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ON THE STABILITY OF THE DIFFERENTIAL PROCESS GENERATED BY COMPLEX INTERPOLATION

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Abstract We study the stability of the differential process of Rochberg and Weiss associated with an analytic family of Banach spaces obtained using the complex interpolation method for families. In the context of Köthe function spaces, we complete earlier results of Kalton (who showed that there is global bounded stability for pairs of Köthe spaces) by showing that there is global (bounded) stability for families of up to three Köthe spaces distributed in arcs on the unit circle while there is no (bounded) stability for families of four or more Köthe spaces. In the context of arbitrary pairs of Banach spaces, we present some local stability results and some global isometric stability results.

Keywords: stability of splitting; twisted sums of Banach spaces; complex interpolation

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Contents

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1. Introduction

Stability problems associated with interpolation processes have been a central topic in the theory since its inception. Stability issues about the differential process associated with an analytic family of Banach spaces have also been considered since the seminal work of Rochberg and Weiss [34]. In this paper, we will start with an interpolation family $(X_\omega)_{\omega\in\partial U}$ on the border of an open subset U of $\mathbb{C}$ conformally equivalent to the open unit disc $\mathbb{D}$, and we will study different stability problems connected to the analytic family of Banach spaces $(X_z)_{z\in U}$ obtained by the complex interpolation methods of [12] and [22].

Analytic families of Banach spaces are relevant to other topics in Banach space theory such as the construction of uniformly convex hereditarily indecomposable spaces [19], the study of θ-Hilbertian spaces [30] introduced by Pisier, or problems about the uniform structure of Banach spaces. Recall that the question of whether the unit sphere of a uniformly convex space is uniformly homeomorphic to the unit sphere of a Hilbert space can be positively answered for Köthe spaces using interpolation methods: if X_0 and X_1 are uniformly convex spaces, then the unit spheres of X_θ and X_ν are uniformly homeomorphic for $0 < \theta, \nu < 1$ by a result of Daher [16]; this fact, together with an extrapolation theorem of Pisier [30], implies that the unit sphere of a uniformly convex Köthe space is uniformly homeomorphic to the unit sphere of the Hilbert space (see also [11]). Thus, an extrapolation theorem for arbitrary uniformly convex spaces would provide a positive answer to the problem.

Starting with $(X_\omega)_{\omega\in\partial U}$, a complex interpolation method constructs a Banach space $\mathcal{F}$ of analytic functions on U with values in a Banach space Σ and, for each $z \in U$, defines a Banach space $X_z = \{f(z) : f \in \mathcal{F}\}$, endowed with the quotient norm in $\mathcal{F}/\ker\delta_z$, where $\delta_z : \mathcal{F} \to \Sigma$ denotes the continuous evaluation map at $z \in U$, and we get the analytic family $(X_z)_{z\in U}$. An important case is the complex method described in [4], in which U is the unit strip $\mathbb{S} = \{z \in \mathbb{C} : 0 < \mathrm{Re} z < 1\}$ and the starting family is just an interpolation pair (X_0, X_1) of Banach spaces. In this case, $X_z = X_{\mathrm{Re}\, z}$; so it is usual to consider only the scale $(X_\theta)_{0<\theta<1}$.

An analytic family $(X_z)_{z\in U}$ of Banach spaces obtained by interpolation generates a differential process $(\Omega_z)_{z\in U}$, where Ω_z is a certain nonlinear map defined on X_z – see formula (3) – which is called the derivation at z. In the context of Köthe spaces, derivations are centralizers in the sense of [21, 22], and therefore can be used in a standard way to generate twisted sums

$$0 \longrightarrow X_z \longrightarrow d_{\Omega_z} X_z \longrightarrow X_z \longrightarrow 0. \tag{1}$$

Rochberg's approach [33], however, contemplates the formation of the so-called derived spaces $dX_z = \{(f'(z), f(z)) : f \in \mathcal{F}\}$ endowed with the obvious quotient norm, and shows that both constructions are isomorphic; i.e., $dX_z \sim d_{\Omega_z} X_z$.

The stability of the differential process associated with an analytic family $(X_z)_{z\in U}$ can be studied at several levels. At the basic level, one considers the stability of isomorphic properties of the spaces X_z either under small perturbations in the parameter z (local stability) or for the whole range of the parameter (global stability). Results of this kind have been obtained by many authors. Let us mention one obtained by Kalton and Ostrovskii [25]: If $d_K(A, B)$ denotes the Kadets distance between two Banach spaces A and B, a property $\mathcal{P}$ is said to be open if for every X having $\mathcal{P}$, there exists $C_X > 0$ such that Y has $\mathcal{P}$ when $d_K(X, Y) < C_X$, while $\mathcal{P}$ is said to be stable if there exists $C > 0$ such that if X has $\mathcal{P}$ and $d_K(X, Y) < C$ then Y has $\mathcal{P}$. Examples of open and stable properties can be found in [1] and [25, § 5]. Kalton and Ostrovskii showed [25, Theorem 4.5] that $d_K(X_t, X_s) \leqslant 2\mathsf{h}(t, s)$, where h is the pseudo-hyperbolic distance on U (see Definition 3.8). Thus, at its basic level, the differential process has local stability with respect to open properties and global stability with respect to stable properties.

At the first level, we consider stability problems for the family $(dX_z)_{z\in U}$ of derived spaces. We will also consider stability problems at level n, i.e., for the families of higher order Rochberg's derived spaces [33] $d^n X_z = \{(\frac{1}{n!} f^{(n)}(z), \dots, f(z)) : f \in \mathcal{F}\}$, which can also be interpreted as twisted sums [5]. As a typical result, we will show a generalized form for the Kalton–Ostrovskii result mentioned before: $d_K(d^n X_z, d^n X_\eta) \leqslant 4(n+1)\mathsf{h}(z, \eta)$, which implies local/global stability for open/stable properties of $d^n X_z$; see Theorem 3.11.

The interpretation of the derived spaces dX_z as twisted sums generated by Ω_z allows one to study the stability of dX_z in terms of the stability of Ω_z (see Definition 4.1), which is what we will mainly do in the paper. Recall that an exact sequence like (1) *splits* when X_z is complemented in dX_z; this happens if and only if Ω_z can be written as the sum of a bounded homogeneous map plus a linear map, and in this case we say that Ω_z is *trivial*. Thus, two derivations Ω_z and Ω'_z are *equivalent* when $\Omega_z - \Omega'_z$ is trivial. Kalton's approach to complex interpolation instead relies on the notion of bounded equivalence: two derivations Ω_z and Ω'_z are said to be boundedly equivalent when $\Omega_z - \Omega'_z$ is bounded.

Probably the first stability results at level one have been those obtained by Cwikel, Jawerth, Milman, and Rochberg [14] for the minimal $(\theta, 1)$-interpolation method applied to an interpolation pair (X_0, X_1). They reinterpret the results of Zafran [36] to show that whenever Ω_θ is bounded for some $0 < \theta < 1$, then all Ω_z are bounded and $X_0 = X_1$ up to a renorming. Kalton obtains in [22] a similar stability result in the context of complex interpolation for pairs of Köthe function spaces: Ω_θ is bounded for some $\theta \in \mathbb{S}$ if and only if Ω_z is bounded for all $z \in \mathbb{S}$ and, in this case, $X_0 = X_1$ up to an equivalent renorming. See Theorem 3.4 for the precise statement.

Kalton's result leaves several questions unanswered, and a good part of this paper is devoted to solving them. We complete Kalton's result by showing: (i) for complex interpolation of pairs (X_0, X_1) of superreflexive Köthe spaces, Ω_θ is trivial if and only if there is a weight w so that $X_0 = X_1(w)$ up to an equivalent renorming, solving the stability problem for splitting for pairs of Köthe spaces; see Theorem 4.4. (ii) The stability results for pairs remain valid for families of up to three Köthe spaces distributed in three arcs of the unit circle $\mathbb{T}$ (see Theorem 4.15) but fail for families of four Köthe spaces (Corollary 4.21), marking the limit of validity for Kalton's theorems. We also give other examples in which stability fails. Related with the results we have just mentioned, Qiu [31] showed that, at the basic level, complex interpolation for families is stable under rearrangements of two spaces, but it is not stable under rearrangements of three spaces. But note that Qiu's results only consider finite-dimensional spaces, while the nonstability we study concerns isomorphic properties.

Regarding Köthe spaces, Theorem 4.22 presents the 1-level interpretation of the classical reiteration result for families of Coifman, Cwikel, Rochberg, Sagher, and Weiss [12, Theorem 5.1]. This result explains, to some extent, the lack of stability in the previous counterexamples and can be used to obtain other natural counterexamples. In the construction of some counterexamples, we use an analogue of Rochberg's concept of flat analytic family [32]. Let $\|\cdot\|$ be a norm on $\mathbb{C}^n$ and let $(T_z)_{z\in\mathbb{D}}$ be a family of bijective linear maps on $\mathbb{C}^n$ depending analytically on z in $\mathbb{D}$. Define $\|x\|_z = \|T_z^{-1}x\|$. Then $(\mathbb{C}^n, \|\cdot\|_z)_{z\in\mathbb{D}}$ is called a *flat analytic family* on $\mathbb{D}$. Proposition 4.17 shows the existence of a nonconstant flat analytic family of Köthe sequence spaces with norms $\|x\|_z = \|e^{-D(z)}x\|_2$ $(z \in \mathbb{D})$ generated by an analytic family $D(z)$ of diagonal operators for which the derivation map Ω_z is linear and does not depend on z.

The existence of local or global stability for the differential process associated with complex interpolation for a couple of arbitrary Banach spaces remains still an open problem: *Assume* (X_0, X_1) *is a pair of Banach spaces such that* Ω_θ *is bounded for some* $0 < \theta < 1$. *Does it follow that* $X_0 = X_1$ *up to equivalence of norms?*

In § 5, we present some isometric stability results valid for couples of Banach spaces. A key role in our analysis is played by the properties of the extremal functions and by some differential estimates for the norm in an interpolation scale. In [14, Theorem 5.2], Cwikel et al. obtained the estimate

$$\frac{d}{d\theta}\|a\|_{\theta,1} \sim \|a\|_{\theta,1} + \|\Omega_\theta a\|_{\theta,1}$$

for the minimal $(\theta, 1)$-method applied to a pair (X_0, X_1) when X_0 is continuously embedded in X_1. Our version of this estimate for the complex method (Lemma 5.5) is

$$\left|\frac{d}{dt}\|a\|_t\right|_{t=\theta^\pm} \leqslant \|\Omega_\theta a\|_\theta,$$

from which we derive a number of stability results for pairs. For interpolation pairs (X_0, X_1) satisfying a coherence condition, including regular interpolation pairs of reflexive spaces that have a common Schauder basis or are rearrangement invariant Köthe spaces, we show in Theorem 5.6 that if $\sup_{\theta_0<t<\theta_1} \|\Omega_t : X_t \to X_t\| < \infty$ for some $0 \leqslant \theta_0 < \theta_1 \leqslant 1$, then $X_0 = X_1$ up to an equivalent norm.

In many cases, derivations are uniquely defined, so it makes sense to study exact stability (instead of up to a bounded plus linear perturbation) problems. We show that exact stability is related to *isometric* characterizations of X_0 and X_1. In particular, Theorem 5.11 provides a complete and explicit characterization of pairs (X_0, X_1) of spaces for which Ω_θ is linear.

2. Preliminary results

For background on the theory of twisted sums and diagrams, we refer to [2, 10]. A *twisted sum of two Banach spaces* Y, Z is a quasi-Banach space X, which has a closed subspace isomorphic to Y such that the quotient X/Y is isomorphic to Z. An exact sequence $0 \to Y \to X \to Z \to 0$ of Banach spaces and continuous operators is a diagram in which the kernel of each arrow coincides with the image of the preceding one. Thus, the open mapping theorem yields that the middle space X is a twisted sum of Y and Z. The simplest exact sequence is obtained taking $X = Y \oplus Z$ with embedding $y \to (y, 0)$ and quotient map $(y, z) \to z$. Two exact sequences $0 \to Y \to X_1 \to Z \to 0$ and $0 \to Y \to X_2 \to Z \to 0$ are said to be *equivalent* if there exists an operator $T : X_1 \to X_2$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X_1 & \longrightarrow & Z & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle T} & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & X_2 & \longrightarrow & Z & \longrightarrow & 0. \end{array}$$

The classical 3-lemma [10, p. 3] shows that T must be an isomorphism. An exact sequence is said to be *trivial* if it is equivalent to $0 \to Y \to Y \oplus Z \to Z \to 0$. In this case, we also say that the exact sequence *splits*. This is equivalent to the subspace Y being complemented in X.

Kalton [21, 22] developed a deep theory connecting derivations and twisted sums for Köthe function spaces that we briefly describe now because it is essential to understand our work. We even present Kalton's definition of Köthe function space since it is slightly different from the standard one [27]. According to [22, p.482], given a σ-finite measure μ on a Polish space, and denoting by $L_0 \equiv L_0(\mu)$ the space of all complex-valued μ-measurable functions endowed with the topology of convergence in measure, an *admissible norm* is a map $\|\cdot\|_X : L_0 \to [0, \infty]$ such that $X = \{f \in L_0 : \|f\|_X < \infty\}$ is a vector subspace of $L_0(\mu)$, $\|\cdot\|_X$ is a norm on X with $\{z \in X : \|z\|_X \leqslant 1\}$ closed in L_0, and there exist strictly positive functions $h, k \in L_0$ such that $\|hf\|_1 \leqslant \|f\|_X \leqslant \|kf\|_\infty$ for every $f \in L_0$. In this case, X is a Banach space continuously embedded in L_0, and it is called an *admissible space*. A Köthe function space is a sublattice of $L_0(\mu)$ endowed with an admissible lattice norm ($|f| \leqslant |g|$ implies $\|f\| \leqslant \|g\|$).

Let X be a Köthe function space. A *centralizer on* X is a homogeneous map $\Omega : X \to L_0(\mu)$ for which there is a constant C such that, given $f \in L_\infty(\mu)$ and $x \in X$, $\Omega(fx) - f\Omega(x) \in X$ and $\|\Omega(fx) - f\Omega(x)\|_X \leqslant C\|f\|_\infty \|x\|_X$. A centralizer Ω on X induces an exact sequence

$$0 \longrightarrow X \xrightarrow{j} X \oplus_\Omega X \xrightarrow{q} X \longrightarrow 0,$$

where $X \oplus_\Omega X = \{(f, x) \in L_0 \times X : f - \Omega x \in X\}$, endowed with the quasi-norm $\|(f, x)\|_\Omega = \|f - \Omega x\|_X + \|x\|_X$, with inclusion $j(y) = (y, 0)$ and quotient map $q(f, x) = x$.

We say that a centralizer Ω is *trivial* if the exact sequence induced by Ω splits. Recall that an homogeneous map $F : X \to X$ is called *bounded* if there is $C > 0$ such that $\|F(x)\| \leqslant C\|x\|$ for every $x \in X$. The following characterization of triviality is essentially known.

Proposition 2.1. *A centralizer $\Omega : X \to L_0(\mu)$ is trivial if and only if there exists a linear map $L : X \to L_0(\mu)$ such that $\Omega - L$ is a bounded map from X to X.*

Proof. If a map L as above exists, then the map $(f, x) \to (f - Lx, 0)$ is a linear bounded projection on $X \oplus_\Omega X$ with range $j(X)$. Indeed, $f - Lx = f - \Omega x + \Omega x - Lx \in X$ and $\|f - Lx\|_X \leqslant \|(f, x)\|_\Omega + \|(\Omega - L)x\|_X$.

