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A CONSTRUCTIVE EXAMPLE FOR ACTIVE REDUNDANCY ALLOCATION IN A K-OUT-OF-N:F SYSTEM UNDER DEPENDENCE CONDITIONS.

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Abstract Bueno and Carmo (2004) considered active redundancy under dependence conditions and analyzed component redundancy allocation in a k out-of-n:F system through compensator transform. In this paper we give a constructive example for that situation.

Keywords: Reliability, martingales, k-out-of-n:F system, active redundancy, reverse rule of order 2.

1.Introduction

The problem of where to allocate a redundant component in a system in order to optimize system lifetime is important in reliability theory.

A common form of redundancy in reliability theory is active redundancy where a spare is put in parallel with the original component at initial time. The problem of where to allocate a spare component, is addressed in Boland et al.(1992) for either active or standby redundancy, in a k-out-of-n:G system of independent components in order to optimize improvement of system reliability.

For a k-out-of-n:G system, Bolland et al.(1992) consider stochastic ordering to shows that for active redundancy it is stochastically optimal always to allocate a spare to the weakest component.

Bueno and Carmo (2004) get a compensator characterization of active redundancy and analyzes the best way to allocate a redundant component in order to optimize the reliability of a k-out-of-n: F system lifetime of dependent components.

The results are interesting by itself and, in this paper, we provide a constructive example for that situation. In this note in Section 2 we define an active redundancy under compensator transform and in Section 3 we formulate a constructive example using the concept of increasing failure rate relative to a family of σ-algebras.

2. Active redundancy in a k-out-of-n:F system

As in Arjas (1981b) we consider a point process formulation of a coherent system. Let the vector $\mathbf{S} = (S_1, S_2, ..., S_n)$ be n component lifetimes which are positive random variables in a complete probability space $(\Omega, \Im, P)$. We assume that the lifetimes can be dependent but simultaneous failures are ruled out.

As introduced in Arjas (1981b), we associate with $\mathbf{S}$ the corresponding failure process, which is defined as follows. Let

$$T_0 = 0, \quad J_0 = \emptyset;$$

$$T_n = \inf\{S_i : 1 \leq i \leq n, S_i > T_{n-1}\};$$

$$J_n = \{i : S_i = T_n\}, \quad n \geq 1;$$

$$N_i(t) = 1_{\{S_i \leq t\}}, \quad t \geq 0, \quad \emptyset \neq i \in \{1, ..., n\};$$

$$\Im_t = \sigma\{N_i(s) : s \leq t, \emptyset \neq i \in \{1, ..., n\}\}.$$

$(\Im_t)_{t \geq 0}$ represents our observations in time and is a family of sub-σ-algebras of $\Im$ which is increasing, right continuous and completed (shortly: satisfies Dellacherie's usual conditions). The marked point process (T_n, J_n) is called the failure process associated to **S**. For each i, let $A_i(t)$ be the $\Im_t$-compensator of the counting process $N_i(t)$. The compensator process is expressed in terms of the conditional probability, given the available information and generalize the classical notion of hazards. Intuitively, this corresponds to producing whether the failure is going to occur now, on the basis of all observations available up to, but not including, the present.We assume that the component's lifetimes are totally inaccessible stopping time and therefore, the $\Im_t$-compensator are absolutely continuous.

An active redundancy of a component's lifetime S_i is the lifetime $S_i \vee S$ where S is independent and identically distributed as S_i and is called spare lifetime .

If the distribution function of S_i is $F_i(t) = 1 - \overline{F}_i(t)$, the resulting lifetime $S_i \vee S$, from an active redundancy operation of component i, has a distribution function $P(S_i \vee S \leq t) = 1 - P(S_i \vee S > t)$, where

$$P(S_i \vee S > t) = 2P(S_i > t) - P(S_i > t)^2 = 2\overline{F}_i(t) - \overline{F}_i(t)^2.$$

In the case of independent components, $A_i(t) = -\ln P(S_i > t | \Im_t)$ on $\{t < S_i\}$, is the $\Im_t$-compensator of component i and we can write $P(S_i \vee S > t | \Im_t) = \exp[-A_i(t)](2 -$

$\exp[-A_i(t)])$ and

$$-\ln P(S_i + S > t|\Im_t) = A_i(t) - \ln(2 - \exp[-A_i(t)]),$$

on $\{t < S_i \vee S\}$, is the $\Im_t$-compensator of $S_i \vee S$.

