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Abstract

Let $(M, \mathcal{D}, g)$ be a sub-Riemannian manifold (i.e. M is a smooth manifold, $\mathcal{D}$ is a smooth distribution on M and g is a smooth metric defined on $\mathcal{D}$) such that the dimension of M is either three or four and $\mathcal{D}$ is a contact or odd-contact distribution, respectively. We construct an adapted connection ∇ on M and use it to study the equivalence problem. Furthermore, we classify the three-dimensional sub-Riemannian manifolds which are sub-homogeneous and show the relation to Cartan's list of homogeneous CR manifolds. Finally, we classify the four-dimensional sub-Riemannian manifolds which are sub-symmetric.

0 Introduction

Sub-Riemannian geometry is concerned with the study of a smooth manifold M equipped with a metric defined only on a subbundle $\mathcal{D}$ of the tangent bundle TM, henceforth a *sub-Riemannian manifold*, and of the related geometric structures in analogy with Riemannian geometry. When $\mathcal{D} = TM$ we recover Riemannian geometry. Despite the similarities between the two geometries, there are new interesting phenomena occuring in sub-Riemannian geometry; see [11] for a survey and references.

It is worth noting that this subject is of more than only formal interest since the several applications and connections range from control theory and mechanics with non-holonomic constraints, sub-Laplacians and hypoelliptic differential equations, to contact geometry and Cauchy-Riemann structures.

Now we come to the subject of this paper. A *sub-Riemannian homogeneous space* is a sub-Riemannian manifold which admits a transitive group of sub-Riemannian isometries. A *sub-Riemannian symmetric space* ([13]) is a homogeneous sub-Riemannian manifold for which there is an involutive isometry which is a central symmetry when restricted to the distribution.

This work is divided into two parts. In the first part, we study 3-dimensional sub-Riemannian homogeneous spaces. We use a connection adapted to the sub-Riemannian structure (the pseudo-Hermitian connection of Webster [15] which was subsequently generalized in [10, 7]) to define geometric invariants and establish an equivalence theorem. Then we classify all the 3-dimensional sub-Riemannian homogeneous spaces by reducing the structure to some algebraic data (see Table 1 for a complete list of spaces and their invariants). This classification adds two new classes of examples (namely, types (7) and (8) in Table 1) to Strichartz's list of 3-dimensional sub-Riemannian symmetric spaces ([13]). Furthermore, we show how our classification is a refinement of Cartan's classification of 3-dimensional homogeneous non-degenerate Cauchy-Riemann manifolds ([4, 5]; see also [3, 6]). In fact, in dimension 3, CR structures are equivalent to conformal sub-Riemannian structures (see [9]), in analogy with the correspondence between complex structures and conformal Riemannian structures in dimension 2.

In the second part of the paper, we consider 4-dimensional sub-Riemannian symmetric spaces. We define a connection adapted to a sub-Riemannian structure of odd-contact type in dimension 4 and then we classify all the 4-dimensional sub-Riemannian symmetric spaces by using a canonical linearization of the structure (see Table 2). This classification is the first step towards a classification of odd-contact sub-Riemannian symmetric spaces in arbitrary dimension, which shall appear in a forthcoming paper ([8]). For an analysis of the case of contact sub-Riemannian symmetric spaces, see [7].

Notably absent from this work is the case of a codimension 2 distribution in a 4-dimensional manifold which is under investigation by S. Namur ([12]).

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1 Sub-Riemannian geometry

A *sub-Riemannian manifold* is a triple $(M, \mathcal{D}, g)$ where M is an oriented manifold, $\mathcal{D}$ is an oriented smooth distribution on M and g is a smoothly varying positive definite symmetric bilinear form defined on $\mathcal{D}$.

In this paper we shall consider only the case of contact and odd-contact distributions on manifolds of dimensions 3 and 4, respectively.

1.1 Dimension 3

Let M be of dimension 3 throughout this subsection. Consider the Levi form

$$\mathcal{L} : \mathcal{D} \times \mathcal{D} \to TM/\mathcal{D}, \quad \mathcal{L}(X,Y) = [\tilde{X}, \tilde{Y}] \mod \mathcal{D} \tag{1}$$

where $\tilde{X}, \tilde{Y}$ are extensions to sections of $\mathcal{D}$. We assume that $\mathcal{D}$ is a *contact* distribution, that is, $\mathcal{L}$ is a *non-degenerate* skew-symmetric bilinear form on $\mathcal{D}$. Let dV be the volume form on $\mathcal{D}$. The (normalized) *contact* form is the 1-form θ on M such that

$$\begin{aligned} \ker\theta &= \mathcal{D}, \\ d\theta|_{\mathcal{D}} &= 2dV. \end{aligned}$$

Observe that M has a canonical orientation given by $\theta \wedge d\theta$ which is independent of the orientation on $\mathcal{D}$.

Since $d\theta$ has rank 2, there is a unique vector field ξ on M such that

$$\begin{aligned} \theta(\xi) &= 1, \\ \iota_\xi d\theta &= 0. \end{aligned}$$

It is called the *characteristic* vector field.

Note that the sub-Riemannian metric g has a natural extension to a Riemannian metric on M by setting ξ to be orthonormal to $\mathcal{D}$.

A canonical connection analogous to the Levi-Civita connection in the case of Riemannian geometry is uniquely defined on M. This connection is in fact defined for a contact sub-Riemannian manifold of arbitrary (odd) dimension; in the 3-dimensional case it is the same as the pseudo-Hermitian connection of Webster ([15]). Let $\underline{TM}$ and $\underline{\mathcal{D}}$ denote respectively the set of sections of TM and of $\mathcal{D}$.

Theorem 1.1 ([10, 7]) *There exists a unique connection* $\nabla : \underline{TM} \to \underline{TM^*} \otimes \underline{TM}$*, called the* adapted connection, *and a unique symmetric tensor* $\tau : \mathcal{D} \to \mathcal{D}$*, called the* sub-torsion, *with the following properties (*T *is the torsion tensor of the connection):*

a. $\nabla_U : \underline{\mathcal{D}} \to \underline{\mathcal{D}}$;

b. $\nabla \xi = 0$;

c. $\nabla g = 0$;

d. $$\begin{aligned} T(X,Y) &= d\theta(X,Y)\xi, \\ T(\xi,X) &= \tau(X); \end{aligned}$$

for $X, Y \in \underline{\mathcal{D}}$, $U \in \underline{TM}$.

The curvature of this connection is given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Observe that

$$< R(X,Y)Z, W >= - < R(X,Y)W, Z >$$

for $Z, W \in \mathcal{D}$.

From the general theory of connections we have the first Bianchi identity

$$\mathfrak{S} R(X,Y)Z = \mathfrak{S} T(T(X,Y),Z) + \mathfrak{S}(\nabla_X T)(Y,Z)$$

where $\mathfrak{S}$ denotes the cyclic summation in X, Y and Z. In the case of the adapted connection we get the following identities

$$\begin{aligned}
&(2) & \mathfrak{S} R(X,Y)Z &= \mathfrak{S} d\theta(X,Y)\tau(Z), \\
&(3) & R(\xi,Y)Z - R(\xi,Z)Y &= (\nabla_Z \tau)(Y) - (\nabla_Y \tau)(Z), \\
&(4) & T(\tau(X),Y) - T(\tau(Y),X) &= -(\nabla_\xi T)(X,Y),
\end{aligned}$$

where $X, Y, Z \in \underline{\mathcal{D}}$.

Consider a local positive orthonormal frame $\{X_1, X_2\}$ on $\mathcal{D}$. Then $T(X_1, X_2) = 2\xi$, identity (2) is trivial and identities (3) and (4) can be rewritten as

$$\begin{aligned}
&(5) & R(\xi,X_1)X_2 - R(\xi,X_2)X_1 &= (\nabla_{X_2}\tau)(X_1) - (\nabla_{X_1}\tau)(X_2), \\
&(6) & T(\tau(X_1),X_2) - T(\tau(X_2),X_1) &= 0.
\end{aligned}$$

We set

$$K = - < R(X_1, X_2)X_1, X_2 >$$

and note that the definition is independent of the chosen local frame.

Define also

$$W_1 = -< R(\xi, X_1)X_2, X_1 > \quad \text{and} \quad W_2 =< R(\xi, X_2)X_1, X_2 >,$$

and

$$W = \sqrt{W_1^2 + W_2^2}.$$

Note that W_1 and W_2 depend on the local frame, but W does not.

Assume now that $\{X_1, X_2\}$ are eigenvectors of the symmetric operator τ. By using identity (6) we can write

$$\tau(X_1) = \tau_0 X_1 \quad \text{and} \quad \tau(X_2) = -\tau_0 X_2,$$

where $\tau_0 \geq 0$.

From now on we suppose that τ_0 is constant. Then identity (5) gives

$$\begin{aligned} R(\xi, X_1)X_2 &= 2\tau_0 \nabla_{X_1} X_2, \\ R(\xi, X_2)X_1 &= -2\tau_0 \nabla_{X_2} X_1. \end{aligned} \tag{7}$$

If $\tau_0 > 0$ we have that the frame of eigenvectors is uniquely defined up to a sign, so W_1 and W_2 are well-defined up to a sign and the sign of $W_1 W_2$ is fixed. It also follows from (7) that in this case

$$[X_1, X_2] = -2\xi - \frac{W_1}{2\tau_0} X_1 + \frac{W_2}{2\tau_0} X_2.$$

If $W_1 \neq 0$, we change, if necessary, the local frame so that the sign of W_1 is positive. This defines a parallelism on the space.