Conversely, if Ω is trivial then there is a bounded linear map $S : X \to X \oplus_\Omega X$ such that qS is the identity on X [10, Lemma 1.1.a]. Then $Sx = (Lx, x)$ for some linear map $L : X \to L_0(\mu)$. Since $\|(Lx, x)\|_\Omega = \|Lx - \Omega x\|_X + \|x\|_X \leqslant \|S\| \cdot \|x\|_X$, L satisfies the required conditions. □

3. Kalton spaces of analytic functions

Here we present the abstract version of the complex interpolation method introduced in [24] and other previous papers of Kalton [21, 22, 25]. Along the section U will be an open subset of $\mathbb{C}$ conformally equivalent to the unit disc $\mathbb{D}$. The boundary of U is denoted by ∂U, and we will write $\mathbb{T} = \partial\mathbb{D}$.

Definition 3.1. A *Kalton space* is a Banach space $\mathcal{F} \equiv (\mathcal{F}(U, \Sigma), \|\cdot\|_\mathcal{F})$ of analytic functions on U with values in a complex Banach space Σ satisfying the following conditions:

(a) For each $z \in U$, the evaluation map $\delta_z : \mathcal{F} \to \Sigma$ is bounded.

(b) If $\varphi : U \to \mathbb{D}$ is a conformal equivalence and $f : U \to \Sigma$ is an analytic map, then $f \in \mathcal{F}$ if and only if $\varphi \cdot f \in \mathcal{F}$, and in this case $\|\varphi \cdot f\|_\mathcal{F} = \|f\|_\mathcal{F}$.

Given a Kalton space $\mathcal{F}(U, \Sigma)$, for each $z \in U$ we define

$$X_z = \{x \in \Sigma : x = f(z) \text{ for some } f \in \mathcal{F}\},$$

which endowed with the norm $\|x\|_z = \inf\{\|f\|_\mathcal{F} : x = f(z)\}$ is isometric to $\mathcal{F}/\ker\delta_z$.

The family $(X_z)_{z\in U}$ is called an *analytic family* of Banach spaces on U, and a function $f_{x,z} \in \mathcal{F}$ such that $f_{x,z}(z) = x$ and $\|f_{x,z}\|_\mathcal{F} \leqslant c\|x\|_z$ is called a c-*extremal* (for x at z).

There are many ways of generating Kalton spaces, and here is where complex interpolation enters the game. We shall mainly consider two cases: $U = \mathbb{S}$, more suitable to handle interpolation pairs [4], and $U = \mathbb{D}$, more suitable for interpolating families [12].

Next we describe the complex interpolation method for pairs as a reference, and later we will introduce other versions of the complex interpolation method when we need them.

Complex interpolation for pairs. An *interpolation pair* (X_0, X_1) is a pair of Banach spaces, both of them linear and continuously contained in a bigger Hausdorff topological vector space Σ, which can be assumed to be $\Sigma = X_0 + X_1$ endowed with the norm $\|x\| = \inf\{\|x_0\|_0 + \|x_1\|_1 : x = x_0 + x_1\ x_j \in X_j \text{ for } j = 0, 1\}$. The pair will be called *regular* if, additionally, $\Delta = X_0 \cap X_1$ is dense in both X_0 and X_1. The space Δ endowed with the norm $\|x\|_\Delta = \max\{\|x\|_{X_0}, \|x\|_{X_1}\}$ is a Banach space, and the inclusions $\Delta \to X_i \to \Sigma$ are contractions.

The *Calderón space* $\mathcal{C} = \mathcal{C}(\mathbb{S}, X_0 + X_1)$ is formed by those bounded continuous functions $F : \overline{\mathbb{S}} \to X_0 + X_1$, which are analytic on $\mathbb{S}$ and such that the maps $t \mapsto F(k + ti) \in X_k$ are continuous and bounded, $k = 0, 1$. Endowed with the norm $\|F\|_{\mathcal{C}} = \sup\{\|F(k + ti)\|_{X_k} : t \in \mathbb{R}, k = 0, 1\} < \infty$, $\mathcal{C}$ is a Kalton space.

Since $X_z = X_{\operatorname{Re} z}$ for $z \in \mathbb{S}$, it is usual to consider only the scale $(X_\theta)_{0<\theta<1}$. Sometimes, for convenience, we will write $(X_0, X_1)_\theta$ instead of X_θ.

3.1. Derivations, centralizers and twisted sums

Given a Kalton space $\mathcal{F}(U, \Sigma)$ and $z \in U$, the evaluation map $\delta'_z : f \in \mathcal{F} \to f'(z) \in \Sigma$ of the derivative at z is bounded for all $z \in U$ (see Lemma 3.5 for a precise estimate of its norm). We also need the following well-known fact, for which we present a proof for the sake of later use.

Proposition 3.2. *For each $z \in U$, the map δ'_z is continuous and surjective from* $\ker \delta_z$ *to X_z.*

Proof. Let $\varphi : U \to \mathbb{D}$ be a conformal equivalence such that $\varphi(z) = 0$. Each $g \in \ker \delta_z$ can be written as $g = \varphi \cdot f$ for some $f \in \mathcal{F}$, and $g'(z) = \varphi'(z) f(z) \in X_z$; thus $\delta'_z(\ker \delta_z) \subset X_z$ and the continuity into X_z follows from the closed graph theorem. Moreover, given $x \in X_z$ and $f \in \mathcal{F}$ with $f(z) = x$, $g = \phi'(z)^{-1}\phi \cdot f \in \ker \delta_z$ and $g'(z) = x$, hence $\delta'_z(\ker \delta_z) = X_z$. □

For each $z \in U$, we consider the space $dX_z = \{(f'(z), f(z)) : f \in \mathcal{F}\}$. The map $\Delta_z : \mathcal{F} \to dX_z$ given by $\Delta_z(f) = (f'(z), f(z))$ is bounded and thus dX_z can be endowed with the quotient norm $\|(a, b)\| = \inf\{\|f\|_{\mathcal{F}} : f \in \mathcal{F},\ f'(z) = a,\ f(z) = b\}$. The space dX_z admits an exact sequence $0 \to X_z \to dX_z \to X_z \to 0$ with inclusion $j_z(x) = (x, 0)$ (thanks to Proposition 3.2) and quotient map $q_z(y, x) = x$. All this yields a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker \delta_z & \longrightarrow & \mathcal{F} & \xrightarrow{\delta_z} & X_z & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \delta'_z} & & \big\downarrow{\scriptstyle \Delta_z} & & \| & & \\ 0 & \longrightarrow & X_z & \xrightarrow[j_z]{} & dX_z & \xrightarrow[q_z]{} & X_z & \longrightarrow & 0 \end{array} \tag{2}$$

Thus we have a method to obtain twisted sums of the spaces X_z associated with a Kalton space. The twisted sum space can be described using the so-called *derivation* *map* given by

$$\Omega_z = \delta'_z B_z, \tag{3}$$

where $B_z : X_z \to \mathcal{F}$ is a homogeneous bounded selection for the evaluation map $\delta_z : \mathcal{F} \to \Sigma$, and the family of maps $(\Omega_z)_{z\in U}$ is the *differential process associated with* $\mathcal{F}$.

We consider the space

$$d_{\Omega_z} X_z = \{(y, x) \in \Sigma \times X_z : y - \Omega_z x \in X_z\}$$

endowed with the quasi-norm

$$\|(y, x)\| = \|y - \Omega_z x\|_z + \|x\|_z$$

so that one has an exact sequence $0 \to X_z \to d_{\Omega_z} X_z \to X_z \to 0$ with inclusion $x \to (x, 0)$ and quotient map $(y, x) \to x$. It is not hard to check [6] that this exact sequence is equivalent to the lower row of (2). Note that different choices of selection B_z lead to different derivations Ω_z, but the difference between two of these derivations is always a bounded map, so both choices produce isomorphic derived spaces and equivalent twisted sums.

The derivation map Ω_z is said to be *trivial* if the associated exact sequence splits. With the proof of Proposition 2.1, we obtain the following result.

Proposition 3.3. *The derivation map* Ω_z *is trivial if and only if there is a linear map* $L : X_z \to \Sigma$ *such that* $\Omega_z - L$ *is a bounded map from* X_z *to* X_z.

A centralizer Ω on a Köthe space of functions X is called *real* if $\Omega(x)$ is real whenever $x \in X$ is real. Kalton's theorem stated below establishes that all real centralizers essentially arise from complex interpolation of an interpolation pair of Köthe spaces.

Theorem 3.4. [21, 22]

(1) *Given an interpolation pair* (X_0, X_1) *of complex Köthe function spaces and* $0 < \theta < 1$, *the derivation* Ω_θ *is a real centralizer on* X_θ.

(2) *For every real centralizer* Ω *on a separable superreflexive Köthe function space* X, *there is a number* $\varepsilon > 0$ *and an interpolation pair* (X_0, X_1) *of Köthe function spaces so that* $X = X_\theta$ *for some* $0 < \theta < 1$ *and* $\varepsilon\Omega - \Omega_\theta : X_\theta \to X_\theta$ *is a bounded map.*

(3) *The derivation* Ω_θ *is bounded as a map* $X_\theta \to X_\theta$ *for some* θ *if and only if* $X_0 = X_1$, *up to an equivalent renorming. In this case,* Ω_θ *is bounded for all* θ.

3.2. Distances and isomorphisms

It is not difficult to translate results (at levels 0 and 1) from the open unit disc $\mathbb{D}$ to the open unit strip $\mathbb{S}$, and conversely. Indeed, if $\varphi : \mathbb{S} \to \mathbb{D}$ is a conformal map and $(X_\omega)_{\omega\in\mathbb{D}}$ is an interpolation family on $\mathbb{D}$, then $Y_z = X_{\varphi(z)}$ provides an interpolation family $(Y_z)_{z\in\mathbb{S}}$ on $\mathbb{S}$. The corresponding derivation maps are related as follows:

$$\Omega_z^{\mathbb{S}} = \varphi'(z)\Omega^{\mathbb{D}}_{\varphi(z)}.$$

Given $s \in U$, we denote by $\varphi_s : U \to \mathbb{D}$ a conformal equivalence taking s to 0. In the case $U = \mathbb{S}$, an example is given by

$$\varphi_s(z) = \frac{\sin(\pi(z-s)/2)}{\sin(\pi(z+s)/2)} \quad (z \in \mathbb{S}) \tag{4}$$

for which $\varphi_s'(s) = \pi/(2\sin\pi s)$. The conformal equivalence φ_s is unique up to a multiplicative constant: any other conformal equivalence ψ_s taking s to 0 can be written as $\psi_s = f \circ \varphi_s$, where $f(z) = e^{i\theta}z$ [3, Lemma 13.14].

Given $\mathcal{F}(U, \Sigma)$ and $z \in U$, we denote by $\delta_z^n : \mathcal{F} \to \Sigma$ the evaluation of the nth derivative at z. We will need the following estimates.

Lemma 3.5. *Let $\mathcal{F}(U, \Sigma)$ be a Kalton space, $s \in U$, and $n \in \mathbb{N}$. Then*

(1) $\|\delta_s^n : \mathcal{F} \to \Sigma\| \leqslant n!/dist(s, \partial U)^n$,

(2) $\|\delta_s' : \ker\delta_s \to X_s\| = \inf\{\|\delta_s' x\| : x \in \ker\delta_s, dist(x, \ker\delta_s') = 1\} = |\varphi_s'(s)|$.

Proof. Given a positively oriented closed rectifiable curve Γ in U for which z belongs to the inside of Γ, the Cauchy integral formula [26, Appendix A3] establishes that, for each $n \in \mathbb{N}_0$,

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(\omega)}{(w-z)^{n+1}}\,dw.$$

We take a number r with $0 < r < \mathrm{dist}(s, \partial U)$ and denote by Γ the boundary of the open disc $\mathbb{D}(s, r)$. By the Cauchy integral formula

$$\|f^{(n)}(s)\| \leqslant \frac{n!}{2\pi}\int_\Gamma \frac{\|f(\omega)\|}{r^{n+1}}\,d|w| \leqslant \frac{n!}{r^n}\|f\|_{\mathcal{F}},$$

and since we can take r arbitrarily close to $\mathrm{dist}(s, \partial U)$, we get estimate (1).

(2) Clearly $\|\delta_s' : \ker\delta_s \to X_s\| \geqslant \inf\{\|\delta_s' x\| : x \in \ker\delta_s, \mathrm{dist}(x, \ker\delta_s') = 1\}$, and given $g \in \ker\delta_s$ the function $f(z) = \varphi_s'(s)\cdot\varphi_s(z)^{-1}g(z)$ is in $\mathcal{F}$ and satisfies $f(s) = g'(s)$ and $\|f\| = |\varphi_s'(s)|\|g\|$. Therefore $\|\delta_s' g\|_s = \|f(s)\|_s \leqslant |\varphi_s'(s)|\|g\|$, and we get $\|\delta_s' : \ker\delta_s \to X_s\| \leqslant |\varphi_s'(s)|$.

Also, given $x \in B_{X_s}$ and $\varepsilon > 0$, we can take $f \in \mathcal{F}$ with $\|f\| < (1+\varepsilon)$ and $f(s) = x$. Then $g(z) = \varphi_s'(s)^{-1}\varphi_s(z)\cdot f(z)$ defines $g \in \ker\delta_s$ with $\|g\| < (1+\varepsilon)/|\varphi_s'(s)|$ and $g'(s) = x$. Hence

$$\delta_s'(B_{\ker\delta_s}) \supset |\varphi_s'(s)|(1+\varepsilon)^{-1}B_{X_s},$$

and we get $\inf\{\|\delta_s' x\| : x \in \ker\delta_s, \mathrm{dist}(x, \ker\delta_s') = 1\} \geqslant |\varphi_s'(s)|$ finishing the proof. □

Part (2) of Lemma 3.5 says that $\delta_s' : \ker\delta_s \to X_s$ is a multiple of a quotient map: the induced injective map $\ker\delta_s/(\ker\delta_s' \cap \ker\delta_s) \to X_s$ is $|\varphi_s'(s)|$ times an isometry.

Lemma 3.6. *For each $f \in \mathcal{F}$ and $s \in U$, we have $\Omega_s(f(s)) - f'(s) \in X_s$ with*

$$\|\Omega_s(f(s)) - f'(s)\|_s \leqslant 2\|\delta_s' : \ker\delta_s \to X_s\|\|f\| \leqslant 2\|f\|/\mathrm{dist}(s, \partial U).$$

Proof. From $\Omega_s(f(s)) - f'(s) = \delta_s'(B_s(f(s)) - f)$ with $B_s(f(s)) - f \in \ker\delta_s$, we get the first part. For the rest, note that the operator $\delta_s' : \ker\delta_s \to X_s$ is bounded by Lemma 3.5. □

Proposition 3.7. *Let $s, t \in U$.*

(1) *The spaces* $\ker \delta_s$ *and* $\mathcal{F}$ *are isometric. Consequently,* $\ker \delta_s$ *and* $\ker \delta_t$ *are isometric.*

(2) *For every $n \in \mathbb{N}$, $\cap_{0 \leqslant k \leqslant n} \ker \delta_s^k$ and $\mathcal{F}$ are isometric.*

Proof. The operator $d_s : \mathcal{F} \to \ker \delta_s$ given by $d_s(f)(z) = f(z)\varphi_s(z)$ is clearly well defined and injective, and it is surjective because each $g \in \ker \delta_s$ can be written as $g = \varphi_s \cdot f$ with $f \in \mathcal{F}$.