In the general case of dependent components, we define

Definition 2.1. An active redundancy of a component's lifetime S_i, with $\Im_t$-compensator process $A_i(t)$ is the lifetime corresponding to the compensator transforms

$$B_j(t) = \int_0^t \alpha_j(s) dA_j(s) \quad where \quad \alpha_i(s) = \frac{2 - 2\exp[-A_i(s)]}{2 - \exp[-A_i(s)]} \quad and \quad \alpha_j(s) = 1 \quad for \quad j \neq i$$

Bueno and Carmo (2004) proves that there exist a probability measure Q , such that, under Q, $B_j(t)$ becomes the $\Im_t$-compensator of $N_j(t)$.

We are concerned with the problem of where to allocate a spare component using active redundancy in a k-out-of-n:F system in order to optimize system reliability improvement under dependence conditions. A k-out-of-n:F system functions if and only if at least $n-k+1$ out of the n components functions. We denote the lifetime of a k-out-of-n:F system by $\tau_{k:F}(\mathbf{S}) = T_k$, where $\mathbf{S} = (S_1, ..., S_n)$ is the random vector of component lifetimes and T_k is the k-th order statistics of S_i, $1 \leq i \leq n$. We denote the system lifetime resulting from an active redundancy operation of component i by $\tau^i_{k:F} = \tau_{k:F}(S_1, ..., S_{i-1}, S_i \vee S, S_{i+1}, ..., S_n)$. We count this system failure through $N^i(t) = 1_{\{\tau^i_{k:F} \leq t\}}$, a counting process with $\Im_t$-compensator $A^i(t)$, $1 \leq i \leq n$.

Observing the component lifetimes after its critical levels, Bueno and Carmo (2004) proves the following result:

Theorem 2.2. The transformation $K(j,t) = 2 - \exp[-A_j(t)], \quad 1 \le j \le n, \quad t \in [0,\infty)$ is reverse rule of order 2 if and only if $N^1(t) \le^{st} N^2(t) \le^{st} ... \le^{st} N^n(t)$.

We note that we are using a theorem from Kwiecinski and Szekli (1991) concerning the $\Im_t$-compensators ordering.

As $A_i(0) = 0$ for all i, if $2 - \exp[-A_j(t)]$ are RR_2 we have $A_i(t) \ge A_j(t)$ for all $i \le j$, in the sense that the propensity for failure of component i is greater than the propensity for failure of component j and we consider component i weaker than component j. Under Theorem 2.2, we understand that it is optimal to perform active redundancy on the weakest component.

As a consequence, in the case of independent components and under the assumption that $2 - \exp[-A_j(t)]$ are RR_2 we can prove that $A_i(t) \ge A_j(t)$ and $S_i \le_{st} S_j$ for all $i \le j$. Its related with the result from Bolland (1992):

Corollary 2.3. Let $\{S_1, ..., S_n\}$ be the stochastically independent components lifetime of a k-out-of-n:G system where $S_1 \le^{st} S_2 \le^{st} ... \le^{st} S_n$. Then $\tau^1_{k:F} \ge^{st} \tau^2_{k:F} \ge^{st} ... \ge^{st} \tau^n_{k:F}$ for $k = 1, ..., n$.

Example 2.4. If the i-th component lifetime S_i in a k-out-of-n:G system has a Gamma distribution with parameters λ and i, $\lambda > 0$ and $i = 1, ..., n$ then $S_1 \le^{st} S_2 \le^{st} ... \le^{st} S_n$, and we choose the first ($i = 1$) component. In this case the first component lifetime increases from S_1 to $S_1 \vee S$ where S is the spare lifetime.