If $\tau_0 = 0$ then the frame of eigenvectors is not uniquely defined, but then identity (7) implies $R(\xi, X)Y = 0$ for all $X, Y \in \underline{\mathcal{D}}$, so $W_1 = W_2 = W = 0$.

From the general theory of connections we have the second Bianchi identity

$$\mathfrak{S}(\nabla_X R)(Y, Z) + \mathfrak{S}R(T(X, Y), Z) = 0$$

from where we get the following identity:

$$-\nabla_\xi K + \nabla_{X_2} W_1 - \nabla_{X_1} W_2 = \frac{W_1^2 - W_2^2}{2\tau_0}.$$

It follows that if K, W_1 and W_2 are constants, then $W_1 = \pm W_2$.

A *local isometry* between two sub-Riemannian manifolds $(M, \mathcal{D}, g)$ and $(M', \mathcal{D}', g')$ is a diffeomorphism between open sets $\psi : U \subset M \to U' \subset M'$ such that $\psi_*(\mathcal{D}) = \mathcal{D}'$ and $\psi^* g' = g$. In the contact case it follows that

that $\psi^*\theta' = \pm\theta$ and $\psi_*\xi = \pm\xi'$ (and therefore ψ will be a local Riemannian isometry relative to the extended Riemannian metrics on M and M'). If ψ is globally defined on M to M', we say simply that ψ is an *isometry*.

Observe that an isometry $\psi : M \to M'$ is affine with respect to the adapted connections, that is, $\nabla'_{\psi_* X}\psi_* Y = \psi_*(\nabla_X Y)$ for $X, Y \in \underline{TM}$.

We can now state the following equivalence theorem which is proved in the special case of null subtorsion in [10].

Theorem 1.2 *If $(M, \mathcal{D}, g)$ and $(M', \mathcal{D}', g')$ are two sub-Riemannian manifolds which have the same constant invariants K, τ_0, W_1, W_2 as defined above, then they are locally isometric.*

Remark 1.1 Assume that the invariants K, τ_0, W_1 and W_2 are constant.

a. The frame was chosen so that $W_1 \geq 0$. It is easy to check that a change of orientation of $\mathcal{D}$ will not affect the invariants W_1 and W_2 (neither K nor τ_0, for that matter).

b. In the case $\tau_0 = 0$, we have also $W_1 = W_2 = 0$. So the only invariant is the sectional curvature K. In the case $\tau_0 \neq 0$ we have the identity

$$\frac{1}{2}\left(K + \frac{W^2}{4\tau_0^2}\right) = -\tau_0 \mathrm{sgn}(W_2).$$

c. In the simply-connected case, if the sub-Riemannian metrics are restrictions of complete Riemannian metrics then the local isometry in Theorem 1.2 can be extended to a global isometry. We will be interested in the homogeneous case, so completeness will hold.

1.2 Dimension 4

Throughout this subsection we assume that the dimension of M is 4. Recall the Levi form $\mathcal{L}$ that was defined in (1). Since $\mathcal{L}$ is a skew-symmetric bilinear form defined on a three-dimensional space, it has at least a one-dimensional kernel. The distribution $\mathcal{D}$ is *odd-contact* if the dimension of the kernel of $\mathcal{L}$ is exactly one. Set $\mathcal{D}_n$ to be the orthogonal complement to $\ker\mathcal{L}$ in $\mathcal{D}$. Since M and $\mathcal{D}$ are oriented, $\mathcal{L}$ induces an orientation on $\mathcal{D}_n$ and an orientation on $\ker\mathcal{L}$. Let dV be the volume form on $\mathcal{D}_n$ and let X_3 be the positive unitary vector in $\ker\mathcal{L}$. The (normalized) *odd-contact* form is the 1-form θ on M such that

$$\begin{aligned} \ker\theta &= \mathcal{D}, \\ d\theta|_{\mathcal{D}_n} &= 2\,dV. \end{aligned}$$

Let $\bar{\mathcal{D}} = [\mathcal{D}_n, \mathcal{D}_n] \oplus \mathcal{D}_n$. Then $\dim \bar{\mathcal{D}} = 3$ and the skew-symmetric bilinear form

$$d\theta : \bar{\mathcal{D}} \times \bar{\mathcal{D}} \to \mathbb{R}$$

has a one-dimensional kernel transversal to $\mathcal{D}_n$. There is a unique vector field ξ on M which is in this kernel and such that $\theta(\xi) = 1$. It is called the *characteristic* vector field.

Note that the sub-Riemannian metric g has a natural extension to a Riemannian metric on M by setting ξ to be orthonormal to $\mathcal{D}$.

A *local isometry* between two sub-Riemannian manifolds $(M, \mathcal{D}, g)$ and $(M', \mathcal{D}', g')$ is a diffeomorphism between open sets $\psi : U \subset M \to U' \subset M'$ such that $\psi_*(\mathcal{D}) = \mathcal{D}'$ and $\psi^* g' = g$. In the odd-contact case it follows that that $\psi^*\theta' = \pm\theta$, $\psi_* X_3 = \pm X_3$ and $\psi_*\xi = \pm\xi'$ (and therefore ψ will be a local Riemannian isometry relative to the extended Riemannian metrics on M and M'). If ψ is globally defined on M to M', we say simply that ψ is an *isometry*.

A canonical connection analogous to the Levi-Civita connection in the case of Riemannian geometry is uniquely defined on M. The definition of this connection is very similar to the adapted connection from Theorem 1.1.

Theorem 1.3 *There exists a unique connection* $\nabla : \underline{TM} \to \underline{TM^*} \otimes \underline{TM}$, *called the* adapted connection, *and a unique symmetric tensor* $\tau : \mathcal{D} \to \mathcal{D}$, *called the* sub-torsion, *with the following properties (T is the torsion tensor of the connection):*

a. $\nabla_U : \underline{\mathcal{D}} \to \underline{\mathcal{D}}$;

b. $\nabla\xi = 0$;

c. $\nabla g = 0$;

d.
$$\begin{aligned} T(X,Y) &= d\theta(X,Y)\xi, \\ T(\xi,X) &= \tau(X) + d\theta(\xi,X)\xi; \end{aligned}$$

for $X, Y \in \underline{\mathcal{D}}$, $U \in \underline{TM}$.

Proof. Let X, Y, $Z \in \underline{\mathcal{D}}$. As is Riemannian geometry, we determine $\nabla_X Y$ by using a., c. and d. and the permutation trick:

$$\begin{aligned} &X < Y, Z > + Y < Z, X > - Z < X, Y > = \\ &\quad 2 < \nabla_X Y, Z > + < Y, [X, Z] + T(X, Z) > \\ &\quad + < X, [Y, Z] + T(Y, Z) > + < Z, [Y, X] + T(Y, X) > . \end{aligned}$$

Because of b., it remains only to define $\nabla_\xi X$. But $\nabla_\xi X - \nabla_X \xi = [\xi, X] + T(\xi, X)$, so

$$\nabla_\xi X = [\xi, X] + \tau(X) + d\theta(\xi, X)\xi.$$

Finally,

$$\begin{aligned}
\xi < X, Y > &= < \nabla_\xi X, Y > + < X, \nabla_\xi Y > \\
&= < [\xi, X] + \tau(X) + d\theta(\xi, X)\xi, Y > \\
&\quad + < X, [\xi, Y] + \tau(Y) + d\theta(\xi, Y)\xi > \\
&= < [\xi, X] + d\theta(\xi, X)\xi, Y > + < [\xi, Y] + d\theta(\xi, Y)\xi, X > \\
&\quad + 2 < \tau(X), Y >
\end{aligned}$$

determines $\tau(X)$ (note that

$$d\theta(\xi, X) = \xi(\theta(X)) - X(\theta(\xi)) - \theta([\xi, X]) = -\theta([\xi, X]),$$

so $[\xi, X] + d\theta(\xi, X)\xi \in \underline{\mathcal{D}}$). □

Corollary 1.1 *The connection* ∇ *has the following properties:*

a. $L_\xi : \underline{\mathcal{D}_n} \to \underline{\mathcal{D}_n}$;

b. $d\theta(X, Y) = \theta(T(X, Y))$;

c. $< \tau(X), Y > = \frac{1}{2} L_\xi g(X, Y)$;

for $X, Y \in \underline{\mathcal{D}}$.

Corollary 1.2 *The characteristic vector field* ξ *is a Killing field on* M *relative to the extended Riemannian metric if and only if* $\tau = 0$.

The curvature of this connection is given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

As before, from the general theory of connections we have the Bianchi identity

$$\mathfrak{S} R(X, Y)Z = \mathfrak{S} T(T(X, Y), Z) + \mathfrak{S}(\nabla_X T)(Y, Z).$$

In the case of the adapted connection we get the following identities

$$\begin{aligned}
\mathfrak{S}R(X,Y)Z &= \mathfrak{S}d\theta(X,Y)\tau(Z),\\
\mathfrak{S}d\theta(X,Y)d\theta(\xi,Z)\xi &= -\mathfrak{S}(\nabla_X T)(Y,Z),\\
R(\xi,X)Y - R(\xi,Y)X &= d\theta(\xi,X)\tau(Y) - d\theta(\xi,Y)\tau(X)\\
&\quad +(\nabla_Y\tau)(X) - (\nabla_X\tau)(Y),\\
T(\tau(X),Y) - T(\tau(Y),X) &= \nabla_\xi T(X,Y) + \{\nabla_X d\theta(\xi,Y) - \nabla_Y d\theta(\xi,X)\}\xi,
\end{aligned}$$

where $X,Y,Z \in \underline{\mathcal{D}}$.

