

The strongest Banach–Stone theorem for $C_0(K, \ell_2^2)$ spaces

Elói Medina Galego 

Department of Mathematics, University of São Paulo, IME, Rua do Matão, São Paulo, Brazil

Correspondence

Elói Medina Galego, Department of Mathematics, University of São Paulo, IME, Rua do Matão 1010, São Paulo, Brazil.

Email: eloi@ime.usp.br

Abstract

As usual denote by ℓ_2^2 the real two-dimensional Hilbert space. We prove that if K and S are locally compact Hausdorff spaces and T is a linear isomorphism from $C_0(K, \ell_2^2)$ onto $C_0(S, \ell_2^2)$ satisfying

$$\|T\| \|T^{-1}\| \leq \sqrt{2.054208},$$

then K and S are homeomorphic.

This theorem is the strongest of all the other vector-valued Banach–Stone theorems known so far in the sense that in none of them the distortion of the isomorphism T , denoted by $\|T\| \|T^{-1}\|$, is as large as $\sqrt{2.054208}$.

Some remarks on the proof method developed here to prove our theorem suggest the conjecture that it is in fact very close to the optimal Banach–Stone theorem for $C_0(K, \ell_2^2)$ spaces, or in more precise words, the exact value of the Banach–Stone constant of ℓ_2^2 is between $\sqrt{2.054208}$ and $\sqrt{2.054209}$.

KEYWORDS

$C_0(K, X)$ spaces, ℓ_2^2 space, Banach–Stone theorem

1 | INTRODUCTION

Let X be a Banach space and K be a locally compact Hausdorff space. The space $C_0(K, X)$ will stand for the Banach space of all continuous X -valued functions on K which vanish at infinity, equipped with the sup norm. We denote by \mathbf{K} the field of real numbers \mathbf{R} or complex numbers \mathbf{C} .

Recall that the Banach space X has the isomorphic Banach–Stone property (in short, IBSP) [2] if there exists $\lambda > 1$ such that for all locally compact Hausdorff spaces K and S and linear isomorphisms T from $C_0(K, X)$ onto $C_0(S, X)$ satisfying $\|T\| \|T^{-1}\| < \lambda$, we have that K and S are homeomorphic.

The Banach–Stone constant of a Banach space X having IBSP (in short, $\text{BS}(X)$) is the largest $\lambda > 1$ satisfying the above statement [5, Definition 2.1].

Amir [1] and Cambern [3] showed that $\text{BS}(\mathbf{K}) \geq 2$. Later on, Cohen [6] proved that $\text{BS}(\mathbf{K}) \leq 2$. Consequently, $\text{BS}(\mathbf{K}) = 2$.

It is well-known that the number 2 is an upper bound for the Banach–Stone constants, that is, $\text{BS}(X) \leq 2$ for every Banach space X having IBSP [5, p. 195]. However, the hard work lies in finding lower bounds for the Banach–Stone constants $\text{BS}(X)$ when the dimension of X is greater than or equal to 2. This happens even for the simplest space \mathbf{R}^2 when equipped with any norm that makes it complete. In particular, it is still an open problem to know the exact value of $\text{BS}(\ell_2^2)$.

In fact, we know very little about $BS(\ell_2^2)$. More specifically in 1976 [4, Main theorem] it was proved that $BS(\ell_2^2) \geq \sqrt{2}$. But it was only in 2019 [10, Theorem 1.2] that it was shown that

$$BS(\ell_2^2) \geq \sqrt{2 + \delta}, \quad (1)$$

where δ is approximately 0.02343.

Therefore, in search of an optimal Banach–Stone theorem for $C_0(K, \ell_2^2)$ spaces, it is natural to ask whether the number δ mentioned in Equation (1) can be increased. The main aim of this work is to give a positive answer to this question by proving that

$$BS(\ell_2^2) > \sqrt{2.054208}.$$

This lower bound for the Banach–Stone constant of ℓ_2^2 follows from Remark 9.2 on the proof of our main result, which is the following theorem.

Theorem 1.1. *Let K and S be locally compact Hausdorff spaces and L a linear isomorphism from $C_0(K, \ell_2^2)$ onto $C_0(S, \ell_2^2)$ satisfying*

$$\|L\| \|L^{-1}\| \leq \sqrt{2.054208},$$

then K and S are homeomorphic.

The above theorem establishes the Banach–Stone theorem for $C_0(K, X)$ spaces, with dimension of X greater than or equal to 2, which is obtained through of linear isomorphisms L with the highest distortion $\|L\| \|L^{-1}\|$ known so far, see [5, 9, 10]. Moreover, our method of proving Theorem 1.1 does not work if we change the digit 8 to 9 in the statement of Theorem 1.1, see Remark 9.1. On the other hand, as mentioned in Remark 9.2, it is possible to increase the number of digits of the decimal part of 2.054208 in such a way that the statement of Theorem 1.1 still holds. This raises the following conjecture about the exact value of $BS(\ell_2^2)$.

Conjecture 1.2. $\sqrt{2.054208} < BS(\ell_2^2) \leq \sqrt{2.054209}$.

The proof of Theorem 1.1 is based on a deeper understanding of the properties of some special subsets of locally compact spaces K introduced in [7] in the study of the geometry of $C_0(K, X)$ spaces, where $X = \mathbf{R}$. These special sets were studied in more detail in [10] in the case where X is a finite-dimensional Hilbert space.

In this paper, we will provide new properties of these special sets when $X = \ell_2^2$. Interestingly, one of the properties of ℓ_2^2 that will play an essential role in the proof of Theorem 1.1 is not valid for the real Hilbert spaces ℓ_2^n , $n \geq 3$, see Proposition 3.2 and Remark 3.4. In particular, the problem of whether it is also possible to increase the lower bound of $BS(\ell_2^3)$ obtained in [10], which is approximately $\sqrt{2.00672}$, still remains open.

