A COMPARISION THEOREM FOR LINEAR VOLTERRA-STIELTJES INTEGRAL EQUATIONS

J. C. Prandini - L. Barbanti
(IME - USP)

1. Introduction

One of the questions that arises in the study of controllability with fixed time is to decide if the equivalence between total controllability and controllability from zero holds. In the ODE context this equivalence is true since we have the invertibility of the resolvent. We shall take up this matter for the linear integral Volterra-Stieltjes equations (K).

We derive also a theorem that gives an estimate of how far a solution of (K) travels from its starting point.

2. Definitions

Given $[0,b] \subset \mathbb{R}$, and X a Banach space, we define the semi-variation of $g: [0,b] \longrightarrow L(X)$ as

$$SV[g] = \sup_{d \in D} \sup\{ \sum_{i=1}^{|d|} \| (g(t_i)-g(t_{i-1})).x_i \| : x_i \in X, \ \| x_i \| \leq 1 \}$$

where D is the set of all partitions

$$d = \{0 = t_0 < t_1 < \ldots < t_{|d|} = b\} ,$$

of the interval $[0,b]$.

If $SV[g] < \infty$ we say that g is of bounded semi-variation, and we write $g \in SV([0,b], L(X))$. Note that SV is a seminorm.

We say that $f: [0,b] \longrightarrow X$ is regulated and write $f \in G([0,b],X)$ if f has only discontinuities of first kind.

For $g \in SV([0,b], L(X))$ and $f \in G([0,b], X)$ there exists the interior (or Dushnik type) integral

$$F_g(f) = \int_0^b .dg(t).f(t) = \lim_{d\in D} \sum_{i=1}^{|d|} (g(t_i)-g(t_{i-1}))f(s_i) \in X,$$

where $s_i \in]t_{i-1}, t_i[$.

Given

$$Q = \{(t,s) \in [a,b] \times [a,b] \; ; \; a \leq s \leq t \leq b\} \subset \mathbb{R}^2 ,$$

and a mapping $K: Q \longrightarrow L(X)$, and putting $K^t(s) = K_s(t) = K(t,s)$, we write $K \in G_0.SV^u(Q,L(X))$ if K satisfies:

(D^0) $\qquad K(t,t) = 0$,

(G^σ) for every $s \in [0,b]$, and all $x \in X$, we have $K_s.x \in G([0,b],X)$, (where we define $K_s.x(t) = K(t,s)x$), and

(SV^u) $\qquad$, $SV^u[K] = \sup_{a\leq t\leq b} SV[K^t] < \infty$.

If instead of (D^0) K satisfies:

(D^I) $\qquad K(t,t) = I_X$,

we write $K \in G_I^\sigma.SV^u(Q,L(X))$

The equation (K) is

$$X(t) + \int_0^t .d_sK(t,s)x(s) = u(t)$$

where $K \in G_0^\sigma SV^u$ and $x,u \in G([o,b],X)$ (see them. 3.4. and 3.8. of Hönig [2]).

We usually consider $u(t) \in \Gamma \subset X$ a fixed subset of X called the constraint set.

Under certain conditions the solution is expressed by a resolvent $R \in G_I^\sigma SV^u$ according to the formula

$$x_u(t) = u(t) - \int_0^t d_s R(t,s) u(s)$$

where R satisfies $\forall\, x \in X$

$$(R^*) \qquad R(t,s)x - x + \int_s^t . d_\sigma K(t,\sigma) R(\sigma,s).x = 0$$

and

$$(R_*) \qquad R(t,s).x - K(t,s).x - x + \int_s^t . d_\sigma R(t,\sigma) K(\sigma,s).x = 0$$

Fixed an $T \in]0,b]$, we say that the process (K) is (exact) <u>controllable in time</u> T (and we denote this property by (C)) if for every $x,y \in X$, there exists an element $u \in G([0,b],X)$, such that the solution x_u satisfies $x_u(0) = x$ and $x_u(T) = y$.

<u>Def. 2</u> - The process (K) is <u>controllable into zero</u> (<u>from zero</u>) in <u>time T</u> if for every $x \in X$, there exists an $u \in G([0,b],X)$ (an $\mu \in G([0,b],X)$ such that $x_u(0) = x$ and $x_u(T) = 0$ ($x_u(0) = 0$ and $x_u(T) = x$).

We denote by $(C)_0$ and $(C)^0$ the property of (K) to be controllable into zero or controllable from zero, in time T, respectively.

By comparing the properties (C), $(C)_0$ and $(C)^0$ we have immediately

$$(C) \Longrightarrow (C)^0$$

A point $x \in X$ for which $\exists\, u \in G([0,b],X)$ such that $x_u(0) = x$ and $x_u(T) = x$, is called a returning point . We denote by [RP] the property that all $x \in \Gamma$ are returning points for the process (K).