3.A constructive example

We propose to find lifetimes S_i^*, $i = 1, ..., n$, with $\Im_t$-compensators $A_i^*(t)$ of $1_{\{S_i^* \leq t\}}$ such that $2 - \exp[-A_j^*(t)]$ has the, *a.s.*, RR_2 property. We consider, as in Arjas (1981a), a lifetime S_i (or its distribution)which is increasing failure rate relative to $\Im_t$. However, as S_i is $\Im_t$-measurable, $P(S_i > t|\Im_t) = 1_{\{T_i > t\}}$ and it is not suitable for our proposal.

Then we consider a lifetime which is increasing failure rate relative to $\Im_t^i$,where

$$\Im_t^i = \sigma\{1_{\{S_j > s\}}, s \leq t, j = 1, ..., n, j \neq i\},$$

shortly S_i is $(IFR|\Im_t^i)$, which means that

$$P((S - t)^+ > s|\Im_t^i) \ \downarrow t \ \ a.s..$$

Clearly we also have

$$\overline{F}_i(t) = P(S_i > t|\Im_t^i) \ \downarrow t \ \ a.s.$$

Let $M_i(t)$ be the cadlag version of the counting process

$$M_i(t) = E[1_{\{S_i \leq t\}}|\Im_t^i].$$

Follows that $M_i(t)$ is a $\Im_t^i$-submartingale with $\Im_t^i$-compensator $C_i(t)$. From Arjas (1981b), if S_i is $(IFR|\Im_t^i)$, $C_i(t)$ is *a.s.* convex on $(0, S_i]$. Now, as $M_i(t) - C_i(t)$ is a $\Im_t^i$-martingale, for $s < t$, we have

$$E[P(S_i \leq t|\Im_t^i) - P(S_i \leq s|\Im_s^i)|\Im_s^i] = E[C_i(t) - C_i(s)|\Im_s^i]$$

and therefore follows from the monotone convergence theorem that

$$\lim_{t \to s} E[P(S_i \leq t|\Im_t^i) - P(S_i \leq s|\Im_s^i)|\Im_s^i] = 0$$

that is

$$\lim_{t\to s}\int_B [P(S_i \le t|\Im_t^i) - P(S_i \le s|\Im_s^i)]dP = 0$$

for all $B \in \Im_t^i$.

As $P(S_i > t|\Im_t) \downarrow t$ *a.s.*, $P(S_i \le t|\Im_t^i)$ is left continuous, *a.s.* and therefore $\Im_t^i$-predictable. Follows that $C_i(t) = P(S_i \le t|\Im_t^i)$ *a.s.*.

We now turn to the $\Im_t$-compensator of $N_i(t) = 1_{\{S_i \le t\}}$ where

$$\Im_t = \Im_t^i \vee \sigma\{1_{\{S_i > s\}}, | s \le t\}.$$

Yashin and Arjas (1988) proves that the $\Im_t$-compensator of $N_i(t)$, $A_i(t)$ is given by

$$A_i(t) = \int_0^t \frac{1_{\{S_i > s\}} dC_i(s)}{\overline{F}_i(s-)} = -\ln(\overline{F}_i(t \wedge S_i))$$

We are looking for a $\Im_t$-compensator of $N_i(t)$, $A_i^*(t)$, a transformation of $A_i(t)$ such that $2 - \exp[-A_j^*(t)]$ has the RR_2 property *a.s.*. As S_i is $(IFR|\Im_t^i)$, we can choose conveniently, $\overline{F}_i(t)$ as a PF_2 function, (a TP_2 function under shift) such that $2 - \exp[-A_j^*(t)]$) has the RR_2 property *a.s.*. We propose the compensator transform

$$A_i^*(t) = \int_0^t \frac{1}{2\exp[-A_i(s)] - 1} dA_i(s)$$

in a suitable domain where it is well defined.

