UNIVERSIDADE DE SÃO PAULO

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ISSN - 0103-2577

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NOTAS DO ICMSC
Série Estatística

São Carlos
Jan./1997

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ABSTRACT

In this paper, the posterior distribution of the Weibull reliability function is obtained by using the Laplace approximation (Tierney and Kadane, 1986). Some numerical comparisons and the precision of the Laplace approximation are considered.

Key words: Weibull distribution, reliability function, Laplace's approximation, posterior distribution, prior distribution.

1. INTRODUCTION

Weibull distribution is a natural generalization of the exponential distribution and is very useful on the study of failure times of electronic components, in biomedical applications, etc.

The Weibull model is defined by the density function

$$f(x \mid p, \theta) = \frac{p}{\theta} x^{p-1} \exp\{-\frac{x^p}{\theta}\}, \tag{1}$$

where $x > 0$, $p > 0$, $\theta > 0$ and p and θ are unknown parameters.

If X denotes the lifetime of a component, the reliability function $R = R(t) = P\{X > t\}$ given by

$$R = R(t) = \exp\{-\frac{t^p}{\theta}\} \quad , \; t > 0 \, , \theta > 0 \, , \tag{2}$$

defines the Weibull reliability of the component or the probability of the component does not fail before t.

By considering $X = (X_1, X_2, ..., X_n)$ as a sample of size n of (1), the likelihood function of (p, θ) is given by:

$$L(p, \theta \,|\, x) = \left(\frac{p}{\theta}\right)^n \left(\prod_{i=1}^{n} x_i\right)^{p-1} \exp\left\{-\frac{\sum_{i=1}^{n} x_i^p}{\theta}\right\}. \tag{3}$$

Let's suppose that we do not have any prior information about p and θ. Under this condition, the prior distribution proposed by Jeffreys (1961),

$$\pi(p, \theta) \propto \frac{1}{p\theta} \tag{4}$$

is more appropriated for our inference problem.

Our purpose in this paper is to determine the posterior distribution of reliability function R given in (2).

In order to achieve this goal, we consider the transformation $(p, \theta) \rightarrow (R, W)$ where $W = p$ and $R = \exp\{-\frac{t^p}{\theta}\}$. So, the likelihood function for (R, W) is given by:

$$L(R, W \mid x) \propto W^n \left(\ln \frac{1}{R}\right)^n \left(\prod_{i=1}^{n} y_i\right)^{W-1} R^{\sum_{i=1}^{n} y_i^W} , \tag{5}$$

where $y_i = \frac{x_i}{t}$, $t > 0$.

It's well known that Jeffreys's prior is invariant under transformation, so, the prior distribution for (R, W) is given by

$$\pi(R, W) \propto \frac{1}{RW \ln(1/R)}. \tag{6}$$

From (5) and (6), the joint posterior distribution of (R, W) is

$$p(R, W \mid x) \propto W^{n-1} \left(\ln \frac{1}{R}\right)^{n-1} \left(\prod_{i=1}^{n} y_i\right)^{W-1} R^{\sum_{i=1}^{n} y_i^W - 1}. \tag{7}$$

Integrating w.r.t. W the marginal posterior of R in (7) we have that

$$p(R \mid x) = \frac{\left(\ln \frac{1}{R}\right)^n \int_0^\infty W^{n-1} \left(\prod_{i=1}^n y_i\right)^{W-1} R^{\sum_{i=1}^n y_i^W - 1} dW}{\Gamma(n) t^n \int_0^\infty W^{n-1} \left(\prod_{i=1}^n x_i\right)^{W-1} \left(\sum_{i=1}^n x_i^W\right)^{-n} dW}. \tag{8}$$

Solving these integrals by numerical integration, we have the value of posterior distribution for each R, but we do not have an expression for the posterior distribution. In the next section we try to use the Laplace Approximation to obtain an approximation for (8).

2. LAPLACE APPROXIMATION FOR THE INTEGRALS INVOLVED THE POSTERIOR DISTRIBUTION OF R.

Laplace approximation for the integrals (Tierney and Kadane, 1986) is a very useful technique in Bayesian Inference to compute marginal posterior, posterior moments and predictive densities.

