CODIMENSION TWO UMBILIC POINTS ON
SURFACES IMMERSED IN $\mathbb{R}^3$

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ABSTRACT. This paper studies the behavior of lines of curvature near umbilic points that appear generically on surfaces depending on two parameters.

1. **Introduction.** The study of principal curvature lines, along which surfaces immersed in $\mathbb{R}^3$ bend maximally and minimally, goes back to the work of Monge [13]. The behavior of principal lines was determined on the triaxial ellipsoids. The analysis around umbilic points, at which surfaces bend equally in all directions, had a preponderant role in this work, which may be regarded as the first one on the subject of foliations with singularities.

In 1896, Darboux [3] carried out a complete local analysis of umbilic points in the class of analytic surfaces, under generic conditions on the third derivatives of the immersions of surfaces at the umbilic points. Darboux established that there are three generic patterns, called here *Darbouxian Umbilics* D_1, D_2 and D_3, illustrated in Fig. 1. This figure represents both families of principal curvature lines, minimal $\mathcal{F}_1$ and maximal $\mathcal{F}_2$, associated to minimal and maximal principal curvatures k_1 and k_2, $k_1 < k_2$, expect at umbilic points. The subscript stands for the number of umbilic separatrices approaching the point. In this case the number is the same for both the minimal and maximal principal curvature foliations.

This result of Darboux was rediscovered by Gutierrez and Sotomayor, [7, 8] in the context of Structural Stability of principal lines on surfaces of class C^r, $r \geq 4$. They showed that the Darbouxian umbilic points are generic, corroborating in C^r, $r \geq 4$, the above mentioned results of Darboux, and characterize the structurally stable ones, under small C^3 deformations of the surface. See also the work of Bruce and Fidal [2]. In Sotomayor [15] comments can be found on the early developments on lines of curvature and their relation with recent works on foliations with singularities geometrically defined on surfaces immersed in $\mathbb{R}^3$.

In [4, 9] the simplest bifurcation patterns of Darbouxian umbilics were established by Garcia, Gutierrez and Sotomayor. This corresponds to the weakening of
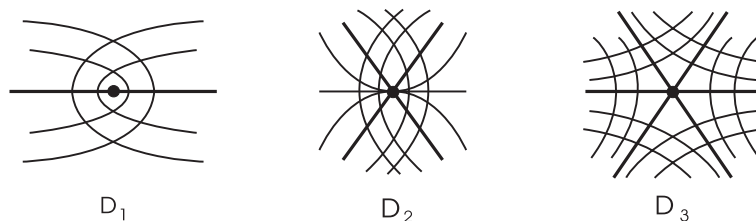
FIGURE 1. Principal curvature lines near the Darbouxian D_i, $i =$ 1, 2, 3, umbilic points and their separatrices

the Darbouxian structural stability conditions in the most stable and generic way, leading to *codimension one umbilic points* and their generic bifurcations. See section 2 for a review of the analytic expressions of these conditions.

The expression *codimension one* means that the umbilic points appear generically on one-parameter families of immersed surfaces. This corresponds to families in general position with respect to a regular part of the bifurcation set –a hypersurface– in the infinite dimensional space of immersions. The two patterns of umbilics of codimension one established in [4, 9], denominated D_2^1 and $D_{2,3}^1$, are illustrated in Fig. 2. The superscript stands for the codimension and the subscripts stand for the number of separatrices approaching the umbilic point. In the first case, the number of separatrices is the same for both the minimal and maximal principal curvature foliations. In the second case, they are not equal and, in our notation, appear separated by a comma. See Theorem 2 in section 2.
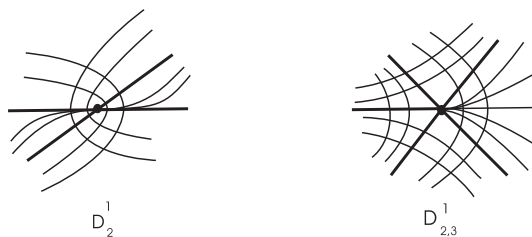

FIGURE 2. Lines of Curvature near Codimension one Umbilic Points D_2^1, left, and $D_{2,3}^1$, right.

This paper pursues the analysis in [4], increasing in one unity the number of parameters on which the immersion of the surface depends. The general *codimension two* umbilic points exhibited by generic bi-parametric families of surfaces immersed in $\mathbb{R}^3$, consisting on four patterns – called D_1^2, D_{2p}^2, D_3^2 and D_{2h}^2 *codimension two* umbilic points – will be established here. The superscript stands for the codimension. The symbols p, for *parabolic*, and h, for *hyperbolic*, have been added to the subscripts above in order to distinguish types that are not discriminated only by the number of separatrices.

The first three patterns can be described as follows: D_1^2 is topologically equivalent to a point D_1; D_{2p}^2 is topologically equivalent to a point D_2; D_3^2 is topologically equivalent to a point D_3. See Fig. 1.

The pattern D_{2h}^2 is illustrated in Fig. 3, where c and B are coefficients defined in equation (1), pertinent to a Monge presentation of the immersion at the umbilic

point. This pattern is new with respect to previous types. Also no proof of its structure has been found in the literature.
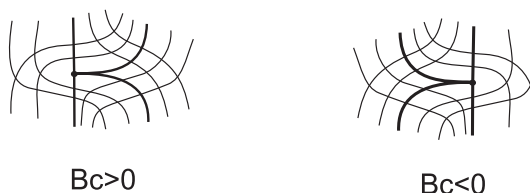

FIGURE 3. Lines of curvature near the umbilic point D_{2h}^2

Section 2 is devoted to review some background on umbilic points of codimension less than two. The local configurations of principal lines at umbilic points of types D_1^2, D_{2p}^2, D_3^2 and D_{2h}^2 are established in Section 3. The fact that these points actually are those of *codimension two*, in a geometric sense, is established in Section 4.

The study of the bifurcation diagrams of lines of principal curvature lines at umbilic points for bi-parametric families of surfaces in $\mathbb{R}^3$, roughly prepared in Section 4, involves some technical and lengthly work, especially in the case of D_{2h}^2, and will be postponed to a future paper [5].

2. **Preliminaries on Umbilic Points.** The following assumptions will hold from now on.