To simplify the notation, from now on the space ℓ_2^2 will be denoted by H and S_H will denote the unit sphere of it.

2 | SPECIAL SETS ASSOCIATED WITH LINEAR ISOMORPHISMS BETWEEN $C_0(K, H)$ SPACES

Fix $0 < \delta < 1$ and let K and S be locally compact Hausdorff spaces and L be a linear isomorphism from $C_0(K, H)$ onto $C_0(S, H)$ satisfying

$$\|L\| \|L^{-1}\| \leq \sqrt{2 + \delta}, \quad (2)$$

In a recent study of these maps [7], it was introduced some classes of subsets $\Gamma_w(k, v)$ and $\Gamma_v(s, w)$ of S and K , respectively, where $k \in K$, $s \in S$ and v and w are suitable elements of \mathbf{R} . As in [8] we are going to work with these sets for $v, w \in H$

instead of \mathbf{R} . In order to prove Theorem 1.1, we will need to state some new properties of these sets. So, in this short preliminary section we will remember some definitions and results already adapted to the context of Theorem 1.1.

Before anything note that if L is a linear isomorphism satisfying the hypothesis of Equation (2) and putting $T = L\|L^{-1}\|$ and $M_\delta = \sqrt[4]{2 + \delta}$, then, for every $f \in C_0(K, H)$,

$$\frac{\|f\|}{M_\delta} \leq \|T(f)\| \leq M_\delta \|f\|, \quad (3)$$

Thus, henceforward, K and S are locally compact spaces, $M_\delta = \sqrt[4]{2 + \delta}$, with $0 < \delta < 1$, T is a linear isomorphism of $C_0(K, H)$ onto $C_0(S, H)$ satisfying Equation (3). Thus, our main task in this work is to show that when $\delta = 0.054208$ the inequalities (3) imply that K and S are homeomorphic and therefore Theorem 1.1 will be proved.

Let $k \in K$, $f \in C_0(K, H)$, and $v \in H$. Following [7, Definition 2.2] we set

$$\omega(k, f, v) = \max\{\|f\|, \|f(k) - v\|\}.$$

Moreover, if v and $w \in H$ with $v \neq 0$ satisfies $\|w\| = \|v\|/M_\delta$, following [7, Definition 3.1], we also set

$$\Gamma_w(k, v) = \{s \in S : \|Tf(s) - w\| \leq M_\delta \omega(k, f, v), \forall f \in C_0(K, H)\}.$$

Analogously, for $s \in S$, w and $v \in H$ with $w \neq 0$ and $\|v\| = \|w\|/M_\delta$, we set

$$\Gamma_v(s, w) = \{k \in K : \|T^{-1}g(k) - v\| \leq M_\delta \omega(s, g, w), \forall g \in C_0(S, H)\}.$$

Let us summarize the results concerning these sets which will be used in this paper. The proof of Proposition 2.1 is quite similar to that of [8, Proposition 2.1].

Proposition 2.1. *Let $k \in K$.*

- (1) *If $v \in H$ and $v \neq 0$ then there is $w \in H$ with $\Gamma_w(k, v) \neq \emptyset$.*
- (2) *Let $v, w \in H$. Then for all $t \neq 0$, we have $\Gamma_w(k, v) = \Gamma_{tw}(k, tv)$.*
- (3) *If $s \in \Gamma_w(k, v)$ and $\Gamma_z(s, w) \neq \emptyset$ for some $v, w, z \in H$ then $\Gamma_z(s, w) = \{k\}$.*
- (4) *If $s \in \Gamma_w(k, v)$ for some $v, w \in H$ then there is $z \in H$ with $\Gamma_z(s, w) = \{k\}$.*

Denote by $\mathcal{P}(K)$ and $\mathcal{P}(S)$ the power set of K and S respectively. Following [8, Section 4] we define two functions $\Phi_\delta : K \rightarrow \mathcal{P}(S)$ and $\Psi_\delta : S \rightarrow \mathcal{P}(K)$ given by

$$\Phi_\delta(k) = \cup\{\Gamma_w(k, v) : v \neq 0 \text{ and } \|w\| = \|v\|/M_\delta\}, \quad (4)$$

and

$$\Psi_\delta(s) = \cup\{\Gamma_v(s, w) : w \neq 0 \text{ and } \|v\| = \|w\|/M_\delta\}. \quad (5)$$

Proposition 2.2. *Suppose that $\Phi_\delta(k)$ and $\Psi_\delta(s)$ are singleton sets for every $k \in K$ and $s \in S$. Then, K and S are homeomorphic.*

Proof. Since $\Phi_\delta(k)$ and $\Psi_\delta(s)$ are singleton sets, we can define two functions $\varphi : K \rightarrow S$ and $\psi : S \rightarrow K$ by

$$\Phi_\delta(k) = \{\varphi(k)\} \text{ and } \Psi_\delta(s) = \{\psi(s)\}.$$

Thus, it suffices to prove that φ and ψ are continuous and $\psi = \varphi^{-1}$. The proof of these facts is direct and follows step by step that of [8, Proposition 6.1], see also [10, Section 8]. \square

Lastly, note that Equation (3) implies that $\|f\|/M_\delta \leq \|T^{-1}(f)\| \leq M_\delta \|f\|$, for any $f \in C_0(S, H)$. So, for any result on $\Gamma_w(k, v)$ or $\Phi_\delta(k)$, a similar result holds for $\Gamma_v(s, w)$ or $\Psi_\delta(s)$, respectively. We will use the same result label to refer to either case.

