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Abstract

We formulate a brachistochrone problem in Lorentzian geometry and we prove a variational principle valid for brachistochrones in stationary manifolds. This variational principle is stated in terms of geodesics in a suitable sub-Riemannian structure on $\mathcal{M}$. Moreover, we prove the regularity of the solutions of our variational problem and we determine a differential equation satisfied by the brachistochrones. Some explicit examples are computed.

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1 Formulation of the Brachistochrone Problem

The classical brachistochrone problem dates back to the end of the seventeenth century, and it is one of the most famous problems of Calculus of Variations. In its original formulation, the problem was to determine the shape of a slide joining two fixed points in the space in such a way that a mass moving along it under the action of the gravity would reach the final point in the shortest time. Several generalization of the problem have been studied, and the most classical Newtonian generalization can be formulated in modern terminology as follows. Given a manifold $\mathcal{M}$ endowed with a Riemannian metric g (the state space) and a smooth function $V : \mathcal{M} \longmapsto I\!R^+$ (the gravitational potential), a brachistochrone of fixed energy E joining the points $x_0, x_1 \in \mathcal{M}$ is a curve $x : [0, T_x] \longmapsto \mathcal{M}$ that minimizes the functional $F(x) = T_x$ in the space of all smooth curves $y : [0, T_y] \longmapsto \mathcal{M}$ such that $y(0) = x_0$, $y(T_y) = x_1$ and satisfying the law of conservation of energy:

$$g(y(t))[\dot{y}(t), \dot{y}(t)] + V(y(t)) \equiv E, \quad \forall t \in [0, T_y]. \tag{1}$$

A well known variational principle states that the curves of fixed energy $E > 0$ that are solutions to the brachistochrone variational problem are minimal geodesics in the manifold $\mathcal{M}$ with respect to a Riemannian metric $\tilde{g}$ which is conformally equivalent to g:

$$\tilde{g}(y) = \frac{g(y)}{E - V(y)}.$$

We will refer to this variational principle as the *classical variational principle for brachistochrones.*

Recently, some generalizations of the brachistochrone problem have been introduced in the context of General Relativity, as in [1, 2, 3]. In particular, in [3], it is defined a brachistochrone problem on a Lorentzian manifold $\mathcal{M}$, under the assumption that the Lorentzian metric g be stationary with respect to a given timelike Killing vector field Y, and that $\mathcal{M}$ admit a global space-time splitting $U \times I\!R$ *adapted to* Y. Roughly speaking, this means that $\mathcal{M}$ admits a global coordinate system $(x_1, x_2, \ldots, x_n, t)$, with $(x_1, x_2, \ldots, x_n) \in U$ open subset of $I\!R^n$, $t \in I\!R$, and $Y = \frac{\partial}{\partial t}$. The Killing property of Y is given by the fact that the coefficients of the metric of $\mathcal{M}$ do not depend on the variable t. Under these hypotheses, the brachistochrone problem is reduced to a purely spatial geodesic problem in U, with the proof of the first relativistic version of the classical variational principle for brachistochrones.

The aim of this paper is to define a general brachistochrone principle in stationary Lorentzian manifolds and to extend the classical variational principle for brachistochrones under intrinsic assumptions on the relativistic space-time. Namely, we do not assume that $\mathcal{M}$ admits a global splitting adapted to a given timelike Killing vector field on $\mathcal{M}$.

For the reduction to a spatial problem, we use the terminology and some techniques of sub-Riemannian geometry. This branch of mathematics has attracted a great deal of interest due to its applications to the Optimal Control Theory (see e.g. [4, 5]), and it is particularly well suited to study dynamical systems of particles subject to non-holonomic constraints.

A general formulation of the brachistochrone problem for stationary space-times may be stated as follows. We will denote by $\mathcal{M}$ an $(n+1)$-dimensional smooth manifold, $n \geq 1$, and by g a Lorentzian metric on $\mathcal{M}$. We will also denote by $\langle\cdot,\cdot\rangle$ the bilinear form induced on the tangent space $T_q\mathcal{M}$ by $g(q)$. Let Y be a smooth timelike Killing vector field on $\mathcal{M}$, i.e. $\langle Y, Y\rangle < 0$ everywhere, and the metric tensor g is invariant by the flow of Y. For the main properties of Lorentzian geometry we refer to the classical textbooks like [6], [7] and [8].

Let $\gamma :]a,b[\longmapsto \mathcal{M}$ be an integral curve of Y and $p \notin \gamma$ be an event of $\mathcal{M}$ (we will also denote by γ the image $\gamma(]a,b[)$ in $\mathcal{M}$).

Let $k \in I\!R^+$ be a fixed constant. We consider the space $\mathcal{B}^+_{p,\gamma}(k)$ of all piecewise smooth future pointing, timelike curves z from p to γ, parameterized by proper time, and satisfying the conservation law $-\langle\dot z, Y\rangle \equiv k > 0$:

$$\mathcal{B}^+_{p,\gamma}(k) = \Big\{ z : [0, T_z] \longmapsto \mathcal{M} : z \text{ of class } C^1, \qquad z(0) = p,\ z(T_z) \in \gamma,\ \langle \dot z, \dot z\rangle \equiv -1, \qquad -\langle \dot z, Y\rangle \equiv k > 0 \text{ (constant)} \Big\}. \tag{2}$$

The curve γ is interpreted as the worldline of an observer; the curves z in $\mathcal{B}^+_{p,\gamma}(k)$ represent the worldline of massive particles starting at the event p and eventually reaching the observer γ. The condition $\langle\dot z, \dot z\rangle = -1$ means that z is parameterized by proper time, so that T_z is the *travel time* of z measured on the particle. The quantity $\langle\dot z, Y\rangle$ is the component along Y of the four-momentum of the particle z; observe that, since Y is Killing, then it is constant on geodesics. The condition $-\langle \dot{z}, Y\rangle \equiv k$ is the analogue of the conservation law of the energy (1) in the Newtonian case. The positivity of k says that z is future pointing with respect to the time orientation induced on $\mathcal{M}$ by Y. A perfectly similar theory may be developed considering past pointing curves. If one modifies properly the definitions, it is easily checked that all the results proven in this paper work equally well in this case.

We denote by $\beta(q)$ the positive smooth function $-\langle Y(q), Y(q)\rangle$. Let's consider the open subset $U_k \subset \mathcal{M}$:

$$U_k = \left\{q \in \mathcal{M} : \beta(q) < k^2\right\}. \tag{3}$$

Observe that, since Y is Killing, then $\langle Y, Y\rangle$ is constant on the flow lines of Y, and so U_k is invariant by the flow of Y. We will also consider the space $\mathcal{U}^+_{p,\gamma}(k)$ consisting of all curves z in $\mathcal{B}^+_{p,\gamma}(k)$ lying in U_k:

$$\mathcal{U}^+_{p,\gamma}(k) = \left\{z \in \mathcal{B}^+_{p,\gamma}(k) : z(s) \in U_k,\ \forall\, s \in [0, T_z]\right\}.$$

Observe that, by the wrong way Schwartz's inequality, if $z \in \mathcal{B}^+_{p,\gamma}(k)$ then z is contained in the closure of U_k. Indeed, $-\langle Y, Y\rangle = \langle \dot{z}, \dot{z}\rangle\langle Y, Y\rangle \leq \langle \dot{z}, Y\rangle^2$. Moreover, if $-\langle \dot{z}, Y\rangle \equiv k$, then z is entirely contained in U_k if and only if $\dot{z}$ is never a multiple of Y.