To prove (2), just note that $(d_s)^{n+1} : \mathcal{F} \to \bigcap_{0 \leqslant k \leqslant n} \ker \delta_s^k$ is also an isometry. □

Let $s, t \in U$. The map $\varphi_s \cdot f \in \ker \delta_s \to \varphi_t \cdot f \in \ker \delta_t$ is a bijective isometry, but we need a more precise description. Note that the map $\varphi_{s,t} : U \to \mathbb{D}$ defined by

$$\varphi_{s,t}(z) = \frac{\varphi_s(z) - \varphi_s(t)}{1 - \overline{\varphi_s(t)}\varphi_s(z)} \quad (z \in U)$$

is a conformal equivalence satisfying $\varphi_{s,t}(t) = 0$. Moreover, denoting $\alpha = \varphi_s(t) \in \mathbb{D}$, one has

$$\begin{aligned}
\|\varphi_s - \varphi_{s,t}\|_\infty &= \sup_{z \in U} \left|\varphi_s(z) - \varphi_{s,t}(z)\right| = \sup_{\lambda \in \mathbb{D}} \left|\lambda - \frac{\lambda - \alpha}{1 - \overline{\alpha}\lambda}\right| \\
&= \sup_{\omega \in \mathbb{T}} \left|\omega - \frac{\omega - \alpha}{1 - \overline{\alpha}\omega}\right| = \sup_{\omega \in \mathbb{T}} \left|\frac{\alpha - \overline{\alpha}\omega^2}{1 - \overline{\alpha}\omega}\right| \\
&= \sup_{\omega \in \mathbb{T}} \left|\frac{\alpha\overline{\omega} - \overline{\alpha}\omega}{\overline{\omega} - \overline{\alpha}}\right| \leqslant 2|\alpha|
\end{aligned}$$

since $|\alpha\overline{\omega} - \overline{\alpha}\omega| \leqslant |\alpha\overline{\omega} - \alpha\overline{\alpha}| + |\alpha\overline{\alpha} - \overline{\alpha}\omega| = 2|\alpha||\overline{\omega} - \overline{\alpha}|$.

Definition 3.8 [25]. The *pseudo-hyperbolic distance* $\mathsf{h}(\cdot, \cdot)$ on U is defined by $\mathsf{h}(s, t) = |\varphi_s(t)|$.

Given two closed subspaces M, N of a Banach space Z, and denoting by B_M the unit ball of M, the *gap* $g(M, N)$ *between* M *and* N is defined as follows:

$$g(M, N) = \max\Big\{\sup_{x \in B_M} \operatorname{dist}(x, B_N),\ \sup_{y \in B_N} \operatorname{dist}(y, B_M)\Big\}.$$

The *Kadets distance* $d_K(X, Y)$ between two Banach spaces X and Y is the infimum of the gap $g(i(X), j(Y))$ taken over all the isometric embeddings of i, j of X, Y into a common superspace. We have the following proposition.

Proposition 3.9 [25, Theorem 4.1]. *Let E and F be closed subspaces of a Banach space Z. Then $d_K(Z/E, Z/F) \leqslant 2g(E, F)$.*

Proposition 3.10. *For each $n \in \mathbb{N} \cup \{0\}$, $g\big(\bigcap_{0 \leqslant k \leqslant n} \ker \delta_s^k, \bigcap_{0 \leqslant k \leqslant n} \ker \delta_t^k\big) \leqslant 2(n+1)\mathsf{h}(s, t)$.*

Proof. We proceed inductively on n. For $n = 0$, we take a norm-one $\varphi_s \cdot f \in \ker \delta_s$. Since $\varphi_{s,t} \cdot f \in \ker \delta_t$ is norm-one and $\|\varphi_s \cdot f - \varphi_{s,t} \cdot f\|_{\mathcal{F}} = \|\varphi_s - \varphi_{s,t}\|_\infty \leqslant 2\mathsf{h}(s, t)$, and we can proceed similarly for each norm-one $\varphi_t \cdot f \in \ker \delta_t$, we get $g\,(\ker \delta_s, \ker \delta_t) \leqslant 2\mathsf{h}(s, t)$.

Moreover, if the estimate holds for $n-1$, then it also holds for n because

$$a^{n+1}-b^{n+1}=a^{n+1}-a^n b+a^n b-b^{n+1}=a^n(a-b)+(a^n-b^n)b. \qquad \square$$

Since $\mathcal{F}/\bigcap_{0\leqslant k\leqslant n}\ker\delta_s^k=d^nX_s$, Propositions 3.9 and 3.10 provide the following result.

Theorem 3.11. *Given $s,t\in U$ and $n\in\mathbb{N}\cup\{0\}$, $d_K(d^nX_s,d^nX_t)\leqslant 4(n+1)\mathsf{h}(s,t)$.*

Corollary 3.12. *Let $\mathcal{P}$ be an open (resp. stable) property. Assume that there is $s\in U$ so that d^nX_s has $\mathcal{P}$. Then d^nX_t has $\mathcal{P}$ for all $t\in U$ (resp. for all t in an open disc centered in s).*

3.3. Bounded stability

Let $\mathcal{F}(U,\Sigma)$ be a Kalton space and let $z\in U$. Then the exact sequence $0\to X_z\to dX_z\to X_z\to 0$ associated with $\Omega_z:X_z\to\Sigma$ splits if and only if there exists a linear map $L:X_z\to\Sigma$ such that $\Omega-L$ takes X_z to X_z and it is bounded (Proposition 3.3). Kalton's work justifies the importance of the case Ω_z bounded in interpolation affairs. Let us accordingly introduce a few related notions.

Definition 3.13. The derivation Ω_z is *bounded* when it takes values in X_z and it is bounded as a map from X_z to X_z. In this case, we will say that the induced exact sequence *boundedly splits.*

Bounded splitting admits the following characterizations.

Theorem 3.14. *Let $\mathcal{F}(U,\Sigma)$ be a Kalton space and let $s\in U$. The following assertions are equivalent:*

(1) $\delta_s:\ker\delta_s'\to X_s$ *is surjective.*

(2) $\mathcal{F}=\ker\delta_s+\ker\delta_s'$.

(3) *There exists $M>0$ such that each $f\in\mathcal{F}$ can be written as $f=g+h$ with $g\in\ker\delta_s$, $h\in\ker\delta_s'$ and $\max\{\|g\|_{\mathcal{F}},\|h\|_{\mathcal{F}}\}\leqslant M\|f\|_{\mathcal{F}}$.*

(4) $\delta_s'(\mathcal{F})\subset X_s$.

(5) $\delta_s':\mathcal{F}\to X_s$ *is bounded.*

(6) $\Omega_s(X_s)\subset X_s$.

(7) $\Omega_s:X_s\to X_s$ *is bounded.*

Proof. Clearly (1) $\Leftarrow$ (2) $\Leftarrow$ (3), (4) $\Leftarrow$ (5), and (6) $\Leftarrow$ (7). Moreover, (4) $\Leftrightarrow$ (6) and (5) $\Leftrightarrow$ (7) follow from Lemma 3.6. We will prove (1) $\Rightarrow$ (3) $\Rightarrow$ (5) and (4) $\Rightarrow$ (2).

(1) $\Rightarrow$ (3): Let $f\in\mathcal{F}$ with $\|f\|=1$. Since $\delta_s:\ker\delta_s'\to X_s$ is surjective, it is open. So there exists $r>0$ such that we can find $h\in\ker\delta_s'$ with $\|h\|\leqslant r\|f(s)\|_s$ and $h(s)=f(s)$. Since $\|f(s)\|_s\leqslant\|f\|$, taking $g=f-h\in\ker\delta_s$ we obtain (3) with $M=r+1$.

(3) $\Rightarrow$ (5): Let $f\in\mathcal{F}$. Then $f=g+h$ with $g\in\ker\delta_s$, $h\in\ker\delta_s'$ and $\|g\|_{\mathcal{F}}\leqslant M\|f\|_{\mathcal{F}}$; hence

$$\|\delta_s'(f)\|_s=\|\delta_s'(g)\|_s\leqslant\|\delta_s':\ker\delta_s\to X_s\|\cdot\|g\|_{\mathcal{F}}\leqslant M\|\delta_s':\ker\delta_s\to X_s\|\cdot\|f\|_{\mathcal{F}}.$$

(4) $\Rightarrow$ (2): We know that the operator $\delta_s' : \ker\delta_s \to X_s$ is surjective. So taking a linear selection $\ell : X_s \to \ker\delta_s$ for δ_s', for each $f \in \mathcal{F}$, $\ell(f'(s)) \in \ker\delta_s$ and $f - \ell(f'(s)) \in \ker\delta_s'$. □

Condition (6) shows that the requirements in Definition 3.13 are redundant. Condition (2) in Theorem 3.14 provides a neat description of how the twisted sum space $d_{\Omega_s} X_s$ splits when Ω_s is bounded. Indeed, since $d_{\Omega_s} X_s = \mathcal{F}/(\ker\delta_s \cap \ker\delta_s')$ and the subspace X_s embeds in $d_{\Omega_s} X_s$ as $\ker\delta_s/(\ker\delta_s \cap \ker\delta_s')$, condition (2) gives

$$\frac{\mathcal{F}}{\ker\delta_s \cap \ker\delta_s'} = \frac{\ker\delta_s + \ker\delta_s'}{\ker\delta_s \cap \ker\delta_s'} = \frac{\ker\delta_s}{\ker\delta_s \cap \ker\delta_s'} \oplus \frac{\ker\delta_s'}{\ker\delta_s \cap \ker\delta_s'}.$$

4. Stability of splitting for Köthe function spaces

We will consider the following notions of stability:

Definition 4.1. We say that a differential process $(\Omega_z)_{z\in U}$

(1) has *local stability* if whenever Ω_{z_0} is trivial, then there is $\varepsilon > 0$ such that Ω_z is trivial for $|z - z_0| < \varepsilon$.

(2) has *local bounded stability* if whenever Ω_{z_0} is bounded, then there is $\varepsilon > 0$ such that Ω_z is bounded for $|z - z_0| < \varepsilon$.

(3) has *global stability* if whenever Ω_{z_0} is trivial, then Ω_z is trivial for all $z \in U$.

(4) has *global bounded stability* if whenever Ω_{z_0} is bounded, then Ω_z is bounded for all $z \in U$.

Theorem 3.4 shows that when an analytic family is generated by an interpolation pair (X_0, X_1) of Köthe spaces, then the differential process is 'rigid', in the sense that whenever Ω_{z_0} is bounded at some point z_0 then $X_0 = X_1$, up to some equivalent renorming. Here we will prove the following:

- The differential process associated with families of up to three Köthe spaces distributed in arcs enjoys global (bounded) stability; in fact, it is 'rigid' in the case of bounded stability and 'rigid' up to weighted versions in the case of stability. See § 4.3.
- The differential process associated with families of four spaces can fail local bounded stability (Proposition 4.17) or local stability (Proposition 4.19).

4.1. Stability for pairs of Köthe spaces

After Kalton's bounded stability theorem (Theorem 3.4), it is a reasonable guess that 'nontrivial scales' of Köthe spaces correspond to 'nontrivial centralizers'. The difficulty is that the nontriviality notion involves uncontrolled linear maps, as we can see in Proposition 3.3. Thus, while Kalton shows [22] that the centralizer Ω_θ associated with the scale $(X_0, X_1)_\theta$ of Köthe function spaces is bounded if and only if $X_0 = X_1$ up to equivalence of norms, the following question remained open: *Does the triviality of Ω_θ imply that X_0 and X_1 are equal, or at least isomorphic?*

We shall now prove global stability for pairs of Köthe spaces. The following sentence in [8, p. 364] clearly suggests that it was known to Kalton, at least in the domain of Köthe sequence spaces: *If* (Z_0, Z_1) *are two superreflexive sequence spaces and* $Z_\theta = (Z_0, Z_1)_\theta$ *for* $0 < \theta < 1$ *is the usual interpolation space by the Calderón method, one can define a derivative* dZ_θ *that is a twisted sum* $Z_\theta \oplus_\Omega Z_\theta$*, which splits if and only if* $Z_1 = wZ_0$ *for some sequence of weights* $w = (w(n))$*, where* $w(n) > 0$ *for all* n*. These remarks follow easily from the methods of* [22].

Next we recall Kalton's formula [22, (3.2)] for the centralizer Ω_θ corresponding to a couple of Köthe function spaces (X_0, X_1) and $0 < \theta < 1$. It is well known [7] that X_θ coincides with the space $X_0^{1-\theta} X_1^{\theta}$, with

$$\|x\|_\theta = \inf\{\|y\|_0^{1-\theta}\|z\|_1^{\theta} : y \in X_0, z \in X_1, |x| = |y|^{1-\theta}|z|^{\theta}\}.$$

We fix $c > 1$. For each $x \in X_\theta$, we write $|x| = |a_0(x)|^{1-\theta}|a_1(x)|^{\theta}$ with $\|a_0(x)\|_0, \|a_1(x)\|_1 \leqslant c\|x\|_\theta$, where a_0 and a_1 are chosen homogeneously.

Then $B_\theta(x)(z) = (\mathrm{sgn} x)|a_0(x)|^{1-z}|a_1(x)|^{z}$ gives an extremal for x at θ, and we obtain

$$\Omega_\theta(x) = \delta'_\theta B_\theta(x) = x \log \frac{|a_1(x)|}{|a_0(x)|}. \tag{5}$$

Given a Köthe function space X of μ-measurable functions, a *weight* w is a positive function in $L_0(\mu)$. We denote by $X(w)$ the space of all measurable scalar functions f such that $wf \in X$, endowed with the norm $\|x\|_w = \|wx\|_X$. From the approach in [9], we get the following general version of a well-known result for L_p-spaces [4, Theorem 5.4.1].

Proposition 4.2. *Let* X *be a Köthe function space with the Radon–Nikodym property, and let* w_0, w_1 *be two weights. Then* $(X(w_0), X(w_1))_\theta = X(w_0^{1-\theta} w_1^{\theta})$ *for* $0 < \theta < 1$*, with associated linear centralizer* $\Omega_\theta(x) = \log(w_1/w_0) \cdot x$ *for* $x \in X$.

Proof. By [24, Theorem 4.6], the space $(X(w_0), X(w_1))_\theta$ is isometric to the space $X(w_0)^{1-\theta} X(w_1)^{\theta}$ endowed with the norm

$$\begin{aligned}\|x\|_\theta &= \inf\{\|a\|_{w_0}^{1-\theta}\|b\|_{w_1}^{\theta} : a \in X(w_0), b \in X(w_1), |x| = |a|^{1-\theta}\,|b|^{\theta}\}\\ &= \inf\{\|w_0 a\|_X^{1-\theta}\|w_1 b\|_X^{\theta} : a \in X(w_0), b \in X(w_1), |x| = |a|^{1-\theta}\,|b|^{\theta}\}.\end{aligned}$$

Standard lattice estimates such as [28, Proposition 1.d.2] imply that

$$\|x\|_\theta \geqslant \inf\{\|w_0^{1-\theta} a^{1-\theta} w_1^{\theta} b^{\theta}\|_X : \quad |x| = |a|^{1-\theta}\,|b|^{\theta}\} = \|x\|_{X(w_0^{1-\theta} w_1^{\theta})},$$

and the reverse inequality can be obtained by using $w_0 a = w_1 b = w_0^{1-\theta} w_1^{\theta} x$.

To obtain Ω_θ on X_θ, observe that $B_\theta(x)(z) = (w_1/w_0)^{\theta - z} x$ is a bounded homogeneous selector for the evaluation map δ_θ. Indeed, $B_\theta(x)(\theta) = x$ while $\|B_\theta x\| = \|x\|_{X_\theta}$ as it follows from

$$\|B_\theta(x)(0+it)\|_{w_0} = \|B_\theta(x)(1+it)\|_{w_1} = \|w_0^{1-\theta} w_1^{\theta} x\|_X = \|x\|_{X_\theta}. \qquad \square$$

Complex interpolation between two Hilbert spaces always yields Hilbert spaces [24]. Let us show that the induced derivation is trivial.