Let p_0 be an umbilic point of an immersion α of an oriented surface $\mathbb{M}$ into $\mathbb{R}^3$, with a once for all fixed orientation. It will be assumed that α is of class C^k, $k \geq 6$. In a local Monge chart near p_0 and a positive adapted $3-$frame, α is given by $\alpha(u,v) = (u, v, h(u,v))$, where

$$
\begin{aligned}
h(u,v) = {}& \frac{k}{2}(u^2 + v^2) + \frac{a}{6}u^3 + \frac{b}{2}uv^2 + \frac{c}{6}v^3 + \frac{A}{24}u^4 + \frac{B}{6}u^3 v \\
& + \frac{C}{4}u^2 v^2 + \frac{D}{6}uv^3 + \frac{E}{24}v^4 + \frac{a_{50}}{120}u^5 + \frac{a_{41}}{24}u^4 v \\
& + \frac{a_{32}}{12}u^3 v^2 + \frac{a_{23}}{12}u^2 v^3 + \frac{a_{14}}{24}uv^4 + \frac{a_{05}}{120}v^5 + h.o.t
\end{aligned}
\tag{1}
$$

Notice that, without loss of generality, the term $u^2 v$ has been eliminated from this expression by means of a rotation in the $(u,v)-$ frame.

According to [16] and [17], the differential equation of lines of curvature in terms of $I = Edu^2 + 2Fdudv + Gdv^2$ and $II = edu^2 + 2fdudv + gdv^2$ around p_0 is:

$$
(Fg - Gf)dv^2 + (Eg - Ge)dudv + (Ef - Fe)du^2 = 0.
\tag{2}
$$

Therefore the functions $L = Fg - Gf$, $M = Eg - Ge$ and $N = Ef - Fe$ are:

$$
L = h_u h_v h_{vv} - (1 + h_v^2)h_{uv}
$$
$$
M = (1 + h_u^2)h_{vv} - (1 + h_v^2)h_{uu}
$$
$$
N = (1 + h_u^2)h_{uv} - h_u h_v h_{uu}.
$$

Extensive calculation gives

$$L = -bv - \frac{1}{2}Bu^2 - (C - k^3)uv - \frac{1}{2}Dv^2 - \frac{a_{41}}{6}u^3$$
$$+ \frac{1}{2}(4bk^2 + k^2a - a_{32})u^2v + \frac{1}{2}(3k^2c - a_{23})uv^2 - \frac{1}{6}(a_{14} + 3bk^2)v^3 + h.o.t$$
$$M = (b - a)u + cv + \frac{1}{2}(C - A + 2k^3)u^2 + (D - B)uv + \frac{1}{2}(E - C - 2k^3)v^2$$
$$+ \frac{1}{6}[6bk^2(a + b) + a_{32} - a_{50}]u^3 + \frac{1}{2}(a_{23} + 2ck^2 - a_{41})u^2v \qquad (3)$$
$$+ \frac{1}{2}[a_{14} - a_{32} - 2k^2(a + b)]uv^2 + \frac{1}{6}(a_{05} - a_{23} - 6ck^2)v^3 + h.o.t$$
$$N = bv + \frac{1}{2}Bu^2 + (C - k^3)uv + \frac{1}{2}Dv^2 + \frac{1}{6}a_{41}u^3$$
$$+ \frac{1}{2}(a_{32} - 3ak^2)u^2v + \frac{1}{2}(a_{23} - k^2c)uv^2 + \frac{1}{6}(a_{14} - 3bk^2)v^3 + h.o.t$$

These expressions appear, up to order two, in Darboux [3], including also the coefficient of u^2v, annihilated here by a rotation in the frame (u, v).

2.1. **Darbouxian Umbilic Points.** For the sake of completeness we will state now the result of Gutierrez and Sotomayor in [7, 8] about the local behavior of curvature lines near a Darbouxian umbilic point.

Theorem 1. *Let p_0 be an umbilic point and consider the Monge chart as in equation (1).*

Suppose that the transversality condition:

T) $b(b - a) \neq 0$

and one the following –Darbouxian– conditions holds.

$D_1)$ $(\frac{c}{2b})^2 - \frac{a}{b} + 2 < 0$
$D_2)$ $(\frac{c}{2b})^2 + 2 > \frac{a}{b} > 1, \quad a \neq 2b$
$D_3)$ $\frac{a}{b} < 1$

Then the behavior of lines of curvature near the umbilic point, separated in two principal foliations $\mathcal{F}_1$ and $\mathcal{F}_2$, with a common singularity at p_0 is as illustrated in Fig. 1.

The local configuration near umbilics is explained in terms of the phase portraits of the singularities of the Lie-Cartan vector field suspension, $X_{\mathcal{F}}$, of the implicit differential equation $\mathcal{F}(u, v, p) = 0$, where,

$$\mathcal{F}(u, v, p) = Lp^2 + Mp + N = 0, \ p = \frac{dv}{du}$$
$$X_{\mathcal{F}} = (\mathcal{F}_p, p\mathcal{F}_p, -(\mathcal{F}_u + p\mathcal{F}_v)). \qquad (4)$$

The projection by $\pi(u, v, p) = (u, v)$ of the integral curves of $X_{\mathcal{F}}$, restricted to $\mathcal{F}(u, v, p) = 0$ are the lines of curvature near the umbilic point. The singularities of $X_{\mathcal{F}}$, restricted to the invariant regular surface $\mathcal{F}(u, v, p) = 0$, are given by $(0, 0, p_i)$ where p_i is a root of the equation $(\mathcal{F}_u + p\mathcal{F}_v)(0, 0, p) = p(bp^2 - cp + a - 2b) = 0$. In the present Darbouxian case, these singularities are hyperbolic saddles and nodes.

In Fig. 4 is illustrated the behavior of $X_{\mathcal{F}}$ in each of the Darbouxian cases. Here only one of the principal foliations has been drawn, for the sake of simplicity. The other one is the projection of the other half cylinder, not sketched in the Figure.
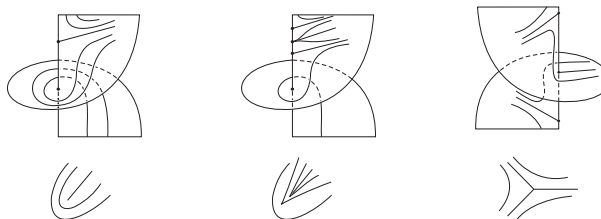
FIGURE 4. Lie-Cartan Suspension of the D_i umbilic points

Remark 1. With reference to the illustrations in Fig. 1 of principal curvature foliations, arises the question of deciding which foliation is minimal and which one is maximal.

The procedure to follow, going back to the basic definitions, is to evaluate the normal curvature function $k_n(p_0, [du : dv]) = II(p_0, [du : dv])/I(p_0, [du : dv])$, on the surface $\mathcal{F}(p_0, [du : dv]) = 0$. Locally, near the projective line $\mathcal{P} = \{(0, 0, p)\}$ the set $\mathcal{F}^{-1}(0) \setminus \mathcal{P}$ has two open connected components defined by $\mathcal{C}_1 = \{(p_0, [du : dv]) : k_n(p_0, [du : dv]) = k_1(p_0)\} \setminus \mathcal{P}$ and $\mathcal{C}_2 = \{(p_0, [du : dv]) : k_n(p_0, [du : dv]) = k_2(p_0)\} \setminus \mathcal{P}$. The projection of the phase portrait of $X_{\mathcal{F}}$ in the component $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) defines the minimal (resp. maximal) principal foliation.