We will assume that γ is a maximal integral curve of Y. For all q in $\mathcal{M}$, γ_q will denote the maximal integral line of Y such that $\gamma_q(0) = q$.

Definition 1.1. A curve $z \in \mathcal{B}^+_{p,\gamma}(k)$ is said to be a *minimal brachistochrone of energy* k between p and γ if z is a minimal point for the *travel time* functional $F : \mathcal{B}^+_{p,\gamma}(k) \to \mathbb{R}^+$ given by

$$F(z) = T_z.$$

A curve $z \in \mathcal{B}^+_{p,\gamma}(k)$ is a *brachistochrone of energy* k if it minimizes locally its travel time, i.e., if for all $r \in [0, T_z[$ and for $\varepsilon \in]0, T_z - r[$ small enough, the curve $\overline{z}(s) = z(s - r)$, defined on the interval $[0, \varepsilon]$, is a minimal brachistochrone between $z(s)$ and $\gamma_{z(s+\varepsilon)}$.

Our definition of brachistochrones extend the definition of the τ-*brachistochrones* given in [3] in the case of a regular stationary manifold. In [3] it is also defined a different brachistochrone problem (in Perlick's terminology: the *t-brachistochrones*), which are curves $z \in \mathcal{B}^+_{p,\gamma}(k)$ minimizing the *arrival time* functional:

$$AT(z) = \gamma^{-1}(z(T_z)), \quad z \in \mathcal{B}^+_{p,\gamma}(k). \tag{4}$$

The duplicate notion of brachistochrones is justified by the fact that, in General Relativity, there is no such a concept as a global time, and there does not exist a universal way to measure the travel time of particles. The two different concepts of brachistochrones are referred to the proper time of the particle in the case of the t-brachistochrones, and the time measured by the observer γ in the case of the τ-brachistochrones.

The two variational problems are essentially different. The arrival time functional, introduced in [9], has been also extensively studied to investigate the structure of causal (timelike or lightlike) geodesics in Lorentzian manifolds (see [10, 9, 11]). A relativistic extension of Fermat's Principle states that the stationary points of AT in suitable spaces are geodesics joining an event with an observer in the space-time (see [10] for the timelike case and [11] for the lightlike case).

In this paper we will only be concerned with the variational problem of Definition 1.1. We prove an extension of the classical variational principle stating that the brachistochrones with fixed energy correspond to the geodesics of a sub-Riemannian structure which is associated to the Killing vector field Y.

For a physical interpretation of our brachistochrone problem, the timelike vector field Y should be related to some observable quantities, i.e., Y should be co-moving with some bodies. For instance, if we are in the solar system and Y is comoving with the planets, the solutions to our brachistochrone problem will give worldline of particles that minimize the travel time among all curves that have a fixed specific energy in the rest system of the planets. If Y is at rest with respect to the sun and to the distant stars, then the brachistochrones will be the worldline of massive objects that minimize the travel time among all curves that have fixed energy in a reference system oriented at distant stars.

It is also possible to return to the original interpretation of of the brachistochrone problem and think of the body guided by a frictionless slide, in which case Y is determined by being the rest system of the slide.

The paper is organized as follows. In Section 2 we give a self-contained exposition of the problem of sub-Riemannian geodesics, which are minimal curves with respect to a positive metric tensor defined on a sub-bundle of the tangent bundle of $\mathcal{M}$. In order to keep the exposition short and self-contained, the problem is willingly not discussed in its full generality, as it will be remarked ahead. In our specific case, we are able to prove that sub-Riemannian geodesics are indeed smooth curves that satisfy the Euler–Lagrange equations associated to the sub-Riemannian energy functional (see Remark 2.4).

In Section 3, under the completeness assumption for the vector field Y, we prove that the brachistochrones can be deformed into geodesics with respect to a suitable sub-Riemannian structure on $\mathcal{M}$, and vice versa, obtaining a relativistic version of the classical variational principle for brachistochrones. Moreover, using the properties of normal sub-Riemannian geodesics, we determine a differential equation satisfied by the brachistochrones.
Finally, in Section 4, we compute explicitly the brachistochrones in a class of stationary Lorentzian manifolds generalizing the Heisenberg space and in the 3-sphere S^3 endowed with its structure of fiber bundle over S^2 with fibers homeomorphic to S^1.

Thanks to the use of sub-Riemannian geometry, the variational principle proven in this paper generalizes the one obtained by V. Perlick in [3]. In that paper, in order to reduce to a spatial problem, the Lorentzian manifold $\mathcal{M}$ was assumed to have a global space-time splitting adapted to Y. From a sub-Riemannian point of view, this can be considered the case where the distribution on which the metric is defined is globally integrable.

In any case, a crucial point in the theory seems to be the fact that the manifold $\mathcal{M}$ and its metric have symmetries or invariance properties with respect to the time. In our case, the symmetry is given by the

Killing property of the vector field Y. This also suggests that further generalizations of the Principle might be obtained by considering more general symmetries, given, for instance, by a conformal timelike field (conformally stationary manifolds), or even by some group action.

2 Sub-Riemannian Geodesics

We denote by Δ the orthogonal distribution to the vector field Y. Since Y is timelike, the wrong way Schwartz's inequality implies that the restriction of the Lorentzian metric g is positive definite. To avoid confusion, we will denote by $g_{\rm sr}$ the restriction of g on Δ; the pair $(\Delta, g_{\rm sr})$ defines a sub-Riemannian structure on $\mathcal{M}$. Observe that, by the Killing property, both $g_{\rm sr}$ and Δ are invariant by the flow of Y, hence the sub-Riemannian structure on $\mathcal{M}$ is Y-invariant.

Since there is no unanimous agreement in the literature on the proper definition of a sub-Riemannian geodesic (see Remark 1 of [4]), in this paper we give a self-contained presentation of the subject in our special case of geodesics joining a point and the integral curve of a Killing vector field. Having one degree of freedom in the choice of the final endpoint, we avoid all the technicalities and the *ab-normalities* that arise in the general treatment of the subject. The reader may find in [4] and in the references therein a detailed discussion and a classification of geodesics for a general sub-Riemannian structure.

We fix once and for all an auxiliary Riemannian metric $g_{\rm R}$ on $\mathcal{M}$ that coincides with the Lorentzian metric g on the distribution Δ. For instance, for $v \in T_q\mathcal{M}$, one can take:

$$g_{\rm R}[v, v] = \langle v, v\rangle_{(\rm R)} = \langle v, v\rangle - 2\frac{\langle v, Y(q)\rangle^2}{\langle Y(q), Y(q)\rangle}. \tag{5}$$

The wrong-way Schwartz's inequality implies that $g_{\rm R}[v, v]$ defines a positive definite, symmetric tensor on $\mathcal{M}$.

Remark 2.1. It is easily checked that the flow of Y preserves $g_{\rm R}$, so that Y is Killing with respect to $g_{\rm R}$ also.

In view to develop an existence theory for sub-Riemannian geodesics, in this Section we define our functional framework by requiring that the maps in our spaces only satisfy a Sobolev-type regularity. From an analyst's viewpoint, this technicality allows to use some results of Calculus of Variations that require the completeness of the functional spaces. However, Corollary 2.7 proven below states that the solutions to our variational problems are smooth curves.

Let ∇ be the covariant derivative with respect to the Levi–Civita connection of g. Using standard notation, for a smooth function $\phi : \mathcal{M} \longmapsto I\!R$ we also denote by $\nabla\phi$ the *Lorentzian gradient* of ϕ, which is the vector field defined by $d\phi(q)[\cdot] = \langle \nabla\phi(q), \cdot \rangle$, for all $q \in \mathcal{M}$.