Corollary 4.3. *Let (H_0, H_1) be an interpolation pair of Hilbert spaces. Then for every $0 < \theta < 1$, the derivation Ω_θ is trivial.*

Proof. It follows from Proposition 4.2, since [17, Lemma 2.2] shows that (H_0, H_1) is equivalent to an interpolation pair $(\ell_2(I), \ell_2(I, w))$, where I is a set and $w : I \to \mathbb{R}$ is a positive weight. □

Next we solve the stability problem for the splitting in the case of a pair of Köthe spaces, completing Theorem 3.4.

Theorem 4.4. *Let (X_0, X_1) be an interpolation pair of superreflexive Köthe function spaces and let $0 < \theta < 1$. Then Ω_θ is trivial if and only if there is a weight w so that $X_1 = X_0(w)$ up to equivalence of norms.*

Proof. Recall that X_0, X_1 are spaces of μ-measurable functions. The proof goes in two steps.

Step 1. If Ω_θ is trivial, then there are weighted versions Y_i of X_i so that if Ψ_θ is the associated derivation, then there is a real function $f \in L_0(\mu)$ so that, denoting also by f the multiplication map by f, $\Psi_\theta(x) - fx \in X_\theta$ and $\Psi_\theta - f$ is a bounded map on a dense subspace of X_θ.

Since we are dealing with interpolation of Köthe function spaces, there is a positive function $k > 0$ such that $\|x\|_{X_j} \leqslant \|kx\|_\infty$ for $j = 0, 1$. Consider the couple (Y_0, Y_1), where $Y_j = X_j(1/k)$, $j = 0, 1$. We denote the derivation induced at θ by this couple by Ψ_θ. Then $Y_\theta = X_\theta(1/k)$ and Ψ_θ is trivial. The advantage of working with Y_θ is that it contains the characteristic functions of measurable sets.

Since Ψ_θ is a centralizer, there is a constant $c > 0$ such that for every $a \in L_\infty(\mu)$ and every $x \in X$ we have $\|\Psi_\theta(ax) - a\Psi_\theta(x)\|_{X_\theta} \leqslant c\|a\|_\infty\|x\|_{Y_\theta}$, and since it is trivial, there is a linear map L so that $\Psi_\theta - L$ takes values in Y_θ and is bounded there. The techniques in [9] (Lemmas 3.10 and 3.13) show that after some averaging it is possible to get a linear map Λ such that $\Psi_\theta - \Lambda$ takes values in Y_θ, is bounded there, and $\Lambda(ux) = u\Lambda x$ for every unit u (every function with $|u| = 1$). Since characteristic functions can be written as the mean of two units, one gets that if $s = \sum_i \lambda_i 1_{A_i}$ is a simple function, then $\Lambda(sx) = s\Lambda(x)$. Now, simple functions are dense in L_∞, so given $a \in L_\infty$ pick a simple s so that $\|a - s\| \leqslant \varepsilon$. Since

$$\Lambda(ax) = \Lambda((a-s)x) + \Lambda(sx) \text{ and } a\Lambda(x) = (a-s)\Lambda(x) + s\Lambda(x),$$

it follows that for some constant K

$$\|\Lambda(ax) - a\Lambda(x)\| = \|\Lambda((a-s)x) - (a-s)\Lambda(x)\| \leqslant K\|a-s\|\|x\| \leqslant K\varepsilon\|x\|,$$

which shows that Λ actually verifies $\Lambda(ax) = a\Lambda(x)$ for every $a \in L_\infty$. It is then a standard fact that Λ must have the form $\Lambda(x) = gx$ on the subspace Y_θ^b of bounded elements of Y_θ.

Since Y_θ is superreflexive, it is σ-order continuous by Theorem 1.a.5 and Proposition 1.a.7 in [28]. So Y_θ^b is dense in Y_θ.

Now, there is also $h > 0$ such that $\|hx\|_{L_1} \leqslant \|x\|_{Y_j}$, $j = 0, 1$. The centralizer Ψ_θ is bounded as a map from Y_θ into $Y_0 + Y_1$, so it is bounded from Y_θ into $L_1(hd\mu)$. The same is true of $\Psi_\theta - \Lambda$, so Λ is bounded from Y_θ into $L_1(hd\mu)$.

Let $x \in Y_\theta$, and take a sequence $(x_n) \subset Y_\theta^b$ such that $x_n \to x$. Then, by the previous considerations, taking limits in $L_1(h d\mu)$,

$$\Lambda(x) = \lim_n \Lambda(x_n) = \lim_n g x_n = g x.$$

So $\Lambda(x) = gx$ for every $x \in Y_\theta$.

Write $g = g_1 + i g_2$, with g_1, g_2 being real functions. Formula (5) shows that the centralizer Ψ_θ is real. So, for every $x \in Y_\theta$ real, we have

$$\|\Psi_\theta(x) - g_1 x\|_{Y_\theta} \leqslant \|\Psi_\theta(x) - gx\|_{Y_\theta} \leqslant C\|x\|_{Y_\theta}$$

for some constant independent of x.

For $x \in Y_\theta$, write $x = x_1 + i x_2$, with x_1, x_2 being real. Then, for some constant C' independent of x,

$$\begin{aligned} \|\Psi_\theta(x) - g_1 x\|_{Y_\theta} &\leqslant \|\Psi_\theta(x) - \Psi_\theta(x_1) - \Psi_\theta(i x_2)\|_{Y_\theta} \\ &\quad + \|\Psi_\theta(x_1) - g_1 x_1\|_{Y_\theta} + \|\Psi_\theta(x_2) - g_1 x_2\|_{Y_\theta} \\ &\leqslant C'(\|x_1\|_{Y_\theta} + \|x_2\|_{Y_\theta}) \\ &\leqslant 2C'\|x\|_{Y_\theta}, \end{aligned}$$

where we have used the quasilinearity of Ψ and the lattice properties of Y_θ. We take $f = g_1$.

Step 2. The spaces Y_0, Y_1 are weighted versions of each other.

Pick $w_0 = e^{\theta f}$ and $w_1 = e^{(\theta-1)f}$. By the previous proposition,

$$(Y_\theta(w_0), Y_\theta(w_1))_\theta = Y_\theta(w_0^{1-\theta} w_1^\theta) = Y_\theta$$

with associated centralizer $\Omega(x) = \log(w_0/w_1)x = fx = \Upsilon(x)$. Thus $\Psi_\theta - \Omega$ is bounded and, by part (3) of Theorem 3.4, we get $Y_0 = X_\theta(w_0)$ and $Y_1 = X_\theta(w_1)$, up to a renorming. □

Theorem 4.4 implies that the map Ω_θ, when trivial, is a bounded perturbation of a multiplication map. This is a consequence of the symmetry properties of the Köthe space. Now we can complete Corollary 4.3 with the following result stating that twisted Hilbert spaces induced by interpolation of Köthe spaces are trivial only in the obvious cases.

Proposition 4.5. *A twisted Hilbert space induced by interpolation at $\theta = 1/2$ between a superreflexive Köthe space X and its dual is trivial if and only if for some weight w we have $X = L_2(w)$ with equivalence of norms.*

Proof. If the twisted space is trivial then since $X_{1/2} = L_2$ (see, e.g., [9]), and since spaces on the whole scale are weighted versions of each other, X and X^* are equal to $L_2(w)$ and $L_2(w^{-1})$ with equivalence of norms, respectively, for some weight. □

4.2. Complex interpolation for families

Here we describe the interpolation method given in [12] with a few modifications introduced in [13], which we will need in the remainder of this section.

We consider an *interpolation family* $(X_\omega)_{\omega \in \mathbb{T}}$ such that each X_ω is continuously embedded in a Banach space Σ, the *containing space*, and there exists a subspace Δ

of $\cap_{\omega\in\mathbb{T}} X_\omega$, the *intersection space*, such that for every $x \in \Delta$ the function $\omega \mapsto \|x\|_\omega$ is measurable and satisfies $\int_{\mathbb{T}} \log^+ \|x\|_\omega \, dt < \infty$. We also suppose that there is a measurable function $k : \mathbb{T} \to [0, \infty)$ satisfying $\int_{\mathbb{T}} \log^+ k(\omega) \, dt < \infty$ and $\|x\|_\Sigma \leqslant k(\omega)\|x\|_\omega$ for every $x \in \Delta$ and every $\omega \in \mathbb{T}$.

We denote by $\mathcal{G}_0$ the space of all analytic functions on $\mathbb{D}$ of the form $g = \sum_{j=1}^n \psi_j x_j$, with ψ_j in the Smirnov class N^+ (see [18, § 2.5]) and $x_j \in \Delta$, such that $\|g\| = \operatorname{ess\,sup}_{\omega\in\mathbb{T}} \|g(\omega)\|_\omega < \infty$. Moreover $\mathcal{G}$ is the completion of $\mathcal{G}_0$.

For each $z_0 \in \mathbb{D}$, we define two spaces. The first one is $X_{\{z_0\}}$, the completion of Δ with respect to the norm $\|x\|_{\{z_0\}} = \inf\{\|g\| : g \in \mathcal{G}_0, g(z_0) = x\}$, and the second one is $X_{[z_0]} = \{f(z_0) : f \in \mathcal{G}\}$ endowed with the natural quotient norm.

By [13, Proposition 1.5], $X_{\{z_0\}} = X_{[z_0]}$ isometrically for every $z_0 \in \mathbb{D}$ when $\mathcal{G} \equiv \mathcal{G}(\mathbb{D}, \Sigma)$ is a Kalton space.

Given $z \in \mathbb{D}$, the Poisson kernel $P_z(\omega)$ on $\mathbb{T}$ (see [12, § 1]) provides the harmonic measure $d\mu_z(\omega) = P_z(\omega)d\omega$ on $\mathbb{T}$, and each function α on $\mathbb{T}$, which is integrable with respect to $d\mu_z$, can be extended to an harmonic function on $\overline{\mathbb{D}}$ by the formula

$$\alpha(z) = \int_{\mathbb{T}} \alpha(\omega) P_z(\omega) \, d\omega. \tag{6}$$

The harmonic conjugate $\tilde{\alpha}$ of α with $\tilde{\alpha}(0) = 0$ is given by $\tilde{\alpha}(z) = \int_{\mathbb{T}} \alpha(\omega) \tilde{P}_z(\omega) \, d\omega$, where $\tilde{P}_z(\omega)$ is the conjugate Poisson kernel. Next we state the reiteration theorem for later use.

Theorem 4.6 [12, Theorem 5.1]. *Let* (X_0, X_1) *be an interpolation pair of Banach spaces, let* $\alpha : \mathbb{T} \to [0, 1]$ *be a measurable function, and let* $X_\omega = (X_0, X_1)_{\alpha(\omega)}$ *for* $\omega \in \mathbb{T}$. *Then* $\{X_\omega\}_{\omega\in\mathbb{T}}$ *is an interpolation family and* $X_{[z]} = (X_0, X_1)_{\alpha(z)}$ *for each* $z \in \mathbb{D}$, $\alpha(z)$ *given by (6), with equality of norms. Moreover, if both* $\inf_{\omega\in\mathbb{T}} \alpha(\omega)$ *and* $\sup_{\omega\in\mathbb{T}} \alpha(\omega)$ *are attained, then* $X_{\{z\}} = X_{[z]}$.

Complex interpolation for admissible families of Köthe spaces. In [22], Kalton considers a variation of the complex interpolation method in [12] for families of Köthe function spaces of μ-measurable functions, where μ is a σ-finite Borel measure on a Polish space. For $U = \mathbb{D}$, he defines the notion of *admissible family* of Köthe function spaces $\{X_\omega\}_{\omega\in\mathbb{T}}$, for which there exist two strictly positive $h, k \in L_0(\mu)$ such that given $x \in L_0(\mu)$, we have $\|xh\|_1 \leqslant \|x\|_\omega \leqslant \|xk\|_\infty$ for every $w \in \mathbb{T}$. The family is *strongly admissible* if, additionally, there exists a countable dimensional subspace V of $L_0(\mu)$ such that $V \cap B_{X_\omega}$ is $L_0(\mu)$-dense in B_{X_ω} for a.e. $\omega \in \mathbb{T}$. These conditions hold in most reasonable situations; for example, if the family is finite and the spaces are separable then the family is strongly admissible. We refer to [22] for the details.

Given an admissible family $\{X_\omega\}_{\omega\in\mathbb{T}}$ of Köthe spaces, the role of Kalton space (Definition 3.1) is played by the space $\mathcal{N}^+(\mathbb{D})$ of functions $f : \mathbb{D} \to L_0(\mu)$ such that

- for μ-almost every s, the function $F_s : \mathbb{D} \to \mathbb{C}$ defined by $F_s(z) = f(z)(s)$ belongs to the Smirnov class N^+ for every $z \in \mathbb{D}$;
- $\|f\| = \operatorname{ess\,sup}_{\omega\in\mathbb{T}} \|f(\omega)\|_\omega < \infty$, where $f(\omega)$ is the radial limit of $f(z)$ in the $L_0(\mu)$-topology (which exists by Fubini's theorem).

The definition of $\mathcal{N}^+(\mathbb{D})$ in [22] does not include the condition $\|f\| = \operatorname{ess\,sup}_{\omega\in\mathbb{T}} \|f(\omega)\|_\omega < \infty$, but we will need it. This amendment is harmless since [22, Proposition 2.4] asserts the existence of extremals in our space, which means that the new space $\mathcal{N}^+(\mathbb{D})$ yields the same spaces X_z.

Remark 1. By [22, Lemma 2.2], each $f \in \mathcal{N}^+(\mathbb{D})$ belongs to the Hardy space $H^1(L_1(hd\mu))$. Hence $\mathcal{N}^+(\mathbb{D})$ consists of analytic functions $f : \mathbb{D} \to L_1(hd\mu)$, the norms of the evaluation maps are at most 1, and the operation of multiplying by a conformal map is isometric on $\mathcal{N}^+(\mathbb{D})$. Moreover, the arguments in [12] allow us to show that $\mathcal{N}^+(\mathbb{D})$ is closed in $H^1(L_1(hd\mu))$. Thus $\mathcal{N}^+(\mathbb{D})$ satisfies the conditions in Definition 3.1 with $\Sigma = L_1(hd\mu)$.

We also need to recall the following notions from [22].

Definition 4.7. A *semi-ideal* is a cone $\mathcal{I} \subset L_1^+$ such that $g \in \mathcal{I}$ and $0 \leqslant f \leqslant g$ imply $f \in \mathcal{I}$. A *strict semi-ideal* is a semi-ideal that contains a strictly positive element.

Given a Köthe function space X, we consider the semi-ideal $\mathcal{I}_X$ of all $f \in L_1^+$ such that $\sup_{x\in B_X} \int f \log^+ |x|\, d\mu < \infty$ and there is $x \in B_X$ such that $\int f\, |\log |x||\, d\mu < \infty$.
The *indicator of* X is the map $\Phi_X : \mathcal{I}_X \to \mathbb{R}$ given by $\Phi_X(f) = \sup\limits_{x\in B_X} \int_S f \log |x|\, d\mu$.

We will need the following result.