3 | ON THE CARDINALITY OF THE SUBSETS $\Phi_\delta(k)$

The main aim of this section is to prove that $\Phi_\delta(k)$ has at most two elements, for any $k \in K$, see Proposition 3.3. Before that we will establish two auxiliary results Propositions 3.1 and 3.2. For each $v \in H$, $v \neq 0$, we will denote $\hat{v} = v/\|v\|$.

Proposition 3.1. *Suppose $s \in \Gamma_w(k, v) \cap \Gamma_{w'}(k', v')$ for some $v, v', w, w' \in H$, with $k \neq k'$. Then*

$$\sqrt{2 - \delta} \leq \|\hat{w} \pm \hat{w}'\| \leq \sqrt{2 + \delta}.$$

Proof. Note that by Proposition 2.1.(2) we may assume that $\|v\| = \|v'\|$. Put $\|v\| = c$. According to the Urysohn's lemma, we can fix $f \in C_0(K, H)$ such that

$$\|f\| = c/2, \quad f(k) = v/2 \text{ and } f(k') = v'/2.$$

It is easy to see that $\omega(k, f, v) = \omega(k', f, v') = c/2$, and since $s \in \Gamma_w(k, v) \cap \Gamma_{w'}(k', v')$, we have

$$\|Tf(s) - w\| \leq cM_\delta/2 \text{ and } \|Tf(s) - w'\| \leq cM_\delta/2. \quad (6)$$

By Equation (6), $\|w - w'\| \leq cM_\delta$ and since $\|w\| = \|w'\| = c/M_\delta$ and $M_\delta^2 = \sqrt{2 + \delta}$, we obtain $\|\hat{w} - \hat{w}'\| \leq M_\delta^2 = \sqrt{2 + \delta}$. Now, by the parallelogram law we know that

$$4 = \|\hat{w} + \hat{w}'\|^2 + \|\hat{w} - \hat{w}'\|^2 \leq \|\hat{w} + \hat{w}'\|^2 + 2 + \delta,$$

consequently $\|\hat{w} + \hat{w}'\| \geq \sqrt{2 - \delta}$.

On the other hand, according to Proposition 2.1.(2) $s \in \Gamma_w(k, v) \cap \Gamma_{-w'}(k', -v')$.

Hence, by repeating the argument replacing v' and w' by $-v'$ and $-w'$, respectively, we deduce $\|\hat{w} + \hat{w}'\| \leq \sqrt{2 + \delta}$, and again by the parallelogram law we conclude that $\|\hat{w} - \hat{w}'\| \geq \sqrt{2 - \delta}$. So, we are done. \square

Proposition 3.2. *Let w_1, w_2 , and w' in S_H satisfying*

$$\sqrt{2 - \delta} \leq \|w_i \pm w'\| \leq \sqrt{2 + \delta}, \quad \forall i \in \{1, 2\},$$

then $\|w_1 - w_2\| \leq \delta$ or $\|w_1 + w_2\| \leq \delta$.

Proof. Initially fix four vectors u_1, u_2, u_3 , and u_4 in S_H as follows. Counterclockwise u_1 and u_2 are such that

$$\|w' - u_1\| = \sqrt{2 - \delta} \text{ and } \|w' - u_2\| = \sqrt{2 + \delta},$$

and clockwise u_3 and u_4 are such that

$$\|w' - u_3\| = \sqrt{2 - \delta} \text{ and } \|w' - u_4\| = \sqrt{2 + \delta}.$$

Now, note that, by our hypotheses each of the $w_i, i \in \{1, 2\}$, is contained either in the arc or in the arc of the circumference S_H .

Next, by a convenient rotation of S_H , we can assume that $w' = e_1 = (1, 0)$. Let α and β be such that $u_1 = (\cos\alpha, \sin\alpha)$ and $u_2 = (\cos\beta, \sin\beta)$. Since

$$\|e_1 - u_1\| = \sqrt{(1 - \cos\alpha)^2 + \sin^2\alpha} = \sqrt{2 - \delta},$$

it follows that $2\cos\alpha = \delta$. Moreover, since

$$\|e_1 - u_2\| = \sqrt{(1 - \cos\beta)^2 + \sin^2\beta} = \sqrt{2 + \delta},$$

we see that $-2\cos\beta = \delta$. Consequently $\cos\beta = -\cos\alpha$. Noting that u_2 is in the second quadrant of S_H , we deduce that $\sin\beta = \sin\alpha$. Therefore, $u_2 = (-\cos\alpha, \sin\alpha)$. Then, $\|u_1 - u_2\| = 2\cos\alpha = \delta$.

Proceeding in the same way with u_3 and u_4 we conclude that $\|u_3 - u_4\| = \delta$.

To finish the proof of the proposition, observe that if and , then

$$\|w_1 - w_2\| \leq \|u_1 - u_2\| \leq \delta.$$

Moreover, in the case where and , then and thus

$$\|w_1 - (-w_2)\| \leq \|u_1 - u_2\| \leq \delta.$$

The other possible locations of w_1 and w_2 are considered analogously. □

Proposition 3.3. *The cardinality of $\Phi_\delta(k)$ is at most 2, for every $k \in K$.*

Proof. Suppose s_1, s_2 , and s_3 are three distinct elements of $\Phi_\delta(k)$. By the definition of $\Phi_\delta(k)$ in Equation (4), $s_i \in \Gamma_{w_i}(k, v_i)$ for some v_i and w_i in H . Next, according to Proposition 2.1.(4) take z_1, z_2 , and z_3 in H such that

$$k \in \Gamma_{z_1}(s_1, w_1) \cap \Gamma_{z_2}(s_2, w_2) \cap \Gamma_{z_3}(s_3, w_3). \quad (7)$$