This procedure applies to all isolated umbilics and will not be repeated for those studied below.

2.2. **Codimension One Umbilic Points.** For future reference, we review a basic result of [4, 9] for umbilics in generic families of surfaces depending on one parameter.

Theorem 2. [4, 9] *Let p_0 be an umbilic point and consider $\alpha(u, v) = (u, v, h(u, v))$ as in equation (1). Suppose the following conditions hold:*

D_2^1) $cb(b - a) \neq 0$ *and either* $(\frac{c}{2b})^2 - \frac{a}{b} + 1 = 0$ *or* $a = 2b$.
$D_{2,3}^1$) $b = a \neq 0$ *and* $\chi = cB - (C - A + 2k^3)b \neq 0$.

Then the behavior of lines of curvature near the umbilic point p_0 in cases D_2^1 and $D_{2,3}^1$, is as illustrated in Fig. 2.

In Fig. 5 is illustrated the behavior of $X_{\mathcal{F}}$ in each of the semi Darbouxian (D_2^1 and $D_{2,3}^1$) cases.
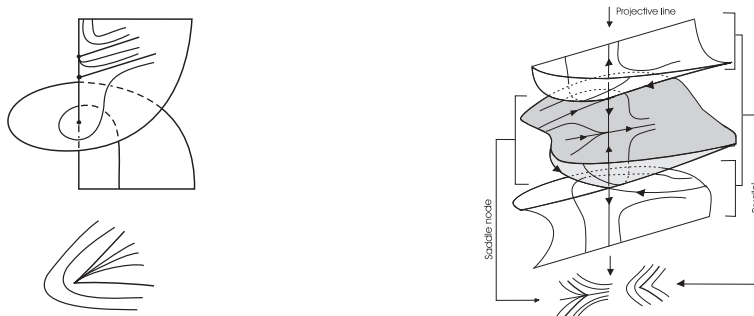
FIGURE 5. Lie-Cartan suspension D_2^1, left, and $D_{2,3}^1$, right.

In this section, umbilic points which appear generically and stably on one-parameter families of surfaces have been described in terms of hyperbolic and partially hyperbolic singularities of vector fields. The analogy between the structurally stable plane vector field singularities and stable umbilic points on surfaces is complete in the Darbouxian case. In the D_2^1 and $D_{2,3}^1$, codimension one umbilics, the analogy is only partial. The full analysis depends both on the classic saddle-node, in the plane, and on the phase portrait of a singularity of a vector field on a conic Morse critical point on the surface $\mathcal{F}^{-1}(0)$.

3. **Patterns of Codimension Two Umbilic Points.** In this section will be studied the four simplest umbilic points, denominated D_1^2, D_{2p}^2, D_3^2 and D_{2h}^2, not discussed previously, assuming that the non-degeneracy conditions imposed in Theorem 2 of section 2, are violated in the mildest possible fashion and are substituted by suitable higher order hypotheses. This procedure goes back to the notions of *higher order stability* due to Andronov and Leontovich [1]. Se also [4] and [6]. Here, referring to equation 1, $j^3h(0)$ does not vanish and the hypotheses are expressed in terms of $j^4h(0)$ and $j^5h(0)$.

3.1. D_1^2 Umbilic Points.

Proposition 1. *Let p_0 be an umbilic point and $\alpha(u,v) = (u,v,h(u,v))$ as in equation (1).*
  If D_1^2): $c = 0$ and $a = 2b \neq 0$,
then the configuration of principal lines near p_0 is topologically equivalent to that of a Darbouxian D_1 umbilic point. See Fig. 1, left.

Proof. The differential equation of curvature lines is given by equation (4), where, for the present case:

$$L = -bv - \frac{1}{2}Bu^2 + (k^3 - C)uv - \frac{1}{2}Dv^2 + h.o.t$$

$$M = -bu + \frac{1}{2}(2k^3 - A + C)u^2 + (D - B)uv + \frac{1}{2}(E - 2k^3 - C)v^2 + h.o.t$$

$$N = bv + \frac{1}{2}Bu^2 + (-k^3 + C)uv + \frac{1}{2}Dv^2 + h.o.t$$

Direct calculation shows that $\mathcal{F}_v(0,0,0) = b \neq 0$. The solution, $v = v(u,p)$ of the implicit equation $\mathcal{F}(u, v(u,p), p) = 0$ near $(0,0,0)$ is given by:

$$v(u,p) = up - \frac{B}{2b}u^2 + O(3).$$

Near the origin, in the chart (u,p), the vector field $X_\mathcal{F}$ is given by

$$\dot{u} = \mathcal{F}_p(u, v(u,p), p) = -bu + O(2)$$

$$\dot{p} = -(\mathcal{F}_u + p\mathcal{F}_v)(u, v(u,p), p) = u[-B + O(1)] + bp^3.$$

Therefore the point 0 is a topological saddle of order three, having the projective line as the unique center manifold.

The analysis of $X_\mathcal{F}$ outside a neighborhood of the $(0,0,0)$ is trivial, since the three singular points on the $p-$axis are concentrated at the origin. In the chart (u,v,q), $q = du/dv$, the correspondent Lie-Cartan vector field is regular at $(0,0,0)$. Therefore, gluing the phase portraits as in Darbouxian D_1 umbilic point, the result follows. See Fig. 4, left. $\square$

3.2. D_{2p}^2 and D_3^2 Umbilic Points.

Proposition 2. *Let p_0 be an umbilic point and $\alpha(u,v) = (u,v,h(u,v))$ be as in equation (1).*
If $a = b \neq 0$, $\chi = cB - b(C - A + 2k^3) = 0$ and
$\xi = 12k^2b^3 + (a_{32} - a_{50})b^2 + (3B^2 - 3BD - ca_{41})b + 3cB(C - k^3) \neq 0$,
then the principal configurations of lines of curvature fall into one of the two cases:

  i) *Case D_{2p}^2: $\xi b < 0$, which is topologically a D_2 umbilic and*
  ii) *Case D_3^2: $\xi b > 0$, which is topologically a D_3 umbilic.*

See Fig. 1, center and right, respectively.