Theorem 4.8 [22, Theorem 4.7]. *Given a strongly admissible family $\{X_\omega\}_{\omega\in\mathbb{T}}$, there is a strict semi-ideal $\mathcal{I}$ such that for each $z_0 \in \mathbb{D}$ and $f \in \mathcal{I}$, we have $\mathcal{I} \subset \mathcal{I}_{X_{z_0}}$, the map $t \mapsto \Phi_{X_{e^{it}}}(f)$ is bounded and measurable, and*

$$\Phi_{X_{z_0}}(f) = \frac{1}{2\pi}\int_0^{2\pi} \Phi_{X_{e^{it}}}(f) P_{z_0}(e^{it})\, dt.$$

The core of Kalton's method is that centralizers on a separable Köthe space X actually live on $L_1(\mu)$. Precisely, given a centralizer Ω on X, then $L_1 = XX^*$ by Lozanovskii's factorization [29]. Note that both X and X^* are spaces of functions. Thus for each $f \in L_1$, there exist $x \in X$ and $x^* \in X^*$ such that $f = xx^*$ and $\|x\|\|x^*\| \leqslant 2\|f\|$, so one can set

$$\Omega^{[1]}(f) = \Omega(x)x^*.$$

This is a centralizer on L_1 and there is a constant $C > 0$ so that, whenever $f = yy^*$ with $y \in X$, $y^* \in X^*$,

$$\|\Omega^{[1]}(f) - \Omega(y)y^*\|_{L_1} \leqslant C\|y\|\|y^*\|.$$

See [21, Theorem 5.1] for details. When $X = L_p$ $(1 < p < \infty)$, $\Omega^{[1]}(f) = u|f|^{1/q}\, \Omega(|f|^{1/p})$, where $u|f|$ is the polar decomposition of f and $p^{-1} + q^{-1} = 1$.

Given a centralizer Ω on a Köthe space X, Kalton considers the strict ideal $\mathcal{I}_\Omega \subset L_1$ of those elements $f \in L_1$ for which $\Omega^{[1]}(f) \in L_1$, and defines on $\mathcal{I}_\Omega$ the functional

$$\Phi^\Omega(f) = \int \Omega^{[1]}(f)\, d\mu.$$

The crucial properties of this functional are established in the next result.

Theorem 4.9 [22, Proposition 7.4]. (1) *Let* (X_0, X_1) *be an interpolation couple of Köthe spaces and let* Ω_θ *be the derivation map associated with* X_θ*. Then on a suitable semi-ideal, one has* $\Phi^{\Omega_\theta} = \Phi_{X_0} - \Phi_{X_1}$.

(2) *Let* $\{X_\omega\}_{\omega\in\mathbb{T}}$ *be a strongly admissible family. If* Ω *is the centralizer associated with* X_z *for* $z = 0$*, then on a suitable strict semi-ideal* $\mathcal{I} \subset \mathcal{I}_\Omega$*, one has that for every* $f \in \mathcal{I}$

$$\Phi^\Omega(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-it}\Phi_{X_{e^{it}}}(f)\,dt.$$

4.3. Stability for families of three Köthe spaces

Here we prove the global stability of both splitting and bounded splitting for interpolation families consisting of three spaces distributed on arcs of $\mathbb{T}$. The starting point is the generalization of the formula $X_\theta = X_0^{1-\theta}X_1^\theta$ for families presented in [22, Theorem 3.3] that Kalton credits to Hernández [20]. Note that the statement in [22, Theorem 3.3] supposes that the family is strongly admissible, but the proof is valid for admissible families.

Definition 4.10. Given Köthe spaces $X(1), \ldots, X(n)$ and positive numbers $a_1, \ldots, a_n$, we define

$$\prod_{j=1}^n X(j)^{a_j} = \left\{ f \in L_0 : |f| \leqslant \prod_{j=1}^n \left|f_j\right|^{a_j},\ f_j \in X(j) \right\}$$

endowed with the norm $\|f\|_\prod = \inf\{\prod_{j=1}^n \|f_j\|_{X(j)}^{a_j}\}$, where the infimum is taken over all choices of $f_j \in X(j)$ so that $|f| \leqslant \prod_{j=1}^n \left|f_j\right|^{a_j}$.

The following result provides the associated derivation map. For the sake of clarity, we have included a streamlined proof of the factorization theorem.

Proposition 4.11. *Let* $\{A_1, \ldots, A_n\}$ *be a partition of* $\mathbb{T}$ *into arcs, and let* $\{X_\omega\}_{\omega\in\mathbb{T}}$ *be the admissible family given by* $X_\omega = X(j)$ *for* $\omega \in A_j$*,* $j = 1, \ldots, n$*. If* μ_{z_0} *is the harmonic measure on* $\mathbb{T}$ *with respect to* z_0*, then* $X_{z_0} = \prod_{j=1}^n X(j)^{\mu_{z_0}(A_j)}$.

In particular, if X *is an admissible Köthe function space,* w_j *are weights, and we take* $X(j) = X(w_j)$*, then the family* $\{X_\omega\}_{\omega\in\mathbb{T}}$ *as above is admissible and* $X_{z_0} = X(\prod w_j^{\mu_{z_0}(A_j)})$ *for* $z_0 \in \mathbb{D}$*, with associated derivation* $\Omega_{z_0}(x) = -\left(\sum_j \psi_j'(z_0)\log w_j\right)x$*, where* ψ_j *is an analytic function on* $\mathbb{D}$ *such that* $Re\,\psi_j = \chi_{A_j}$ *on* $\mathbb{T}$ *and* $Re\,\psi_j(z_0) = \mu_{z_0}(A_j)$.

Proof. Pick $f \in X_{z_0}$. In order to apply [22, Lemma 3.2 and Theorem 3.3], recall that if $\mathcal{E}$ denotes the Köthe function space on $\mathbb{D}\times\mathbb{T}$ with norm $\|\phi\|_\mathcal{E} = \operatorname{ess\,sup}\|\phi(\cdot,\omega)\|_{X_\omega}$, then there is $\phi \in \mathcal{E}$ so that $\|\phi\|_\mathcal{E} = \|f\|_{X_{z_0}}$ and

$$|f(s)| = \exp\left(\int_\mathbb{T} P_{z_0}(\omega)\log\phi(s,\omega)\,d\omega\right).$$

By Jensen's inequality,

$$|f(s)| \leqslant \prod_{j=1}^{n} \left(\frac{1}{\mu_{z_0}(A_j)} \int_{A_j} \phi(s,\omega) P_{z_0}(\omega)\, d\omega \right)^{\mu_{z_0}(A_j)}.$$

Set $f_j(s) = \frac{1}{\mu_{z_0}(A_j)} \int_{A_j} \phi(s,\omega) P_{z_0}(\omega)\, d\omega$ so that

$$\begin{aligned} \|f_j\|_{X(j)} &= \left\| \frac{1}{\mu_{z_0}(A_j)} \int_{A_j} \phi(\cdot,\omega) P_{z_0}(z)\, d\omega \right\|_{X(j)} \\ &\leqslant \frac{1}{\mu_{z_0}(A_j)} \int_{A_j} \|\phi(\cdot,\omega)\|_{X(j)} P_{z_0}(\omega)\, d\omega \leqslant \|\phi\|_{\mathcal{E}}. \end{aligned}$$

Then $f_j \in X(j)$ and $|f| \leqslant \prod |f_j|^{\mu_{z_0}(A_j)}$, and thus $\|f\|_{\prod} \leqslant \prod \|f_j\|_{X(j)}^{\mu(A_j)} \leqslant \|\phi\|$. So $\|f\|_{\prod} \leqslant \|f\|_{X_{z_0}}$. Assume now that $|f| \leqslant \prod |f_j|^{\mu_{z_0}(A_j)}$, and let ϕ be given by $\phi(s,\omega) = \prod |f_j(s)|^{\varphi_j(\omega)}$, where φ_j is a harmonic function that coincides with χ_{A_j} on $\mathbb{T}$, $j = 1, \ldots, n$. Then $\varphi_j(z_0) = \mu_{z_0}(A_j)$, and

$$\begin{aligned} |f(s)| &\leqslant \prod |f_j(s)|^{\varphi_j(z_0)} = \exp\left(\log \prod |f_j(s)|^{\varphi_j(z_0)}\right) \\ &= \exp\left(\sum \log |f_j(s)|^{\varphi_j(z_0)}\right) = \exp\left(\sum \int_{\mathbb{T}} \log |f_j(s)|^{\varphi_j(\omega)} P_{z_0}(\omega)\, d\omega\right) \\ &= \exp\left(\sum \int_{A_j} \log |f_j(s)|^{\varphi_j(\omega)} P_{z_0}(\omega)\, d\omega\right) \\ &= \exp\left(\sum \int_{A_j} \log \prod_k |f_k(s)|^{\varphi_k(\omega)} P_{z_0}(\omega)\, d\omega\right) \\ &= \exp\left(\int_{\mathbb{T}} \log \prod |f_j(s)|^{\varphi_j(\omega)} P_{z_0}(\omega)\, d\omega\right) = \exp\left(\int_{\mathbb{T}} \log \phi(s,\omega) P_{z_0}(\omega)\, d\omega\right). \end{aligned}$$

Therefore, $\|f\|_{X_{z_0}} \leqslant \|\phi\| = \max \|f_j\|_{X(j)}$. If we multiply each f_j by $\frac{\prod \|f_i\|_{X(i)}^{\mu_{z_0}(A_i)}}{\|f_i\|_{X(i)}}$, then we still have that $|f(s)| \leqslant \prod |f_j(s)|^{\mu_{z_0}(A_j)}$ and $\|f\|_{X_{z_0}} \leqslant \prod \|f_j\|_{X(j)}^{\mu_{z_0}(A_j)}$. Since the functions f_j are arbitrary, we get $\|f\|_{X_{z_0}} \leqslant \|f\|_{\prod}$.

For the second part, let h_1 and k_1 be functions that show that X is admissible. We set $h = h_1 \min w_j$ and $k = k_1 \max w_j$. Then h and k are such that $\|xh\|_1 \leqslant \|x\|_z \leqslant \|xk\|_\infty$ for every $x \in X$ and $z \in \mathbb{T}$. Also $\|xw_jh_1\| \leqslant \|x\|_{X(w_j)} \leqslant \|xw_jk_1\|_\infty$. Since it is clear that $B_{X(w_j)}$ is closed in L_0, each space $X(w_j)$ is admissible.

Since $X_{z_0} = \prod X(j)^{\mu_{z_0}(A_j)}$, for every $x \in X_{z_0}$, one has

$$\begin{aligned} \|x\|_{z_0} &= \inf\left\{ \prod \|x_j\|_{X(j)}^{\mu_{z_0}(A_j)} : |x| \leqslant \prod |x_j|^{\mu_{z_0}(A_j)} \right\} \\ &= \inf\left\{ \prod \|w_j x_j\|_{X}^{\mu_{z_0}(A_j)} : |x| \leqslant \prod |x_j|^{\mu_{z_0}(A_j)} \right\} \\ &\geqslant \inf\left\{ \| \prod (w_j x_j)^{\mu_{z_0}(A_j)} \|_X : |x| \leqslant \prod |x_j|^{\mu_{z_0}(A_j)} \right\} \end{aligned}$$

$$\geqslant \inf \left\{ \| \prod w_j^{\mu_{z_0}(A_j)} x \|_X : |x| \leqslant \prod |x_j|^{\mu_{z_0}(A_j)} \right\}$$
$$= \|x\|_{X(\prod w_j^{\mu_{z_0}(A_j)})},$$

where the first inequality follows from $\|x^\theta y^{1-\theta}\| \leqslant \|x\|^\theta \|y\|^{1-\theta}$ and an induction argument, and the second one from $|x| \leqslant \prod |x_j|^{\mu_{z_0}(A_j)}$. Moreover, taking $f_j = \prod_k w_k^{\mu_{z_0}(A_k)} w_j^{-1} x$, we get the reverse inequality. Now let

$$F(z) = \prod_k w_k^{\mu_{z_0}(A_k)} \frac{x}{\prod_j w_j^{\psi_j(z)}}.$$

Then $F \in \mathcal{N}^+(\mathcal{H})$, $F(z_0) = x$, and for $\omega \in A_j$ one has

$$\|F(\omega)\|_{X_j} = \left\| w_j \prod w_k^{\mu_{z_0}(A_k)} \frac{x}{w_j} \right\|_X = \|x\|_{X_{z_0}}.$$

Therefore F is a 1-extremal function for x. Moreover

$$\Omega_{z_0}(x) = F'(z_0) = -\prod w_j^{\mu_{z_0}(A_j)} x \prod_k w_k^{-\psi_k(z_0)} \sum_j \psi_j'(z_0) \log w_j = -\Big(\sum_j \psi_j'(z_0) \log w_j\Big) x,$$

and the proof is done. □

We now pass to the study of the (bounded) stability. We first observe that the extension to $\overline{\mathbb{D}}$ of a conformal transformation on $\mathbb{D}$ taking z_0 to 0 takes an arc of $\mathbb{T}$ onto an arc of $\mathbb{T}$. So we can assume without loss of generality that $z_0 = 0$, and so we will do throughout this section.

Lemma 4.12. *Let $A_0 = [\theta_0, \theta_1)$, $A_1 = [\theta_1, \theta_2)$, and $A_2 = [\theta_2, \theta_0)$ be a partition of $[0, 2\pi)$. For $j = 0, 1, 2$, set $\alpha_j = \frac{1}{2\pi} \int_{A_j} P_0(e^{it})\, dt$ and $\beta_j = \frac{1}{2\pi} \int_{A_j} e^{-it}\, dt$. Then the vectors $a = (\alpha_0, \alpha_1, \alpha_2)$, $b = (Re(\beta_0), Re(\beta_1), Re(\beta_2))$, and $c = (Im(\beta_0), Im(\beta_1), Im(\beta_2))$ are linearly independent in $\mathbb{R}^3$. Consequently, we can find $a_j \in \mathbb{C}$ such that $\sum a_j \alpha_j = 0$ and $\sum a_j \beta_j = -1$.*

Proof. We begin by noting that $\sum \alpha_j = 1$ and $\sum \beta_j = 0$. So a cannot be written as a linear combination of b and c. Also, the only way for $\{a, b, c\}$ to be linearly dependent is if b is a multiple of c. We have

$$\frac{-i}{2}\beta_0 = -\sin\frac{\theta_1 - \theta_0}{2} \sin\frac{\theta_1 + \theta_0}{2} + i \sin\frac{\theta_0 - \theta_1}{2} \cos\frac{\theta_0 + \theta_1}{2} \quad \text{and}$$
$$\frac{-i}{2}\beta_1 = -\sin\frac{\theta_2 - \theta_1}{2} \sin\frac{\theta_2 + \theta_1}{2} + i \sin\frac{\theta_1 - \theta_2}{2} \cos\frac{\theta_1 + \theta_2}{2}.$$

So, if we consider the matrix with lines b and c, up to a factor of $\frac{-i}{2}$ the determinant of the first two columns is

$$-\sin\frac{\theta_1 - \theta_0}{2} \sin\frac{\theta_1 + \theta_0}{2} \sin\frac{\theta_1 - \theta_2}{2} \cos\frac{\theta_1 + \theta_2}{2}$$
$$+\sin\frac{\theta_0 - \theta_1}{2} \cos\frac{\theta_0 + \theta_1}{2} \sin\frac{\theta_2 - \theta_1}{2} \sin\frac{\theta_2 + \theta_1}{2}$$

$$= \sin\frac{\theta_0-\theta_1}{2}\sin\frac{\theta_2-\theta_1}{2}\left(\cos\frac{\theta_0+\theta_1}{2}\sin\frac{\theta_2+\theta_1}{2}-\sin\frac{\theta_1+\theta_0}{2}\cos\frac{\theta_1+\theta_2}{2}\right)$$
$$= \sin\frac{\theta_0-\theta_1}{2}\sin\frac{\theta_2-\theta_1}{2}\sin\frac{\theta_2-\theta_0}{2},$$

which is zero if and only if two of the $\theta_j's$ are equal, which is not the case. □

The expressions for α_j and β_j are provided by Theorems 4.8 and 4.9.