Let $\{X_1, X_2\}$ and $\{X_3\}$ be, respectively, positive orthonormal bases of $\mathcal{D}_n$ and $\ker\mathcal{L}$. The above identities translate into

$$\begin{aligned}
R(X_1,X_2)X_3 + R(X_3,X_1)X_2 + R(X_2,X_3)X_1 &= 2\tau(X_3),\\
2d\theta(\xi,X_3)\xi &= -\mathfrak{S}(\nabla_{X_1}T)(X_2,X_3),\\
R(\xi,X_1)X_2 - R(\xi,X_2)X_1 &= (\nabla_{X_2}\tau)(X_1)\\
&\quad -(\nabla_{X_1}\tau)(X_2),\\
R(\xi,X_1)X_3 - R(\xi,X_3)X_1 &= -d\theta(\xi,X_3)\tau(X_1)\\
&\quad +(\nabla_{X_3}\tau)(X_1)\\
&\quad -(\nabla_{X_1}\tau)(X_3),\\
T(\tau(X_1),X_2) - T(\tau(X_2),X_1) &= 0,\\
-T(X_1,\nabla_\xi X_3) - d\theta(\xi,\nabla_{X_3}X_1)\xi &= T(\tau(X_3),X_1).
\end{aligned}$$

Observe that an isometry $\psi : M \to M'$ is affine with respect to the adapted connections, that is, $\nabla'_{\psi_* X}\psi_* Y = \psi_*(\nabla_X Y)$ for $X, Y \in \underline{TM}$.

2 Sub-Riemannian homogeneous and symmetric spaces

A *sub-Riemannian homogeneous space* (or *sub-homogeneous space*, for short) is a sub-Riemannian manifold $(M,\mathcal{D},g)$ which admits a transitive Lie group of isometries acting smoothly on M. A *sub-Riemannian symmetric space* (or *sub-symmetric space*) is a sub-homogeneous space $(M,\mathcal{D},g)$ such that for every point $x_0 \in M$ there is an isometry ψ, called the *sub-symmetry* at x_0, with $\psi(x_0) = x_0$ and $\psi_*|_{\mathcal{D}_{x_0}} = -1$ (recall that $\mathcal{D}$ is either contact or odd-contact; for a more general definition see [13]).

In the three-dimensional (resp., four-dimensional) case the sub-symmetry ψ preserves the orientation of $\mathcal{D}$ (resp., $\mathcal{D}_n$), so $\psi^*\theta = \theta$. Then we must have $\psi_*\xi_{x_0} = \xi_{x_0}$, since $(\psi^*\theta)(\xi) = \theta(\psi_*\xi)$.

It is easy to see that the sub-symmetry at a point x_0 must be unique; in fact, it is given by $\exp_{x_0}(X + a\xi_{x_0}) \mapsto \exp_{x_0}(-X + a\xi_{x_0})$, where exp is the affine exponential map associated to the adapted connection and $X \in \mathcal{D}_{x_0}$, $a \in \mathbb{R}$. Moreover, by homogeneity it is enough to check the existence of the sub-symmetry at one single point of M.

Let $p : \tilde{M} \to M$ be the universal covering of a smooth manifold M. Then it is easy to see that a sub-Riemannian (resp., sub-homogeneous, sub-symmetric) structure on M lifts to a unique sub-Riemannian (resp., sub-homogeneous, sub-symmetric) structure on $\tilde{M}$ such that p is a local sub-Riemannian isometry.

3 The classification of 3-dimensional sub-homogeneous spaces

The 3-dimensional simply-connected sub-symmetric spaces were classified by Stritchartz in [13]. They fall into six classes which include Lie groups of semisimple, nilpotent and solvable type. More precisely, they are the universal coverings (to be denoted with a tilde) of the Heisenberg group H^3, the Euclidean proper motion group $Euc_2^+ = SO(2) \ltimes \mathbb{R}^2$, the Poincaré orthochronological proper group $Poinc_2^+ = SO(1,1) \ltimes \mathbb{R}^2$, the special linear group $SL_2\mathbb{R} \cong SU(1,1)$ (this space admits two distinct distributions; one of them is given by the usual Cartan involution; we distinguish the space equipped with the other distribution with a "prime") and the sphere group $S^3 \cong SU(2)$. In this section we shall see that there are exactly two other 3-dimensional sub-homogeneous spaces which are not sub-symmetric (the detailed list of sub-homogeneous spaces with the respective distributions and metrics is in Table 1).

Proposition 3.1 *Let $(M^3, \mathcal{D}^2, g)$ be a simply-connected three-dimensional sub-homogeneous space. Then:*

a. *there is a connected, simply-connected Lie group G of sub-Riemannian isometries of M which acts simply transitively on M;*

b. *the Lie algebra $\mathfrak{g}$ of G has a decomposition $\mathfrak{g} = [\mathfrak{p}, \mathfrak{p}] + \mathfrak{p}$ where $\mathfrak{p}$ is the subspace of $\mathfrak{g}$ corresponding to $\mathcal{D}_{x_0}$ under the identification of $\mathfrak{g}$ with*

$T_{x_0}M$ for a chosen base-point x_0, and $\mathfrak{p}$ does not depend on the chosen x_0;

c. *the inner product B induced on $\mathfrak{p}$ by the identification of $\mathfrak{p}$ with $\mathcal{D}_{x_0}$ does not depend on the chosen $x_0 \in M$.*

Proof. As observed in [13], if the space is sub-symmetric, then the sub-symmetry ψ induces an automorphism $s = \mathrm{Ad}_\psi$ of the Lie algebra $\mathfrak{g}'$ of the group G' of all sub-Riemannian isometries of M, and so there is a decomposition $\mathfrak{g}' = \mathfrak{h} + \mathfrak{p}$ into the ± 1-eigenspaces of s. Now $[\mathfrak{p}, \mathfrak{p}] + \mathfrak{p}$ is a subalgebra of $\mathfrak{g}'$ and the corresponding 3-dimensional subgroup G of G' will do.

If the space is not sub-symmetric, then proceed as follows. Write $M = G/K$ where G is the *connected* (OK, as long as M is connected) Lie group of sub-Riemannian isometries of M and K is the isotropy subgroup at a chosen base-point $x_0 \in M$. Let $\mathfrak{g}, \mathfrak{k}$ denote the respective Lie algebras of G, K. Then K is compact, since G is a group of Riemannian isometries, so there is an Ad_K-invariant decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$. Consider the projection $\pi : G \to M$, $\pi(g) = g(x_0)$. Then π_* identifies $T_{x_0}M$ with $\mathfrak{m}$. Let $\mathfrak{p}$ be the inverse image in $\mathfrak{m}$ of $\mathcal{D}_{x_0}$ under π_*. Since $\mathcal{D}$ is contact, we have $\mathfrak{g} = \mathfrak{k} + [\mathfrak{p}, \mathfrak{p}] + \mathfrak{p}$, Ad_K-invariant decomposition. Let $\mathfrak{h} = \mathfrak{k} + [\mathfrak{p}, \mathfrak{p}]$. Then $\mathfrak{h} = \ker d\theta^*$ for $\theta^* = \pi^*(\theta)$ and so $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ and $\mathfrak{h}$ contains $\mathfrak{k}$ as an ideal (see [1, 7]). Let B be g_{x_0} lifted to $\mathfrak{p}$ by π. Then B is an Ad_K-invariant inner product on $\mathfrak{p}$ and $\dim \mathfrak{k}$ is at most one.

We want to show that $\mathfrak{k} = 0$. It is enough to show that $[\mathfrak{k}, \mathfrak{p}] = 0$, as Ad_K must be effective on $\mathfrak{p}$ (because $\mathfrak{p}$ modulo $\mathfrak{k}$ generates $\mathfrak{g}/\mathfrak{k}$ and and Ad_K is effective on $\mathfrak{g}/\mathfrak{k}$). We apply the Jacobi identity to get

$$[\mathfrak{k}, [\mathfrak{p}, \mathfrak{p}]] \subset [[\mathfrak{k}, \mathfrak{p}], \mathfrak{p}] \subset [\mathfrak{p}, \mathfrak{p}].$$

But $[\mathfrak{k}, [\mathfrak{p}, \mathfrak{p}]] \subset \mathfrak{k}$ because $\mathfrak{k}$ is an ideal of $\mathfrak{h}$. Thus $[\mathfrak{k}, [\mathfrak{p}, \mathfrak{p}]] = 0$ and $\mathfrak{h}$ is abelian. Using Jacobi again,

$$[[\mathfrak{p}, \mathfrak{p}], [\mathfrak{k}, \mathfrak{p}]] \subset [\mathfrak{k}, [[\mathfrak{p}, \mathfrak{p}], \mathfrak{p}]] \subset [\mathfrak{k}, \mathfrak{h}] + [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}.$$

If, to the contrary, $[\mathfrak{k}, \mathfrak{p}] \neq 0$, then $[\mathfrak{k}, \mathfrak{p}] = \mathfrak{p}$ as $\mathrm{ad}_{\mathfrak{k}}$ is skew-symmetric on $\mathfrak{p}$. Thus $[[\mathfrak{p}, \mathfrak{p}], \mathfrak{p}] \subset \mathfrak{p}$ and so the endomorphism s of $\mathfrak{g}$ which is $+1$ on $\mathfrak{h}$ and -1 on $\mathfrak{p}$ is an automorphism. Therefore it descends to an automorphism σ of the corresponding simply-connected group $\tilde{G}$. Let $\tilde{K}$ be the connected subgroup of $\tilde{G}$ for $\mathfrak{k}$. Now $\tilde{G}/\tilde{K}$ is a simply-connected sub-symmetric space (the sub-symmetry at the base-point is conjugation with σ) which covers M, hence it *is* M. But we assumed M not to be sub-symmetric. contradiction.