Hence applying Proposition 3.1 twice in Equation (7) we have

$$\sqrt{2 - \delta} \leq \|\hat{z}_1 \pm \hat{z}_2\| \leq \sqrt{2 + \delta} \text{ and } \sqrt{2 - \delta} \leq \|\hat{z}_2 \pm \hat{z}_3\| \leq \sqrt{2 + \delta},$$

and by Proposition 3.2 we deduce that there exists $\lambda \in \{-1, 1\}$ satisfying

$$\|\hat{z}_1 + \lambda\hat{z}_3\| \leq \delta. \quad (8)$$

But, by Equation (7) we know that $k \in \Gamma_{z_1}(s_1, w_1) \cap \Gamma_{z_3}(s_2, w_3)$. Then, by Proposition 3.1,

$$\sqrt{2 - \delta} \leq \|\hat{z}_1 + \lambda\hat{z}_3\|. \quad (9)$$

So, by Equations (8) and (9) we see that $2 - \delta \leq \delta^2$, that is, $\delta \leq -2$ or $\delta \geq 1$, absurd. □

Remark 3.4. The statement of Proposition 3.2 is not true when we exchange H for ℓ_2^n with $n \geq 3$. Indeed take the following vectors from the canonical basis of ℓ_2^n : $w_1 = e_1, w_2 = e_2$ and $w' = e_3$. Thus,

$$\|w_i \pm w'\| = \sqrt{2}, \forall i \in \{1, 2\}, \text{ but } \|w_1 - w_2\| = \|w_1 + w_2\| = \sqrt{2}.$$

4 | ON THE SUBSETS $\Phi_\delta(k)$ WHICH ARE NOT SINGLETON SETS

In this section, we show that if $\Phi_\delta(k)$ is not a singleton set, then either there are three convenient special sets $\Gamma_w(k, v)$ with nonempty intersection or there are three convenient special sets $\Gamma_v(s, w)$ with nonempty intersection. This result

will be fundamental in the next section in order to obtain a sufficient condition on δ for $\Phi_\delta(k)$ to be a singleton set, for any $k \in K$. More specifically, here we will prove:

Proposition 4.1. *Let $k \in K$ and suppose that $\Phi_\delta(k)$ is not a singleton set. Then, one of the following two possibilities holds:*

$$\Gamma_{w_1}(k, v_1) \cap \Gamma_{w_2}(k, v_2) \cap \Gamma_{w_3}(k', v_3) \neq \emptyset,$$

for some $k' \in K$, $k \neq k'$, $v_1, v_2, v_3, w_1, w_2, w_3$ in H with v_1 orthogonal to v_2 , or

$$\Gamma_{v_4}(s, w_4) \cap \Gamma_{v_5}(s, w_5) \cap \Gamma_{v_6}(s', w_6) \neq \emptyset,$$

for some $s, s' \in S$, $s \neq s'$, $v_4, v_5, v_6, w_4, w_5, w_6$ in H with w_4 orthogonal to w_5 .

Proof. Since $\Phi_\delta(k)$ is not a singleton set, by Proposition 3.3 there are s and s' in S with $s \neq s'$ such that $\Phi_\delta(k) = \{s, s'\}$. Put

$$V = \{v \in S_H : s \in \Gamma_w(k, v) \text{ for some } w\},$$

and

$$V' = \{v \in S_H : s' \in \Gamma_w(k, v) \text{ for some } w\}.$$

According to Proposition 2.1.(1) we see that $S_H = V \cup V'$. Moreover, V and V' are closed sets. Indeed, by symmetry, it is enough to prove that V is a closed set. Let $(v_j)_{j \in \mathbf{N}}$ be a sequence in V such that $v_j \rightarrow v$. For each $j, \in \mathbf{N}$ fix $w_j \in H$ such that

$$s_1 \in \Gamma_{w_j}(k, v_j). \quad (10)$$

Since the sequence $(w_j)_{j \in \mathbf{N}}$ is bounded, it admit a subsequence convergent. Thus, we may assume that $w_j \rightarrow w$, for some $w \in H$. Given $f \in C_0(K, H)$, by Equation (10) we deduce that

$$\|Tf(s_1) - w_j\| \leq M_\delta \omega(k, f, v_j), \quad \forall j \in \mathbf{N},$$

and therefore

$$\|Tf(s_1) - w\| \leq M_\delta \omega(k, f, v).$$

That is, $s_1 \in \Gamma_w(k, v)$. Consequently, $v \in V$ and we are done.

Therefore, since S_H is connected, there is $v_1 \in V \cap V'$. Fix v_2 in S_H orthogonal to v_1 . Hence, v_2 belongs to V or V' . Without loss of generality, we can assume that $v_2 \in V$. Thus,

$$s \in \Gamma_{w_1}(k, v_1) \cap \Gamma_{w_2}(k, v_2), \quad (11)$$

for some $w_1, w_2 \in H$. Next, we distinguish two cases:

Case 1: $\Psi_\delta(s)$ is not a singleton set. Then, there exists $k' \in \Psi_\delta(s)$ with $k' \neq k$. By the definition of $\Psi_\delta(s)$ in Equation (5), $k' \in \Gamma_{v_3}(s, w)$ for some v_3 and w in H . Now, by Proposition 2.1.(4), take w_3 in H such that $\Gamma_{w_3}(k', v_3) = \{s\}$. Then, by Equation (11),

$$s \in \Gamma_{w_1}(k, v_1) \cap \Gamma_{w_2}(k, v_2) \cap \Gamma_{w_3}(k', v_3),$$

and we are done.