It follows from Kalton's Theorem 3.4 that given two interpolation couples (X_0, X_1) and (Y_0, Y_1) such that $(X_0, X_1)_\theta = (Y_0, Y_1)_\theta$ (up to renorming) and $\Omega_\theta = \Upsilon_\theta$ (up to a bounded map) for some $0 < \theta < 1$, then $X_0 = Y_0$ and $X_1 = Y_1$ (up to renorming). Next we give the version for *three spaces on arcs* of that result. The proof is essentially an adaptation of the proof of the uniqueness part of Theorem 7.6 of [22].

Theorem 4.13. *Let $\{X_\omega : \omega \in \mathbb{T}\}$ and $\{Y_\omega : \omega \in \mathbb{T}\}$ be two strongly admissible families with $X_\omega = X(j)$ and $Y_\omega = Y(j)$ for $\omega \in A_j$, $j = 0, 1, 2$, where $\{A_0, A_1, A_2\}$ is a partition of $\mathbb{T}$ into arcs. Let Ω_0 and Ψ_0 be the corresponding derivations at $z_0 = 0$. If $X_0 = Y_0$ (up to renorming) and $\Omega_0 - \Psi_0$ is bounded, then $X(j) = Y(j)$ (up to renorming) for $j = 0, 1, 2$. Moreover, $\Omega_z - \Psi_z$ is bounded for every $z \in \mathbb{D}$.*

Proof. Since Ω_0 and Ψ_0 are equivalent, so are $\Omega_0^{[1]}$ and $\Psi_0^{[1]}$ by definition, and then

$$d(\Phi^{\Omega_0}, \Phi^{\Psi_0}) = \sup_{\|f\|\leqslant 1, f\in\mathcal{I}} \left|\Phi^{\Omega_0}(f) - \Phi^{\Psi_0}(f)\right| < \infty.$$

We can use now Theorems 4.8, 4.9 and Lemma 4.12 to get equations that determine $\Phi_{X(j)}$ in terms of Φ_{X_0}, $Re(\Phi^{\Omega_0})$, and $Im(\Phi^{\Omega_0})$; and the same for $\Phi_{Y(j)}$ in terms of $\Phi_{Y_0} = \Phi_{X_0}$, $Re(\Phi^{\Psi_0})$, and $Im(\Phi^{\Psi_0})$. More specifically, on a suitable strict semi-ideal one has

$$\begin{aligned}
\alpha_0\Phi_{X(0)} + \alpha_1\Phi_{X(1)} + \alpha_2\Phi_{X(2)} &= \Phi_{X_0}\\
Re(\beta_0)\Phi_{X(0)} + Re(\beta_1)\Phi_{X(1)} + Re(\beta_1)\Phi_{X(2)} &= Re(\Phi^{\Omega_0})\\
Im(\beta_0)\Phi_{X(0)} + Im(\beta_1)\Phi_{X(1)} + Im(\beta_1)\Phi_{X(2)} &= Im(\Phi^{\Omega_0})\\
\alpha_0\Phi_{Y(0)} + \alpha_1\Phi_{Y(1)} + \alpha_2\Phi_{Y(2)} &= \Phi_{Y_0}\\
Re(\beta_0)\Phi_{Y(0)} + Re(\beta_1)\Phi_{Y(1)} + Re(\beta_1)\Phi_{Y(2)} &= Re(\Phi^{\Psi_0})\\
Im(\beta_0)\Phi_{Y(0)} + Im(\beta_1)\Phi_{Y(1)} + Im(\beta_1)\Phi_{Y(2)} &= Im(\Phi^{\Psi_0}).
\end{aligned}$$

Lemma 4.12 establishes that there is a unique solution for the numerical system

$$\begin{aligned}
\alpha_0 x + \alpha_1 y + \alpha_2 z &= a\\
Re(\beta_0)x + Re(\beta_1)y + Re(\beta_1)z &= b\\
Im(\beta_0)x + Im(\beta_1)y + Im(\beta_1)z &= c
\end{aligned}$$

and two uniformly bounded sets of data $(a(x), b(x), c(x))$ and $(a'(x), b'(x), c'(x))$ with bounded difference will produce two solutions $\Phi_{X(j)}$ and $\Phi_{Y(j)}$ with bounded difference. So we can use [22, Proposition 4.5] to conclude that $X(j) = Y(j)$ (up to renorming) for $j = 0, 1, 2$. □

Now, after a preparatory lemma, we consider the stability results for three Köthe spaces.

Lemma 4.14. *Let X be a Köthe function space, let $\{A_0, A_1, A_2\}$ be a partition of $\mathbb{T}$ into arcs, and let $f \in L_0(\mu)$. Then there are weights ω_j such that taking $Y_\omega = X(\omega_j)$ for $\omega \in A_j$ and $j = 0, 1, 2$, the admissible family $\{Y_\omega : \omega \in \mathbb{T}\}$ yields $Y_0 = X$ with derivation map $\Omega_0(x) = f \cdot x$, a (linear) multiplication map.*

Proof. Write $f = f_1 + i f_2$. By Lemma 4.12, there are real numbers a_0, a_1, a_2 such that $\sum a_j \alpha_j = 0$ and $\sum a_j \beta_j = -1$, and there are real numbers b_0, b_1, b_2 such that $\sum b_j \alpha_j = 0$ and $\sum b_j \beta_j = -i$. Set $w_j = e^{a_j f_1 + b_j f_2}$. Then $Y_0 = X(w_0^{\alpha_0} w_1^{\alpha_1} w_2^{\alpha_2}) = X$ and $\Omega_0 = -\sum \beta_j (a_j f_1 + b_j f_2) = f$. □

Theorem 4.15. *Let $\{X_\omega : \omega \in \mathbb{T}\}$ be the strongly admissible family with $X_\omega = X(j)$ for $\omega \in A_j$ and $j = 0, 1, 2$, where $\{A_0, A_1, A_2\}$ is a partition of $\mathbb{T}$ into arcs. If the derivation map Ω_0 is trivial, then there are weights w_j such that $X(j) = X_0(w_j)$ with equivalence of norms. In particular, Ω_z is trivial for every $z \in \mathbb{D}$.*

Proof. If Ω_0 is trivial, then arguing as in Theorem 4.4 (up to passing to weighted versions of the spaces), we get $f \in L_0(\mu)$ so that $\Lambda(x) = f \cdot x$ is a linear map, and $\Omega_0 - \Lambda$ takes X_0 into X_0 and it is bounded on X_0. We take $X = X_0$ in Lemma 4.14 so that we obtain a new family $\{Y_\omega : \omega \in \mathbb{T}\}$ which is strongly admissible, since X_w is so, and whose induced centralizer at 0 is Λ. By Theorem 4.13, we obtain that $X(j) = X_0(w_j)$ with equivalence of norms for some suitable weights.

To see that Ω_z is trivial for every $z \in \mathbb{D}$, observe that the Kalton spaces $\mathcal{N}^+(\{X_\omega\})$ and $\mathcal{N}^+(\{Y_\omega\})$ associated with the families $\{X_\omega\}$ and $\{Y_\omega\}$ coincide, with equivalence of norms. In particular, $X_z = Y_z$ for all $z \in \mathbb{D}$, with equivalence of norms.

Let Λ_z be the trivial centralizer induced by $\{Y_\omega\}$ at z. Given $x \in V$, we fix $(1+\epsilon)$-extremals $F_x \in \mathcal{N}^+(\{X_\omega\})$ and $G_x \in \mathcal{N}^+(\{Y_\omega\})$ so that $\Omega_z(x) = \delta'_z F_x$ and $\Lambda_z(x) = \delta'_z G_x$. Thus there are some constants C, C' such that, for all $x \in V$, one has

$$\begin{aligned} \|\Omega_z(x) - \Lambda_z(x)\|_{X_z} &= \|\delta'_z(F_x - G_x)\|_{X_z} \\ &\leqslant \|\delta'_z : \ker \delta_z \to X_z\| \|F_x - G_x\|_{\mathcal{N}^+} \\ &\leqslant C(\|F_x\|_{\mathcal{N}^+(\{X_\omega\})} + \|G_x\|_{\mathcal{N}^+(\{Y_\omega\})}) \\ &\leqslant C(1+\epsilon)(\|x\|_{X_z} + \|x\|_{Y_z}) \\ &\leqslant C'\|x\|_{X_z}. \end{aligned}$$

Since Λ_z is trivial and V is dense in X_z, the derivation Ω_z is trivial. □

Observe that our reasoning does not work for general families of three spaces not distributed in arcs. This should be compared with the results of [31], previously obtained in [15].

4.4. (Bounded) stability fails for families of four Köthe spaces

Here we show that the statement of Theorem 3.4 is no longer true for arbitrary families of Köthe spaces.

Let c_{00} denote the space of finitely nonzero sequences of scalars. A sequence of functions (φ_n) that are continuous on $\overline{\mathbb{D}}$ and analytic on $\mathbb{D}$ induces a family of diagonal linear maps $D(z) : c_{00} \to c_{00}$ $(z \in \overline{\mathbb{D}})$ given by $D(z)(x_n) = (\varphi_n(z)x_n)$. We define a family of Banach spaces $\{X_s : s \in \mathbb{T}\}$ by taking as X_s the completion of c_{00} with respect to the norm $\|x\|_s = \|e^{-D(s)}x\|_2$. Moreover, for $x \in c_{00}$, we define $\|x\|_\Sigma = \inf\{\|x_1\|_{z_1} + \cdots + \|x_n\|_{z_n}\}$, where the infimum is taken over all $n \in \mathbb{N}$, $z_i \in \mathbb{T}$, and $x_i \in c_{00}$ such that $x = x_1 + \cdots + x_n$.

We claim that $\|.\|_\Sigma$ is a norm on c_{00}. Indeed, the only difficulty is to show that $\|x\|_\Sigma = 0$ implies $x = 0$. Let $x = (a^j) \in c_{00}$ with $a^k \neq 0$. Note that $e^{-D(z)}$ is the multiplication operator associated with the sequence $(e^{-\varphi_n(z)})$. If $|\varphi_k(z)| \leqslant M$, then $\left|e^{-\varphi_k(z)}\right| = e^{-Re(\varphi_k(z))} \geqslant e^{-M}$. Therefore

$$\sum \|x_j\|_{z_j} = \sum \|e^{-D(z_j)}x_j\|_2 \geqslant e^{-M}\left|a^k\right|,$$

and we conclude that $\|x\|_\Sigma > 0$.

Let Σ be the completion of c_{00} with respect to $\|.\|_\Sigma$. Then for each $\omega \in \mathbb{T}$, we have $X_\omega \subset \Sigma$ with inclusion having norm at most 1. Note also that the projection P_n onto the first n coordinates is a norm-one operator on X_ω for each $\omega \in \mathbb{T}$, and also on Σ.

Proposition 4.16. *The above defined family $(X_\omega)_{\omega\in\mathbb{T}}$ is an interpolation family with containing space Σ and intersection space $\Delta = c_{00}$. Moreover, for every z_0 in $\mathbb{D}$, one has the following:*

1. $X_{\{z_0\}} = X_{[z_0]}$. *Thus we can denote $X_{z_0} = X_{\{z_0\}} = X_{[z_0]}$.*
2. *The space X_{z_0} is the completion of c_{00} with respect to the norm $\|x\|_{z_0} = \|e^{-D(z_0)}x\|_2$.*
3. $\Omega_{z_0}x = D'(z_0)x$ *for $x \in c_{00}$.*

Proof. (1) Let $x \in c_{00}$. Clearly $\|x\|_{[z_0]} \leqslant \|x\|_{\{z_0\}}$. Let $f \in \mathcal{F}$ (see § 3) be such that $f(z_0) = x$. Take n such that $P_n(x) = x$ and define $g(z) = P_n(f(z))$. Then $g(z) = \sum_{j=1}^n \psi_j(z)e_j$, where (e_j) is the canonical basis of ℓ_2. Since $\psi_j(z)e_j = (P_j - P_{j-1})f(z)$ and f is analytic when viewed as a Σ-valued function, we get that ψ_j is analytic. If $z \in \mathbb{D}$, then

$$\left|\psi_j(z)\right| \|e_j\|_\Sigma = \|(P_j - P_{j-1})f(z)\|_\Sigma \leqslant 2\|f(z)\|_\Sigma \leqslant 2\|f\|_\mathcal{F}.$$

Hence $\psi_j \in H^\infty$, the space of bounded analytic functions on $\mathbb{D}$, which is contained in the Smirnov class N^+. Also, for almost every $z \in \mathbb{T}$, we have $\|g(z)\|_{X_z} = \|P_n(f(z))\|_{X_z} \leqslant \|f(z)\|_{X_z} \leqslant \|f\|_\mathcal{F}$. Thus $g \in \mathcal{G}$ and $\|g\|_\mathcal{G} \leqslant \|f\|_\mathcal{G}$. Since $g(z_0) = P_n(f(z_0)) = x$, we get $\|x\|_{\{z_0\}} = \|x\|_{[z_0]}$.

To prove (2), let $x \in c_{00}$ and let $g(z) = e^{D(z)-D(z_0)}x \in \mathcal{G}$. Then $g(z_0) = x$, and for $z \in \mathbb{T}$ we have $\|g(z)\|_{X_z} = \|e^{-D(z_0)}x\|_2$. Thus $\|x\|_{z_0} \leqslant \|e^{-D(z_0)}x\|_2$. Take $f \in \mathcal{G}$ such that $f(z_0) = x$. Given a nonzero $y \in c_{00}$, define $h(z) = \langle e^{-D(z)}f(z), y\rangle$.

It follows that $h \in H^\infty$. Indeed, f may be written as a finite sum $\sum f_i x_i$, with $f_i \in N^+$ and $x_i \in c_{00}$. Since $e^{-D(z)}$ is bounded, we have that $e^{-D(z)}f_i(z) \in N^+$. This implies that $h \in N^+$, and since it is bounded on $\mathbb{T}$, $h \in H^\infty$ [18, Theorem 2.11]. Moreover, $\|h\|_{H^\infty} \leqslant \|f\|_\mathcal{G}\|y\|_2$. Hence $|h(z_0)| = \left|\langle e^{-D(z_0)}x, y\rangle\right| \leqslant \|f\|_\mathcal{G}\|y\|_2$. Since $\|f\|_\mathcal{G}$ can be taken arbitrarily close to $\|x\|_{z_0}$ and y is arbitrary, $\|e^{-D(z_0)}x\|_2 \leqslant \|x\|_{z_0}$.

(3) We have shown that the following function g is an extremal function for $x = (a^j)$ at z_0:

$$g(z) = e^{D(z)-D(z_0)}x = (e^{\varphi_1(z)-\varphi_1(z_0)}a^1, e^{\varphi_2(z)-\varphi_2(z_0)}a^2, \ldots).$$

Then $g'(z) = \left(\varphi_n'(z)e^{\varphi_n(z)-\varphi_n(z_0)}a^n\right)$; hence $\Omega_{z_0}x = g'(z_0) = \left(\varphi_n'(z_0)a^n\right) = D'(z_0)x$. □

By Proposition 4.16, there is no local bounded stability for arbitrary families of Köthe spaces.