Proof. The differential equation of curvature lines is given by $\mathcal{F}(u,v,p) = Lp^2 + Mp + N = 0$, $p = \frac{dv}{du}$ where the coefficients are given from equation (3) as follows:

$$L = -bv - \frac{1}{2}Bu^2 - (C - k^3)uv - \frac{1}{2}Dv^2 - \frac{1}{6}a_{41}u^3 + \frac{1}{2}(5bk^2 - a_{32})u^2v$$
$$+ \frac{1}{2}(3k^2c - a_{23})uv^2 - \frac{1}{6}(a_{14} + 3bk^2)v^3 + h.o.t$$
$$M = cv + \frac{1}{2}(C - A + 2k^3)u^2 + (D - B)uv + \frac{1}{2}(E - C - 2k^3)v^2$$
$$+ \frac{1}{6}[12bk^2 + a_{32} - a_{50}]u^3 + \frac{1}{2}(a_{23} + 2ck^2 - a_{41})u^2v \qquad (5)$$
$$+ \frac{1}{2}[a_{14} - a_{32} - 4k^2b]uv^2 + \frac{1}{6}(a_{05} - a_{23} - 6ck^2)v^3 + h.o.t$$
$$N = bv + \frac{1}{2}Bu^2 + (C - k^3)uv + \frac{1}{2}Dv^2 + \frac{1}{6}a_{41}u^3 + \frac{1}{2}(a_{32} - 3bk^2)u^2v$$
$$+ \frac{1}{2}(a_{23} - k^2c)uv^2 + \frac{1}{6}(a_{14} - 3bk^2)v^3 + h.o.t$$

Direct calculation shows that $\mathcal{F}_v(0,0,0) = b \neq 0$. The solution $v = v(u,p)$ of equation $\mathcal{F}(u, v(u,p), p) = 0$ is given by:

$$v(u,p) = -\frac{B}{2b}u^2 + [\frac{3B(C - k^3) - a_{41}b}{6b^2}]u^3 + [\frac{cB - (C - A + 2k^3)b}{2b^2}]u^2p + O(4).$$

Therefore, in the chart (u,p), the Lie-Cartan vector field $X_\mathcal{F}$ is given by

$$\dot{u} = \mathcal{F}_p(u, v(u,p), p)$$
$$\dot{p} = -(\mathcal{F}_u + p\mathcal{F}_v)(u, v(u,p), p),$$

which amounts to

$$\dot{u} = -\frac{\chi}{2b}u^2 + \frac{\xi}{6b^2}u^3 + \frac{c\chi}{2b^2}u^2p + O(4)$$
$$\dot{p} = -Bu - bp + O(2), \qquad (6)$$

where $\xi = 12k^2b^3 + (a_{32} - a_{50})b^2 + (3B^2 - 3BD - ca_{41})b + 3cB(C - k^3)$.

As $\chi = cB - b(C - A + 2k^3) = 0$, the singular point 0 is a topological saddle of order three when $b\xi > 0$ and is a topological node when $b\xi < 0$.

The other singular points of the vector field $X_\mathcal{F} = (\mathcal{F}_p, p\mathcal{F}_p, -(\mathcal{F}_u + p\mathcal{F}_v))$ are $(0,0,p_1)$ and $(0,0,p_2)$, where p_i are the roots of the equation $bp^2 - cp - b = 0$, which are topological saddles. In fact,

$$DX_{\mathcal{F}}(0,0,p_i) = \begin{bmatrix} 0 & -2bp_i + c & 0 \\ 0 & p_i(2bp_i - c) & 0 \\ b_1 & b_2 & 3bp_i^2 - 2cp_i - b \end{bmatrix},$$

where $b_1 = 2Bp_i^2 - 2p_iC - p_ik^3 + p_iA - B - p_i^3k^3 + p_i^3C - Dp_i^2$ and $b_2 = p_i^2k^3 + 2p_i^2C - 2p_iD + p_iB - C + k^3 + Dp_i^3 - p_i^2E$.

The non zero eigenvalues of $DX_{\mathcal{F}}(0,0,p_i)$ are $\lambda_1 = -2bp_i^2 + cp_i = -(cp_i + 2b) = -b(p_i^2 + 1)$ and $\lambda_2 = 3bp_i^2 - 2cp_i - b = cp_i + 2b = b(p_i^2 + 1)$.

The implicit surface $\mathcal{F}(u,v,p) = 0$ has two critical points, one at $(0,0,p_1)$ and the other at $(0,0,p_2)$. They are of Morse conic type if and only if $\chi \neq 0$. Otherwise, they have quadratic co-rank 1 and Morse index 1.

Below it will be established that, at these points, the surface is locally a topological disk. This will follow from the smooth right equivalence of $\mathcal{F}$ with $x^2 - y^2 + z^3 = 0$, provided $\xi \neq 0$. The smoothness class (≥ 1) is explained below as a consequence of [11].

In fact, at the points p_1 and p_2, the Hessian of $\mathcal{F}$ is given by:

$$Hess(\mathcal{F}(0,0,p_i)) = \begin{bmatrix} 0 & b_{12} & 0 \\ b_{12} & -\frac{p_i(cB - Eb + Dc + Ab)}{b} & c - 2bp_i \\ 0 & c - 2bp_i & 0 \end{bmatrix},$$

where $b_{12} = \frac{p_i}{b}\left((k^3 - C)c + (D - B)b\right)$.

The eigenvalues of $Hess(\mathcal{F}(0,0,p_i))$ are $0 < \mu_1$, $\mu_2 < 0$, with $\mu_1\mu_2 = -[b_{12}^2 + (c - 2bp_i)^2] < 0$, and $\mu_3 = 0$. The eigenspace associated to $\mu_3 = 0$ is spanned by $\ell_3 = \left((2bp_i - c)b, 0, p_i((k^3 - C)c + (D - B)b)\right)$.

Consider the implicit equation

$$(\mathcal{F}_v(u,v,p_i + q), \mathcal{F}_p(u,v,p_i + q)) = (0,0).$$

By the Implicit Function Theorem, the solution of the equation above is a regular curve c. A regular parametrization of c is given by

$$c(s) = (b(2bp_i - c)s, 0(2), p_i + p_i[(k^3 - C)c + (D - B)b]s + 0(2)).$$

Therefore, $\gamma(s) = \mathcal{F}(c(s))$ is such that $\gamma(0) = \mathcal{F}(0,0,p_i) = 0$, $\gamma'(0) = 0$, $\gamma''(0) = -p_ib(c^2 + 4b^2)\chi = 0$ and $\gamma'''(0) = p_i(2bp_i - c)^3\xi \neq 0$. This follows from the fact that p_i are solutions of $bp^2 - cp + b = 0$.

The calculation leading to the expression for $\gamma'''(0)$, linear in ξ, has been corroborated by Computer Algebra.

Therefore, by the Generalized Morse Lemma – see Thom [18], pp. 75, and Lopez de Medrano [11]– there exist coordinates (u_1, v_1, w_1) such that $\mathcal{F}(u_1, v_1, w_1) = \mu_1 u_1^2 + \mu_2 v_1^2 + w_1^3$.

