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ELECTRODYNAMICS IN A BACKGROUND CHIRAL FIELD

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We consider the one-loop effective action in four-dimensional Euclidean space for a background chiral field coupled to a spinor field. It proves possible to find an exact expression for this action if the mass m of the spinor vanishes. If m does not vanish, one can make a perturbative expansion in powers of the axial field that contributes to the chiral field, while treating the contribution of the vector field exactly when it is a constant. The analogous problem in two dimensions is also discussed.

Keywords: Effective action; axial field; Schwinger expansion.

1. Introduction

Parity violating interactions with a spinor field yield several interesting consequences, among them an anomalous divergence in the axial current[1–3] and the absence of bound states in a "Coulomb" axial potential.[4,5] In this paper we consider the one-loop effective action for a spinor field in the presence of a constant background chiral vector field. The analogous situation in which the interaction is parity conserving is well known.[1,6–8]

2. Effective Action

If a spinor ψ is in the presence of a background vector field V^μ and a background axial field A^μ in four-dimensional Euclidean space, we have the Lagrangian

$$\mathcal{L} = \psi^\dagger[(\not{p} - \not{W}_+ P_+ - \not{W}_- P_-) - mt]\psi\,, \tag{1}$$

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where $p = -i\partial$ and $W_\pm = V \pm A$ are chiral fields. (The notation used is listed in the Appendix.) The effective action is then given by the one-loop expression

$$\Gamma_4 = \ln\det(\not{p} - \not{W}_+ P_+ - \not{W}_- P_- - m)\,. \tag{2}$$

We now rewrite Eq. (2) as

$$\Gamma_4 = \left[\ln\det(\not{p} - \not{W}_+ P_+ - \not{W}_- P_-) + \ln\det\left(1 - \frac{m}{\not{p} - \not{W}_+ P_+ - \not{W}_- P_-}\right)\right] \tag{3}$$

and then expand the second term in Eq. (3) so that

$$\ln\det\left(1 - \frac{m}{\not{p} - \not{W}_+ P_+ - \not{W}_- P_-}\right) = -\mathrm{tr}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{m}{\not{p} - \not{W}_+ P_+ - \not{W}_- P_-}\right)^n. \tag{4}$$

We now rewrite

$$\begin{aligned}\frac{1}{\not{p} - \not{W}_+ P_+ - \not{W}_- P_-} &= \frac{1}{\not{p}}\frac{1}{1 - \frac{1}{\not{p}}(\not{W}_+ P_+ + \not{W}_- P_-)}\\ &= \frac{1}{\not{p}}\sum_{n=0}^{\infty}\left[\frac{1}{\not{p}}(\not{W}_+ P_+ + \not{W}_- P_-)\right]^n\end{aligned}$$

which by the properties of the projection operators $P_\pm$ becomes

$$\begin{aligned}&= \frac{1}{\not{p}}\sum_{n=0}^{\infty}\left[\left(\frac{1}{\not{p}}\not{W}_+\right)^n P_+ + \left(\frac{1}{\not{p}}\not{W}_-\right)^n P_-\right]\\ &= \frac{1}{\not{p} - \not{W}_+}P_+ + \frac{1}{\not{p} - \not{W}_-}P_-\,.\end{aligned} \tag{5}$$

Similarly, we have for the first term in Eq. (3)

$$\begin{aligned}\ln\det(\not{p} - \not{W}_+ P_+ - \not{W}_- P_-) &= \mathrm{tr}\left[\ln\not{p} - \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{\not{p}}\not{W}_+ P_+ + \frac{1}{\not{p}}\not{W}_- P_-\right)^n\right]\\ &= \mathrm{tr}[(\ln(\not{p} - \not{W}_+))P_+ + (\ln(\not{p} - \not{W}_-))P_-]\,.\end{aligned} \tag{6}$$

Together, Eqs. (3)–(6) show that

$$\begin{aligned}\Gamma_4 = \mathrm{tr}\Bigg[&(\ln\not{\Pi}_+)P_+ + (\ln\not{\Pi}_-)P_- - \frac{m}{1}\left(\frac{1}{\not{\Pi}_+}P_+ + \frac{1}{\not{\Pi}_-}P_-\right)\\ &- \frac{m^2}{2}\left(\frac{1}{\not{\Pi}_-}\frac{1}{\not{\Pi}_+}P_+ + \frac{1}{\not{\Pi}_+}\frac{1}{\not{\Pi}_-}P_-\right)\\ &- \frac{m^3}{3}\left(\frac{1}{\not{\Pi}_+}\frac{1}{\not{\Pi}_-}\frac{1}{\not{\Pi}_+}P_+ + \frac{1}{\not{\Pi}_-}\frac{1}{\not{\Pi}_+}\frac{1}{\not{\Pi}_-}P_-\right) - \cdots\Bigg],\end{aligned} \tag{7}$$

where $\Pi_\pm \equiv p - W_\pm$.