Since Y is Killing with respect to g, then for every pair v_1, v_2 of smooth vector fields on $\mathcal{M}$, it is (see [8]):

$$\langle \nabla_{v_1} Y, v_2 \rangle + \langle \nabla_{v_2} Y, v_1 \rangle = 0. \tag{6}$$

The characterization (6) of the Killing property of Y will be used implicitly or explicitly throughout the paper.

Using the Riemannian metric, we define the space $L^2([0,1], T\mathcal{M})$, consisting of all measurable maps $\zeta : [0,1] \longmapsto T\mathcal{M}$ such that $\langle \zeta, \zeta \rangle_{(\mathrm{R})}$ is integrable on $[0,1]$. The Sobolev space $H^1([0,1], \mathcal{M})$ is defined as the set of all absolutely continuous curves $z : [0,1] \longmapsto \mathcal{M}$ such that $\dot{z}$ is in $L^2([0,1], T\mathcal{M})$. By a classical result of Global Analysis on Manifolds, the spaces $L^2([0,1], T\mathcal{M})$ and $H^1([0,1], \mathcal{M})$ actually do not depend on the choice of the Riemannian metric on $\mathcal{M}$ (see [12]). Our functional framework consists of the following spaces:

$$\Omega_{p,\gamma} = \Big\{ w \in H^1([0,1], \mathcal{M}) : w(0) = p, \ w(1) \in \gamma \Big\}, \tag{7}$$

$$\Omega_{p,\gamma}(\Delta) = \Big\{ w \in \Omega_{p,\gamma} : \dot{w}(s) \in \Delta \text{ a.e. on } [0,1] \Big\}. \tag{8}$$

Moreover, for $q \in \gamma$, we denote by $\Omega_{p,q}$ the space of all curves $w \in \Omega_{p,\gamma}$ such that $w(1) = q$.

It is well known that $\Omega_{p,\gamma}$ has the structure of a smooth Hilbertian manifold; for $w \in \Omega_{p,\gamma}$ the tangent space $T_w\Omega_{p,\gamma}$ is identified with the

Hilbert space:

$$T_w\Omega_{p,\gamma} = \Big\{\zeta : [0,1] \longmapsto T\mathcal{M} : \zeta \text{ absolutely continuous}, \quad \zeta(0)=0,\ \zeta(1) \parallel Y(w(1)), \quad \nabla_{\dot w}\zeta \in L^2([0,1],T\mathcal{M}), \quad \zeta(s)\in T_{w(s)}\mathcal{M}\ \forall s\Big\}, \tag{9}$$

endowed with the Hilbert norm:

$$\|\zeta\|_w = \left(\int_0^1 \langle \nabla_{\dot w}\zeta, \nabla_{\dot w}\zeta\rangle_{(R)}\,\mathrm{d}s\right)^{\frac{1}{2}}.$$

For all $q\in\gamma$, $\Omega_{p,q}$ is a closed submanifold of $\Omega_{p,\gamma}$, and for $w\in\Omega_{p,q}$ the tangent space $T_w\Omega_{p,q}$ is given by:

$$T_w\Omega_{p,q} = \Big\{\zeta \in T_w\Omega_{p,\gamma} : \zeta(1) = 0\Big\}. \tag{10}$$

Lemma 2.2. *$\Omega_{p,\gamma}(\Delta)$ is a smooth submanifold of $\Omega_{p,\gamma}$.*
For $w\in\Omega_{p,\gamma}(\Delta)$, the tangent space $T_w\Omega_{p,\gamma}(\Delta)$ is identified with the Hilbert subspace of $T_w\Omega_{p,\gamma}$:

$$T_w\Omega_{p,\gamma}(\Delta) = \Big\{\zeta\in T_w\Omega_{p,\gamma} : \langle\nabla_{\dot w}\zeta, Y\rangle + \langle \dot w, \nabla_\zeta Y\rangle \equiv 0\Big\}. \tag{11}$$

Proof. We consider the map $\Psi : \Omega_{p,\gamma} \longmapsto L^2([0,1],\mathbb{R})$ given by $\Psi(w) = \langle \dot w, Y\rangle$. It is easily seen that Ψ is smooth, and, for $\zeta\in T_w\Omega_{p,\gamma}$, the Gateaux derivative $\Psi'(w)[\zeta]$ is given by:

$$\Psi'(w)[\zeta] = \langle\nabla_{\dot w}\zeta, Y\rangle + \langle \dot w, \nabla_\zeta Y\rangle.$$

Using the Implicit Function Theorem (see [13]), it suffices to prove that $\Psi'(w)$ is surjective for all $w\in\Omega_{p,\gamma}(\Delta)$. To see this, for all $h\in L^2([0,1],\mathbb{R})$ we set:

$$\mu_h(s) = \int_0^s \frac{h(r)}{\langle Y(w(r)), Y(w(r))\rangle}\,\mathrm{d}r.$$

Clearly $\zeta_h = \mu_h\cdot Y$ is in $T_w\Omega_{p,\gamma}$, and, recalling (6), a straightforward calculation gives $\Psi'(w)[\zeta] = h$. This gives the surjectivity of $\Psi'(w)$ and concludes the proof. ∎

It is possible to give a more geometric proof of Lemma 2.2 by showing that the *endpoint mapping*, which is the map $\mathfrak{end} : \Omega_{p,\gamma} \longmapsto \mathcal{M}$ that assigns to each curve w its final point $w(1)$, is transversal to the curve γ (see [5]).

The regularity result proven in Lemma 2.2 allows to give the following definition of sub-Riemannian geodesic.

Definition 2.3. Let $\phi : \mathcal{M} \longmapsto I\!R^+$ be a smooth map which is Y-invariant, i.e. $\langle Y, \nabla\phi \rangle \equiv 0$. Consider the sub-Riemannian metric $\tilde{g}_{sr} = \phi \cdot g_{sr}$ on Δ (observe that, by the wrong-way Schwartz's inequality, the orthogonal space to a timelike vector is always spacelike). A *normal geodesic* between p and γ with respect to $\tilde{g}_{sr}$ is a curve w which is a smooth critical point in $\Omega_{p,\gamma}(\Delta)$ for the energy functional E_ϕ:

$$E_\phi(w) = \frac{1}{2}\int_0^1 \phi(w) \cdot \langle \dot{w}, \dot{w} \rangle \, ds. \tag{12}$$

A normal geodesic is said to be *minimal* if it is a minimal point for E_ϕ.

Remark 2.4. A sub-Riemannian manifold consists of a triple $(\mathcal{M}, \Delta, g_{sr})$, where $\mathcal{M}$ is a smooth manifold, Δ is a smooth distribution on $\mathcal{M}$ and g_{sr} is a positive definite metric tensor defined only on Δ.

A (piecewise) C^1-curve z is said to be horizontal if $\dot{z}(s)$ belongs to Δ for (almost) all s. We remark here that the choice of the adjective *normal* given to the critical points of E_ϕ derives from the fact that, in general sub-Riemannian manifolds, there exist the so called *ab-normal* geodesics, which are singular points in the set of horizontal curves joining two fixed points (see e.g. [14]). Such geodesics do not in general satisfy the geodesic equation, and they cannot be recovered as critical points of the sub-Riemannian energy functional.