The subspace $\mathfrak{p}$ of $\mathfrak{g}$ and the inner product B on $\mathfrak{p}$ induced by g_{x_0} do not depend on x_0 since G acts on M preserving $\mathcal{D}$ and g. Finally, $\dim G = 3$ and M is simply-connected, so the action is simple transitive. □

Conversely, given a three-dimensional Lie algebra $\mathfrak{g}$, a subspace $\mathfrak{p}$ of codimension one such that $\mathfrak{g} = [\mathfrak{p}, \mathfrak{p}] + \mathfrak{p}$ and an inner product B on $\mathfrak{p}$, we can construct a simply-connected three-dimensional sub-homogeneous space by taking the simply-connected Lie group G with Lie algebra $\mathfrak{g}$ and the G-invariant distribution $\mathcal{D}$ and metric g determined by $\mathfrak{p}$ and B, respectively. Note that a sufficient (but *not* necessary) condition for G to be sub-symmetric is that $[[\mathfrak{p}, \mathfrak{p}], \mathfrak{p}] \subset \mathfrak{p}$.

Now we take up the classification of three-dimensional simply-connected sub-homogeneous spaces. According to Proposition 3.1 it is enough to classify the three-dimensional simply-connected Lie groups equipped with a left-invariant sub-Riemannian structure $(G, \mathcal{D}, g)$. Consider the associated algebraic data $(\mathfrak{g}, \mathfrak{p}, B)$ as above. Let $\{X_1, X_2\}$ be an orthonormal basis of $\mathfrak{p}$ and let $Y = [X_1, X_2]$. We may assume that $\mathfrak{p}$ is not ad_Y-invariant, for otherwise the space is sub-symmetric and these spaces have already been classified. Consider the matrix of $\mathrm{ad}_Y : \mathfrak{p} \to \mathfrak{g}$ relative to $\{X_1, X_2, Y\}$:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \tag{8}$$

The Jacobi identity in $\mathfrak{g}$ is equivalent to the following relations:

$$\begin{cases} a_{11} + a_{22} & = & 0 \\ a_{13}a_{21} - a_{23}a_{11} & = & 0 \\ a_{13}a_{22} - a_{23}a_{12} & = & 0 \end{cases} \tag{9}$$

If we take a different basis $\{\tilde{X}_1, \tilde{X}_2, \tilde{Y} = [\tilde{X}_1, \tilde{X}_2]\}$ for $\mathfrak{g}$ and the change of basis is given by the matrix

$$M = \begin{pmatrix} N & & 0 \\ & & 0 \\ 0 & 0 & \det N \end{pmatrix}, \qquad N \in O(2),$$

then A is transformed to $\bar{A} = (\det N) N A M^{-1}$.

Observe that $a_{13}^2 + a_{23}^2 \neq 0$ since $\mathfrak{p}$ is not ad_Y-invariant. Therefore, relations (9) imply that the minor $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is singular. Now the same relations yield that $\mathrm{ad}_Y|_{\mathfrak{p}}$ is singular and so there is an orthonormal basis of $\mathfrak{p}$ such that

$$A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \end{pmatrix}, \qquad b \neq 0.$$

Hence the normal forms for A are:

FIRST CASE:

$$\begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}, \qquad b > 0.$$

In this case the transformation

$$\begin{cases} X_1' &= \frac{1}{b} X_1 \\ X_2' &= Y + bX_2 \\ Y' &= Y \end{cases}$$

maps $\mathfrak{g}$ onto the direct product of the two-dimensional non-abelian Lie algebra spanned by Y', X_1' (i.e., $[Y', X_1'] = Y'$) and the one-dimensional Lie algebra spanned by X_2'; and it maps $\mathfrak{p}$ onto the subspace spanned by the orthonormal vectors bX_1' and $\frac{1}{b}(X_2' - Y')$. Denote with $\Sigma_0(b)$ the corresponding simply-connected (solvable) group. We shall see below that this sub-homogeneous space is isometric to a sub-symmetric space whose underlying group is $\widetilde{SL_2\mathbb{R}} \cong \widetilde{SU(1,1)}$.

SECOND CASE:

$$\begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \end{pmatrix}, \qquad a \neq 0, \quad b > 0.$$

In this case the transformation

$$\begin{cases} X_1' &= \frac{1}{|a|^{1/2}} X_1 \\ X_2' &= X_2 \\ Y' &= \frac{1}{|a|^{1/2}} Y \end{cases}$$

maps $\mathfrak{g}$ onto the semidirect product of the one-dimensional Lie algebra spanned by X_1' with the two dimensional abelian Lie algebra spanned by X_2' and Y' relative to the X_1'-action: $X_2' \mapsto Y'$, $Y' \mapsto -\mathrm{sgn}(a)X_2' - \frac{b}{|a|^{1/2}}Y'$; and it maps $\mathfrak{p}$ onto the subspace spanned by the orthonormal vectors $|a|^{1/2}X_1'$ and X_2'. Denote with $\Sigma_+(b')$ (resp., $\Sigma_-(b')$), $b' = b/|a|^{1/2} > 0$, the corresponding simply-connected (solvable) group if $a > 0$ (resp., $a < 0$). Observe that these families of solvable groups generalize the groups $SO(2) \ltimes \mathbb{R}^2$ and $SO(1,1) \ltimes \mathbb{R}^2$.

The adapted connection

In this subsection we shall compute the adapted connection and its associated invariants for a three-dimensional Lie group equipped with a left-invariant sub-Riemannian structure $(G, \mathcal{D}, g)$. According to earlier results, this includes all three dimensional sub-homogeneous manifolds (of sub-symmetric and non-sub-symmetric type).

Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{p} = \mathcal{D}_1$ and $B = g_1$. Then $\mathfrak{g} = [\mathfrak{p}, \mathfrak{p}] + \mathfrak{p}$. Consider the normalized contact (left-invariant) 1-form θ on G and the (left-invariant) characteristic vector field ξ. Choose a positive orthonormal basis $\{X_1, X_2\}$ of $\mathfrak{p}$ and let $Y = [X_1, X_2]$. Note that the Lie algebra structure of $\mathfrak{g}$ is completely determined by the linear map $\mathrm{ad}_Y : \mathfrak{p} \to \mathfrak{g}$. Let A as in (8) be its matrix relative to the basis $\{X_1, X_2, Y\}$. We can express the adapted connection in terms of the entries of A:

$$
\begin{aligned}
\xi &= -\frac{1}{2}(Y - a_{23}X_1 + a_{13}X_2); \\
T(X_1, X_2) &= -Y + a_{23}X_1 - a_{13}X_2; \\
(\tau) &= \frac{1}{2}\begin{pmatrix} a_{11} & \frac{a_{12}+a_{21}}{2} \\ \frac{a_{12}+a_{21}}{2} & a_{22} \end{pmatrix}; \\
\nabla_{X_1}X_1 &= -a_{23}X_2; \\
\nabla_{X_2}X_2 &= -a_{13}X_1; \\
\nabla_{X_1}X_2 &= a_{23}X_1; \\
\nabla_{X_2}X_1 &= a_{13}X_2; \\
\nabla_{X_1}Y &= -a_{23}^2X_2 - a_{13}a_{23}X_1; \\
\nabla_{X_2}Y &= a_{13}^2X_1 + a_{13}a_{23}X_2; \\
\nabla_Y X_1 &= [\frac{1}{2}(a_{12} - a_{21}) - (a_{13}^2 + a_{23}^2)]X_2; \\
\nabla_Y X_2 &= [\frac{1}{2}(a_{21} - a_{12}) + (a_{13}^2 + a_{23}^2)]X_1.
\end{aligned}
$$

The invariants are then:

$$
\begin{aligned}
K &= \frac{1}{2}(a_{12} - a_{21}) - (a_{13}^2 + a_{23}^2);\\
\tau_0 &= \frac{1}{4}\sqrt{4a_{11}^2 + (a_{12} + a_{21})^2};\\
W_1 &= \sqrt{\frac{3a_{11}^2(a_{13}^2 + a_{23}^2) + a_{12}^2 a_{13}^2 + a_{21}^2 a_{23}^2}{8}};\\
W_2 &= \epsilon W_1;\\
\epsilon &= -\frac{1}{2}\left(\frac{K}{\tau_0} + \frac{W_1^2}{2\tau_0^3}\right).
\end{aligned}
$$

The complete classification is summarized in Table 1. In the first column we list representatives G for the three-dimensional groups which are sub-homogeneous; in the second column we describe the Lie algebra structure of $\mathfrak{g}$: there is a basis $\{X_1', X_2'\}$ of $\mathfrak{p}$ such that $Y' = [X_1', X_2']$ and A' is the matrix of $\mathrm{ad}_{Y'}|_{\mathfrak{p}}$ relative to the basis $\{X_1', X_2', Y'\}$; in the third column we give the matrix of inner products $B' = (< X_i', X_j' >)$; the remaining columns list the invariants: τ_0, K, W_1 and W_2.