If we now use the identity

$$\operatorname{tr} X = \frac{1}{2}\operatorname{tr}[X + \gamma^5 X \gamma^5]\,, \tag{8}$$

then we see that terms in Eq. (7) with odd powers of m vanish. This reduces Eq. (7) to

$$\Gamma_4 = \frac{1}{2}\operatorname{tr}\left\{\left[\ln\left(\not{\Pi}_+^2\left(1 - \frac{m^2}{\not{\Pi}_-\not{\Pi}_+}\right)\right)\right]P_+ + \left[\ln\left(\not{\Pi}_-^2\left(1 - \frac{m^2}{\not{\Pi}_+\not{\Pi}_-}\right)\right)\right]P_-\right\}. \tag{9}$$

Under "charge conjugation" we find that

$$C^{-1}(\not{p} - \not{W}_+ P_+ - \not{W}_- P_- - m)C = [\not{p} + \not{W}_+ P_- + \not{W}_- P_+ - m]^T \tag{10}$$

and so Eq. (2) is symmetric under the replacement $W_\pm \to -W_\mp$. (In Ref. 9 the fact that $p^{\mu T} = -p^\mu$ was ignored.)

3. Explicit Evaluation of the Effective Action

Evaluation of Γ in Eq. (9) in closed form when $m^2 \neq 0$ involves having to determine $\operatorname{tr}\ln(\not{\Pi}_\pm\not{\Pi}_\mp - m^2)$. If $\not{W}_\pm \neq \not{W}_\mp$ this is prohibitively difficult, even if $W_\pm = \pm A$. In this case we must consider

$$\operatorname{tr}\ln[(\not{p} \pm \not{A})(\not{p} \mp \not{A}) - m^2] = \operatorname{tr}\ln[(p^\mu \mp i\sigma^{\mu\nu}A^\nu)^2 + 2A^2 \pm iA^\lambda_{,\lambda} - m^2] \tag{11}$$

which, though it is well suited for a perturbative expansion in powers of A^μ,[10,11] does not lend itself to being evaluated even when A^μ corresponds to there being a constant field strength.

However, if $m^2 = 0$, or if Eq. (9) were expanded to some finite order in powers of m^2, then one is faced with evaluation of only $\frac{1}{2}(\Lambda_+ + \Lambda_-)$ where $\Lambda_\pm = \operatorname{tr}[\ln \not{\Pi}_\pm^2]P_\pm$. In Refs. 1, 6 and 7, it is shown that since $(\not{p} - \not{V})^2 = (p^\mu - V^\mu)^2 - \frac{1}{2}\sigma^{\mu\nu}F^{\mu\nu}$ ($F = \partial \wedge V$) the gamma matrix trace occurring in $\Lambda_\pm$ involves

$$\begin{aligned}\operatorname{tr} e^{\frac{1}{2}F^{\mu\nu}\sigma^{\mu\nu}t}P_\pm &= \operatorname{tr}\left\{\cosh K_- P_+ + \cosh K_+ P_-\right.\\ &\qquad \left. + \frac{t}{2}\sigma^{\mu\nu}F^{\mu\nu}\left(\frac{\sinh K_-}{K_-}P_+ + \frac{\sinh K_+}{K_+}P_-\right)\right\}P_\pm \\ &= 4\cosh K_\mp\,,\end{aligned} \tag{12}$$

where $K_\pm^2 = \frac{t^2}{2}[F^{\mu\nu}F^{\mu\nu} \pm F^{\mu\nu}F^{*\mu\nu}]$. We thus see that the presence of the chiral projection operator $P_\pm$ in Eq. (9) serves to eliminate the contribution of $\cosh K_\pm$ as well as $\sinh K_+$ and $\sinh K_-$, leaving only $4\cosh K_\mp$.