Remark 2.5. Observe that, since E is a smooth functional on $\Omega_{p,\gamma}$, then, by Lemma 2.2, its restriction to $\Omega_{p,\gamma}(\Delta)$ is also smooth. For $\zeta \in T_w\Omega_{p,\gamma}(\Delta)$ the Gateaux derivative $E'_\phi(w)[\zeta]$ is easily computed as:

$$E'_\phi(w)[\zeta] = \int_0^1 \Big(\phi(w)\langle \nabla_{\dot{w}}\zeta, \dot{w}\rangle + \frac{1}{2}\langle \nabla\phi(w), \zeta\rangle\langle \dot{w}, \dot{w}\rangle\Big)\, ds. \tag{13}$$

Here, the gradient $\nabla\phi$ is computed with respect to the Lorentzian metric g.

We denote by $H^1_*([0,1],I\!R)$ the space of all absolutely continuous functions $\mu : [0,1] \longmapsto I\!R$ with square integrable derivative and satisfying $\mu(0) = 0$.

Lemma 2.6. *Let $w \in \Omega_{p,\gamma}(\Delta)$ and $q = w(1)$. The following are equivalent:*

(1) w is a normal geodesic between p and γ with respect to $\tilde{g}_{\rm sr}$;

(2) w is a critical point of the functional E_ϕ in the space $\Omega_{p,\gamma}$;

(3) w is a critical point of the functional:

$$E_{\rm R}(w) = \frac{1}{2}\int_0^1 \phi(w)\cdot g_{\rm R}(z)[\dot z, \dot z]\,{\rm d}s,$$

on $\Omega_{p,q}$, where $g_{\rm R}$ is given by (5).

Proof. Let $w \in \Omega_{p,\gamma}(\Delta)$ be fixed. For all $\mu \in H^1_*([0,1],I\!R)$, let ζ_μ denote the vector field along w given by:

$$\zeta_\mu(s) = \mu(s)\cdot Y(w(s)).$$

Since $\langle\nabla\phi(w), Y\rangle \equiv 0$, $\langle\dot w, Y\rangle \equiv 0$ and Y is Killing ($\langle\nabla_{\dot w}Y, \dot w\rangle \equiv 0$), from (13) one computes immediately:

$$E'_\phi(w)[\zeta_\mu] = 0, \quad \forall\, \mu \in H^1_*([0,1],I\!R).$$

To prove the arrow (1) $\Longrightarrow$ (2), it suffices to show that $T_w\Omega_{p,\gamma}(\Delta)$ and the vectors ζ_μ, $\mu \in H^1_*([0,1],I\!R)$ generate the space $T_w\Omega_{p,\gamma}$. To this aim, let $\tilde\zeta$ be any vector in $T_z\Omega_{p,\gamma}$. Then, an easy computation shows that $\tilde\zeta$ is written as $\tilde\zeta = \zeta + \zeta_\mu$, where $\zeta \in T_w\Omega_{p,\gamma}(\Delta)$ and μ is given by:

$$\mu(s) = \int_0^s \left(\frac{\langle\nabla_{\dot w}\tilde\zeta, Y\rangle - \langle\tilde\zeta, \nabla_{\dot w}Y\rangle}{\langle Y, Y\rangle}\right){\rm d}r.$$

To prove the implication (2) $\Longrightarrow$ (3), observe that, by definition of $g_{\rm R}$, we can write:

$$E_{\rm R}(w) = E_\phi(w) - G(w),$$

where

$$G(w) = \int_0^1 \phi(w) \frac{\langle \dot{w}, Y \rangle^2}{\langle Y, Y \rangle} \, ds.$$

Since $E'_\phi(w)[\zeta] = 0$ for all $\zeta \in T_w\Omega_{p,\gamma}(\Delta)$, and in particular for all $\zeta \in T_w\Omega_{p,q}$, in order to prove that w is a critical point for E_{R} it suffices to show that w is a critical point for G in $\Omega_{p,q}$. To see this, we observe that G is a smooth functional on $\Omega_{p,q}$, whose derivative $G'(w)[\zeta]$ in the direction $\zeta \in T_w\Omega_{p,q}$ is easily computed as:

$$G'(w)[\zeta] = \int_0^1 \left[\langle \nabla\phi(w), \zeta \rangle \frac{\langle \dot{w}, Y \rangle^2}{\langle Y, Y \rangle} + 2\,\phi(w) \frac{\langle \dot{w}, Y \rangle \langle \nabla_{\dot{w}}\zeta, Y \rangle}{\langle Y, Y \rangle} \right] ds.$$

Since $\langle \dot{w}, Y \rangle \equiv 0$, it follows that $G'(w) = 0$ on $T_w\Omega_{p,q}$ and (3) is proven.

To pass from (3) to (1), observe that if w is a critical point of E_{R} on $\Omega_{p,q}$, by standard regularity results, w is a smooth curve. Moreover, since $\langle Y, \dot{w} \rangle \equiv 0$, by (5) we have that the derivative $E'_\phi(w)[\zeta]$ vanishes for all variations ζ that satisfy $\zeta(0) = \zeta(1) = 0$. The computation above shows that $E'_\phi(w)[\zeta]$ vanishes also for all variations $\zeta = \zeta_\mu = \mu \cdot Y$, hence E'_ϕ vanishes in $T_w\Omega_{p,\gamma}(\Delta)$, and we are done. □

As a Corollary to Lemma 2.6, we obtain that the normal sub-Riemannian geodesics between p and γ are regular curves that satisfy the Euler–Lagrange equation of a smooth functional:

Corollary 2.7. *Let $w \in \Omega_{p,\gamma}$ be a normal geodesic with respect to $\tilde{g}_{\mathrm{sr}}$. Then, w is a smooth curve that satisfies the differential equation:*

$$\nabla_{\dot{w}}\dot{w} + \langle \frac{\nabla\phi(w)}{\phi(w)}, \dot{w} \rangle \, \dot{w} - \frac{1}{2} \frac{\nabla\phi(w)}{\phi(w)} \langle \dot{w}, \dot{w} \rangle = 0. \tag{14}$$

Moreover, the quantity $E_\phi(w) = \frac{1}{2}\phi(w(s))\langle \dot{w}(s), \dot{w}(s) \rangle$ is constant along w.

Conversely, every smooth curve $w : [0,1] \longmapsto \mathcal{M}$ joining p with γ, satisfying (14) and $\langle \dot{w}(0), Y(w(0)) \rangle = 0$ is a normal sub-Riemannian geodesic with respect to $\tilde{g}_{\mathrm{sr}}$.

Proof. Since w is a geodesic for the Riemannian metric $\tilde{g}_{\mathrm{R}}$, then w is smooth. It satisfies the equation $E'_\phi(w)[\zeta] = 0$ for all $\zeta \in T_w\Omega_{p,q}$; in particular $\zeta(0) = \zeta(1) = 0$ and we can integrate by parts the formula (13), obtaining:

$$\begin{aligned} 0 &= E'_\phi(w)[\zeta] = \frac{1}{2}\int_0^1 \langle\nabla\phi(w),\zeta\rangle\,\langle\dot{w},\dot{w}\rangle\,\mathrm{d}s + \\ &= \int_0^1 \left(-\phi(w)\langle\zeta,\nabla_{\dot{w}}\dot{w}\rangle - \langle\nabla\phi(w),\dot{w}\rangle\,\langle\zeta,\dot{w}\rangle\right)\mathrm{d}s, \end{aligned} \tag{15}$$

for all $\zeta \in T_w\Omega_{p,\gamma}(\Delta)$. Passing to a strong equality in (15), we obtain that w satisfies the equation (14).