It is easy to compute that the invariants for the space $\Sigma_0(b)$ are $K = -b^2$, $\tau_0 = W_1 = W_2 = 0$, namely, exactly the same as the ones for space $(4; b^2, b^2)$ in Table 1. Hence, these spaces are isometric as sub-Riemannian manifolds by Theorem 1.2. In particular, we get

Remark 3.1 There are left-invariant Riemannian metrics on $\mathbb{R}$ cross the affine group of $\mathbb{R}$ and on the universal covering of $SL(2,\mathbb{R})$ which turn them into isometric, but not isomorphic, Lie groups.

Spaces (1) through (6) in Table 1 are the sub-symmetric spaces. By looking at the invariants in Table 1 and applying Theorem 1.2 we conclude that spaces (7) and (8) are (the only) sub-homogeneous spaces which are not sub-symmetric.

4 Homogeneous CR manifolds

A *CR manifold* is a triple $(M, \mathcal{D}, J)$ where M is a smooth manifold, $\mathcal{D}$ is a smooth distribution on M and J is a smoothly varying complex structure defined on $\mathcal{D}$.

We shall always assume that $\dim M = 3$ and the CR structure is *non-degenerate*, i.e. $\mathcal{D}$ is a contact distribution on M.

type	G	A'	B'	τ_0	K	W_1	W_2
(1)	H^3	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	0	0	0	0
(2a)	$\widetilde{Euc_2^+}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{pmatrix}$	$\frac{a}{4}$	$\frac{a}{2}$	0	0
(3a)	$Poinc_2^+$	$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{pmatrix}$	$\frac{a}{4}$	$-\frac{a}{2}$	0	0
(4ad)	$\widetilde{SU(1,1)}$	$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{d} \end{pmatrix}$	$\frac{a-d}{4}$	$-\frac{a+d}{2}$	0	0
(5ad)	$\widetilde{SU(1,1)}'$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{d} \end{pmatrix}$	$\frac{a+d}{4}$	$\frac{a-d}{2}$	0	0
(6ad)	$SU(2)$	$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{d} \end{pmatrix}$	$\frac{a-d}{4}$	$\frac{a+d}{2}$	0	0
(7ab)	$\Sigma_+(b)$	$\begin{pmatrix} 0 & 1 & -b \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{pmatrix}$	$\frac{a}{4}$	$a(\frac{1}{2} - b^2)$	$a^{3/2}b\frac{\sqrt{2}}{4}$	$-a^{3/2}b\frac{\sqrt{2}}{4}$
(8ab)	$\Sigma_-(b)$	$\begin{pmatrix} 0 & -1 & b \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{pmatrix}$	$\frac{a}{4}$	$-a(\frac{1}{2} + b^2)$	$a^{3/2}b\frac{\sqrt{2}}{4}$	$a^{3/2}b\frac{\sqrt{2}}{4}$

Table 1: Three-dimensional sub-homogeneous spaces: $\{X_1', X_2', Y' = [X_1', X_2']\}$ is a basis of the Lie algebra of G, $\{X_1', X_2'\}$ is a basis of the distribution and A' is the matrix of $\mathrm{ad}_{Y'}$ restricted to the distribution; B' is the matrix of the inner product on the distribution; τ_0, K, W_1 and W_2 are invariants; a, b and d are positive parameters; we may assume $a \geq d$ for types (4) and (6).

In $TM \otimes \mathbb{C}$ we have $\mathcal{D} \otimes \mathbb{C} = T_{1,0} \oplus T_{0,1}$ where $T_{1,0}$ and $T_{0,1}$ are respectively the i and $-i$ eigenvalues of J. In particular, $T_{1,0} = \bar{T}_{0,1}$. Conversely, a complex line bundle $L \subset TM \otimes \mathbb{C}$ such that $L \neq \bar{L}$ determines a subbundle $\mathcal{D} \subset TM$ with a complex structure on it. The non-degeneracy condition is equivalent to $[\zeta, \bar{\zeta}]$ being everywhere transversal to $L \oplus \bar{L}$ for a nowhere-vanishing section ζ of L.

A CR manifold $(M, \mathcal{D}, J)$ is *homogeneous* if its group of CR automorphisms (i.e. diffeomorphisms $\psi : M \to M$ such that $\psi_*(\mathcal{D}) = \mathcal{D}$ and $\psi_* J = J\psi_*$), acts transitively on M. A CR manifold $(M, \mathcal{D}, J)$ is *locally homogeneous* if the universal covering CR manifold $(\tilde{M}, \tilde{\mathcal{D}}, \tilde{J})$ is homogeneous (this is stronger than requiring that any two points in M have CR-isomorphic neighborhoods). A CR manifold locally isomorphic to S^3 (equipped with its natural CR structure inherited from $\mathbb{C}^2$) is called *spherical*. A locally homogeneous, non-degenerate CR manifold which is not spherical is called *aspherical*.

There is a classical equivalence between complex structures and oriented conformal Riemannian structures on 2-dimensional manifolds. In analogy with that equivalence, there is an equivalence between non-degenerate CR structures $(M, \mathcal{D}, J)$ and *conformal sub-Riemannian* (or *sub-conformal*, for short) structures $(M, \mathcal{D}, [g])$ on a 3-dimensional manifold M, where $[g]$ is the conformal class of a sub-Riemannian metric g on $\mathcal{D}$. Namely, giving the endomorphism J on $\mathcal{D}$ is the same as giving a ninety-degree rotation on $\mathcal{D}$ (see [9] for further developments in this direction).

Proposition 4.1 *Under the above equivalence, a non-degenerate CR structure on a 3-dimensional manifold is homogeneous if and only if there is a homogeneous sub-Riemannian metric in the conformal class of the corresponding sub-conformal structure.*

Proof. We have to show that given a homogeneous sub-conformal structure $(M, \mathcal{D}, [g])$, there is a sub-Riemannian metric $\bar{g}$ in the class $[g]$ such that $(M, \mathcal{D}, \bar{g})$ is a homogeneous sub-Riemannian manifold. Now it turns out that all non-degenerate, homogeneous 3-dimensional CR manifolds are isomorphic to Lie groups with left-invariant CR structures on them ([4, 5]). Therefore we know that there is a Lie group G of sub-conformal transformations acting simply transitively on M, so if we take the metric g at the base-point of M and translate it to a G-invariant sub-Riemannian metric $\bar{g}$ on M, then we get a homogeneous sub-Riemannian manifold $(M, \mathcal{D}, \bar{g})$ such that $\bar{g}$ is in the sub-conformal class $[g]$ because G acts on M by sub-conformal transformations relative to $[g]$. □

For a sub-conformal structure $(M, \mathcal{D}, [g])$ there is an invariant C which is equivalent to Cartan's CR invariant ([4, 5]) under the equivalence between sub-conformal structures and CR-structures in dimension 3, and so it vanishes precisely when the CR structure is spherical. In the homogeneous case C can be expressed in terms of the invariants τ_0, K, W_1 of the associated homogeneous sub-Riemannian structure $(M, \mathcal{D}, \bar{g})$ as follows:

$$C = \begin{cases} \frac{3K}{4\tau_0} + \frac{W_1^2}{3\tau_0^3} & \text{if } \tau_0 \neq 0; \\ 0 & \text{if } \tau_0 = 0. \end{cases}$$

In order to get C, we recall the parallelism obtained by Cartan for aspherical CR structures. This means that there exists a canonical sub-Riemannian metric in the sub-conformal class defined by an aspherical CR structure. Use Webster's embedding of the pseudo-Hermitian connection into the Cartan connection for a CR structure (see [15] and its correction in [2]) to write C in terms of the sub-Riemannian data. Details will appear elsewhere.

Proposition 4.1 implies that our classification of simply-connected 3-dimensional sub-homogeneous manifolds (see Section 3) is a refinement of Cartan's classification of simply-connected, homogeneous, non-degenerate 3-dimensional CR manifolds ([4, 5]; see also [3, 6]). So we now construct all the simply-connected, homogeneous, non-degenerate 3-dimensional CR manifolds from our Table 1. The notation is taken from there.

Type (1)

The Heisenberg group H^3 has a unique CR structure given by the complex line bundle $L \subset TH^3 \otimes \mathbb{C}$ spanned by $X_1' + iX_2'$. The structure is spherical.

Type (2a)

For each $a > 0$, the universal covering of the Euclidean proper motion group, $\widetilde{Euc_2^+}$, has a CR structure given by the complex line bundle spanned by $\sqrt{a}X_1' + iX_2'$. As it turns out, this CR structure is isomorphic to the one corresponding to $X_1' + iX_2'$ under the map $X_1' \mapsto X_1'$, $X_2' \mapsto \frac{1}{\sqrt{a}}X_2'$, $Y' \mapsto \frac{1}{\sqrt{a}}Y'$. So $\widetilde{Euc_2^+}$ gets a unique CR structure. It is aspherical.

Type (3a)

Similarly to type (2a), the Poincaré orthochronological proper group $Poinc_2^+$ has a unique CR structure given by $X_1' + iX_2'$ which is aspherical.

Type $(6ad)$

The sphere $S^3 \cong SU(2)$ has a one-parameter family of CR structures given by the complex line bundle $< \sqrt{a}X_1' + i\sqrt{d}X_2' >=< X_1' + isX_2' >$, where $s = \sqrt{d/a} \in (0,1]$. The structure is spherical if and only if $s = 1$.