Case 2: $\Psi_\delta(s)$ is a singleton set. In this case, according to Equation (11) we know that $s \in \Gamma_{w_1}(k, v_1)$. Now, by Propositions 2.1.(4) take $z \in H$ such that $\Gamma_z(s, w_1) = \{k\}$. Thus, $k \in \Psi_\delta(s)$ and consequently $\Psi_\delta(s) = \{k\}$. Next, pick $\{w_4, w_5\}$ an

orthogonal basis of H , and by Propositions 2.1.(1), $\{v_4, v_5\}$ in H such that

$$k \in \Gamma_{v_4}(s, w_4) \cap \Gamma_{v_5}(s, w_5). \quad (12)$$

On the other hand, since $s' \in \Phi_\delta(k)$, it follows from the definition of $\Phi_\delta(k)$ in Equation (4), that $s' \in \Gamma_{w_6}(k, v)$ for some v and w_6 in H . Now, by Proposition 2.1.(4), let v_6 in H such that $\Gamma_{v_6}(s', w_6) = \{k\}$. Thus, according to Equation (12)

$$k \in \Gamma_{v_4}(s, w_4) \cap \Gamma_{v_5}(s, w_5) \cap \Gamma_{v_6}(s', w_6),$$

and we are also done. □

5 | A SUFFICIENT CONDITION ON δ FOR $\Phi_\delta(k)$ TO BE A SINGLETON SET

In this section, we are going to introduce a subset \mathbf{A} of \mathbf{R} that will allow us, through Proposition 5.2, to obtain a condition on δ for $\Phi_\delta(k)$ be a singleton set.

Definition 5.1. Let \mathbf{A} be the set of all $0 < \delta < 1$ such that does not there exist u_1, u_2, u' in H and w_1, w_2, w' in S_H satisfying all the inequalities below.

- (a) $\|u' \pm (u_i - w_i)\| \leq M_\delta^2/2, \forall i \in \{1, 2\};$
- (b) $\|u_i \pm (u' - w')\| \leq M_\delta^2/2, \forall i \in \{1, 2\};$
- (c) $\|\sqrt{2}u' + \lambda u_1 + \theta u_2\| \leq M_\delta^2/\sqrt{2}, \forall \lambda, \theta \in \{-1, 1\}.$

Proposition 5.2. If $\delta \in \mathbf{A}$ then $\Phi_\delta(k)$ is a singleton set for any $k \in K$.

Proof. Assume, by contradiction, that $\Phi_\delta(k)$ is not a singleton set for some $k \in K$. Then, Proposition 4.1 states that there are two possibilities. We will consider the first of these. The proof of using the second possibility is analogous and is obtained by working with T^{-1} instead of T . Thus, suppose that there exists $s \in S$ such that

$$s \in \Gamma_{w_1}(k, v_1) \cap \Gamma_{w_2}(k, v_2) \cap \Gamma_{w'}(k', v'), \quad (13)$$

for some $k' \in K, k \neq k', v_1, v_2, v', w_1, w_2, w'$ in H with v_1 orthogonal to v_2 .

By using the Proposition 2.1.(2), we may assume that $\|v_i\| = \|v'\| = M_\delta$, for every $i \in \{1, 2\}$. Then $\|w_i\| = \|w'\| = 1$ for every $i \in \{1, 2\}$.

Now proceeding as in the proof of [8, Proposition 5.1], the desired contradiction is achieved. □

6 | ON THE NUMBERS $\delta > 0$ WHICH DO NOT BELONG TO \mathbf{A}

The main objective of this section is to establish four inequalities involving numbers $\delta > 0$ not belonging to \mathbf{A} and the respective vectors of H involved in Definition 5.1. These inequalities will play an important role in the next section where we will show that the number 0.054208 belongs to \mathbf{A} .

Proposition 6.1. Suppose that $\delta \notin \mathbf{A}$ and let $u_1, u_2, u', w_1, w_2, w'$ in H satisfying the inequalities (a), (b) and (c) of Definition 5.1. Put $z_i = u_i - w_i/2$, for $i \in \{1, 2\}$ and $z' = u' - w'/2$. We have

- (1) There exists $\lambda \in \{-1, 1\}$ such that $\|w_1 + \lambda w_2\| \geq \sqrt{4 - \delta^2}$.
- (2) $2\|z'\|^2 + 2\|z_i\|^2 \leq \delta/2$, for all $i \in \{1, 2\}$.
- (3) $8\|u'\|^2 + 4\|u_1 + \lambda u_2\|^2 \leq 4 + 2\delta$, for any $\lambda \in \{-1, 1\}$.
- (4) $\|u'\|^2 \leq \left(\delta^2 - 2\delta + 4\sqrt{4\delta - \delta^3}\right)/8$.

Proof.

(1) Note that by the inequalities (a) and (b) of Definition 5.1 we have

$$\|w' - w_i\| \leq \|w' - (u' + u_i)\| + \|u' + u_i - w_i\| \leq \sqrt{2 + \delta}, \quad (14)$$

and

$$\|w' + w_i\| \leq \|w' + u_i - u'\| + \|-(u_i - u') + w_i\| \leq \sqrt{2 + \delta}. \quad (15)$$

Since $\|w'\| = \|w_i\| = 1$ by the parallelogram law we deduce from Equation (14) that $\sqrt{2 - \delta} \leq \|w' + w_i\|$ and from Equation (15) that $\sqrt{2 - \delta} \leq \|w' - w_i\|$. Thus, again by the parallelogram law and by Proposition 3.2 we conclude that

$$\|w_1 + w_2\| \geq \sqrt{4 - \delta^2} \text{ or } \|w_1 - w_2\| \geq \sqrt{4 - \delta^2}.$$

(2) Fix $i \in \{1, 2\}$. It follows from (a) and (b) of Definition 5.1 that

$$\|u' + u_i - w_i\| \leq M_\delta^2/2 \text{ and } \|u' + u_i - w'\| \leq M_\delta^2/2.$$