The background field strength $W_\pm$ in the gauge $x \cdot W_\pm = 0$ can be expanded in powers of the field strength $F_\pm$,[12–14]

$$W_{\pm}^{\mu} = \sum_{n=0}^{\infty} \frac{-1}{n!(n+2)} x^{\nu} x^{\lambda_1} \cdots x^{\lambda_n} F_{\pm}^{\mu\nu,\lambda_1\cdots\lambda_n}(0) . \tag{13}$$

The first term in Eq. (13) corresponds to a constant background field as discussed in Refs. 1, 6 and 7; higher contributions are dealt with in Refs. 8, 15–17. Other special background field configurations have been considered.[1,8,18–20]

If $m^2 = 0$ and $W_{\pm} = \pm A$, then we have a purely axial coupling and

$$\Gamma_A^{(0)} = \frac{1}{2}\,\mathrm{tr}[(\ln(\not{p} - \not{A})^2)P_+ + (\ln(\not{p} + \not{A})^2)P_-] . \tag{14}$$

If A^μ is in the gauge $x \cdot A = 0$ so that it is expressed in the form of Eq. (13) then gauge invariance is manifestly preserved since A^μ is expressed in terms of the field strength. If we then expand $\Gamma_A^{(0)}$ with this background field using the Schwinger expansion as in Refs. 1 and 21, then the three-point function $\langle AAA \rangle$ vanishes. However, again computing $\langle AAA \rangle$ but with plane wave background axial fields, the three-point function is consistent with the axial anomaly.[1–3]

If $m^2 \neq 0$ when $W_{\pm} = \pm A$ then Eq. (9) reduces to

$$\Gamma_A = \frac{1}{2}\,\mathrm{tr}\Big\{[\ln((\not{p} + \not{A})(\not{p} - \not{A}) - m^2)]P_+ + [\ln((\not{p} - \not{A})(\not{p} + \not{A}) - m^2)]P_- + \frac{1}{2}[\ln(\not{p} - \not{A})^2 - \ln(\not{p} + \not{A})^2]\gamma_5\Big\} . \tag{15}$$

There does not appear to be a way of evaluating this in closed form when even $A^\mu = -\frac{1}{2}F^{\mu\nu}x^\nu$ if $m^2 \neq 0$, though with this background field $\langle AAA \rangle = 0$. With a plane wave background field the axial anomaly can however be recovered[21] when $\langle AAA \rangle$ is computed by applying the Schwinger expansion[1] to Eq. (15).

Although it does not appear to be feasible to compute Γ_4 when there is a constant strength $\partial^\mu A^\nu - \partial^\nu A^\mu$ in Eq. (1), we can consider the case in which Γ_4 is restricted to being linear in the external axial field and the vector field is taken to be constant. In this case we begin by using Eq. (8) to write

$$\Gamma_4 = \frac{1}{2}\ln\det[(\not{p} - \not{V} - \not{A}\gamma^5)^2 - m^2] . \tag{16}$$

Dropping those terms in Eq. (12) that cannot contribute to the contribution to Γ_4 that are linear in A_μ, we see that upon letting $m^2 \to -m^2$,

$$\Gamma_4 \approx \frac{1}{2}\ln\det\Big[(p - V)^2 + m^2 - \frac{1}{2}F^{\mu\nu}\sigma^{\mu\nu} + iA^{\mu,\mu}\gamma^5 + i\sigma^{\mu\nu}\Big(2A^\mu p^\nu + \frac{i}{2}G^{\mu\nu} - 2A^\mu V^\nu\Big)\gamma^5\Big] , \tag{17}$$

where $F^{\mu\nu} = \partial^\mu V^\nu - \partial^\nu V^\mu$ and $G^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. If we now employ operator regularization to expand Γ_4 in Eq. (17) to the term linear in A_μ, we need the equations[21]

$$\frac{1}{2}\ln\det(H_0+H_1) = -\frac{1}{2}\frac{d}{ds}\bigg|_0 \operatorname{tr}\frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\, e^{-(H_0+H_1)t}$$

$$= -\frac{1}{2}\frac{d}{ds}\bigg|_0 \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\operatorname{tr}\bigg[e^{-H_0 t} + \frac{(-t)}{1}e^{-H_0 t}H_1$$

$$+ \frac{(-t)^2}{2}\int_0^1 du\, e^{-(1-u)H_0 t}H_1 e^{-uH_0 t}H_1 + \cdots\bigg]. \tag{18}$$

Upon using Eq. (18), Eq. (17) reduces to

$$\Gamma_4 \approx \frac{1}{2}\frac{d}{ds}\bigg|_0 \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^s \operatorname{tr} e^{-[(p-V)^2+m^2-\frac{1}{2}F^{\mu\nu}\sigma^{\mu\nu}]t}$$

$$\times\left[iA^{\mu}_{,\mu} + i\sigma^{\lambda\sigma}\left(2A^\lambda p^\sigma + \frac{i}{2}G^{\lambda\sigma} - 2A^\lambda V^\sigma\right)\right]\gamma^5. \tag{19}$$