Types $(4ad)$ and $(5ad)$

The group $\widetilde{SL(2,\mathbb{R})} \cong \widetilde{SU(1,1)}$ has two distinct one-parameter families of CR structures. In order to describe them, let X_1', X_2', Y' be a basis of its Lie algebra such that $[X_1', X_2'] = Y'$, $[X_2', Y'] = X_1'$, $[X_2', Y'] = X_1'$. Then they are given by the complex line bundles $< X_1' + isX_2' >$ and $< Y' + isX_2' >$ for $s \in (0,1]$. In each case, the value $s = 1$ correspond to a spherical CR structure on the group (and the other values correspond to aspherical strctures). Note that the former spherical CR structure comes from a sub-torsionless sub-Riemannian structure on the group, but not the latter one.

Type $(8ab)$

For each $b > 0$, the group $\Sigma_-(b)$ has a unique CR structure given by $< X_1' + iX_2' >$. It can be checked to be aspherical.

Type $(7ab)$

The group $\Sigma_+(b)$ has a unique CR structure given by $< X_1' + iX_2' >$. It can be checked to be spherical for $b = 3/\sqrt{2}$ and aspherical otherwise (in fact, $C = 3/2 - b^2/3$). The spherical case is exactly the universal covering of the CR structure on $D = \{(z,w) \in \mathbb{C}^2 : \Im w = |z|^2, \Im z > 0\}$ with group of CR automorphisms isomorphic to $\mathbb{R} \ltimes \mathbb{R}^2$ and $t \in \mathbb{R}$ acting on $\mathbb{R}^2$ by $t(x,y) = (e^t x, e^{2t} y)$.

5 Sub-orthogonal involutive Lie algebras

In this section we assume $\dim M = 4$ and we construct a linear object associated to a four-dimensional sub-symmetric space: its sub-OIL algebra. We also show how to recover a four-dimensional sub-symmetric space from an abstract sub-OIL algebra. The construction can be substantially generalized, see e. g. [13, 7, 8], but we do not pursue this matter here since we have no use for it.

Proposition 5.1 ([13]) *Let $(M^4, \mathcal{D}^3, g)$ be a simply-connected sub-symmetric space of dimension four. Then:*

a. *there is a connected, simply-connected Lie group G of sub-Riemannian isometries of M which acts simply transitively on M;*

b. *there is an involution s of the Lie algebra $\mathfrak{g}$ of G such that $\mathfrak{g} = [\mathfrak{p}, \mathfrak{p}] + \mathfrak{p}$ is the decomposition of $\mathfrak{g}$ into the ± 1-eigenspaces of s, where $\mathfrak{p}$ corresponds to $\mathcal{D}_{x_0}$ under the identification of G with M for a chosen base-point x_0, and $\mathfrak{p}$ does not depend on the chosen x_0;*

c. *the inner product B induced on $\mathfrak{p}$ by the identification of $\mathfrak{p}$ with $\mathcal{D}_{x_0}$ does not depend on the chosen x_0.*

Proof. Let G' be the Lie group of all sub-Riemannian isometries of M, choose $x_0 \in M$, let K be the isotropy subgroup at x_0, and let $\psi \in K$ be the sub-symmetry at x_0. Let $\mathfrak{g}'$ and $\mathfrak{k}$ denote the respective Lie algebras of G' and K and let $\mathfrak{g}' = \mathfrak{h} + \mathfrak{p}$ be the decomposition of $\mathfrak{g}'$ into the ± 1-eigenspaces of the involution $s' = \mathrm{Ad}_\psi$ of $\mathfrak{g}'$. Then M is represented as the coset space G'/K. K is a compact subgroup of G', since G' is a group of Riemannian isometries relative to the canonical extended Riemannian metric on M. For any $k \in K$, Ad_k factors through a linear map on $\mathfrak{g}/\mathfrak{k}$, and because K is compact we can find a complementary Ad_K-invariant space $\mathfrak{m}$. Now π_* identifies $\mathfrak{m}$ with the tangent space $T_{x_0}M$ and is easily seen to be an equivalence between the Ad_K-action on $\mathfrak{m}$ and the K-action on $T_{x_0}M$. Define $\mathfrak{p}_0$ to be the inverse image of $\mathcal{D}_{x_0}$ in $\mathfrak{m}$ under π_*. Then $\mathfrak{p}_0 \subset \mathfrak{p}$ because $\psi_*|_{\mathcal{D}_{x_0}} = -1$. $\mathfrak{k}$ contains no nonzero ideal of $\mathfrak{g}$ because Ad_K is effective on $\mathfrak{m}$ (because K is effective on $T_{x_0}M$). In fact, Ad_K is effective on $\mathfrak{p}_0$ as $\mathfrak{p}_0$ modulo $\mathfrak{k}$ generates $\mathfrak{g}/\mathfrak{k}$ by the odd-contact condition. Let $X \in \mathfrak{k}$ and $Y \in \mathfrak{p}_0$. Then $[X - s'X, Y] \in \mathfrak{p}_0$, so

$$-[X - s'X, Y] = s'[X - s'X, Y] = [s'X - X, s'Y] = [X - s'X, Y]$$

and so $\mathrm{ad}_{X - s'X}[\mathfrak{p}_0] = 0$. But the centralizer of $\mathfrak{p}_0$ in $\mathfrak{k}$ is zero since Ad_K is effective on $\mathfrak{p}_0$. Thus we have $X - s'X = 0$ and $\mathfrak{k} \subset \mathfrak{h}$. Now $\mathfrak{h} = \mathfrak{k} + [\mathfrak{p}_0, \mathfrak{p}_0]$ and $\mathfrak{p} = \mathfrak{p}_0$. Since $\mathfrak{g} = [\mathfrak{p}, \mathfrak{p}] + \mathfrak{p}$ is a subalgebra, it defines a connected subgroup G of G'. Let us show that $\dim G = \dim \mathfrak{g} = 4$. It is enough to prove that $\dim[\mathfrak{p}, \mathfrak{p}] = 1$. In fact, decompose $\mathfrak{p} = \mathfrak{p}_n \oplus \ker \Theta$ where $\mathfrak{p}_n$ is a 2-dimensional space and $\Theta : \mathfrak{p} \times \mathfrak{p} \to \mathfrak{h}/\mathfrak{k}$ is the skew-symmetric bilinear form $\Theta(A, B) = [A, B] \mod \mathfrak{k}$. Then the Jacobi identity gives

$$[[\mathfrak{k}, \mathfrak{p}_n], \ker\Theta] \subset [[\mathfrak{k}, \ker\Theta], \mathfrak{p}_n] + [[\ker\Theta, \mathfrak{p}_n], \mathfrak{k}]. \tag{10}$$

Now $\mathrm{ad}_{\mathfrak{k}}$ is effective and skew-symmetric on $\mathfrak{p}$, so $[\mathfrak{k}, \mathfrak{p}_n] = \mathfrak{p}_n$, $\dim \mathfrak{k} \leq 1$ and $[\mathfrak{k}, \ker\Theta] = 0$. Since $[\ker\Theta, \mathfrak{p}_n] \subset \mathfrak{k}$, (10) implies that $[\mathfrak{p}_n, \ker\Theta] = 0$, and so $\dim[\mathfrak{p},\mathfrak{p}] = 1$. Let $s = s'|_{\mathfrak{g}}$. The subspace $\mathfrak{p}$ of $\mathfrak{g}$ and the inner product B on $\mathfrak{p}$ induced by g_{x_0} do not depend on x_0 since G' acts on M preserving $\mathcal{D}$ and g. Finally, $\dim G = 4$ and M is simply-connected, so the action is simply transitive. □

To a four-dimensional sub-symmetric space $(M, \mathcal{D}, g)$ we have now associated a triple $(\mathfrak{g}, \mathfrak{p}, B)$ where $\mathfrak{g}$ is a four-dimensional Lie algebra, $\mathfrak{p}$ is a three-dimensional subspace of $\mathfrak{g}$ such that $\mathfrak{g} = [\mathfrak{p},\mathfrak{p}] + \mathfrak{p}$, $[[\mathfrak{p},\mathfrak{p}],\mathfrak{p}] \subset \mathfrak{p}$ and B is an inner product on $\mathfrak{p}$. The triple is called the *sub-orthogonal involutive Lie (sub-OIL) algebra* of $(M, \mathcal{D}, g)$.

An *abstract sub-orthogonal involutive Lie algebra* is defined to be a triple $(\mathfrak{g}, \mathfrak{p}, B)$ with the properties in the above paragraph. Given an abstract sub-OIL algebra $(\mathfrak{g}, \mathfrak{p}, B)$ we can construct a simply-connected sub-symmetric space as follows. Let G be the simply-connected group with Lie algebra $\mathfrak{g}$. Then $\mathfrak{g} = [\mathfrak{p},\mathfrak{p}] + \mathfrak{p}$ so that $\mathfrak{p}$ translates to a G-invariant odd-contact distribution $\mathcal{D}$ on G such that $\mathcal{D}_1 = \mathfrak{p}$, and B translates to a G-invariant metric g on $\mathcal{D}$ such that $g_1 = B$. The involutive automorphism s of $\mathfrak{g}$ which is $+1$ on $[\mathfrak{p},\mathfrak{p}]$ and is -1 on $\mathfrak{p}$ induces an automorphism ψ of G. ψ is an isometry of G and $\psi_* = s$, so ψ fixes 1 and induces -1 on $\mathcal{D}_1$. Thus ψ is the sub-symmetry at 1 and $g\psi g^{-1}$ is the sub-symmetry at g. We have proved that G is sub-symmetric. In other words,

Proposition 5.2 *Let $(\mathfrak{g}, \mathfrak{p}, B)$ be an abstract sub-OIL algebra. Let $(G, \mathcal{D}, g)$ be the simply-connected sub-symmetric space of dimension four constructed above from $(\mathfrak{g}, \mathfrak{p}, B)$. Then $(\mathfrak{g}, \mathfrak{p}, B)$ is the sub-OIL algebra associated to $(G, \mathcal{D}, g)$.*

6 The classification of 4-dimensional sub-symmetric spaces

Let $(G, \mathcal{D}, g)$ be a simply-connected sub-symmetric space of dimension 4 and odd-contact distribution and consider its associated sub-OIL algebra $(\mathfrak{g}, \mathfrak{p}, B)$. Then $\dim \mathfrak{g} = 4$, $\dim \mathfrak{p} = 3$, $\mathfrak{g} = [\mathfrak{p},\mathfrak{p}] + \mathfrak{p}$, $[[\mathfrak{p},\mathfrak{p}],\mathfrak{p}] \subset \mathfrak{p}$ and B is an inner product on $\mathfrak{p}$. The Levi form $\mathcal{L}$ on $\mathcal{D}$ induces a skew-symmetric bilinear form Θ on $\mathfrak{p}$ which is exactly the bracket of vectors on $\mathfrak{p}$, i.e. $\Theta(X,Y) = [X,Y]$.