So, by the parallelogram law

$$\|2u' + 2u_i - (w' + w_i)\|^2 + \|w' - w_i\|^2 = 2\|u' + u_i - w_i\|^2 + 2\|u' + u_i - w'\|^2 \leq M_\delta^4,$$

that is,

$$\|u' + u_i - (w' + w_i)/2\| \leq \sqrt{M_\delta^4 - \|w_i - w'\|^2}/2. \quad (16)$$

Next, according to (a) and (b) of Definition 5.1, we also have

$$\|u' - u_i + w_i\| \leq M_\delta^2/2 \text{ and } \|u' - u_i - w'\| \leq M_\delta^2/2.$$

Then, again by the parallelogram law

$$\|2u' - 2u_i + (w_i - w')\|^2 + \|w' + w_i\|^2 = 2\|u' - u_i + w_i\|^2 + 2\|u' - u_i - w'\|^2 \leq M_\delta^4,$$

that is,

$$\|u' - u_i - (w' - w_i)/2\| \leq \sqrt{M_\delta^4 - \|w_i + w'\|^2}/2. \quad (17)$$

Finally, once again by the parallelogram law together with (16) and (17) we have

$$\begin{aligned} 2\|z'\|^2 + 2\|z_i\|^2 &= \|z' + z_i\|^2 + \|z' - z_i\|^2 \\ &= \|u' + u_i - (w' + w_i)/2\|^2 + \|u' - u_i - (w' - w_i)/2\|^2 \\ &\leq (M_\delta^4 - \|w_i - w'\|^2)/4 + (M_\delta^4 - \|w_i + w'\|^2)/4 \\ &= M_\delta^4/2 - (\|w_i - w'\|^2 - \|w_i + w'\|^2)/4 \\ &= M_\delta^4/2 - (2\|w_i\|^2 + 2\|w'\|^2)/4 \\ &= (2 + \delta)/2 - 1 = \delta/2. \end{aligned}$$

(3) By the parallelogram law

$$\|2\sqrt{2}u'\|^2 + \|2(u_1 + \lambda u_2)\|^2 = 2\|\sqrt{2}u' + (u_1 + \lambda u_2)\|^2 + 2\|\sqrt{2}u' - (u_1 + \lambda u_2)\|^2.$$

So, according to (c) of Definition 5.1 we deduce

$$8\|u'\|^2 + 4\|u_1 + \lambda u_2\|^2 \leq 4\left(\sqrt{\frac{(2+\delta)}{2}}\right)^2 = 4 + 2\delta.$$

(4) According to Proposition 6.1.(1) fix $\lambda \in \{-1, 1\}$ such that

$$\frac{\sqrt{4-\delta^2}}{2} \leq \left\| \frac{w_1 + \lambda w_2}{2} \right\|. \quad (18)$$

From the triangular inequality, it is easy to see that

$$\left\| \frac{w_1 + \lambda w_2}{2} \right\| \leq \left\| \frac{w_1}{2} - u_1 \right\| + \left\| \frac{w_2}{2} - u_2 \right\| + \|u_1 + \lambda u_2\|. \quad (19)$$

On the other hand, by Proposition 6.1.(2) we know that $2\|z_i\|^2 \leq \delta/2$, for all $i \in \{1, 2\}$. Hence by Equations (18) and (19) we infer

$$\frac{\sqrt{4-\delta^2}}{2} \leq \left\| \frac{w_1 + \lambda w_2}{2} \right\| \leq \frac{2\sqrt{\delta}}{2} + \|u_1 + \lambda u_2\|,$$

that is,

$$\|u_1 + \lambda u_2\| \geq \frac{\sqrt{4-\delta^2}}{2} - \sqrt{\delta}.$$

Therefore, by Proposition 6.1.(3)

$$8\|u'\|^2 + 4\left(\frac{\sqrt{4-\delta^2}}{2} - \sqrt{\delta}\right)^2 \leq 4 + 2\delta,$$

that is,

$$8\|u'\|^2 \leq 4 + 2\delta - 4\left(\frac{4-\delta^2}{4} + \delta - \sqrt{4\delta - \delta^3}\right) = \delta^2 - 2\delta + 4\sqrt{4\delta - \delta^3},$$

and we are done. □

7 | AN AUXILIAR CONTINUOUS FUNCTION g

Before finishing the proof of Theorem 1.1 in the next section, it is convenient to introduce the function g mentioned in the following lemma.

Lemma 7.1. Consider the continuous function $g : [1, 15] \rightarrow \mathbf{R}$ given by

$$g(x) = \frac{\delta^2}{8} - \frac{\delta}{4} \left(\frac{x-2}{x} \right) + \frac{1}{2} \sqrt{(4\delta - \delta^3) \left(\frac{x-1}{x} \right)}.$$

Then,

- (a) $g(x) > 0$ for all $x \in [1, 15]$.
 (b) $g(x) < 1/4$ for all $x \in [1, 15]$.

Proof.

- (a) Computing the derivative of the function g on the half-open interval $(1, 15]$ we get

$$g'(x) = \frac{1}{x^2} \left(-\frac{\delta}{2} + \frac{1}{4} \sqrt{(4\delta - \delta^3)} \sqrt{\frac{x}{x-1}} \right).$$

So, $g'(x) > 0$ if and only if

$$\frac{1}{x} > 1 - \left(\frac{4\delta - \delta^3}{4\delta^2} \right),$$

and this inequality is true because

$$\frac{4\delta - \delta^3}{4\delta^2} < 18.5.$$

Therefore, the continuous functions g is strictly increasing on $(1, 15]$ and satisfies $g(x) > 0$ because

$$g(1) = \frac{\delta^2}{8} + \frac{\delta}{4} > 0.$$

- (b) Put $h(x) = 1/4 - g(x)$ and notice that the derivative of h is equal to $-g'(x)$ for all $x \in (1, 15]$. Hence, h is strictly decreasing on $[1, 15]$ and calculating the value of h at 15 we see that $h(15) > 0.03$. This implies that $h(x) > 0$ for all $x \in [1, 15]$, and thus we are done. \square