If $F^{\mu\nu}$ is constant, then by Eqs. (A.1) and (A.2) this becomes

$$= \frac{1}{2}\frac{d}{ds}\bigg|_0 \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^s \operatorname{tr} e^{[(p-V)^2+m^2]t}\bigg[(\cosh K_-)P_+ + (\cosh K_+)P_-$$

$$+\left(\frac{\sinh K_-}{K_-}P_+ + \frac{\sinh K_+}{K_+}P_-\right)w^{\mu\nu}\sigma^{\mu\nu}\bigg]$$

$$\cdot\left[iA^{\lambda}_{,\lambda} + i\sigma^{\lambda\sigma}\left(2A^\lambda p^\sigma + \frac{i}{2}G^{\lambda\sigma} - 2A^\lambda V^\sigma\right)\right]\gamma_5\,, \tag{20}$$

where $w^{\mu\nu} = \frac{1}{2}F^{\mu\nu}t$ and $K_\pm^2 = 2(w^{\alpha\beta}w^{\alpha\beta} \pm w^{*\alpha\beta}w^{\alpha\beta})$.

Evaluating the γ-matrix traces in Eq. (20) leads to

$$= \frac{d}{ds}\bigg|_0 \frac{i}{\Gamma(s)}\int_0^\infty dt\, t^s \operatorname{tr} e^{-[(p-V)^2+m^2]t}\bigg\{(\cosh K_- - \cosh K_+)A^\lambda_{,\lambda}$$

$$+2\left[\left(\frac{\sinh K_-}{K_-} - \frac{\sinh K_+}{K_+}\right)w^{\lambda\sigma} - 2\left(\frac{\sinh K_-}{K_-} + \frac{\sinh K_+}{K_+}\right)w^{*\lambda\sigma}\right]$$

$$\times\left[2A^\lambda p^\sigma + \frac{i}{2}G^{\lambda\sigma} - 2A^\lambda V^\sigma\right]\bigg\}. \tag{21}$$

When $V^\mu = -\frac{1}{2}F^{\mu\nu}x^\nu$, the result of Schwinger[1] gives

$$\langle x|e^{-(p-V)^2 t}|y\rangle = \frac{i}{(4\pi t)^2}\exp\left(i\int_y^x dz\cdot V(z)\right)e^{-L(t)}$$

$$\times\exp\left(-\frac{1}{4}(x-y)\cdot F\cdot\cot(Ft)\cdot(x-y)\right) \tag{22}$$

can be used to compute the functional trace in Eq. (21). (Here we have $L(t) = \frac{1}{2}\mathrm{tr}\ln((Ft)^{-1}\sin(Ft))$.) In particular, it follows from Eq. (22) that

$$\begin{aligned}\mathrm{tr}\, e^{-(p-V)^2 t} A^\lambda p^\sigma &= \mathrm{tr}\int dz \langle x|e^{-(p-V)^2 t}|z\rangle i\partial_y^\sigma \langle z|A^\sigma|y\rangle \\ &= \int dx \int dy\, \delta(x-y) i\partial_y^\sigma \left[\frac{i}{(4\pi t)^2}\exp\left(i\int_y^x dz\, V(z)\right) e^{-L(t)}\right. \\ &\quad \left.\times \exp\left(-\frac{1}{4}(x-y)\cdot F\cdot\cot(Ft)\cdot(x-y)\right) A^\lambda(y)\right] \\ &= \frac{i}{(4\pi t)^2} e^{-L(t)} \int dx [V^\sigma(x) A^\lambda(x) + i\partial_x^\sigma A^\lambda(x)]\,. \end{aligned} \tag{23}$$

Substitution Eqs. (22) and (23) into Eq. (21) leads to

$$\begin{aligned}\Gamma_4 \approx \frac{-1}{(4\pi)^2}\frac{d}{ds}\bigg|_0 \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-2}\, e^{-L(t)-m^2 t}\int dx \bigg\{ (\cosh K_- - \cosh K_+) A^{\mu,\mu}(x) \\ - \frac{i}{2} G^{\lambda\sigma}(x) t \left[\left(\frac{\sinh K_-}{K_-} - \frac{\sinh K_+}{K_+}\right) F^{\lambda\sigma} - \left(\frac{\sinh K_-}{K_-} + \frac{\sinh K_+}{K_+}\right) F^{*\lambda\sigma}\right]\bigg\}\,. \end{aligned} \tag{24}$$