Direct examination of proof in this paper gives that the coordinates are of smoothness class $k - 2 - 3 \geq 1$, $k - 2$ in the class of $\mathcal{F}$.

The phase portrait of $X_{\mathcal{F}}$ near the singular point $(0,0,p_i)$ is shown in Fig. 6.

This picture illustrates the orientation reversing projection of the surface into the plane, which occurs in D_3^2. To get the orientation preserving case it suffices to look at this picture upside down.

To conclude the proof it must be noticed that for $X_{\mathcal{F}}$ the singular point 0 is a node when $b\xi < 0$ and it is a saddle when $b\xi > 0$, which follows from equation (6), while p_i $(i = 1, 2)$ are always saddles. This leads to D_{2p}^2 in the first case and to D_3^2 in the second case.   □
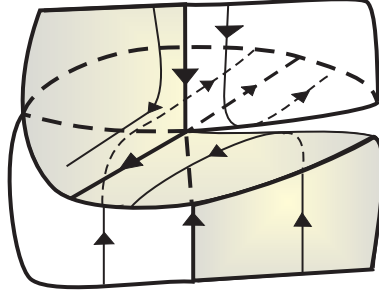
FIGURE 6. Phase portrait of $X_{\mathcal{F}}$ in a neighborhood of a critical point of corank 1.

3.3. D_{2h}^2 Umbilic Points.

Proposition 3. *Let p_0 be an umbilic point and $\alpha(u, v) = (u, v, h(u, v))$ be as in equation (1).*

If $a = b = 0$ and $cB \neq 0$ or if $b = c = 0$ and $aD \neq 0$, then the principal configuration near p_0 is as in Fig. 3.

This umbilic point is denoted by D_{2h}^2.

Proof. Only the case $a = b = 0$ and $cB \neq 0$ will be considered. The other case ($b = c = 0$ and $aD \neq 0$) reduces to this one by an appropriated change of coordinates - a rotation of angle $\pi/2$ in the plane uv. Under these hypotheses, there are two singular points for the Lie-Cartan vector field (4); one is located at $p = 0$ and the other at $p = \infty$, that is at $q = 0$.

Consider first the implicit differential equation of curvature lines

$$
\begin{aligned}
\mathcal{G}(u, v, q) = &\left[-\frac{1}{2}Bu^2 + (k^3 - C)uv - \frac{1}{2}Dv^2 + O(3)\right] \\
&+ \left[cv + \frac{1}{2}(C + 2k^3 - A)u^2 + (D - B)uv + \frac{1}{2}(E - 2k^3 - C)v^2 + O(3)\right]q \quad (7) \\
&+ \left[\frac{1}{2}Bu^2 - (k^3 - C)uv + \frac{1}{2}Dv^2 + O(3)\right]q^2 = 0.
\end{aligned}
$$

in the projective chart (u, v, q), where $q = du/dv$. This implicit surface has a singularity of Morse type at $(0, 0, 0)$ provided $cB \neq 0$. In such a case, it has the topological structure of a cone.

In fact,

$$
Hess(\mathcal{G})(0) = \begin{bmatrix} -B & k^3 - C & 0 \\ k^3 - C & -D & c \\ 0 & c & 0 \end{bmatrix}.
$$

Therefore, $det(Hess(\mathcal{G})(0)) = c^2 B \neq 0$. The Lie-Cartan vector field

$$
Y = (q\mathcal{G}_q, \mathcal{G}_q, -(\mathcal{G}_u q + \mathcal{G}_v))
$$

is given by:

$$u' = q[cv + \frac{1}{2}(C + 2k^3 - A)u^2 + (D - B)uv + \frac{1}{2}(E - 2k^3 - C)v^2 + O(3)]$$

$$v' = cv + \frac{1}{2}(C + 2k^3 - A)u^2 + (D - B)uv + \frac{1}{2}(E - 2k^3 - C)v^2 + O(3)$$

$$q' = (C - k^3)u + Dv - cq + \frac{1}{2}cu^2 + (c - 3k^2c)uv$$

$$+\frac{1}{2}a_{14}v^2 + (2B - D)uq + (2C - E + k^3)vq + O(3)$$

Therefore,

$$DY(0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & c & 0 \\ C - k^3 & D & -c \end{bmatrix}.$$

So, the nonzero eigenvalues of $DY(0)$ are $\lambda_1 = c$ and $\lambda_2 = -c$ and 0 is a saddle point for $Y_{\mathcal{G}}$. The phase portrait of $Y_{\mathcal{G}}$ is as shown in Fig. 7 below.
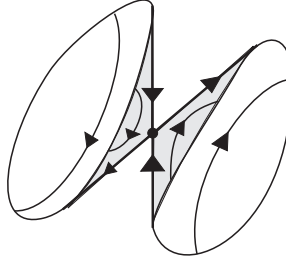

FIGURE 7.   Cone at $p = \infty$, $q = 0$.

Consider now the implicit differential equation

$$\mathcal{F}(u, v, p) = [-\frac{1}{2}Bu^2 + (k^3 - C)uv - \frac{1}{2}Dv^2 - \frac{1}{6}a_{41}u^3$$

$$-\frac{1}{2}a_{32}u^2v + \frac{1}{2}(3k^2c - a_{23})uv^2) - \frac{1}{6}a_{14}v^3 + O(4)]p^2$$

$$+[cv + \frac{1}{2}(C + 2k^3 - A)u^2 + (D - B)uv + \frac{1}{2}(E - 2k^3 - C)v^2$$

$$+\frac{1}{6}(a_{32} - a_{50})u^3 + \frac{1}{2}(2k^2c + a_{23} - a_{41})u^2v + \frac{1}{2}(a_{14} - a_{32})uv^2 \qquad (8)$$

$$+\frac{1}{6}(a_{05} - 6k^2c - a_{23})v^3 + O(4)]p$$

$$+\frac{1}{2}Bu^2 - (k^3 - C)uv + \frac{1}{2}Dv^2 + \frac{1}{6}a_{41}u^3$$

$$+\frac{1}{2}a_{32}u^2v + \frac{1}{2}(a_{23} - k^2c)uv^2 + \frac{1}{6}a_{14}v^3 + O(4) = 0.$$

in the projective chart (u, v, p), where $p = dv/du$.

This surface also has a singular point of Morse type at $(0, 0, 0)$ with the topological structure of a cone, provided $cB \neq 0$.