8 | COMPLETING THE PROOF OF THE MAIN THEOREM

The goal in this section is to complete the proof of Theorem 1.1. Thanks to Propositions 2.2 and 5.2, it suffices to show that the number 0.054208 belongs to \mathbf{A} . So, throughout this section $\delta = 0.054208$. Arguing by contradiction, assume that $\delta \notin \mathbf{A}$ and fix $u_1, u_2, u', w_1, w_2, w'$ in H satisfying the inequalities (a), (b), and (c) of Definition 5.1. Put $z_i = u_i - w_i/2$, for $i \in \{1, 2\}$ and $z' = u' - w'/2$. We will establish some properties of the vector z' and the desired contradiction will appear in Proposition 8.5, see Remark 8.6.

Proposition 8.1. *A following inequality is true*

$$\frac{\delta}{15} < 4\|z'\|^2.$$

Proof. Assume on the contrary that $4\|z'\|^2 \leq \delta/15$. Then

$$\frac{1}{2} = \left\| \frac{w'}{2} \right\| = \|u' - z'\| \leq \|u'\| + \|z'\| \leq \|u'\| + \sqrt{\frac{\delta}{60}}.$$

Consequently,

$$\left(\frac{1}{2} - \sqrt{\frac{\delta}{60}} \right)^2 \leq \|u'\|^2. \tag{20}$$

Thus, since $\delta = 0.054028$, it follows from Equation (20) that $0.2208 < \|u'\|^2$, which is a contradiction, because by Proposition 6.1.(3)

$$\|u'\|^2 \leq \frac{\delta^2 - 2\delta + 4\sqrt{4\delta - \delta^3}}{8} < 0.2196.$$

□

Next, we will introduce a convenient function G and establish in Propositions 8.2 and 8.3 two properties of it that will be fundamental in the proof of Proposition 8.4. Observe that Lemma 7.1 (a) allows us to define $G : [1, 15] \rightarrow \mathbf{R}$ by putting:

$$G(x) = 4 \left(\frac{1}{2} - \sqrt{\frac{\delta^2}{8} - \frac{\delta}{4} \left(\frac{x-2}{x} \right) + \frac{1}{2} \sqrt{(4\delta - \delta^3) \left(\frac{x-1}{x} \right)}} \right)^2. \quad (21)$$

Proposition 8.2. *Suppose that $\delta/x < 4\|z'\|^2$ for some $1 \leq x \leq 15$. Then*

$$G(x) < 4\|z'\|^2.$$

Proof. By Proposition 6.1.(1) we know that $2\|z'\|^2 + \|z_i\|^2 \leq \delta/2$, for all $i \in \{1, 2\}$. So, as a consequence of our hypothesis, we have

$$\frac{\delta}{2x} + 2\|z_i\|^2 \leq \frac{\delta}{2},$$

that is,

$$\|z_i\| < \sqrt{\frac{\delta}{4} \left(\frac{x-1}{x} \right)}.$$

Fix λ as in Proposition 6.1.(1). So, having in mind the inequality (19), we have

$$\frac{\sqrt{4 - \delta^2}}{2} < 2\sqrt{\frac{\delta}{4} \left(\frac{x-1}{x} \right)} + \|u_1 + \lambda u_2\|,$$

that is,

$$\|u_1 + \lambda u_2\| > \frac{\sqrt{4 - \delta^2}}{2} - \sqrt{\delta \left(\frac{x-1}{x} \right)}.$$

Then, by Proposition 6.1.(2)

$$8\|u'\|^2 + 4 \left(\frac{\sqrt{4 - \delta^2}}{2} - \sqrt{\delta \left(\frac{x-1}{x} \right)} \right)^2 < 4 + 2\delta,$$

Thus,

$$8\|u'\|^2 < 4 + 2\delta - 4 \left(\frac{4 - \delta^2}{4} + \delta \left(\frac{x-1}{x} \right) - \sqrt{(4\delta - \delta^3) \left(\frac{x-1}{x} \right)} \right),$$

that is,

$$8\|u'\|^2 < \delta^2 - \delta\left(\frac{2x-4}{x}\right) + 4\sqrt{(4\delta - \delta^3)\left(\frac{x-1}{x}\right)}.$$

Consequently,

$$\|u'\| < \sqrt{\frac{\delta^2}{8} - \frac{\delta}{4}\left(\frac{x-2}{x}\right) + \frac{1}{2}\sqrt{(4\delta - \delta^3)\left(\frac{x-1}{x}\right)}},$$

and since

$$\frac{1}{2} = \left\| \frac{w'}{2} \right\| = \|u' - z'\| \leq \|u'\| + \|z'\|,$$

we deduce

$$\frac{1}{2} - \sqrt{\frac{\delta^2}{8} - \frac{\delta}{4}\left(\frac{x-2}{x}\right) + \frac{1}{2}\sqrt{(4\delta - \delta^3)\left(\frac{x-1}{x}\right)}} < \|z'\|. \quad (22)$$

Now, note that according to Lemma 7.1 (b) the first member of the above inequality is positive for all $1 \leq x \leq 15$. So, by the definition of G in Equations (21) and (22) we have $G(x) < 4\|z'\|^2$. \square

It is worth mentioning here that the hardest part of the proposition below was figuring out its statement involving the number 0.0001. This was done using a computer program.

Proposition 8.3. *For all $1 \leq x \leq 15$, we have*

$$\frac{\delta}{x - 0.0001} < G(x),$$

where the function G is defined in Equation (21).