Expanding Eq. (24) to lowest order in $F^{\lambda\sigma}$ results in

$$\begin{aligned}\Gamma_4 &\approx \frac{1}{(4\pi)^2}\frac{d}{ds}\bigg|_0 \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-2}\, e^{-m^2 t}\int dx \left[\frac{1}{2} t^2 F^{\lambda\sigma} F^{*\lambda\sigma} A^{\mu,\mu}(x)\right. \\ &\quad \left. - itG^{\lambda\sigma}(x) F^{*\lambda\sigma}\right] \\ &= \frac{1}{(4\pi)^2}\int dx \left[\frac{1}{m^2} F^{\lambda\sigma} F^{*\lambda\sigma} A^{\mu,\mu} + i(\ln m^2) G^{\lambda\sigma} F^{*\lambda\sigma}\right]. \end{aligned} \tag{25}$$

Neither term in Eq. (25) would arise from the calculation of one-loop Feynman diagrams with plane wave external fields. For $F_{\mu\nu}$ being a constant field, the first term in Eq. (25) is a total derivative. When either F or G (or both) are non-constant, the second term is also a total derivative.

4. The Two-Dimensional Limit

The two-dimensional limit of massive electrodynamics has been considered in Refs. 22 and 23. If there is an axial coupling between the spinor and an external axial field, this leads to the one-loop effective action

$$\Gamma_2 = \ln\det(\not{p} - \not{A}\sigma^3 - m)\,, \qquad p \equiv -i\partial\,. \tag{26}$$

However, as $\gamma^\mu \sigma^3 = \epsilon^{\mu\nu}\gamma_\nu$, this becomes

$$\Gamma_2 = \ln\det(\not{p} - A_\mu \epsilon^{\mu\nu}\gamma_\nu - m)\,. \tag{27}$$

Consequently, if the background field A_μ corresponds to a constant field strength $A_\mu = -\frac{1}{2}F_{\mu\nu}x^\nu = -\frac{f}{2}\epsilon_{\mu\nu}x^\nu$, then Eq. (27) reduces to

$$\Gamma_2 = \ln\det\left(\not{p} - \frac{f}{2}\not{x} - m\right) \tag{28}$$

which is what would be obtained if there were a parity conserving coupling with an external vector field $V_\mu = \frac{1}{4}f\partial_\mu(x^2)$ which corresponds to a pure gauge field. This effective action should thus be independent of f, which we will show explicitly by using Schwinger's technique.[1]

If now

$$\Pi_\mu = p_\mu - \frac{f}{2}x_\mu\,, \tag{29}$$

then Eq. (28) becomes

$$\Gamma_2 = \ln\det{}^{1/2}(\not{\Pi} + m)(\not{\Pi} - m) = \frac{1}{2}\ln\det(\Pi^2 - m^2) \tag{30}$$

upon using the two-dimensional analogue of Eq. (8) and

$$[\Pi_\mu, \Pi_\nu] = 0\,. \tag{31}$$

Regulating Γ_2 using the ζ-function[24,25] we have

$$\Gamma_2 = -\frac{1}{2}\frac{d}{ds}\bigg|_0 \frac{1}{\Gamma(s)}\,\mathrm{tr}\int_0^\infty d\,it(it)^{s-1}\,e^{i(m^2-\Pi^2)t}\,. \tag{32}$$

To evaluate the functional trace in Eq. (32), we use the Hamiltonian approach of Ref. 1, defining

$$\langle x(t)|y(0)\rangle = \langle x|e^{-iHt}|y\rangle \tag{33}$$

with

$$H = -\Pi^2\,. \tag{34}$$

The equations

$$i\frac{\partial \Pi^\mu(t)}{\partial t} = [\Pi^\mu(t), H]\,, \tag{35a}$$

$$i\frac{\partial x^\mu}{\partial t} = [x^\mu(t), H]\,, \tag{35b}$$

can be integrated to give

$$\Pi^\mu(t) = \Pi^\mu(0)\,, \tag{36a}$$

$$x^\mu(t) = -2\Pi_\mu(0)\,. \tag{36b}$$

Since Eq. (36) is identical to the equations that arise if $f = 0$, we see that the effective action in two dimensions for a spinor in the presence of a constant background axial field is just that of a free field.