Clearly, the quantity $E_\phi(w) = \frac{1}{2}\phi(w)\langle\dot{w},\dot{w}\rangle$ is constant along every geodesic in the Riemannian metric $\tilde{g}_{\mathrm{R}}$.

Conversely, if w is a smooth curve joining p and γ satisfying (14), then w is a geodesic with respect to the metric $\tilde{g} = \phi \cdot g$. Since Y is Killing with respect to $\tilde{g}$, then $\phi(w)\langle\dot{w}(s), Y(w(s))\rangle \equiv \phi(w)\langle\dot{w}(0), Y(w(0))\rangle = 0$, so that $w \in \Omega_{p,\gamma}(\Delta)$ and w is a normal sub-Riemannian geodesic between p and γ with respect to $\tilde{g}_{\mathrm{sr}}$. □

We also have the following immediate Corollary:

Corollary 2.8. *The curve $w \in \Omega_{p,\gamma}(\Delta)$ is a normal geodesic with respect to $\tilde{g}_{\mathrm{sr}}$ if and only if w minimizes* locally *the energy functional E_ϕ in $\Omega_{p,\gamma}(\Delta)$.* □

3 The Variational Principle

Let $\mathcal{M}$ be a stationary Lorentzian manifold and Y a timelike Killing vector field on $\mathcal{M}$, which is assumed to be complete. We denote by

$$\psi : \mathcal{M} \times \mathbb{R} \longmapsto \mathcal{M}$$

the flow of Y. The Killing property of Y is expressed by the fact that, for all t, the map $x \longmapsto \psi(x,t)$ is an isometry of $\mathcal{M}$. We recall that, denoting by d_x and d_t the differential operators with respect to the variables x and t respectively, by definition of flow, for all $(x,t) \in \mathcal{M} \times \mathbb{R}$ it is:

(1) $\mathrm{d}_t\psi(x,t) = Y(\psi(x,t))$;

(2) $\mathrm{d}_x\psi(x,t)[Y(x)] = Y(\psi(x,t))$.

Moreover, since Y is Killing, $\mathrm{d}_x\psi(x,t) : T_x\mathcal{M} \longmapsto T_{\psi(x,t)}\mathcal{M}$ is an isometry. Observe that, in particular, the quantity $\langle Y, Y\rangle$ is constant along the flow lines of Y:

$$\langle Y(\psi(x,t)), Y(\psi(x,t))\rangle = \langle \mathrm{d}_x\psi(x,t)[Y(x)], \mathrm{d}_x\psi(x,t)[Y(x)]\rangle = \langle Y(x), Y(x)\rangle.$$

We use the flow of Y to define a map from the space $\mathcal{B}^+_{p,\gamma}(k)$ to the space $\Omega_{p,\gamma}(\Delta)$ as follows. First of all, we introduce the space $\tilde{\mathcal{B}}^+_{p,\gamma}(k)$ consisting of all curves in $\mathcal{B}^+_{p,\gamma}(k)$ suitably reparameterized on the interval $[0,1]$:

$$\tilde{\mathcal{B}}^+_{p,\gamma}(k) = \Big\{\sigma : [0,1] \longmapsto \mathcal{M} \text{ of class } C^1 \ \Big| \ \sigma(0) = p,\ \sigma(1) \in \gamma,\ \exists T_\sigma, \text{ such that } \langle\dot\sigma,\dot\sigma\rangle \equiv -T_\sigma^2,\ \frac{\langle\dot\sigma, Y(\sigma)\rangle}{T_\sigma} \equiv -k.\Big\} \tag{16}$$

Clearly, the map $R : \mathcal{B}^+_{p,\gamma}(k) \longmapsto \tilde{\mathcal{B}}^+_{p,\gamma}(k)$ defined by $R(z) = \sigma$, where $\sigma(s) = z(T_z \cdot s)$, gives a bijection between $\mathcal{B}^+_{p,\gamma}(k)$ and $\tilde{\mathcal{B}}^+_{p,\gamma}(k)$.

The map R is used to identify the spaces $\mathcal{B}^+_{p,\gamma}(k)$ and $\tilde{\mathcal{B}}^+_{p,\gamma}(k)$; a curve $\sigma \in \tilde{\mathcal{B}}^+_{p,\gamma}(k)$ will be called a brachistochrone of energy k if $R^{-1}(\sigma)$ is a brachistochrone of energy k.

We introduce the following map $\mathcal{F} : \tilde{\mathcal{B}}^+_{p,\gamma}(k) \longmapsto \Omega_{p,\gamma}(\Delta)$:

$$\mathcal{F}(\sigma)(s) = w_\sigma(s) = \psi(\sigma(s), t_\sigma(s)), \tag{17}$$

where

$$t_\sigma(s) = -\int_0^s \frac{\langle\dot\sigma, Y(\sigma)\rangle}{\langle Y(\sigma), Y(\sigma)\rangle}\,\mathrm{d}r = k \cdot T_\sigma \int_0^s \frac{\mathrm{d}r}{\langle Y(\sigma), Y(\sigma)\rangle}. \tag{18}$$

The curve w_σ is of class C^1, moreover, the following calculation shows that w_σ is in $\Omega_{p,\gamma}(\Delta)$:

$$\langle \dot w_\sigma, Y\rangle = \langle \mathrm{d}_x\psi[\dot\sigma], Y\rangle + \dot t_\sigma\langle Y, Y\rangle = \langle\dot\sigma, Y\rangle + \dot t_\sigma\langle Y, Y\rangle = 0.$$

Observe that, if for some positive k the curve $\sigma \in \tilde{\mathcal{B}}^{+}_{p,\gamma}(k)$ has image in the open set U_k of (3), then, since U_k is Y-invariant, also $\mathcal{F}(\sigma)$ has image in U_k.

Our variational principle states that the map $\mathcal{F}$ is a bijection between the set of brachistochrones of fixed energy k joining p with γ and the set of normal sub-Riemannian geodesics with respect to a suitable metric which is conformally equivalent to g_{sr}. For $k > 0$, let's consider the smooth function:

$$\phi_k = -\frac{\langle Y, Y\rangle}{k^2 + \langle Y, Y\rangle}. \tag{19}$$

The map ϕ_k is defined and positive in the open set U_k. We are ready to state and prove our variational principle for brachistochrones:

Proposition 3.1. *Let k be a fixed positive number. If σ is a brachistochrone of energy k between p and γ, then $w_\sigma = \mathcal{F}(\sigma)$ is a normal geodesic joining p and γ in U_k with respect to the metric $\phi_k \cdot g_{\mathrm{sr}}$.*

Conversely, if w is a normal geodesic in U_k between p and γ with respect to $\phi_k \cdot g_{\mathrm{sr}}$, then there exists a unique brachistochrone σ of energy k between p and γ such that $w = \mathcal{F}(\sigma)$. Moreover, in either case, w is minimal if and only of σ is minimal.