Let $h : R \to R$ a differentiable function of ϕ with $-h$ having a unique mode $\hat{\phi}$. Laplace's method gives an approximation to the integral of the form,

$$I = \int \exp\{-nh(\phi)\} d\phi \tag{9}$$

by using a Taylor expansion of h on $\hat{\phi}$,and obtaining the following result:

$$\int \exp\{-nh(\phi)\} d\phi \simeq \sqrt{\frac{2\pi\sigma^2}{n}} \exp\left\{-nh(\hat{\phi})\right\}, \tag{10}$$

where $\sigma = \left(h''(\hat{\phi})\right)^{-1/2}$.

So, applying Laplace's method to integrals in (8), we get

$$p(R \mid x) = \frac{\left(\ln \frac{1}{R}\right)^{n-1} \sigma_1 \exp\{-nh_1(\hat{W}_R)\}}{\Gamma(n) t^n \sigma_2 \exp\{-nh_2(\hat{W})\}} \tag{11}$$

where $\sigma_1 = \left(h_1"(\hat{W}_R)\right)^{-1/2}$, $\sigma_2 = \left(h_2"(\hat{W})\right)^{-1/2}$ with $\hat{W}_R$ and $\hat{W}$ the modes of

$$h_1(W) = -\frac{n-1}{n}.\ln(W) - \frac{w-1}{n}\sum_{i=1}^{n}\ln(y_i) - \frac{\sum_{i=1}^{n} y_i^W - 1}{n}\ln(R),$$

and

$$h_2(W) = -\frac{n-1}{n}.\ln(W) - \frac{w-1}{n}\sum_{i=1}^{n}\ln(x_i) + \ln(\sum_{i=1}^{n} x_i^W)$$

respectively.

It is important to observe that $\hat{W}_R$ is the root of the equation $h_1^{(1)}(W) = 0$ which depends on R.

By using a Taylor expansion for the integrals in (8), Achcar (1990) introduced a formula to verify the precision of Laplace approximation in (11) in the following way:

$$\frac{\left(\ln \frac{1}{R}\right)^n \int_0^\infty W^{n-1} \left(\prod y_i\right)^{W-1} R^{\sum y_i^W - 1} dW}{\Gamma(n) t^n \int_0^\infty W^{n-1} \left(\prod x_i\right)^{W-1} \left(\sum x_i^W\right)^{-n} dW} = \frac{\left(\ln \frac{1}{R}\right)^{n-1} \sigma_1 \exp\{-nh_1(\hat{W}_R)\}}{\Gamma(n) t^n \sigma_2 \exp\{-nh_2(\hat{W})\}} \varepsilon(R,n)$$

where

$$\varepsilon(R,n) = \frac{1 + \frac{15.(h_1^{(3)}(W_R))^2}{72.n.(h_1^{(2)}(W_R))^3}}{1 + \frac{15.(h_2^{(3)}(W))^2}{72.n.(h_2^{(2)}(W))^3}}. \tag{12}$$

Observe that for ε near 1, mean that (11) is appropriated for (8).

In section 4, we present a numerical illustration where we verify the precision of this approximation.

4-NUMERICAL ILLUSTRATION

To verify the results obtained by Laplace approximation, let's consider three simulated samples of size 5, 10 and 25 of the Weibull distribution with $p = 2$, $\theta = 4$ and $t = 2$.

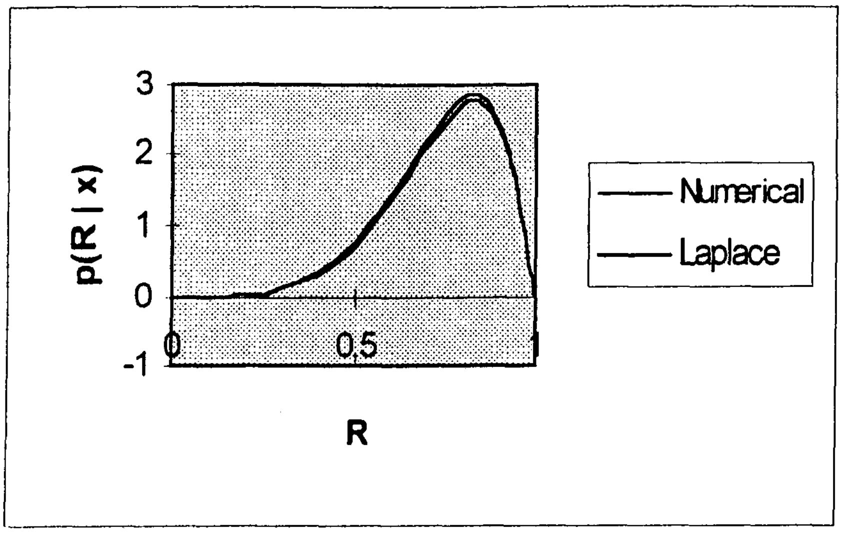

Figure 1: Posterior density of reliability for n = 5.