5. Conclusions

We thus see that the one-loop effective action for a spinor in the presence of a constant background chiral field is closely related to that of considered in Refs. 1, 6–8 provided $m^2 = 0$. The case in which $m^2 \neq 0$ in four dimensions has not as yet been given in closed form. Higher order calculations, or those involving non-constant background fields are currently being considered, as is that all-orders approach in the presence of a weak background field.[26,27]

We note the use of projection operators in conjunction with background gauge fields in Ref. 28.

Appendix

In four-dimensional Euclidean space we have the conventions

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}\,, \qquad [\gamma^\mu, \gamma^\nu] = 2i\sigma^{\mu\nu}\,,$$

$$[\sigma^{\mu\nu}, \sigma^{\lambda\sigma}] = 2i(\delta^{\mu\lambda}\sigma^{\nu\sigma} - \delta^{\mu\sigma}\sigma^{\nu\lambda} + \delta^{\nu\sigma}\sigma^{\mu\lambda} - \delta^{\nu\lambda}\sigma^{\mu\sigma})\,,$$

$$\{\sigma^{\mu\nu}, \sigma^{\lambda\sigma}\} = 2(\delta^{\mu\lambda}\delta^{\nu\sigma} - \delta^{\mu\sigma}\delta^{\nu\lambda}) - 2\epsilon^{\mu\nu\lambda\sigma}\gamma^5\,,$$

$$\gamma^\alpha\gamma^\beta\gamma^\lambda = \delta^{\alpha\beta}\gamma^\lambda - \delta^{\alpha\lambda}\gamma^\beta + \delta^{\beta\lambda}\gamma^\alpha - \epsilon^{\alpha\beta\lambda\rho}\gamma^\rho\gamma^5\,,$$

$$\epsilon^{1234} = 1\,, \qquad \gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4\,, \qquad \mathrm{tr}\,\gamma^5 = 0\,,$$

$$\sigma^{\mu\nu}\gamma^5 = \epsilon^{\mu\nu\lambda\sigma}\sigma^{\lambda\sigma}\,.$$

$$P_\pm = \frac{1 \pm \gamma^5}{2}\,, \qquad (P_\pm)^2 = P_\pm\,, \qquad P_\pm P_\mp = 0\,,$$

$$P_\pm\gamma^\mu = \gamma^\mu P_\mp\,, \qquad P_\pm\gamma^5 = \gamma^5 P_\pm\,.$$

These show that if

$$e^{\lambda w^{\mu\nu}\sigma^{\mu\nu}} = (A_+(\lambda)P_+ + A_-(\lambda)P_-) + (B_+(\lambda)P_+ + B_-(\lambda)P_-)w^{\mu\nu}\sigma^{\mu\nu}\,, \quad \text{(A.1)}$$

then the differential equation

$$\frac{d}{d\lambda}e^{\lambda w^{\mu\nu}\sigma^{\mu\nu}} = w^{\mu\nu}\sigma^{\mu\nu}e^{\lambda\sigma^{\mu\nu}w^{\mu\nu}}$$

leads to

$$\dot{A}_\pm = K_\mp^2 B_\pm\,, \qquad \dot{B}_\pm = A_\pm \qquad (A_\pm(0) = 1,\, B_\pm(0) = 0)\,,$$

where $K_\pm^2 = 2(w^{\mu\nu}w^{\mu\nu} \pm w^{\mu\nu}w^{*\mu\nu})$ and $w^{*\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma}w^{\lambda\sigma}$. These have the solution when $\lambda = 1$

$$A_\pm = \cosh K_\mp\,, \qquad B_\pm = \frac{\sinh K_\mp}{K_\mp}\,. \quad \text{(A.2)}$$

The "charge conjugation" matrix C satisfies $C^{-1}\gamma^\mu C = -\gamma^{\mu T}, C^{-1}\gamma^5 C = \gamma^{5T}$.

In two-dimensional Minkowski space, we take

$$g^{00} = 1 = -g^{11} \quad \text{and} \quad \gamma^0 = \sigma^1\,, \quad \gamma^1 = i\sigma^2 \quad \text{so that}$$

$$\text{if} \quad \epsilon^{01} = 1 = \epsilon_{10}\,,\ \text{then}\ \gamma^\mu\gamma^\nu = g^{\mu\nu} - \epsilon^{\mu\nu}\sigma^3 \quad \text{and} \quad \gamma^\mu\sigma^3 = \epsilon^{\mu\nu}\gamma_\nu\,,$$

(where σ^i is a Pauli spin matrix).

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