Proof. Observe that the function ϕ_k defined in (19) is Y-invariant, because $\langle Y, Y\rangle$ is constant on the flow lines of Y. Let σ be any curve in $\tilde{\mathcal{B}}^{+}_{p,\gamma}(k)$ with image in U_k and $w_\sigma = \mathcal{F}(\sigma)$. Recalling (18), we have:

$$\begin{aligned}
&\phi_k(w_\sigma)\langle \dot{w}_\sigma, \dot{w}_\sigma\rangle = \\
&\qquad = -\frac{\langle Y, Y\rangle}{k^2 + \langle Y, Y\rangle}\langle \mathrm{d}_x\psi[\dot\sigma] + \dot{t}_\sigma Y, \mathrm{d}_x\psi[\dot\sigma] + \dot{t}_\sigma Y\rangle = \\
&\qquad = -\frac{\langle Y, Y\rangle}{k^2 + \langle Y, Y\rangle}(\langle\dot\sigma, \dot\sigma\rangle + 2\,\dot{t}_\sigma\langle Y, \dot\sigma\rangle + \dot{t}_\sigma^2\langle Y, Y\rangle) = \qquad (20)\\
&\qquad = -\frac{\langle Y, Y\rangle}{k^2 + \langle Y, Y\rangle}(\langle\dot\sigma, \dot\sigma\rangle - \frac{\langle\dot\sigma, Y\rangle^2}{\langle Y, Y\rangle}) = \\
&\qquad = -\frac{\langle Y, Y\rangle}{k^2 + \langle Y, Y\rangle}(-T_\sigma^2 - T_\sigma^2\frac{k^2}{\langle Y, Y\rangle}) = T_\sigma^2 \equiv -\langle\dot\sigma, \dot\sigma\rangle.
\end{aligned}$$

If σ is a brachistochrone of energy k, then σ minimizes locally the quantity T_σ. It follows that w_σ minimizes locally the functional $E_{\phi_k}(w)$ (see 12), hence w is a normal geodesic with respect to $\phi_k \cdot g_{\rm sr}$.

Conversely, let w be any curve in $\Omega_{p,\gamma}(\Delta)$. Let $L_{\phi_k}(w)$ be the length of w with respect to the metric $\phi_k \cdot g_{\rm sr}$:

$$L_{\phi_k}(w) = \int_0^1 \sqrt{\phi_k(w)\langle \dot w, \dot w\rangle}\, {\rm d}s.$$

Define the curve σ_w by:

$$\sigma_w(s) = \psi(w(s), t_w(s)),$$

where t_w is given by:

$$t_w(s) = -kL_{\phi_k}(w)\int_0^s \frac{{\rm d}r}{\langle Y(w), Y(w)\rangle}. \tag{21}$$

Since

$$\langle {\rm d}_x\psi(w,t_w)[\dot w], Y(\sigma_w)\rangle = \langle {\rm d}_x\psi(w,t_w)[\dot w], {\rm d}_x\psi(w)[Y(w)]\rangle = \langle \dot w, Y(w)\rangle = 0,$$

an easy computation shows that

$$\langle \dot\sigma_w, Y(\sigma_w)\rangle \equiv -kL_{\phi_k}(w)$$

is constant. Set $T_{\sigma_w} = L_{\phi_k}(w)$. If w is a normal geodesic in U_k with respect to $\phi_k \cdot g_{\rm sr}$, then w is smooth, and thus also σ is smooth. Moreover, the quantity $2\,E_{\phi_k}(w) = \phi_k(w)\langle \dot w, \dot w\rangle = L_{\phi_k}(w)^2$ is constant, so:

$$\begin{aligned}
\langle \dot\sigma_w, \dot\sigma_w\rangle &= \langle \dot w, \dot w\rangle + 2\dot t_w\langle \dot w, Y\rangle + \dot t_w^2\langle Y, Y\rangle = \\
&= \langle \dot w, \dot w\rangle + \dot t_w^2\langle Y, Y\rangle = \langle \dot w, \dot w\rangle + 2k^2\,\frac{E_{\phi_k}(w)}{\langle Y, Y\rangle} = \\
&= 2E_{\phi_k}(w)\Big[-\frac{k^2 + \langle Y, Y\rangle}{\langle Y, Y\rangle} + \frac{k^2}{\langle Y, Y\rangle}\Big] = \\
&= -2E_{\phi_k}(w) = -T_{\sigma_w}^2.
\end{aligned} \tag{22}$$

Hence, σ_w is in $\tilde{\mathcal{B}}^+_{p,\gamma}(k)$. Since w minimizes locally the functional E_{ϕ_k}, then, by (22), σ_w minimizes locally T_{σ_w}, and it is a brachistochrone of energy k.

Let w be a normal geodesic between p and γ with respect to $\phi_k \cdot g_{sr}$. Suppose that $\sigma_1, \sigma_2 \in \tilde{\mathcal{B}}^+_{p,\gamma}(k)$ are two brachistochrone of energy k such that $\mathcal{F}(\sigma_1) = \mathcal{F}(\sigma_2) = w$; then, by (20), it is $T_{\sigma_1} = T_{\sigma_2}$. By definition of the space $\tilde{\mathcal{B}}^+_{p,\gamma}(k)$ (see (16)), this implies that $\langle \dot\sigma_1, Y\rangle \equiv \langle \dot\sigma_2, Y\rangle$. But $\sigma_1(s)$ and $\sigma_2(s)$ belong to the same integral line of Y for all s, while $\sigma_1(0) = \sigma_2(0) = p$. Then, there exists a smooth function $\tau(s)$ such that:

$$\sigma_2(s) = \psi(\sigma_1(s), \tau(s)), \quad \forall\, s \in [0,1],$$

and $\tau(0) = 0$. Then, recalling the main properties of a Killing vector field, it is:

$$\tau' = \frac{\langle \dot\sigma_2, Y(\sigma_2)\rangle - \langle \mathrm{d}_x\psi[\dot\sigma_1], Y(\sigma_2)\rangle}{\langle Y(\sigma_2), Y(\sigma_2)\rangle} = \frac{\langle \dot\sigma_2, Y\rangle - \langle \dot\sigma_1, Y\rangle}{\langle Y, Y\rangle} \equiv 0,$$

hence, it must be $\tau \equiv 0$, i.e. $\sigma_1 \equiv \sigma_2$ and the uniqueness is proven.

Finally, the equalities (20) and (22) imply immediately that σ is a minimal brachistochrone if and only if $w = \mathcal{F}(\sigma)$ is a minimal geodesic. This concludes the proof. □

Using the variational principle of Proposition 3.1 and the Euler–Lagrange equation (14) satisfied by the sub-Riemannian geodesics, we can write the differential equation satisfied by the relativistic brachistochrones.

Corollary 3.2. *Let z be a curve in $\mathcal{B}^+_{p,\gamma}(k)$ with image in U_k. Then, z is a brachistochrone of energy k joining p and γ if and only if z is a smooth curve satisfying the differential equation:*

$$\begin{aligned}\nabla_{\dot z}\dot z + \frac{2k}{\langle Y, Y\rangle}\nabla_{\dot z}Y \;&+\; 2k^2\frac{\langle \nabla_{\dot z}Y, Y\rangle}{\langle Y, Y\rangle\,(k^2 + \langle Y, Y\rangle)}\,\dot z \;+ \\ &-\; 2k\frac{\langle \nabla_{\dot z}Y, Y\rangle}{\langle Y, Y\rangle\,(k^2 + \langle Y, Y\rangle)}\,Y = 0, \qquad (23)\end{aligned}$$

where ∇ is the covariant derivative with respect to the Levi-Civita connection of the Lorentzian metric g.

Proof. Let $z \in \mathcal{U}^+_{p,\gamma}(k)$ be fixed and T_z its travel time. We consider the curves $\sigma = R(z) \in \tilde{\mathcal{B}}^+_{p,\gamma}(k)$, given by $\sigma(s) = z(T_z \cdot s)$ and $w = \mathcal{F}(\sigma) \in \Omega_{p,\gamma}(\Delta)$, given by $w(s) = \psi(\sigma(s), t(s))$, where $t(s)$ satisfies (18).