As above, the Lie-Cartan vector field $Z = (\mathcal{F}_p, p\mathcal{F}_p, -(\mathcal{F}_u + p\mathcal{F}_v))$ is given by:

$$u' = cv + \frac{1}{2}(C + 2k^3 - A)u^2 + (D - B)uv + \frac{1}{2}(E - 2k^3 - C)v^2 + O(3).$$

$$v' = p[cv + \frac{1}{2}(C + 2k^3 - A)u^2 + (D - B)uv + \frac{1}{2}(E - 2k^3 - C)v^2 + O(3)]$$

$$p' = -Bu + (k^3 - C)v - cp^2 + (A - k^3 - 2C)up + (B - 2D)vp$$

$$\qquad - \frac{1}{2}a_{41}u^2 - a_{32}uv + \frac{1}{2}(k^2c - a_{23})v^2 + O(3)$$

(9)

In this case the singular point $(0,0,0)$ is not hyperbolic for the vector field $Z = Z_{\mathcal{F}}$. In fact,

$$DZ(0) = \begin{bmatrix} 0 & c & 0 \\ 0 & 0 & 0 \\ -B & k^3 - C & 0 \end{bmatrix},$$

and the three eigenvalues are zero.

In order to describe the phase portrait of Z near zero, observe that the sheets of the cone, with the origin removed, are located at $p > 0$ and $p < 0$.

To describe the phase portrait of $Z_{\mathcal{F}}$ near 0 it is suitable to perform the following weighted rescaling of variables

$$u = r^2\bar{u}, \qquad v = r^3\bar{v}, \qquad p = r\bar{p}, \qquad \bar{u}^2 + \bar{v}^2 + \bar{p}^2 = 1, \qquad r \geq 0. \qquad (10)$$

Then, from equation (8) it follows that

$$H(\bar{u}, \bar{v}, \bar{p}) = \mathcal{F}(r^2\bar{u}, r^3\bar{v}, r\bar{p}) = c\bar{v}\bar{p} + \frac{1}{2}B\bar{u}^2$$

$$\qquad\qquad + r[(C - k^3)\bar{u}\bar{v} + \bar{u}^2\bar{p}(k^3 + \frac{1}{2}C)] + O(r^2). \qquad (11)$$

So, from equations (9) and (11), dropping the bars to simplify the notation and dividing by appropriate nonnegative functions, the following vector field is obtained: $X = X_u \frac{\partial}{\partial u} + X_v \frac{\partial}{\partial v} + X_p \frac{\partial}{\partial p} + X_r \frac{\partial}{\partial r}$, with

$$u' = X_u = -2cv^2up + 3cv^3 + cp^2v + 2Bu^2p + 2cup^3 + rF_1(u, v, p, r)$$

$$v' = X_v = v(2cu^2p + 3Bup - 3cuv + 4cp^3) + rF_2(u, v, p, r)$$

$$p' = X_p = -cuvp - 2Bu^3 - 2cu^2p^2 - 3Buv^2 - 4cv^2p^2 + rF_3(u, v, p, r)$$

$$r' = X_r = r[cuv - Bup - cp^3 + cv^2p + rF_4(u, v, p, r)].$$

In the expression above $F_i(u, v, p, 0) = 0$, $(i = 1, \ldots, 4)$.

The restriction of X to the unitary sphere $u^2 + v^2 + p^2 = 1$ is given by $X_0 = X(u, v, p, 0)$ which has exactly six singular points, defined by the intersection of the curves

$$c_1(p) = (0, 0, p), \quad c_2(p) = (-\frac{cp^2}{B}, 0, p), \quad c_3(p) = (-\frac{4cp^2}{3B}, -\frac{8cp^3}{9B}, p)$$

with the unitary sphere.

A long, but straightforward, calculation leads to the following conclusions:

i) The non zero eigenvalues of $DX_0(c_1(p))$ are $2cp^3$ and $4cp^3$;

ii) The non zero eigenvalues of $DX_0(c_2(p))$ are
$$-2cp^3(2c^2p^2 + B^2)/B^2 \quad \text{and} \quad cp^3(2c^2p^2 + B^2)/B^2;$$

iii) The non zero eigenvalues of $DX_0(c_3(p))$ are

$$-\frac{4}{81}\frac{cp^3}{B^2}(27B^2 + 96c^2p^2 + 64c^2p^4) \quad \text{and} \quad -\frac{2}{27}\frac{cp^3}{B^2}(27B^2 + 96c^2p^2 + 64c^2p^4).$$

Therefore, under the hypothesis $cB \neq 0$, the singular points at the curves c_1 and c_3 are hyperbolic (sinks and sources) and the singularities at c_2 are hyperbolic saddles.

Also, the quadratic cone $\mathcal{Q}_2(u, v, p) = cvp + \frac{B}{2}u^2 = 0$ is invariant by X_0, $\mathcal{Q}_2(c_1(p)) = \mathcal{Q}_2(c_3(p)) = 0$ and $\mathcal{Q}_2(c_2(p)) \neq 0$.

Summarizing, the phase portrait of X_0 can be described as follows.

The intersection of the sphere with the cone $\mathcal{Q}_2^{-1}(0)$ decomposes the sphere into two topological disks $\bar{D}_1 \cup \bar{D}_2$ and one cylinder $\bar{C}$.

Each curve bounding the disks $\bar{D}_1$ and $\bar{D}_2$ is invariant by the flow of X_0 and on it X_0 has two hyperbolic singular points (one attractor and one repeller). Therefore in each disk the phase portrait of X_0 has only one parallel canonical region. In the cylinder $\bar{C}$, X_0 has two hyperbolic saddles and the limit set of ordinary (i.e. non closed) orbits, perhaps with the exception of the saddle separatrices, are the singularities contained in the bounding curves.

From the conclusions above, proceed to the analysis of the phase portrait of X in the coordinates (u, v, p, r).

The four singular points of X on $(c_1(p), 0)$ and $(c_3(p), 0)$ are hyperbolic saddles of stability index 2 or 1; two of each type. In fact the normal derivatives of X in direction r, evaluated at the singularities above, are given by:

$$-cp^3, \quad \frac{1}{81}\frac{p^3c(27B^2 + 64p^4c^2 + 96c^2p^2)}{B^2}.$$

The singular points $(c_2(p), 0)$ are not hyperbolic for X, but for the purposes of our analysis it is not necessary to examine their topological type. The phase portrait of X at the conic critical point is illustrated in Fig. 8.
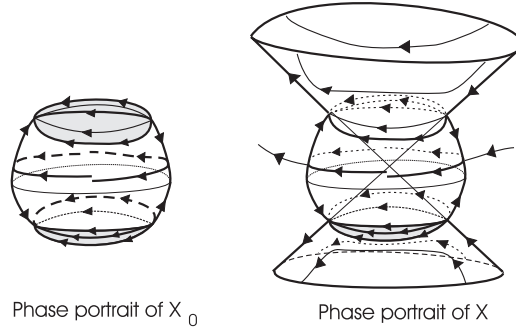

Phase portrait of X_0 Phase portrait of X

FIGURE 8.   Blow up at the conic singular point for D_{2h}^2

Therefore, near the projective line, the phase portrait of $Z = Z_{\mathcal{F}}$, is restricted to the implicit surface $\mathcal{F}^{-1}(0)$, is as in Fig. 9. Here the point at ∞ and that at 0 have been moved to finite conic points p_0 and p_∞, with $p_0 p_\infty = -1$, for better illustration.