Choose a positive orthonormal basis $\{X_1, X_2, X_3\}$ of $\mathfrak{p}$ such that X_3 spans the kernel of Θ. Then X_3 centralizes X_1 and X_2 and the characteristic field $\xi = -\frac{1}{2}[X_1, X_2] \in [\mathfrak{p}, \mathfrak{p}]$. Let $Y = [X_1, X_2]$ and consider the matrix of $\mathrm{ad}_Y : \mathfrak{p} \to \mathfrak{p}$ relative to the basis $\{X_1, X_2, X_3\}$:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Now the Jacobi identity is equivalent to the last row of A being zero (i.e., X_3 central in $\mathfrak{g}$) and A being traceless. If we take a different basis for $\mathfrak{p}$ given by $\tilde{X} = MX$ for

$$(11) \qquad M = \begin{pmatrix} \multicolumn{2}{c}{N} & 0 \\ & & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \qquad N \in O(2), \quad \lambda = \pm 1,$$

then A is transformed to $\tilde{A} = (\det N) MAM^{-1}$. Next we classify the (unoriented) isomorphism classes of $(\mathfrak{g}, \mathfrak{p}, B)$; let $\mathfrak{p}_n$ be the subspace of $\mathfrak{p}$ generated by X_1 and X_2:

$\mathfrak{p}_n$ is ad_Y-invariant

Then $a_{13} = a_{23} = 0$ and $\mathfrak{g}$ is the direct product of a three-dimensional Lie algebra generated by Y, X_1, X_2 with the one-dimensional Lie algebra generated by X_3. We distinguish three cases.

$\mathrm{ad}_Y|\mathfrak{p}_n$ is null

In this case the matrix A is null and $\mathfrak{g}$ is the sub-OIL algebra of $H^3 \times \mathbb{R}$.

$\mathrm{ad}_Y|\mathfrak{p}_n$ is singular and non-null

In this case there is an orthonormal basis of $\mathfrak{p}$ such that the matrix A has the form

$$\begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad a \neq 0.$$

The transformation

$$\begin{aligned} X_1' &= \frac{1}{|a|^{1/2}} X_1 \\ X_2' &= X_2 \\ X_3' &= X_3 \\ Y' &= \frac{1}{|a|^{1/2}} Y \end{aligned}$$

is an isomorphism onto the sub-OIL algebra of

$$\begin{cases} \widetilde{Euc_2^+} \times \mathbb{R} & \text{if } a > 0; \\ Poinc_2^+ \times \mathbb{R} & \text{if } a < 0 \end{cases}$$

spanned by $X_1', X_2', X_3', Y' = [X_1', X_2']$ and distribution spanned by the X_i's. The matrix of $\mathrm{ad}_{Y'}$ relative to $\{X_1', X_2', X_3'\}$ is

$$\begin{pmatrix} 0 & \mathrm{sgn}(a) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

and the matrix of the inner product on the distribution relative to the same basis is

$$\begin{pmatrix} \frac{1}{|a|} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\mathrm{ad}_Y|\mathfrak{p}_*$ is non-singular

In this case there is an orthonormal basis of $\mathfrak{p}$ such that the matrix A has the form

$$\begin{pmatrix} 0 & a & 0 \\ d & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad a, d \neq 0.$$

The transformation

$$\begin{aligned} X_1' &= \frac{1}{|a|^{1/2}} X_1 \\ X_2' &= \frac{1}{|d|^{1/2}} X_2 \\ X_3' &= X_3 \\ Y' &= \frac{1}{|ad|^{1/2}} Y \end{aligned}$$

is an isomorphism onto the sub-OIL algebra of

$$\begin{cases} \widetilde{SU(1,1)} \times \mathbb{R} & \text{if } a < 0, d > 0; \\ \widetilde{SU(1,1)}' \times \mathbb{R} & \text{if } ad > 0; \\ SU(2) \times \mathbb{R} & \text{if } a > 0, d < 0. \end{cases}$$

The matrix of $\mathrm{ad}_{Y'}$ is

$$\begin{pmatrix} 0 & \mathrm{sgn}(a) & 0 \\ \mathrm{sgn}(d) & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

and the matrix of the inner product on the distribution is

$$\begin{pmatrix} \frac{1}{|a|} & 0 & 0 \\ 0 & \frac{1}{|d|} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note the following isomorphisms: $\widetilde{SU(1,1)} \times \mathbb{R} \cong \widetilde{U(1,1)}$ and $U(1,1)$ is the full sub-Riemannian isometry group of the subtorsionless $\widetilde{SL_2\mathbb{R}} \cong \widetilde{SU(1,1)}$ and $SU(2) \times \mathbb{R} \cong \widetilde{U(2)}$ and $U(2)$ is the full sub-Riemannian isometry group of the subtorsionless $S^3 \cong SU(2)$.

Remark 6.1 The sub-symmetric spaces constructed above are exactly the ones that we get by taking the direct product of a three-dimensional sub-symmetric space with $\mathbb{R}$.

$\mathfrak{p}_n$ is not ad_Y-invariant and $\mathrm{ad}_Y|_{\mathfrak{p}_n}$ is singular

Then there is an orthonormal basis of $\mathfrak{p}$ such that the matrix A has the form

$$\begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad a \in \mathbb{R}, \quad b \neq 0.$$

Hence the normal forms for A are:

FIRST CASE:

$$\begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad b > 0.$$

In this case the transformation

$$\begin{cases} X_1' &= X_1 \\ X_2' &= X_2 \\ X_3' &= -bX_3 \\ Y' &= Y \end{cases}$$

is an isomorphism onto the sub-OIL algebra of the Engel group E^4. Here the matrix of $\mathrm{ad}_{Y'}$ is

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

and the matrix of the inner product on the distribution is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b^2 \end{pmatrix}$$

SECOND CASE:

$$\begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad a \neq 0, \quad b > 0.$$

In this case the transformation

$$\begin{cases} X_1' &= \frac{1}{|a|^{1/2}} X_1 \\ X_2' &= X_2 + \frac{b}{a} X_3 \\ X_3' &= X_3 \\ Y' &= \frac{1}{|a|^{1/2}} Y \end{cases}$$

is an isomorphism onto the sub-OIL algebra of

$$\begin{cases} \widetilde{Euc_2^+} \times \mathbb{R} & \text{if } a > 0; \\ Poinc_2^+ \times \mathbb{R} & \text{if } a < 0. \end{cases}$$

Here the matrix of the inner product on the distribution is

$$\begin{pmatrix} \frac{1}{|a|} & 0 & 0 \\ 0 & 1 + \frac{b^2}{a^2} & \frac{b}{a} \\ 0 & \frac{b}{a} & 1 \end{pmatrix}$$

$\mathfrak{p}_n$ is not ad_Y-invariant and $\mathrm{ad}_Y|_{\mathfrak{p}_n}$ is nonsingular

Let $\pi : \mathfrak{p} \to \mathfrak{p}_n$ be the orthogonal projection. We consider two cases separately.

$\pi \circ \mathrm{ad}_Y$ is singular

In this case there is an orthonormal basis of $\mathfrak{p}$ such that the matrix A has the form

$$\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}, \quad a, c \neq 0, \quad b \in \mathbb{R}$$

Hence the normal forms are:

$$\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}, \quad a \neq 0, \quad c > 0, \quad b \geq 0.$$

The transformation

$$\begin{cases} X_1' &= \frac{1}{|a|^{1/2}} X_1 \\ X_2' &= \frac{|a|^{1/4}}{c^{1/2}} (X_2 + \frac{b}{a} X_3) \\ X_3' &= X_3 \\ Y' &= \frac{1}{|a|^{1/4} c^{1/2}} Y \end{cases}$$

is an isomorphism onto a sub-OIL algebra spanned by X_1', X_2', X_3', $Y' = [X_1', X_2']$ and distribution spanned by the X_i's. Here the matrix of $\mathrm{ad}_{Y'}$ is

$$\begin{pmatrix} 0 & \mathrm{sgn}(a) & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and the matrix of the inner product on the distribution is

$$\begin{pmatrix} \frac{1}{|a|} & 0 & 0 \\ 0 & \frac{|a|^{1/2}}{c}(1 + \frac{b^2}{a^2}) & \frac{|a|^{1/4}}{c^{1/2}} \frac{b}{a} \\ 0 & \frac{|a|^{1/4}}{c^{1/2}} \frac{b}{a} & 1 \end{pmatrix}$$

The corresponding groups are the universal coverings of the semidirect products

$$\begin{cases} SO(2) \ltimes H^3 & \text{if } a > 0; \\ SO(1,1) \ltimes H^3 & \text{if } a < 0. \end{cases}$$

Note that these groups are respectively isomorphic to the full sub-Riemannian isometry group of H^3, and to the full sub-Lorentzian isometry group of H^3 (i.e. we replace the definite metric on the distribution by an indefinite one).