By the Proposition 3.1, z is a brachistochrone of energy k if and only if w is a normal sub-Riemannian geodesic with respect to the metric $\phi_k \cdot g_{sr}$. Then, by Corollary 2.7, z is a brachistochrone of energy k if and only if w is a smooth curve that solves the differential equation (14) (it satisfies automatically $\langle \dot{w}, Y(w)\rangle \equiv 0$).

In order to translate the differential equation (14) in terms of z, we argue as follows. We consider the map $F : [0,1] \times I\!R \longmapsto \mathcal{M}$ given by:

$$F(s,t) = \psi(\sigma(s), t),$$

where $\sigma(s) = z(T_z \cdot s)$. Denoting by $T(s,t)$ the vector field along F given by:

$$T = \frac{\partial F}{\partial s},$$

since $Y = \dfrac{\partial F}{\partial t}$, a standard argument in calculus of connections (see for instance Proposition 6.9 of [15]) shows that:

$$\nabla_Y T - \nabla_T Y = 0. \tag{24}$$

Since $w(s) = F(s, t(s))$, it is:

$$\dot{w} = T(w) + \dot{t}\, Y(w)$$

and so, using (24), we compute:

$$\begin{aligned} \nabla_{\dot{w}} \dot{w} &= \nabla_{\dot{w}}(T + \dot{t}\, Y) = \nabla_{\dot{w}} T + \ddot{t}\, Y + \dot{t}\, \nabla_{\dot{w}} Y = \\ &= \nabla_{T + \dot{t} Y} T + \ddot{t}\, Y + \dot{t}\, \nabla_{T + \dot{t} Y} Y = \\ &= \nabla_T T + 2\dot{t}\, \nabla_T Y + \dot{t}^2\, \nabla_Y Y + \ddot{t}\, Y. \end{aligned} \tag{25}$$

It is:

$$T(s,t) = \mathrm{d}_x \psi(\sigma(s), t)[\dot{\sigma}(s)] \quad \text{and} \quad Y(w(s)) = \mathrm{d}_x \psi(\sigma(s), t)[Y(\sigma)]. \tag{26}$$

Considering that $\mathrm{d}_x\psi$ is a isometry, then for every pair of smooth vector fields v_1 and v_2 on $\mathcal{M}$ it is:

$$\nabla_{\mathrm{d}_x\psi[v_1]} \mathrm{d}_x \psi[v_2] = \mathrm{d}_x \psi\left[\nabla_{v_1} v_2\right]. \tag{27}$$

Putting together (25), (26) and using (27), we get:

$$\nabla_{\dot w}\dot w = \mathrm{d}_x\psi\Big[\nabla_{\dot\sigma}\dot\sigma + 2\,\dot t\,\nabla_{\dot\sigma}Y(\sigma) + \dot t^2\,\nabla_{Y(\sigma)}Y(\sigma) + \ddot t\,Y(\sigma)\Big]. \tag{28}$$

From (19), recalling that $\langle\nabla_v Y, Y\rangle = -\langle\nabla_Y Y, v\rangle$, the gradient $\nabla\phi_k$ is computed easily as:

$$\nabla\phi_k = \frac{2k^2}{(\langle Y,Y\rangle + k^2)^2}\,\nabla_Y Y, \tag{29}$$

namely, for every vector field v on $\mathcal{M}$, it is:

$$\langle\nabla\phi_k, v\rangle = \nabla_v\phi_k = \frac{-2k^2}{(\langle Y,Y\rangle + k^2)^2}\langle\nabla_v Y, Y\rangle = \langle\frac{2k^2}{(\langle Y,Y\rangle + k^2)^2}\nabla_Y Y, v\rangle.$$

Recalling the conservation law (20), we have:

$$\phi_k(w)\langle\dot w,\dot w\rangle \equiv 2\,E_{\phi_k}(w) = T_z^2; \tag{30}$$

moreover, it is easily computed:

$$\begin{aligned}\frac{\nabla\phi_k(w)}{\phi_k(w)} &= -\frac{2k^2}{\langle Y,Y\rangle\,(k^2 + \langle Y,Y\rangle)}\,\nabla_{Y(w)}Y(w) = \\ &= -\frac{2k^2}{\langle Y,Y\rangle\,(k^2 + \langle Y,Y\rangle)}\,\mathrm{d}_x\psi[\nabla_{Y(\sigma)}Y(\sigma)]; &(31)\\ \dot w &= \mathrm{d}_x\psi[\dot\sigma] + \dot t\,Y(w) = \mathrm{d}_x\psi[\dot\sigma + \dot t\,Y(\sigma)]; &(32)\end{aligned}$$

and, from (18):

$$\dot t = \frac{k\,T_z}{\langle Y,Y\rangle},\quad \ddot t = -2k\,T_z\frac{\langle\nabla_{\dot\sigma}Y, Y\rangle}{\langle Y,Y\rangle^2}. \tag{33}$$

Recalling that $\phi_k(w)\langle\dot w,\dot w\rangle \equiv T_z^2$ and $\langle\nabla_Y Y, Y\rangle \equiv 0$, patiently substituting (28), (29), (30), (31), (32) and (33) into (14) gives:

$$\begin{aligned}\mathrm{d}_x\psi\Big[\,\nabla_{\dot\sigma}\dot\sigma + 2\frac{k\,T_z}{\langle Y,Y\rangle}\nabla_{\dot\sigma}Y + 2k^2\,\frac{\langle\nabla_{\dot\sigma}Y, Y\rangle}{\langle Y,Y\rangle\,(k^2 + \langle Y,Y\rangle)}\,\dot\sigma + \quad (34)\\ -2k\,T_z\frac{\langle\nabla_{\dot\sigma}Y, Y\rangle}{\langle Y,Y\rangle\,(k^2 + \langle Y,Y\rangle)}Y\Big] = 0.\end{aligned}$$

Since $d_x\psi$ is an isometry, this is equivalent to the vanishing of the argument inside the square bracket of (34).

Finally, since $\dot\sigma = T_z \cdot \dot z$, we obtain easily (23). □

Thus, the differential equation (23) characterizes the brachistochrones in $\mathcal{U}^+_{p,\gamma}(k)$.

4 Examples

To illustrate the results obtained, in this section we discuss two examples of stationary space–times and their brachistochrones. Our first example concerns a generalized version of the Heisenberg group, which is a manifold diffeomorphic to $\mathbb{R}^3$ such that the orthogonal distribution to the Killing vector field $Y = \frac{\partial}{\partial t}$ is not integrable. The family of Lorentzian metrics in this example are parameterized by a smooth function v of the space variables x and y. If one chooses $v = x$, then the projection onto the xy-plane of this metric coincides with the projection of the metric of the *Rinsler's model* (see [16]), and we recover a result by V. Perlick obtained in [3].

In our second example we study a model of stationary manifold built over the manifold S^3, considered as a bundle over S^2 with fiber diffeomorphic to circles (*Hopf fibration* $S^1 \longmapsto S^3 \longmapsto S^2$). This model gives a stationary space-time which is *not* regular, i.e. it does not admit a global space–time splitting adapted to the Killing vector field Y.