Near the origin, the projective line is a center manifold with a quadratic semi-stable singularity. The other separatrix, asymptotic to the origin, sketched in Fig. 9,

also behaves topologically as a semi-stable singularity. This is obtained by blowing down the phase portrait of X restricted to the cone.
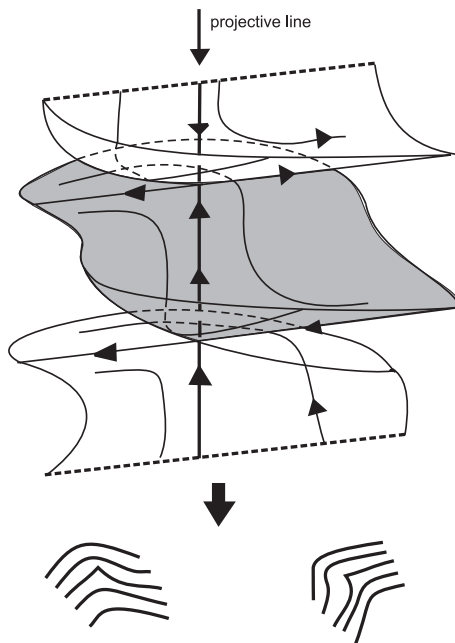

FIGURE 9.   Lie-Cartan suspension of an umbilic point of type D_{2h}^2

Gluing together the phase portraits of $Z = Z_{\mathcal{F}}$ near the projective line, in the charts (u, v, p) and (u, v, q), ends the proof.   ☐

4. **A Summarizing Theorem.** A *bi-parametric family of immersed surfaces* will be a family α_λ of immersions of an oriented surface $\mathbb{M}$ into oriented $\mathbb{R}^3$, indexed by a parameter λ ranging in another surface $\mathbb{L}$, of *parameters*. The families under consideration will be *smooth* in the sense that $\alpha(m, \lambda) = \alpha_\lambda(m)$ is of class C^∞ in the product manifold $\mathbb{M} \times \mathbb{L}$. The space of families, denoted $\mathcal{F}_{\mathbb{M} \times \mathbb{L}}$ will be endowed with the Whitney Fine Topology. See Levine [10] and Mather [12].

The term *generic*, referring to a property of families, means that it is valid for a collection of families which contains the intersection of countably many open dense sets in $\mathcal{F}_{\mathbb{M} \times \mathbb{L}}$. By the appropriate version of Baire Theorem, these collections are dense. See [10], [12], where can be found the basic terminology and concepts that are standard in Singularity Theory. Another reference on this subject, pertinent to geometric singularities, is Porteous [14].

For the sake of simplicity, from now on, the term *smooth* applied to mappings and manifolds will mean *of class C^∞*. See [10, 12].

Consider the space $\mathbb{J}^r(\mathbb{M}, \mathbb{R}^3)$ of r-jets of immersions, α, of $\mathbb{M}$ into $\mathbb{R}^3$, endowed with the structure of Principal Fiber Bundle. The base space is $\mathbb{M} \times \mathbb{R}^3$; the fiber is the space $\mathcal{J}^r(2, 3)$, where $\mathcal{J}^r(2, 3)$ is the space of r-jets of immersions of $\mathbb{R}^2$ to $\mathbb{R}^3$, preserving the respective origins. The structure group, $\mathbb{A}_+^r$, is the product of the group $\mathcal{L}_+^r(2, 2)$ of r-jets of origin and orientation preserving diffeomorphisms of

$\mathbb{R}^2$, acting on the right by coordinate changes, and the group $\mathcal{O}_+(3,3)$ of positive isometries; the action on the left consists of a positive rotation of $\mathbb{R}^3$.

Denote by $\Pi_{r,s}$, $r \leq s$ the projection of $\mathcal{J}^s(2,3)$ onto $\mathcal{J}^r(2,3)$. It is well known that the group action commutes with projections, [10]. For the present needs, take r so that $2 \leq r \leq 5$.

Recall from Section 2 that each 5-jet of an immersion at an umbilic point is of the form (p,P,z), with $(p,P) \in \mathbb{M} \times \mathbb{R}^3$ and z is in the orbit of a polynomial immersion $(u,v,h(u,v))$, where

$$
\begin{aligned}
h(u,v) = {} & \frac{k}{2}(u^2+v^2) + \frac{a}{6}u^3 + \frac{b}{2}uv^2 + \frac{c}{6}v^3 + \frac{A}{24}u^4 + \frac{B}{6}u^3v \\
& + \frac{C}{4}u^2v^2 + \frac{D}{6}uv^3 + \frac{E}{24}v^4 + \frac{a_{50}}{120}u^5 + \frac{a_{41}}{24}u^4v \qquad (12)\\
& + \frac{a_{32}}{12}u^3v^2 + \frac{a_{23}}{12}u^2v^3 + \frac{a_{14}}{24}uv^4 + \frac{a_{05}}{120}v^5
\end{aligned}
$$

Recall that the general quadratic part of h has the form $\frac{k_{11}}{2}u^2 + k_{12}uv + \frac{k_{22}}{2}v^2$. Therefore, the umbilic conditions $k_{11} = k_{22}$, $k_{12} = 0$, imposed on jets, define a closed submanifold $\mathcal{U}^5$, of *umbilic jets*, of codimension 2 in $\mathcal{J}^5(2,3)$.

Recall also that the general cubic part has also a term of the form $\frac{b'}{2}u^2v$. The expression in equation (12), is a representative of the orbit of umbilic jets in the space (a,b,c), with $b' = 0$.

Define below the *canonic umbilic stratification* of $\mathcal{J}^5(2,3)$. The term *canonic* means that the strata are invariant under the action of the group $\mathbb{A}_+^5 = \mathcal{O}_+(3,3) \times \mathcal{L}_+^5(2,2)$. It is to the orbits of this action that reference is made below.

1) *Umbilic Jets*: $\mathcal{U}^5$, those in the orbits of $j^5(u,v,h)$, where $h = h(u,v)$, is as in equation (12), with $k \in \mathbb{R}$. It is a closed submanifold of codimension 2.