$\pi \circ \mathrm{ad}_Y$ is non-singular

In this case there is an orthonormal basis of $\mathfrak{p}$ such that the matrix A has the form

$$\begin{pmatrix} 0 & a & b \\ d & 0 & c \\ 0 & 0 & 0 \end{pmatrix}, \quad a, d \neq 0, \quad b^2 + c^2 \neq 0.$$

Hence the normal forms are:

$$\begin{pmatrix} 0 & a & b \\ d & 0 & c \\ 0 & 0 & 0 \end{pmatrix}, \quad a, d \neq 0, \quad b^2 + c^2 \neq 0, \quad b, c \geq 0.$$

The transformation

$$\begin{cases} X_1' &= \frac{1}{|a|^{1/2}}(X_1 + \frac{c}{d}X_3) \\ X_2' &= \frac{1}{|d|^{1/2}}(X_2 + \frac{b}{a}X_3) \\ X_3' &= X_3 \\ Y' &= \frac{1}{|ad|^{1/2}}Y \end{cases}$$

is an isomorphism onto the sub-OIL algebra of

$$\begin{cases} \widetilde{U(1,1)} & \text{if } a < 0, d > 0; \\ \widetilde{U(1,1)'} & \text{if } ad > 0; \\ \widetilde{U(2)} & \text{if } a > 0, d < 0. \end{cases}$$

Here the matrix of the inner product on the distribution is

$$\begin{pmatrix} \frac{1}{|a|}(1 + \frac{c^2}{d^2}) & \frac{1}{|ad|^{1/2}}\frac{bc}{ad} & \frac{1}{|a|^{1/2}}\frac{c}{d} \\ \frac{1}{|ad|^{1/2}}\frac{bc}{ad} & \frac{1}{|d|}(1 + \frac{b^2}{a^2}) & \frac{1}{|d|^{1/2}}\frac{b}{a} \\ \frac{1}{|a|^{1/2}}\frac{c}{d} & \frac{1}{|d|^{1/2}}\frac{b}{a} & 1 \end{pmatrix}.$$

The adapted connection

Next we compute the adapted connection and its associated invariants in each of the above examples. We get:

$$\begin{aligned}
\xi &= -\frac{1}{2}Y; \\
T(X_1, X_2) &= -Y; \\
(\tau) &= \frac{1}{2}\begin{pmatrix} a_{11} & \frac{a_{12}+a_{21}}{2} & \frac{a_{13}}{2} \\ \frac{a_{12}+a_{21}}{2} & a_{22} & \frac{a_{23}}{2} \\ \frac{a_{13}}{2} & \frac{a_{23}}{2} & 0 \end{pmatrix}; \\
\nabla_X X_1 &= \nabla_X X_2 = \nabla_X X_3 = 0 \quad \text{for } X \in \mathfrak{p}; \\
\nabla_\xi X_1 &= \frac{1}{4}(a_{21} - a_{12})X_2 - \frac{1}{2}a_{13}X_3; \\
\nabla_\xi X_2 &= \frac{1}{4}(a_{12} - a_{21})X_1 - \frac{1}{2}a_{23}X_3; \\
\nabla_\xi X_3 &= \frac{1}{4}(a_{13}X_1 + a_{23}X_2); \\
R(X, X_3) &= R(X, \xi) = 0 \quad \text{for } X \in \mathfrak{g}; \\
R(X_1, X_2) &: \begin{cases} X_1 \mapsto \frac{1}{2}(a_{21} - a_{12})X_2 - a_{13}X_3, \\ X_2 \mapsto \frac{1}{2}(-a_{21} + a_{12})X_1 - a_{23}X_3, \\ X_3 \mapsto \frac{1}{2}a_{13}X_1 + \frac{1}{2}a_{23}X_2; \end{cases} \\
K(X_1, X_2) &= - < R(X_1, X_2)X_1, X_2 >= \frac{1}{2}(a_{12} - a_{21}).
\end{aligned}$$

We organize the classification in Table 2. In the first column we list representatives G for the four-dimensional groups which are sub-symmetric; in the second column we describe the Lie algebra structure of $\mathfrak{g}$: there is a basis $\{X_1', X_2', X_3'\}$ of $\mathfrak{p}$ such that X_3' is central, $Y' = [X_1', X_2']$ and A' is the matrix of $\mathrm{ad}_{Y'}$ relative to that basis; in the last column we give the matrix of inner products $B' = (< X_i', X_j' >)$.

The full group of isometries

Finally we compute the full group of isometries for each of these spaces. It suffices to find the group K of all isometries that preserve the identity, since these together with G generate all isometries. To do this, we need to find all orthogonal transformations of $(\mathfrak{p}, B)$ which extend to automorphisms of $\mathfrak{g}$. If we identify the transformation with the 3×3 matrix M (cf. (11)), we require $(\det N)MAM^{-1} = A$ for M to extend (via $Y \mapsto (\det N)Y$) to an

type	G	A'	B'
(1)	$H^3 \times \mathbb{R}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$(2ab)$	$\widetilde{Euc_2^+} \times \mathbb{R}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{a} & 0 & 0 \\ 0 & 1+\frac{b^2}{a^2} & \frac{b}{a} \\ 0 & \frac{b}{a} & 1 \end{pmatrix}$
$(3ab)$	$Poinc_2^+ \times \mathbb{R}$	$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{a} & 0 & 0 \\ 0 & 1+\frac{b^2}{a^2} & -\frac{b}{a} \\ 0 & -\frac{b}{a} & 1 \end{pmatrix}$
$(4abcd)$	$\widetilde{U(1,1)}$	$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{a}(1+\frac{c^2}{d^2}) & -\frac{1}{(ad)^{1/2}}\frac{bc}{ad} & \frac{1}{a^{1/2}}\frac{c}{d} \\ -\frac{1}{(ad)^{1/2}}\frac{bc}{ad} & \frac{1}{d}(1+\frac{b^2}{a^2}) & -\frac{1}{d^{1/2}}\frac{b}{a} \\ \frac{1}{a^{1/2}}\frac{c}{d} & -\frac{1}{d^{1/2}}\frac{b}{a} & 1 \end{pmatrix}$
$(5abcd)$	$\widetilde{U(1,1)'}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{a}(1+\frac{c^2}{d^2}) & \frac{1}{(ad)^{1/2}}\frac{bc}{ad} & \frac{1}{a^{1/2}}\frac{c}{d} \\ \frac{1}{(ad)^{1/2}}\frac{bc}{ad} & \frac{1}{d}(1+\frac{b^2}{a^2}) & \frac{1}{d^{1/2}}\frac{b}{a} \\ \frac{1}{a^{1/2}}\frac{c}{d} & \frac{1}{d^{1/2}}\frac{b}{a} & 1 \end{pmatrix}$
$(6abcd)$	$\widetilde{U(2)}$	$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{a}(1+\frac{c^2}{d^2}) & -\frac{1}{(ad)^{1/2}}\frac{bc}{ad} & -\frac{1}{a^{1/2}}\frac{c}{d} \\ -\frac{1}{(ad)^{1/2}}\frac{bc}{ad} & \frac{1}{d}(1+\frac{b^2}{a^2}) & \frac{1}{d^{1/2}}\frac{b}{a} \\ -\frac{1}{a^{1/2}}\frac{c}{d} & \frac{1}{d^{1/2}}\frac{b}{a} & 1 \end{pmatrix}$
$(7d)$	E^4	$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d^2 \end{pmatrix}$
$(8abd)$	$\widetilde{SO(2)} \ltimes H^3$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{a^{1/2}}{d}(1+\frac{b^2}{a^2}) & \frac{b}{a^{3/4}d^{1/2}} \\ 0 & \frac{b}{a^{3/4}d^{1/2}} & 1 \end{pmatrix}$
$(9abd)$	$SO(1,1) \ltimes H^3$	$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{a^{1/2}}{d}(1+\frac{b^2}{a^2}) & -\frac{b}{a^{3/4}d^{1/2}} \\ 0 & -\frac{b}{a^{3/4}d^{1/2}} & 1 \end{pmatrix}$

Table 2: Four-dimensional sub-symmetric spaces: $\{X_1', X_2', X_3', Y' = [X_1', X_2']\}$ is a basis of the Lie algebra of G, $\{X_1', X_2', X_3'\}$ is a basis of the distribution such that X_3' is central and A' is the matrix of $\mathrm{ad}_{Y'}$ restricted to the distribution; B' is the matrix of the inner product on the distribution; a, b, c, d are parameters such that a, $d > 0$ and b, $c \geq 0$; we may assume $a \geq d$ for types (4) and (6).

automorphism of $\mathfrak{g}$. A simple computation shows that K is discrete, except in the cases (1), $(4 : a, 0, 0, a)$ and $(6 : a, 0, 0, a)$, where any M of the form

$$\begin{pmatrix} & N & 0 \\ & & 0 \\ 0 & 0 & \epsilon \end{pmatrix}, \qquad N \in O(2), \quad \epsilon = \pm 1,$$

is allowed.

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