4.1 Brachistochrones in the Generalized Heisenberg Space.

We consider in $\mathcal{M} = \mathbb{R}^3$, with coordinates (x, y, t), the Lorentzian metric

$$g = dx^2 + dy^2 - v^2(dt - y\,dx + x\,dy)^2,$$

where $v = v(x, y)$ is a smooth map on $\mathbb{R}^2$. As the coefficients of the metric do not depend on the variable t, the timelike vector field $Y = \frac{\partial}{\partial t}$ is a Killing vector field in $(\mathcal{M}, g)$. Setting $\theta^1 = dx$, $\theta^2 = dy$

and $\theta^3 = v(\mathrm{d}t - y\,\mathrm{d}x + x\,\mathrm{d}y)$, so that $g = (\theta^1)^2 + (\theta^2)^2 - (\theta^3)^2$, then the Riemannian metric g_{R} in $\mathcal{M}$ is:

$$g_{\mathrm{R}} = (\theta^1)^2 + (\theta^2)^2 + (\theta^3)^2.$$

If (e_1, e_2, e_3) denotes the dual basis of $(\theta^1, \theta^2, \theta^3)$, given by:

$$e_1 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial t}, \quad e_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial t}, \quad e_3 = \frac{1}{v}\frac{\partial}{\partial t},$$

then the Levi–Civita connection of g_{R} is given by:

$$\begin{aligned} \nabla e_1 &= v\theta^3 e_2 + (v\theta^2 + \frac{v_x}{v}\theta^3)e_3, \\ \nabla e_2 &= -v\theta^3 e_1 - (v\theta^1 - \frac{v_y}{v}\theta^3)e_3, \\ \nabla e_3 &= -(v\theta^2 + \frac{v_x}{v}\theta^3)e_1 + (v\theta^1 - \frac{v_y}{v}\theta^3)e_3. \end{aligned}$$

If $w(s) = (x(s), y(s), t(s))$, then $\dot{w} = \dot{x}\,e_1 + \dot{y}\,e_2 + v(\dot{t} - y\dot{x} + x\dot{y})e_3$; the normality condition $\dot{t} - y\dot{x} + x\dot{y} = 0$ gives:

$$\dot{w} = \dot{x}\,e_1 + \dot{y}\,e_,$$

hence:

$$\nabla_{\dot{w}}\dot{w} = \ddot{x}\,e_1 + \ddot{y}\,e_2.$$

Considering the function:

$$\phi_k = \frac{v^2}{k^2 - v^2},$$

the equation (14) for the normal geodesics in the metric $\phi_k \cdot g_{\mathrm{sr}}$ is given by the system:

$$\begin{cases} \ddot{x} + \frac{k^2}{v(k^2-v^2)}(v_x(\dot{x}^2 - \dot{y}^2) + 2v_y\dot{x}\dot{y}) = 0, \\ \\ \ddot{y} + \frac{k^2}{v(k^2-v^2)}(v_y(\dot{y}^2 - \dot{x}^2) + 2v_x\dot{x}\dot{y}) = 0, \\ \\ \dot{t} - y\dot{x} + x\dot{y} = 0. \end{cases} \tag{35}$$

Each solution of the system (35) gives a brachistochrone σ of energy k on $\mathcal{M}$, given by $\sigma(s) = (x(s), y(s), t(s)+\tau(s))$, with $\tau(s) = k\int_0^s v^{-2}\,\mathrm{d}r$.

Choosing $v \equiv 1$, the manifold $(\mathcal{M}, g_{\mathrm{R}})$ coincides with the Heisenberg space. This is the Lie subgroup of $SL(3,I\!\!R)$ consisting of upper triangular matrices $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$; the metrics g, g_{R} and the vector field Y are left invariant. In this case the equations (35) reduce to:

$$\ddot{x} = 0, \quad \ddot{y} = 0, \quad \dot{t} = 0,$$

whose solutions are straight lines $w(s) = (as, bs, 0)$, and the brachistochrones of energy k are of the form $\sigma(s) = (as, bs, ks)$.

Observe also that, if we take $v = x$, the projection onto the xy plane of the metric g coincides with the projection of the metric of the Rindler's model given by $\mathrm{d}x^2 + \mathrm{d}y^2 - x^2\mathrm{d}t^2$. The spatial part of the brachistochrones in these two models are the same; it is easily checked that the first two equations of (35) are equivalent to formulas (60) and (61) of [3].

4.2 Brachistochrones in the 3-sphere S^3

In $\mathbb{C}^2$, with coordinates $z = (z_1, z_2)$, we consider the 3-sphere

$$S^3 = \left\{ z \in \mathbb{C}^2 : z_1\overline{z}_1 + z_2\overline{z}_2 = 1 \right\}.$$

For $z = (z_1, z_2)$, we write $z^\perp = (-\overline{z}_2, \overline{z}_1)$; the triple $(iz, z^\perp, iz^\perp)$ is a basis for T_zS^3 as a real vector subspace of $\mathbb{C}^2$.

On S^3 we consider the 1-forms:

$$\theta = -i(\overline{z}_1\,\mathrm{d}z_1 + \overline{z}_2\,\mathrm{d}z_2), \quad \text{and} \quad \varphi = -z_2\,\mathrm{d}z_1 + z_1\,\mathrm{d}z_2,$$

and the Lorentz metric $g = \varphi \otimes \overline{\varphi} - \theta^2$ (observe that $\theta = \overline{\theta}$).

Since $\varphi(z^\perp) = 1$, $\varphi(iz) = 0$, then, writing $\varphi = \theta^1 + i\theta^2$, $(\theta^1, \theta^2, \theta)$ is the dual basis of $(z^\perp, iz^\perp, iz)$.

The metric g_{R} is given by

$$g_{\mathrm{R}} = \varphi \otimes \overline{\varphi} + \theta^2,$$

and the covariant derivative associated to $g_{\mathbb{R}}$ is given by:

$$\nabla z^{\perp} = -\theta\, iz^{\perp} + \theta^2\, iz,$$
$$\nabla iz^{\perp} = \theta\, z^{\perp} - \theta^1\, iz,$$
$$\nabla iz = -\theta^2\, z^{\perp} + \theta^1\, iz.$$

The vector field $Y(z) = iz$ is timelike and Killing, and $\langle Y, Y\rangle \equiv -1$, hence, in the notation of Section 3, we have:

$$\phi_k \equiv \frac{1}{k^2-1}, \quad \nabla\phi_k \equiv 0.$$

The integral curves of Y are circles and the quotient space is diffeomorphic to the 2-sphere S^2, giving rise to the Hopf fibration of S^2 by S^1.

The orthogonal distribution Δ to Y is the kernel of the 1-form θ. The normal geodesics satisfy the system of equations:

$$\begin{cases} \overline{z}_1\dot{z}_1 + \overline{z}_2\dot{z}_2 = 0, \\ z_1\ddot{z}_2 - z_2\ddot{z}_1 = 0, \\ \langle \dot{z}, z\rangle = 0,\ \langle \ddot{z}, \overline{z}^{\perp}\rangle = 0. \end{cases} \tag{36}$$

For each $z_0 \in S^3$, the solutions of (36) through z_0 are given by:

$$z(t) = z_0 \cos t + \alpha\, z_0^{\perp} \sin t, \quad |\alpha| = 1,$$

they are maximal circles on S^2.

The flow of the vector field Y is $\psi(z,t) = e^{it}z$; moreover

$$\tau(s) = -k\int_0^s \frac{dr}{\langle Y, Y\rangle} = -ks.$$

Hence, the brachistochrones through z_0 are given by:

$$\sigma(s) = e^{-iks}\left(z_0 \cos s + \alpha\, z_0^{\perp} \sin s\right).$$

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