2) *Umbilic Jets with Rotationally Symmetric cubic part*: $\mathcal{Q}^5$, those in the orbit of $j^5(u,v,h)$, with h in expression (12) having $a = b = b' = c = 0$. It is a closed submanifold of codimension 4 inside $\mathcal{U}^5$ and codimension 6 in $\mathcal{J}^5(2,3)$.

3) *Non-Darbouxian Umbilic Jets*: $(\mathcal{ND})^5$, are those umbilic jets outside $\mathcal{Q}^5$, which are in the orbits of those whose expression (12) satisfy:

   3.1) $\tau = b(b-a) = 0$ or

   3.2) $b(b-a) \neq 0$ and conditions D_1 or D_2 in Theorem 1 fail. This is written $\delta_1 = a - 2b = 0$ and $\delta = c^2 - 4b(a-2b) = 0$.

See Fig. 10 for an illustration of a spherical section the stratification of the intersection of $(\mathcal{ND})^3$, with the 3 space (a,b,c), in the notation of equation (12).

The stratification on this sphere into points, curves and open sets, by the zeros of the homogeneous linear form $\delta_1 = a - 2b$ and quadratic forms $\tau = b(a-b)$ and $\delta = c^2 - 4b(a-2b) = 0$ and their complements is transferred to $(\mathcal{ND})^5$ by the orbits of the group action, preserving the codimension of the strata inside $\mathcal{U}^5$. This codimension will be called *umbilic codimension*, to it 2 must be added to get the actual codimension inside $\mathcal{J}^5(2,3)$.
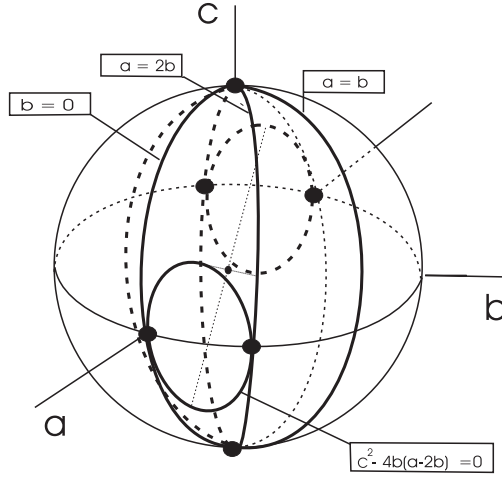
4) The variety $(\mathcal{ND})^5$ of *Non-Darbouxian jets*, is further partitioned into

   4.1) $(\mathcal{D}_{2,3}^1)^5$, defined by the orbits of jets with $a = b \neq 0$, $\chi \neq 0$. See Theorem 2.

   4.2) $(\mathcal{D}_{2p}^2)^5$, defined by the orbits of jets with $a = b \neq 0$, $\chi = 0$, $b\xi < 0$. See Proposition 2.

   4.3) $(\mathcal{D}_3^2)^5$, defined by the orbits of jets with $a = b \neq 0$, $\chi = 0$, $b\xi > 0$

FIGURE 10.   Stratification of a section of the fiber of $(\mathcal{ND})^3$.

4.4) $(\mathcal{D}_{2h}^2)^5$, defined by the orbits of jets with $a = b = 0$, $cB \neq 0$. This is a manifold of umbilic codimension 2. See Proposition 3.

4.5) $(\mathcal{D}_2^1)^5$ defined by the orbits of jets with $c\tau \neq 0$ and $\delta = 0$ or $\delta_1 = 0$. See Theorem 2. This is a manifold of umbilic codimension 1.

4.6) $(\mathcal{D}_1^2)^5$ defined by the orbits of jets with $c = 0$, $b \neq 0$ and $\delta_1 = 0$. See Proposition 1. This is a manifold of umbilic codimension 2.

4.7) $\mathcal{Z}^5$, defined by the orbits of jets with $a = b \neq 0$, $\chi = 0$, $\xi = 0$, which is a subvariety of umbilic codimension 3.

4.8) $\mathcal{W}^5$, defined by the orbits of jets with $a = b \neq 0$, $B = 0$, which is a subvariety of umbilic codimension 3.

5) The orbits passing through the open sets in Fig. 10 correspond to the Darbouxian umbilic jets: $(\mathcal{D}_i)^5$, $i = 1, 2, 3$.

The *canonic stratification* of $\mathcal{J}^5(2,3)$ induces a canonic stratification of $\mathbb{J}^5(\mathbb{M}, \mathbb{R}^3)$ whose strata are principal sub-bundles, with codimension equal to that of their fibers, which are the canonic strata of $\mathcal{J}^5(2,3)$, as defined above in items $1, 2, 3, 4$ and 5.

The collection of sub-bundles which stratify $\mathbb{J}^5(\mathbb{M}, \mathbb{R}^3)$ will be called *umbilic stratification*. The strata are: $\mathbb{U}^5(\mathbb{M}, \mathbb{R}^3)$, corresponding to $\mathcal{U}^5$; $\mathbb{D}_i{}^5(\mathbb{M}, \mathbb{R}^3)$, $i = 1, 2, 3$, corresponding to the strata of Darbouxian umbilic jets $(\mathcal{D}_i)^5$, $i = 1, 2, 3$, and so on, one bundle for each of the strata in the items above.

Theorem 3. *The following properties are generic for smooth bi-parametric families, α_λ, of immersions in $\mathcal{F}_{\mathbb{M} \times \mathbb{L}}$.*

The set $\mathbb{U}(\alpha_\lambda)$ to $\mathbb{L}$ points (m, λ) in $\mathbb{M} \times \mathbb{L}$ such that m is an umbilic point of α_λ form a smooth regular surface, which is stratified as follows:

Darbouxian umbilics, D_1, D_2 and D_3, occur on regular surfaces,

Codimension one umbilics, D_2^1 and $D_{2,3}^1$, occur along regular curves,

Codimension two umbilics, D_1^2, D_{2p}^2, D_3^2 and D_{2h}^2, occur at isolated points.

Proof. Apply Thom Transversality Theorem in Jet Spaces, to make the jet extension $j^5\alpha_\lambda$, as a mapping of $\mathbb{M} \times \mathbb{L}$, transversal to the canonic stratification of $\mathbb{J}^5(\mathbb{M}, \mathbb{R}^3)$. See Levine [10] and Mather [12].

Notice that the strata not listed in the statement of the Theorem have codimensions larger than 4 and therefore are avoided by transversality. While the other strata, listed there, give the right structure of regular surfaces, curves and points claimed in the Theorem, since they are obtained by the pull back of the canonic strata by a $j^5\alpha_\lambda$ transversal to them.   □

Remark 2. In [5], the curves and points above in Theorem 3 will be better organized in terms of Whitney folds and cusps of the projection of $\mathbb{U}(\alpha_\lambda)$ to $\mathbb{L}$. See [10].

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