the proof of Lemma 5.2. $\qquad\square$

Proposition 9. *When $\beta = 0$ and $\alpha \neq 0$ system* (3) *does not have polynomial first integral.*

Proof. We proceed as in the proof of Proposition 8, using the same arguments as in the proof of Proposition 7, together with Lemma 5.4. $\qquad\square$

In order to conclude the proof of Theorem 1.3 we have only to show that system (3) does not have an exponential factor and consequently a Darboux first integral.

Let $E = \exp(f/g) \notin \mathbb{C}$ an exponential factor of system (3) with cofactor $L = L_0 + L_1 x + L_2 y + L_3 z$, where $f, g \in \mathbb{C}[x, y, z]$, with $(f, g) = 1$. As the system verifies Theorem 1.3, it follows from Lemma 2.1 that $E = \exp(f)$, with $f = f(x, y, z) \in \mathbb{C}[x, y, z] \setminus \mathbb{C}$. Also, it follows from equation (10) that f satisfies

$$yz\frac{\partial f}{\partial x} + (x - y)\frac{\partial f}{\partial y} + (1 - \alpha xy - \beta x^2)\frac{\partial f}{\partial z} = L_0 + L_1 x + L_2 y + L_3 z, \tag{49}$$

where we have simplified the common factor $\exp(f)$. Let n be the degree of f, write $f = \sum_{i=0}^{n} f_i$, where f_i is a homogeneous polynomial of degree i and, without loss of generality, assume that $f_n \neq 0$. Taking $n > 1$ and evaluating the terms of degree $n + 1$ in (49), we arrive at the following PDE

$$yz\frac{\partial f_n}{\partial x} - (\alpha xy + \beta x^2)\frac{\partial f_n}{\partial z} = 0. \tag{50}$$

Observe that if PDE (50) admits solution then system (3) admits a polynomial first integral, contradicting Theorem 1.3. Therefore, system (3) does not have an exponential factor and consequently, a Darboux first integral.

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