Now we let

$$L_i(t) = (\exp[A_i(S_i)])^{N(t)} \exp[A_i(t) - A_i^*(t)] =$$

$$1 - \int_0^t \exp[-\int_0^s \frac{(2\exp[-A_i(u)] - 2)}{(2\exp[-A_i(u)] - 1)} dA_i(u)][\frac{(2\exp[-A_i(s)] - 2)}{(2\exp[-A_i(s)] - 1)}] d[N_i(s) - A_i(s)]$$

$L_i(t)$ is a local martingale since that $N_i(s) - A_i(s)$ is an $\Im_t$-martingale and the integrand is $\Im_t$-predictable. We suppose that $L_i(t)$ is uniformly integrable.

However, $E[L_i(t)] = 1$, $L_i(t)$ can be considered as a density function and we can define a measure Q_i by the Radon Nikodyn derivative $\frac{dQ_i}{dP} = L_i(S_i)$. Therefore, applying Girsanov Theorem (Bremaud (1981)) we have that $A_i^*(t)$ is the $\Im_t$-compensator of $N_i(t)$ under the measure Q_i.

Follows that

$$2 - \exp([-A_j^*(t)]) = 2 - \exp[\int_0^t \frac{1}{2\exp[-A_i(s)] - 1} dA_i(s)] = \exp[A_i(t)] = \frac{1}{\overline{F}_i(t)}$$

which has the RR_2 property $a.s.$.

We define the finite component lifetimes S_i^* by :

$$Q(S_i^* > t|\Im_t) = \exp[-A_i^*(t)], \quad 1 \leq i \leq n$$

on $\{t < S_i^*\}$.

To give a practical example we consider the ordered lifetimes with a conditional survival function given by:

$$\overline{F}(t_i|t_1, t_2, ...t_{i-1}) = \exp[-(\frac{t_i - \eta_i}{\theta})^\beta + (\frac{t_{i-1} - \eta_i}{\theta})^\beta]$$

for $\eta_i \vee t_{i-1} < t_i$, where t_i are the ordered observations and density functions :

$$f(t_1, t_2, ..., t_n) = \prod_{i=1}^{n} f(t_i|t_1, t_2, ...t_{i-1}) = (\frac{\beta}{\theta})(\frac{t_1 - \eta_1}{\theta})^{\beta-1} \exp[(\frac{t_1 - \eta_1}{\theta})^\beta]$$

$$\prod_{i=2}^{n} (\frac{\beta}{\theta})(\frac{t_i - \eta_i}{\theta})^{\beta-1} \exp[-(\frac{t_i - \eta_i}{\theta})^\beta + (\frac{t_{i-1} - \eta_i}{\theta})^\beta].$$

Follows that

$$dA_i(t|t_1,t_2,...t_{i-1}) = \frac{f(t|t_1,t_2,...t_{i-1})}{\overline{F}(|t_1,t_2,...t_{i-1})} = (\frac{\beta}{\theta})(\frac{t-\eta_i}{\theta})^{\beta-1}, \quad T_{i-1} \le t < T_i, \quad T_0 = 0.$$

In particular we can take $\beta = 2$ in which case S_i is a conveniently $(IFR|\Im_t^i)$ such that $\overline{F}(t_i|t_1,t_2,...t_{i-1})$ is TP_2 (under shift) on t_i and η_i.

Therefore

$$A_i(t) = \frac{2}{\theta^2}\int_{\eta_i}^{t}(s-\eta_i)ds = (\frac{t-\eta_i}{\theta})^2, \quad t > \eta_i,, \quad T_{i-1} \le t < T_i, \quad T_0 = 0.$$

and

$$A_i^*(t) = \int_0^t \frac{1}{2\exp[-A_i(s)]-1} dA_i(s) = -\ln(2-\exp[(\frac{t-\eta_i}{\theta})^2]), \quad t > \eta_i, \quad T_{i-1} \le t < T_i.$$

Follows that we can define the component life times S_i^* by :

$$Q(S_i^* > t|\Im_t) = 2 - \exp[(\frac{t-\eta_i}{\theta})^2], \quad \eta_i + 0,83\theta > t \ge \eta_i, \quad i = 1,...,n.$$

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