Proof. Consider the continuous function $D : [1, 15]$ given by

$$D(x) = G(x) - \frac{\delta}{x - 0.0001}.$$

Once again by using to the classic GeoGebra application package, we deduce that D has no real roots and since

$$D(1) = 4\left(\frac{1}{2} - \sqrt{\frac{\delta^2}{8} + \frac{\delta}{4}}\right)^2 - \frac{\delta}{0.9999} > 0,$$

we conclude that $D(x) > 0$ for $1 \leq x \leq 15$. \square

Proposition 8.4. *For all $n \in \mathbf{N}$, $n \leq 140,001$, we have*

$$\frac{\delta}{15 - 0.0001 n} < 4\|z'\|^2.$$

Proof. The proof will be made by induction on n . By Proposition 8.1, the statement of this proposition holds when $n = 0$. Assume the statement of the proposition holds for $n \leq 140,000$, that is:

$$\frac{\delta}{15 - 0.0001 n} < 4\|z'\|^2. \quad (23)$$

Since $1 \leq 15 - 0.0001 n \leq 15$, it follows from Equation (23) and Propositions 8.2 with $x = 15 - 0.0001 n$, that

$$G(15 - 0.0001 n) < 4\|z'\|^2. \quad (24)$$

Moreover, by Equation (23) and Proposition 8.3 with $x = (15 - 0.0001 n)$, we see that

$$\frac{\delta}{(15 - 0.0001 n) - 0.0001} < G(15 - 0.0001 n). \quad (25)$$

Now, joining this inequality with that of Equation (24) we obtain

$$\frac{\delta}{15 - 0.0001 (n + 1)} < G(15 - 0.0001 n) < 4\|z'\|^2,$$

and the proposition is proved. □

Proposition 8.5. *The following inequality holds*

$$\delta < 4\|z'\|^2.$$

Proof. It suffices to observe that by Proposition 8.4 with $n = 140001$ we have

$$\frac{\delta}{0.9999} \leq 4\|z'\|^2. \quad \square$$

Remark 8.6. Proposition 8.5 contradicts Proposition 6.1.(1), because it follows from it that $4\|z'\|^2 \leq \delta$.

9 | FINAL REMARKS ON THE SET A

Remark 9.1. We do not know if 0.054209 belongs to **A**. However, we point out that although the statements of Propositions 8.1 and 8.2 remain true when we replace 0.054208 by 0.054209, the statement of Proposition 8.4 for the correspondent function G associated with 0.054209 does not hold, even replacing 0.0001 by any positive number. To show this, it suffices to observe that if

$$\frac{0.054209}{c} < G(4.1999), \quad (26)$$

for some $c > 0$, then $c > 4.1999$. Indeed, in the case where $\delta = 0.054209$, the calculation of $G(4.1999)$ shows that $G(4.1999) < 0.12907$. Therefore, if $c > 0$ satisfies the inequality (26), then

$$\frac{0.054209}{c} < 0.12907.$$

Consequently, $4.19996 < c$.

Remark 9.2. By using the method developed in this work, the number of digits in 0.054208 can be increased in such a way that the statements of the results obtained still remain true. For example, if we replace the number 0.0001 with 0.00001

in the statement of Proposition 8.3 and the number 140,001 with 1400,001 in the statement of Proposition 8.4, we can replace the number 0.054208 with 0.0542085 throughout the rest of the paper. So, 0.0542085 also belongs to **A**.

Remark 9.3. There is a number close to 0.054208 that do not belong to **A**. Indeed, let $\delta = 34 - 24\sqrt{2}$ the smallest root of $p(x) = x^2 - 68x + 4$. Then, δ is approximately 0.058874 and we will show that it does not belong to **A**. Put

$$a = \frac{1}{2} - \sqrt{\frac{2 + \delta}{8} - \frac{1}{4}},$$

and take $w' = (1, 0)$, $u' = (a, 0)$, $w_1 = w_2 = (0, 1)$ and $u_1 = u_2 = (0, a)$. First of all notice that $a^2 - a = (\delta - 2)/8$. Then,

(a) For each $i \in \{1, 2\}$, we have

$$\|u' \pm (u_i - w_i)\|^2 = a^2 + (a - 1)^2 = 2a^2 - 2a + 1 = (2 + \delta)/4.$$

(b) For each $i \in \{1, 2\}$, we have

$$\|u_i \pm (u' - w')\|^2 = a^2 + (a - 1)^2 = (2 + \delta)/4.$$

(c) For each $\lambda, \theta \in \{1, 2\}$, we have

$$\|\lambda u_1 + \theta u_2\| \leq \|u_1\| + \|u_2\| = 2a.$$

Consequently, putting $D = \|\sqrt{2}u' + \lambda u_1 + \theta u_2\|^2$, it follows that

$$\begin{aligned} D &\leq \|\sqrt{2}u'\|^2 + \|\lambda u_1 + \theta u_2\|^2 \leq 6a^2 = 6\left(\frac{2 + \delta}{8} - \sqrt{\frac{2 + \delta}{8} - \frac{1}{4}}\right) \\ &= 6\left(\frac{9}{2} - 3\sqrt{2} - \sqrt{\frac{17}{4} - 3\sqrt{2}}\right) = 6\left(\frac{9}{2} - 3\sqrt{2} - \sqrt{\left(\frac{3}{2} - \sqrt{2}\right)^2}\right) \\ &= 6\left(\frac{9}{2} - 3\sqrt{2} - \left(\frac{3}{2} - \sqrt{2}\right)\right) = 6(3 - 2\sqrt{2}) = \frac{2 + \delta}{2}. \end{aligned}$$

Therefore, in view of (a), (b), and (c), we see that $\delta \notin \mathbf{A}$.

ORCID

Elói Medina Galego  <https://orcid.org/0000-0002-1117-3869>

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