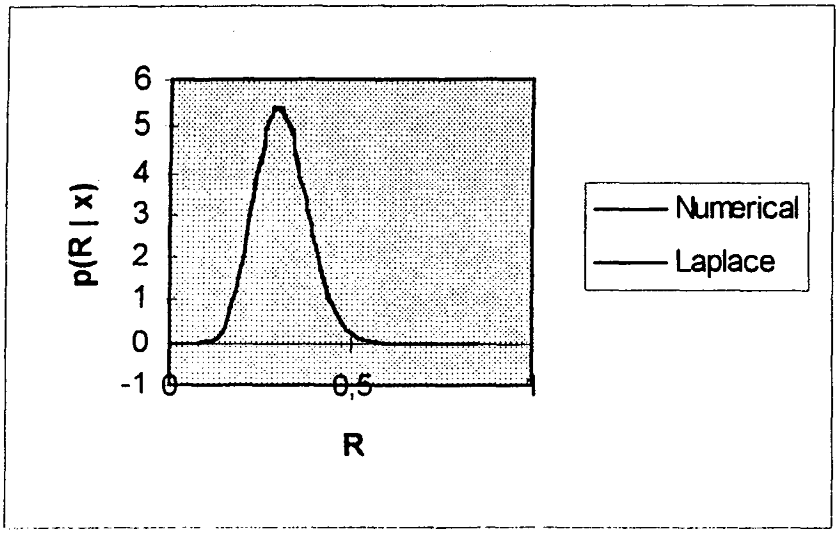

Figure 2: Posterior density of reliability for n = 25.

Table 1 shows the results for the posterior moments and MLE via Laplace Approximation. Also, a numerical integration using the software "Maple" is presented. The reliability function for $t = 2$ is given by $R = 0.3679$.

Table 1: Posterior moments and maximum-likelihood estimators (MLE)

n	E(R | x)		Mode		Var(R | x)		MLE
	Numerical	Laplace	Numerical	Laplace	Numerical	Laplace	
5	0.7386	0.7389	0.8300	0.8300	0.0222	0.0220	0.7531
10	0.3540	0.3545	0.3300	0.3300	0.0137	0.0137	0.3294
25	0.3085	0.3086	0.2970	0.2970	0.0054	0.0054	0.2974

In general, the statistician is very concerned about the precision of the Laplace approximation. So, Table 2 presents a numerical illustration of (12) for different values of R and n.

Table 2: Errors $|1-\epsilon|$ for the Laplace approximation for posterior of R.

R	n = 5	n = 10	n = 25
0.01	0.032880	0.020463	0.006349
0.05	0.054840	0.014053	0.005799
0.1	0.065306	0.009532	0.005355
0.2	0.075248	0.002669	0.004589
0.3	0.079676	0.003547	0.003799
0.4	0.081185	0.010027	0.002885
0.5	0.080529	0.017405	0.001742
0.6	0.077883	0.026467	0.000212
0.7	0.073049	0.038504	0.002000
0.8	0.065308	0.056035	0.005530
0.9	0.052422	0.084889	0.012030
0.95	0.041389	0.107874	0.017760
1	0.110197	0.142143	0.006635

Table 3: Errors $|1-\epsilon|$ for the posteriors moments.

Moments	n = 5	n = 10	n = 25
E(R | x)	0.01315981	0.03491625	0.00009
$E(R^2 \mid x)$	0.02266468	0.0346659	0.00019

Table 2 and 3 show very clearly that the Laplace Approximation is improved for large samples even for R closed to one.

5. CONCLUSION

For the data set considered in this study we obtain a useful Laplace approximation in the sense that we have a minimization problem instead of a numerical integration which in some situations could be hard to obtain. Also, it was verified that the Laplace approximation is very accurate for large sample sizes and stable with respect to R.

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RESUMO: Neste artigo, a distribuição a posteriori da função de confiabilidade do modelo Weibull é obtida usando a Aproximação de Laplace (Tierney and Kadane, 1986). Algumas comparações numéricas e a precisão da Aproximação de Laplace são consideradas.

UNITERMOS: Distribuição de Weibull, função de confiabilidade, Aproximação de Laplace, distribuição a posteriori, distribuição a priori.

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