Proposition 4.17. *Let $D(x_n) = (w_n x_n)$ be an unbounded diagonal operator on c_{00}.*

(1) *The choice $D(z) = zD$ yields an analytic family for which $\Omega_z = D$ for every $z \in \mathbb{D}$.*

(2) *The choice $D(z) = z^2 D$ yields an analytic family such that $\Omega_z = 2zD$ for every $z \in \mathbb{D}$. Therefore $\Omega_0 = 0$ while Ω_z is unbounded for $z \neq 0$.*

We pass now to show that there is no local stability for families of Köthe spaces.

Proposition 4.18. *Let $p : \mathbb{T} \to [1, \infty)$ be a measurable function and let $\alpha : \overline{\mathbb{D}} \to \mathbb{C}$ be an analytic function on $\mathbb{D}$ satisfying $Re\left(\alpha(z)^{-1}\right) = p(z)^{-1}$ on $\mathbb{T}$, and such that* $\inf Re(\omega)$ *and* $\sup Re(\omega)$ *are attained on $\mathbb{T}$. We consider the interpolation family $(\ell_{p(\omega)})_{\omega\in\mathbb{T}}$. Given $z_0 \in \mathbb{D}$ with $\alpha(z_0) \in \mathbb{R}$, the interpolation space at z_0 is $\ell_{p(z_0)}$ with derivation*

$$\Omega_{z_0}((x_n)) = -\frac{\alpha'(z_0)}{\alpha(z_0)}\left(x_n \log \frac{|x_n|}{\|x\|_{\ell_{p(z_0)}}}\right).$$

Proof. The containing space for the family is ℓ_∞, and the intersection space may be taken as c_{00}. We first check that

$$f(z) = \left(|x_n|^{\frac{\alpha(z_0)}{\alpha(z)}} \frac{x_n}{|x_n|}\right)$$

is a 1-extremal for $x = (x_n) \in c_{00}$ with $\|x\|_{p(z_0)} = 1$. The function f is analytic, $f(z_0) = x$, and $f \in \mathcal{G}$ because for every $z \in \mathbb{D}$

$$\|f(z)\|_{p(z)}^{p(z)} = \sum |x_n|^{Re(\frac{\alpha(z_0)}{\alpha(z)})p(z)} = \sum |x_n|^{\alpha(z_0)} = \sum |x_n|^{p(z_0)} = 1.$$

Therefore, $\|x\|_{z_0} \leqslant \|x\|_{p(z_0)}$. Hence $\|x\|_{z_0} = \|x\|_{p(z_0)}$ by Theorem 4.6. Moreover, for nonzero $x = (x_n) \in \ell_{p(z_0)}$, one has

$$\begin{aligned}\Omega_{z_0}(x) &= \|x\|_{\ell_{p(z_0)}} \Omega_{z_0}\left(\frac{x}{\|x\|_{\ell_{p(z_0)}}}\right)\\ &= \|x\|_{\ell_{p(z_0)}}\left(-\frac{x_n}{|x_n|}\left(\frac{|x_n|}{\|x\|_{\ell_{p(z_0)}}}\right)^{\frac{\alpha(z_0)}{\alpha(z_0)}} \frac{\alpha(z_0)}{\alpha(z_0)^2}\alpha'(z_0)\log\frac{|x_n|}{\|x\|_{\ell_{p(z_0)}}}\right)\\ &= -\frac{\alpha'(z_0)}{\alpha(z_0)}\left(x_n \log\frac{|x_n|}{\|x\|_{\ell_{p(z_0)}}}\right),\end{aligned}$$

and the proof is complete. □

An exact sequence is *singular* when the quotient map q is strictly singular; i.e., no restriction of q to an infinite-dimensional subspace is an isomorphism. A derivation is said to be *singular* if the induced exact sequence is singular [6]. Obviously, singular derivations are not trivial.

Proposition 4.19. *The family* $(\ell_{p(z)})_{z\in\mathbb{T}}$ *with* $p(z)^{-1} = Re\left((z^2+2)^{-1}\right)$ *yields* $\Omega_0 = 0$ *and* Ω_z *singular for* $0 \neq z \in \mathbb{D}$.

Proof. Since $p(z)^{-1} \in [1/3, 1]$, it turns out that $p(z) \in [1, 3]$. We thus set $\alpha(z) = z^2 + 2$ on $\mathbb{D}$. In that case, we get $\alpha(z) \in \mathbb{R}$ if and only if $z = t$ or $z = it$, $t \in \mathbb{R}$. By the previous lemma, $\Omega_0 = 0$, and for $z = t$ and $z = it$, $t \neq 0$, Ω_z is a nonzero multiple of the Kalton–Peck map on $\ell_{p(z_0)}$, and therefore it is singular. Moreover, the choice $\alpha_{z_0}(z) = z^2 + 2 - i\mathrm{Im}(\alpha(z_0))$ yields that Ω_{z_0} is a nonzero multiple of the Kalton–Peck map for any $z_0 \in \mathbb{D}$, $z_0 \neq 0$. □

The moral of all this. We can present two explanations for the fact that families of two or three Köthe spaces have global (bounded) stability and are even rigid in different senses while families of four spaces do not. The first one emerges from the proof of Theorem 4.13: any point in the interior of the convex hull of three points admits a unique representation as a convex combination of them, but this is false for four points. The second one arises from the reiteration theorem for families [12]. Using that result to set the initial configuration, one gets the following theorem.

Theorem 4.20. *Let* α *and* $(X_0, X_1)_{\alpha(\omega)}$ *for* $\omega \in \mathbb{T}$ *be as in Theorem 4.6, let* Ω_s *denote the derivation corresponding to* $(X_0, X_1)_s$ *for* $0 < s < 1$*, and let* $z_0 \in \mathbb{D}$*. Then the derivation corresponding to the family* $(X_0, X_1)_{\alpha(z)}$ *at* z_0 *is* $\Phi_{z_0} = w'(z_0)\Omega_{\alpha(z_0)}$*, where* $w = \alpha + i\tilde{\alpha}$ *and* $\tilde{\alpha}$ *is the harmonic conjugate of* α *with* $\tilde{\alpha}(z_0) = 0$.

Proof. Fix $z \in \mathbb{D}$ and $x \in X_0 \cap X_1$, and take a c-extremal f for x at $\alpha(z)$ in the Calderón space $\mathcal{C}(X_0, X_1)$. By [4, Lemma 4.2.3], we may assume that f is a linear combination of functions with values in $X_0 \cap X_1$. Included in the proof of [12, Theorem 5.1] is the fact that $g = f \circ w$ is an extremal for x at z with respect to the family $(X_0, X_1)_{\alpha(z)}$, and $\|g\| \leqslant \|f\|$. Therefore $\Phi_z(x) = (f \circ w)'(z) = w'(z)\Omega_{w(z)}(x)$. Finally $\Omega_{w(z)}$ may be chosen as $\Omega_{\alpha(z)}$ by vertical symmetry. □

Theorem 4.20 can be seen as the 1-level version of Theorem 4.6. It shows that the derivation maps of $\left((X_0, X_1)_{\alpha(\omega)}\right)_{\omega\in\mathbb{T}}$ are always multiples of the derivation maps of the initial pair.

Corollary 4.21. *Let* (X_0, X_1) *be an interpolation pair, let* Ω_θ *be the derivation at* $\theta \in (0, 1)$*, let* $B_0 = \{e^{i\theta} : \theta \in [0, \frac{\pi}{2}] \cup [\pi, \frac{3\pi}{2}]\}$*, and let* $\alpha = \chi_{B_0} : \mathbb{T} \to [0, 1]$*. Consider the interpolation family* $\{(X_0, X_1)_{\alpha(w)} : w \in \mathbb{T}\}$*. Then for* $z = t$ *or* $z = it$*,* $t \in (-1, 1)$*, we get* $X_z = (X_0, X_1)_{\frac{1}{2}}$ *with derivations* $\Phi_0 = 0$ *and* Φ_z *equal to a multiple of* Ω_θ *with* $\theta = \alpha(z)$ *for* $0 \neq z \in \mathbb{D}$.

A case similar to that in Proposition 4.19 can be obtained with two spaces distributed on four arcs on $\mathbb{T}$ as above: just consider $X_0 = \ell_\infty$ and $X_1 = \ell_1$, which produces $X_z = \ell_2$ for every $z = t$, $z = it$, $t \in (-1, 1)$, and $\Phi_0 = 0$ while Φ_z is a nonzero multiple of the

Kalton–Peck map on X_z for values of $z \in \mathbb{D}$ arbitrarily close to 0. Thus, the differential process lacks local stability.

The next result explains, to some extent, the exceptional character of the previous examples. It is a direct consequence of Theorem 4.20.

Theorem 4.22. *Let (X_0, X_1) be an interpolation pair of Köthe function spaces, and let us consider the notation of Theorem 4.20.*

(1) *If the derivation Φ_{z_0} is bounded for some $z_0 \in \mathbb{D}$ such that $0 < \alpha(z_0) < 1$ and $w'(z_0) \neq 0$, then $X_0 = X_1$ with equivalence of norms and Φ_z is bounded for each $z \in \mathbb{D}$.*

(2) *If the derivation Φ_{z_0} is trivial for some $z_0 \in \mathbb{D}$ such that $0 < \alpha(z_0) < 1$ and $w'(z_0) \neq 0$, then X_0 is a weighted version of X_1 and Φ_z is trivial for each $z \in \mathbb{D}$.*

5. Stability of splitting for pairs of Banach spaces

For general Banach spaces, the problem of existence of local or global stability remains open. Here we give some positive results for pairs of sequence spaces with a common basis and pairs of Köthe function spaces. In the latter case, they provide more information than the results given before, which have an isomorphic nature, while Theorem 5.11, under the additional hypotheses it imposes, yields isometric uniqueness and stability.

In this section, we will use an alternative description of the complex interpolation method for pairs, which is given in [16]. Let $\overline{X} = (X_0, X_1)$ be an interpolation pair of Banach spaces and, for $z \in \mathbb{S}$, let $P_z^{\mathbb{S}}$ be the Poisson kernel on $\partial\mathbb{S}$ at z.

For $0 < \theta < 1$ and $1 \leqslant p < \infty$, we consider the spaces $\mathcal{F}_\theta^p(\overline{X})$ and $\mathcal{F}^\infty(\overline{X})$ of all functions $F : \overline{\mathbb{S}} \to \Sigma$, which are analytic on $\mathbb{S}$, such that $F(j+it) \in X_j$ for $j = 0, 1$ and $t \in \mathbb{R}$, the maps $f_j : t \in \mathbb{R} \to F(j+it) \in X_j$ $(j = 0, 1)$ are Bochner measurable, F has a Poisson representation $F(z) = \int_{\partial\mathbb{S}} F(w) P_\theta^{\mathbb{S}}(w)\, dw$ for $z \in \mathbb{S}$, and

$$\|F\|^p_{\mathcal{F}_\theta^p(\overline{X})} = \int_{\mathbb{R}} \|f_0(t)\|^p_{X_0} P_\theta^{\mathbb{S}}(t)\, dt + \int_{\mathbb{R}} \|f_1(t)\|^p_{X_1} P_\theta^{\mathbb{S}}(1+it)\, dt < \infty$$

for $1 \leqslant p < \infty$, or $\|F\|_{\mathcal{F}^\infty(\overline{X})} = \max_{j=0,1} \|f_j\|_{L_\infty(\mathbb{R},X_j)} < \infty$ for $p = \infty$.

It is not difficult to check that $\mathcal{F}^\infty(\overline{X})$ endowed with the norm $\|\cdot\|_{\mathcal{F}^\infty(\overline{X})}$ is a Kalton space of analytic functions on $\mathbb{S}$. Moreover, the associated spaces X_θ coincide (with equality of norms) with the spaces obtained in § 3 using the Calderón space $\mathcal{C}$ [16].

Given $0 < \theta < 1$ and $t \in \mathbb{R}$, the invariance under vertical translations of $\mathbb{S}$ implies that given f in the Calderón space $\mathcal{C}$ such that $f(\theta) = x$, the function $g(z) = f(z - it)$ is in $\mathcal{C}$ and satisfies $\|f\|_{\mathcal{C}} = \|g\|_{\mathcal{C}}$ and $g(\theta + it) = x$; the same is true for $\mathcal{F}^\infty(\overline{X})$. Thus $X_\theta = X_{\theta+it}$ isometrically, and it is enough to study the scale $(X_\theta)_{0<\theta<1}$.

Recall that an interpolation pair (X_0, X_1) is *regular* if Δ is dense in both X_0 and X_1. We need the following properties of the map $\theta \to \|\cdot\|_\theta$.

Lemma 5.1. *Let (X_0, X_1) be a regular interpolation pair and let $0 \leqslant \theta_0 < \theta_1 \leqslant 1$. For every $x \in X_{\theta_0} \cap X_{\theta_1}$, the map $\theta \mapsto \|x\|_\theta \in \mathbb{R}$ is log-convex on (θ_0, θ_1); it is therefore continuous with right and left derivatives on any point of (θ_0, θ_1).*

Proof. For each $\theta \in [\theta_0, \theta_1]$, one has $\|x\|_\theta \leqslant \|x\|_{\theta_0}^{1-t}\|x\|_{\theta_1}^{t}$ when $\theta = (1-t)\theta_0 + t\theta_1$: the case $\theta_0 = 0$, $\theta_1 = 1$ is well known, and the general case is a consequence of the reiteration theorem for complex interpolation [4, Theorem 4.6.1]. From this, it follows that the map $\theta \mapsto \log\|x\|_\theta$ is convex on $[\theta_0, \theta_1]$, and therefore continuous with right and left derivatives at every point of (θ_0, θ_1). □

5.1. Local bounded stability for coherent pairs

We begin by introducing the special interpolation pairs we study in this section.

Definition 5.2. We say that an interpolation pair (X_0, X_1) is *coherent* if there exists an increasing sequence (E_n) of finite-dimensional subspaces of $\Delta = X_0 \cap X_1$ such that $\Delta_0 = \cup_{n\in\mathbb{N}} E_n$ is dense in Δ, and for every $x \in E_n$ we can select a 1-extremal $f_{x,\theta}$ so that the corresponding derivation map Ω_θ takes E_n into E_n.

Note that the restriction of a derivation map to a finite-dimensional subspace is always bounded. The following result provides examples of coherent pairs.

Proposition 5.3. *Let (X_0, X_1) be a regular interpolation pair of reflexive spaces. Suppose that*

1. *X_0 and X_1 have a common monotone basis (e_n), or*
2. *X_0 and X_1 are rearrangement invariant spaces on $[0, 1]$.*

Then (X_0, X_1) is coherent.

Proof. Given $0 < \theta < 1$ and $x \in X_\theta$, there exists a 1-extremal $g_{x,\theta}$ by [16, Proposition 3].

(1) Take $E_n = [e_1, \ldots, e_n]$ and denote by P_n the natural norm-one projection from Σ onto E_n. For $x \in E_n$, if $g_{x,\theta}$ is a 1-extremal then $f_{x,\theta}(z) = P_n\left(g_{x,\theta}(z)\right)$ defines a 1-extremal that satisfies the remaining conditions because all norms are equivalent on E_n and for $y \in E_n$

$$\Omega_\theta(y) = f'_{y,\theta}(\theta) = \left(P_n g_{y,\theta}\right)'(\theta) = P_n\left(g'_{y,\theta}(\theta)\right).$$

(2) The proof is similar: For each $n \in \mathbb{N}$, we take as E_n the subspace generated by the characteristic functions of the intervals $\left((k-1)/2^n, k/2^n\right)$, $k = 1, \ldots, 2^n$. The arguments in the proof of [28, Theorem 2.a.4] show that

$$P_n f = \sum_{k=1}^{2^n} 2^n \left(\int_0^1 f\,\chi_{n,k}\,dt\right)\chi_{n,k}$$

define a norm-one projection from Σ onto E_n. □

The proof of the following result is a part of the proof of the main Theorem of [16]. We include some details for the convenience of the reader.