3. Results

Lemma 1 - For $\Gamma = X$, $[C]^0 + [RP] \Longrightarrow [C]$.

Proof - Let u,v be such that

$$x_u(0) = 0 \quad , \quad x_u(T) = x-y \qquad \text{and}$$

$$x_v(0) = y \quad , \quad x_v(T) = y \; .$$

Then $x_{u+v}(0) = y$ and $x_{u+v}(T) = x$.

This shows the importance of knowing what the returning points of (K) are.

One kind of answer is obtainded by the consideration of the controls which generate the constant solutions of (K).

Prop 1. - A control u generates a constant solution iff

$$u(t) = [I-K(t,0)]u(0) \qquad \forall\, t \; .$$

The proof is immediately read off the equation (K).

So when $\Gamma = X$ one has [RP] hence

Prop. 2. - $[C] \Longleftrightarrow [C]^0$.

Another kind of question that has been investigated is to estimate how much a solution is apart from its initial condition as the time goes by.

Results of this kind were obtained by Kobayashi & Shimemura [3] for a linear system with constant coefficients in a finite dimensional context and by Henriques [1] for the infinite dimensional context.

Lemma 2 - Suppose that either $\forall\ (t,s)\ K(t,s)X \subset Z$ a closed subspace of X or that $K(t,s)Y \subset Z$, where Y is the closure of the linear span of Γ, $K(t,s)Z \subset Z$, and the resolvent is given by a Neumann series (see Def 3.3 and Them. 3.9 of [2]). Then

$$\forall\ t \qquad \int_0^t ds\ R(t_s s)u(s) \in Z\ .$$

Proof - It suffices to notice that

$$\forall\ y \in Y$$

$$\{R(t,s) - R(t,s')\}.y = \int_{s'}^{t} .d_\sigma K(t,\sigma)R(\sigma,s')y +$$

$$\int_{s'}^{t} .d_\sigma K(t,\sigma)R(\sigma,s)y$$

the other case follows after showing by induction that

$$K^{(n)}(t,s)y \in Z \quad \text{where}$$

$$K^{(1)}(t,s) = K(t,s)$$

$$K^{(n+1)}(t,s).x = \int_s^t .d_s K(t,\sigma)K^{(n)}(\sigma,s).x$$

Prop. 3 - Under the same hypothesis of Lemma 2 with the additional assumption that $\exists\ C > 0$ such that $\forall\ \gamma\ \forall\ \gamma' \in \Gamma, \exists\ \eta \in Z^0, |\eta| = 1$ such that

$$C\|\gamma - \gamma'\| \le \langle \gamma-\gamma' , \eta\rangle$$

Then

$$C\| u(t)-u(0)\| \le \| x(t) - x(0)\| .$$

Proof - Using the lemma 2 and the formula of the solution x_u one gets

$$\langle x(t) - u(0),\eta\rangle = \langle u(t) - u(0), u\rangle$$

and this is enough.

4. An Example

$$\text{Consider} \quad \dot{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} t \\ t \\ t \end{pmatrix} \quad \text{in } \mathbb{R}^3$$

$$x(0) = 0$$

The corresponding equation (K) is

$$x(t) + \int_0^t d_s \begin{pmatrix} -s & 0 & 0 \\ 0 & -2s & 0 \\ 0 & 0 & 0 \end{pmatrix} x(s) = \begin{pmatrix} t^2/2 \\ t^2/2 \\ t^2/2 \end{pmatrix}$$

Here $Y = \mathbb{R}.(1,1,1)$; the only η is $(0,0,1)$, $C=1$, and we certainly have

$$\langle x(t),\eta\rangle = t^2/2 = \langle u(t),\eta\rangle \quad \text{for the solution is}$$

$$x(t) = \begin{pmatrix} t & -e^t & +1 \\ t/2 & +\frac{e^{2t}}{4} & -\frac{1}{4} \\ & \frac{t^2}{2} & \end{pmatrix} , \quad \text{then}$$

$$\| x(t)\| \ge \| u(t)\| , \quad \forall\, t \ge 0 .$$

REFERENCES

[1] H. Henríquez - Una propriedad geometrica de los sistemas de control en espacios de Banach preprint (1984).

[2] C.S. Hönig - Equations Integrales Generalisees et applications - Publications Mathematiques d'Orsay 83-01 (1983).

[3] H. Kobayashi & E. Shimemura - Note on a property of linear control systems - Inter. J. Control., 23, 6 , p. 1171-1176 (1981).

o0o