Lemma 5.4. *Given (X_0, X_1) a regular interpolation pair with X_0 reflexive, $x \in \Delta$, $\theta \in (0, 1)$, and a 1-extremal $f_{x,\theta}$ one has $\|f_{x,\theta}(z)\|_z = \|x\|_\theta$ for every $z \in \mathbb{S}$.*

Proof. It is enough to prove the result when $\|x\|_\theta = \|f_{x,\theta}\|_{\mathcal{F}^\infty(\overline{X})} = 1$. We select $x^* \in (X_\theta)^* \equiv (X^*)_\theta$ such that $\|x^*\| = \langle x, x^*\rangle = 1$. As in Daher's proof, we select $f^* \in \mathcal{F}^2_\theta(\overline{X^*})$ with $f^*(\theta) = x^*$ and $\|f^*\|_{\mathcal{F}^2_\theta(\overline{X^*})} = 1$.

Using [4, Lemma 4.2.3], we can show that $g(z) = \langle f_{x,\theta}(z), f^*(z)\rangle$ defines an analytic function. Since $|g(z)| \leqslant 1$ for every $z \in \mathbb{S}$ and $g(\theta) = 1$, the maximum principle for analytic functions implies that $g(z) = 1$ for every $z \in \mathbb{S}$. In particular, $\|f_{x,\theta}(z)\|_z \geqslant 1$; hence $\|f_{x,\theta}(z)\|_z = 1$. □

Lemma 5.5. *Let (X_0, X_1) be a regular interpolation pair with X_0 reflexive, let $x \in \Delta$, and let $0 \leqslant \theta_0 < \theta < \theta_1 \leqslant 1$. Suppose that there is a 1-extremal $f_{x,\theta}$, which is derivable at $z = \theta$ as a function with values in both spaces X_{θ_i} $(i = 0, 1)$, and consider the derivation $\Omega_\theta(x) = f'_{x,\theta}(\theta)$. Then the right and left derivatives of $t \mapsto \|x\|_t$ at θ are bounded in modulus by $\|\Omega_\theta(x)\|_\theta$.*

Proof. By Lemma 5.4, $\|x\|_\theta = \|f_{x,\theta}(\theta+\varepsilon)\|_{\theta+\varepsilon}$. Hence

$$\begin{aligned}\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\big|\|x\|_{\theta+\varepsilon} - \|x\|_\theta\big| &\leqslant \limsup_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\left(\|f_{x,\theta}(\theta+\varepsilon) - x\|_{\theta+\varepsilon}\right)\\ &\leqslant \limsup_{\varepsilon\to 0^+}\left(\|\Omega_\theta(x)\|_{\theta+\varepsilon} + \|\frac{1}{\varepsilon}(f_{x,\theta}(\theta+\varepsilon) - x) - \Omega_\theta(x)\|_{\theta+\varepsilon}\right)\\ &\leqslant \limsup_{\varepsilon\to 0^+}\left(\|\Omega_\theta(x)\|_{\theta+\varepsilon} + \max_{i=0,1}\|\frac{1}{\varepsilon}(f_{x,\theta}(\theta+\varepsilon) - x) - \Omega_\theta(x)\|_{\theta_i}\right).\end{aligned}$$

Note that $\Omega_\theta(x)$ belongs to $X_{\theta_0} \cap X_{\theta_1}$ by hypothesis. So, by Lemma 5.1, we have that $\|\Omega_\theta(x)\|_{\theta+\varepsilon}$ tends to $\|\Omega_\theta(x)\|_\theta$. Since $\frac{1}{\varepsilon}(f_{x,\theta}(\theta+\varepsilon) - x)$ tends to $\Omega_\theta(x)$ in X_{θ_i}, $i = 0, 1$, we get

$$\left|\frac{d\|x\|_t}{dt}|_{t=\theta^\pm}\right| \leqslant \|\Omega_\theta(x)\|_\theta\,. \tag{7}$$

□

Next we give some conditions on an interpolation pair (X_0, X_1) implying $X_0 = X_1$ up to an equivalent renorming.

Theorem 5.6 (Local bounded stability). *Let (X_0, X_1) be a coherent interpolation pair of reflexive spaces and let $0 \leqslant \theta_0 < \theta_1 \leqslant 1$. Suppose that $\sup_{\theta_0<t<\theta_1}\|\Omega_t : X_t \to X_t\| < \infty$. Then $X_0 = X_1$, up to an equivalent renorming.*

Proof. Fix $x \in \Delta_0 = \cup_{n\in\mathbb{N}} E_n$. For $\theta_0 < s < \theta_1$, one has

$$\left|\frac{d\|x\|_t}{dt}\right|_{t=s^+} \leqslant \|\Omega_s(x)\|_s \leqslant M\|x\|_s.$$

If we set $g(s) = e^{Ms}\|x\|_s$, then

$$\left(\frac{dg}{dt}\right)_{t=s^+} = e^{Ms}\left(M\|x\|_s + \left(\frac{d\|x\|_t}{dt}\right)_{t=s^+}\right) \geqslant 0.$$

Since g is continuous, it is nondecreasing on (θ_0, θ_1). Therefore, whenever $[\theta-\varepsilon, \theta+\varepsilon] \subset (\theta_0, \theta_1)$ one has $g(\theta+\varepsilon) \geqslant g(\theta-\varepsilon)$, which implies

$$\|x\|_{\theta+\varepsilon} \geqslant e^{-M(\theta+\varepsilon)} e^{M(\theta-\varepsilon)} \|x\|_{\theta-\varepsilon} = e^{-2M\varepsilon} \|x\|_{\theta-\varepsilon}.$$

Working with $e^{-Ms}\|x\|_s$ instead we obtain $\|x\|_{\theta+\varepsilon} \leqslant e^{2M\varepsilon}\|x\|_{\theta-\varepsilon}$.

By density, we get $X_{\theta+\varepsilon} = X_{\theta-\varepsilon}$. Thus $X_s = X_\theta$ with equivalence of norms for $|\theta - s| \leqslant \varepsilon$, and a result of Stafney [35, Theorem 1.7] implies that $X_0 = X_1$ with equivalence of norms. □

5.2. Isometric rigidity of linear derivations for optimal interpolation pairs

As we remarked in the Introduction, [14, Theorem 5.2] proves the estimate

$$\frac{d}{d\theta}\|x\|_{\theta,1} \sim \|x\|_{\theta,1} + \|\Omega_\theta(x)\|_{\theta,1}$$

for the real $(\theta, 1)$-method of interpolation. From this fact, an analogue of Theorem 5.6 is derived [14, Theorem 5.16]: If the maps Ω_θ are uniformly bounded for all $|\theta-\theta_0| < \varepsilon$, then $X_0 = X_1$. Moreover, [14, Theorem 5.17] shows that the (θ, q)-method has a stronger stability property: if Ω_θ is bounded for some $0 < \theta < 1$ then $X_0 = X_1$. A similar result for the complex interpolation method is still unknown in general, but we will give here some partial positive results.

From now on, we sometimes denote $B_\theta(x)(z) = f_{x,\theta}(z)$ $(z \in \mathbb{S})$ for convenience.

Definition 5.7. An interpolation pair (X_0, X_1) will be called *optimal* if, for every $0 < \theta < 1$ and each $x \in X_\theta$, there exists a unique 1-extremal $f_{x,\theta}$.

Daher [16, Proposition 3] showed that a regular interpolation pair of reflexive spaces is optimal when one of the spaces is strictly convex. He also essentially observed the following result.

Lemma 5.8. *Let (X_0, X_1) be an optimal interpolation pair with X_0 reflexive. For all $x \in \Delta$ and $t, z \in \mathbb{S}$, we have*

(1) $\|B_t(x)(z)\|_z = \|x\|_t$,
(2) $B_t(x) = B_z(B_t(x)(z))$,
(3) $B_t(x)'(z) = \Omega_z(B_t(x)(z))$.

Proof. (1) was proved in Lemma 5.4, (2) follows from the uniqueness of the extremals, since both functions have the same norm and take the value $B_t(x)(z)$ at z, and (3) follows from (2) and $B_t(x)'(z) = \Omega_z(x)$. □

Lemma 5.9. *Let (X_0, X_1) be an optimal interpolation pair. For all $0 < \theta < 1$ and $t \in \mathbb{R}$, one has $\Omega_{\theta+it} = \Omega_\theta$.*

Proof. Observe that $B_\theta(x)(z - it) = B_{\theta+it}(x)(z)$ since both are extremals for x at $z = \theta + it$. Hence $\Omega_{\theta+it}(x) = B_{\theta+it}(x)'(\theta+it) = B_\theta(x)'(z) = \Omega_\theta(x)$. □

We are ready to obtain some stability results when Ω_θ is linear and bounded. We start with the simplest case $\Omega_\theta = 0$. Note that the following result is new even for Köthe function spaces.

Proposition 5.10. *Let (X_0, X_1) be an optimal interpolation pair with X_0 reflexive. Then $\Omega_\theta = 0$ for some $0 < \theta < 1$ if and only if $X_0 = X_1$ isometrically.*

Proof. The if part is well known, and it easily follows from $B_\theta(x)(z) = x$ for $x \in \Delta$. As for the converse, consider the function $F : \mathbb{R} \to \Sigma$ defined by $F(t) = B_\theta(x)(\theta + it)$.

This function is constant since $F'(t) = \Omega_{\theta+it}\,(B_\theta(x)(\theta + it)) = 0$. Thus the analytic function $B_\theta(x)$ is constant on the vertical line through θ, hence constant on $\mathbb{S}$. In particular, $\|x\|_\theta = \|B_\theta(x)(\theta)\|_\theta = \|B_\theta(x)(z)\|_z = \|x\|_z$ for each z. □

Recall that an operator T acting on a Banach space X is said to be *Hermitian* when e^{itT} is an isometry on X for all $t \in \mathbb{R}$ (see [23]).

Theorem 5.11. *Let (X_0, X_1) be a coherent and optimal interpolation pair of Banach spaces. Suppose that $\Omega_\theta : X_\theta \to \Sigma$ is linear for some $0 < \theta < 1$. Then, we have the following:*

(1) *$\Omega_z(x) = \Omega_\theta(x)$ for all $z \in \mathbb{S}$ and all $x \in \Delta_0$.*

(2) *For every $0 < s < 1$, the map $x \in \Delta_0 \mapsto e^{s\Omega_\theta}x$ induces an isometry between X_0 and X_s, which gives $\|x\|_s = \|e^{-s\Omega_\theta}x\|_0$.*

(3) *Ω_z is a Hermitian operator on X_z for all $z \in \mathbb{S}$.*

Proof. (1) Since $B_\theta(x)'(\theta + it) = \Omega_{\theta+it}\,(B_\theta(x)(\theta + it)) = \Omega_\theta\,(B_\theta(x)(\theta + it))$ for all $t \in \mathbb{R}$, the function $t \to B_\theta(x)(\theta + it)$ satisfies the differential equation

$$f'(t) = i\Omega_\theta(f(t)). \tag{8}$$

Equivalently, $B_\theta(x)$ satisfies the equation $f'(z) = \Omega_\theta(f(z))$ for $z \in \mathbb{S}_\theta = \{z \in \mathbb{S} : Re(z) = \theta\}$. Since $B_\theta(x) : \mathbb{S} \to \Sigma$ is the unique 1-extremal and $x \in \Delta_0$, it is analytic as a map into Δ. When Ω_θ is linear, $\Omega_\theta \circ B_\theta(x) : \mathbb{S} \to \Sigma$ is analytic and takes values in Δ for $x \in \Delta_0$, and the derivative $B_\theta(x)' : \mathbb{S} \to \Sigma$ is of course analytic. Since both functions coincide on $\mathbb{S}_\theta$, they coincide on $\mathbb{S}$; thus $B_\theta(x)$ solves the equation $f'(z) = \Omega_\theta(f(z))$ on $\mathbb{S}$ and we get

$$\begin{aligned}\Omega_\theta(x) &= \Omega_\theta\,(B_z(x)(z)) = \Omega_\theta\,(B_\theta(B_z(x)(\theta))(z)) \\ &= B_\theta\,(B_z(x)(\theta))'\,(z) = B_z(x)'(z) = \Omega_z(x).\end{aligned}$$

To prove (2), we need to make sense of the function $G(t) = e^{-it\Omega_\theta}B_\theta(x)(\theta + it)$ for $x \in \Delta_0$.

Pick $n \in \mathbb{N}$ such that $x \in E_n$. Since $\Omega_\theta(E_n) \subset E_n$, the iterations Ω_θ^k are operators on E_n, so that G is well defined. Now, since $\Omega_\theta : X_\theta \to \Sigma$ is linear and bounded,

$$\begin{aligned}G'(t) &= e^{-it\Omega_\theta}iB_\theta(x)'(\theta + it) - e^{-it\Omega_\theta}i\Omega_\theta\,(B_\theta(x)(\theta + it)) \\ &= e^{-it\Omega_\theta}\Big(i\Omega_\theta\,(B_\theta(x)(\theta + it)) - i\Omega_\theta\,(B_\theta(x)(\theta + it))\Big) = 0.\end{aligned}$$

So the function $G(t)$ is constant and equal to $G(0) = x$; thus $B_\theta(x)(\theta + it) = e^{it\Omega_\theta}x$. This means that for any z in the vertical line through θ, $B_\theta(x)(z) = e^{(z-\theta)\Omega_\theta}x$. Since both functions are analytic on $\mathbb{S}$, it turns out that $B_\theta(x)(z) = e^{(z-\theta)\Omega_\theta}x$ for all $z \in \mathbb{S}$ and $x \in \Delta_0$. Since the functions are equal on $\mathbb{S}$, they have the same radial limits a.e. on the border.

So $B_\theta(x)(z) = e^{(z-\theta)\Omega_\theta}x$ for a.e. z on the border of $\mathbb{S}$. Thus $1 = \|B_\theta(x)(it)\|_0 = \|e^{(it-\theta)\Omega_\theta}x\|_0$ for a.e. $t \in \mathbb{R}$. By continuity, $\|e^{(it-\theta)\Omega_\theta}x\|_0 = \|x\|_\theta$ for every $t \in \mathbb{R}$. Clearly the same reasoning works for $1 + it$ instead of it.

Thus $\|x\|_\theta = \|B_\theta(x)(s)\|_s = \|e^{(s-\theta)\Omega_\theta}x\|_s$ for each $s \in [0, 1]$ and $x \in \Delta_0$. Taking $y = e^{\theta\Omega_\theta}x$, we get $\|x\|_0 = \|e^{s\Omega}\Omega_\theta x\|_s$ for every $s \in [0, 1]$ and every $x \in \Delta_0$, which is dense in both X_0 and X_s. Hence the map $x \to e^{-s\Omega_\theta}x$ extends to an isometry between X_s and X_0, and $\|x\|_s = \|e^{-s\Omega_\theta}x\|_0$.

(3) Since $\|x\|_z = \|B_z(x)(z + it)\|_{z+it} = \|B_z(x)(z + it)\|_z = \|e^{it\Omega_\theta}x\|_z$ and the norm $\|B_\theta(x)(z)\|_z$ is constant and equal to $\|x\|_\theta$ for z in the vertical line through θ, we get that $\{e^{it\Omega_\theta}\}_{t\in\mathbb{R}}$ is a group of linear isometries on X_z. □

We can compare this result to Theorem 4.4, in which the Ω_θ trivial implies that X_1 is a weighted version X_0 and Ω is the operator acting as multiplication by $-\log w$.

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