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Process Parameters Estimation in the Taguchi On-line Quality Monitoring Procedure for Attributes

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Abstract

In this work, estimators for the parameter vector of the statistical model proposed by Nayebpour and Woodall (1993) for the Taguchi on-line quality monitoring procedure for attributes are derived. For the maximum likelihood method, the likelihood function is generated by the time the first defective item is found under the monitoring strategy. Under the bayesian standpoint, the posterior probability distributions for the parameters are determined by taking into account independent beta densities as prior distributions.

1. Introduction

In Taguchi [1-3] and Taguchi et al. [4] an economically designed on-line quality monitoring procedure for attributes is proposed. The setting is that of a production process in which items are produced one-by- one, independently, and whose fraction defective shifts at some point from the initial value of 0 to a value θ, $0 \leq \theta \leq 1$. The monitoring procedure consists of inspecting a single item after every m produced and carrying out, as soon as a defective item is detected, a process adjustment routine that restores process fraction defective to its initial value. Taking into account cost factors associated with the production process under the above described monitoring procedure, mathematical expressions to assess the inspection interval, m^*, that minimizes the procedure's long run expected cost per item produced are developed separately in two situations: case 1, in which the process fraction defective shifts from 0 (no defective items are produced) to $\theta = 1$ (all items produced are defective); and case 2, in which the process fraction defective shifts from 0 to θ, $0 \leq \theta \leq 1$.

The problem of assessing the inspection interval, m*, that minimizes Taguchi's monitoring procedure long run expected cost per item produced was later reexamined in Nayebpour and Woodall (1993), where a geometric waiting time to model the process failure mechanism was introduced into Taguchi's original setting, and, more recently, in Borges, Ho and Turnes (2001) where, in addition to that, the possibility of random inspection errors is taken into account. In both of these approaches, mathematical expressions for the optimal inspection interval are derived from renewal theory considerations. These expressions depend, however, not only on the cost factors involved, but also on the unknown values of the parameters of the underlying statistical models. The estimation of these parameters from process data is, therefore, imperative if one is willing to determine the optimal diagnosis interval from the aforementioned mathematical expressions.

In this work, estimators for the parameter vector of the statistical model proposed by Nayebpour and Woodall (1993) are derived by both the maximum likelihood method and the bayesian standpoint. In the former approach, the likelihood function considered is generated by the time the first defective item is found (a single observation) under the monitoring strategy described earlier. In the latter, the posterior probability distributions for the parameters are determined by taking into account independent beta densities as prior distributions.

To the authors knowledge this estimation problem has not been thoroughly investigated in the literature. Considering data from previous cycles, a brief discussion appears in Nayebpour and Woodall (1993) which assert that the method of moments and the maximum likelihood method yield the same estimators. Also, as they did not conceive all the underlying unknown quantities as a single parameter vector to be estimated as a whole, the estimators there obtained depend not only on the value of the diagnosis interval used in the procedure, but also on other generally unknown quantities that should also be considered as parameter components. Furthermore, as pointed out by those authors, the application of these estimators still requires historical data on retrospective inspections, the additional sampling costs of which seems to have been neglected. Hence a deeper study of this subject, concerning not only the methods of estimation properly but also some mathematical features of the underlying statistical model, is made herein.

For the reader's guidance, this paper is organized as follows: in section 2, the mathematical setup and the associated statistical model is introduced; in section 3 the maximum likelihood estimator is obtained; in section 4 the Bayesian paradigm is developed and section 5 is devoted to final comments and conclusions.

2. The Mathematical Setup

In this section, the statistical model introduced by Nayebpour and Woodall (1993), incorporating a geometric distribution for the waiting time for process shift, in case 2 presented earlier, is reviewed. It should also be mentioned that the possibility of diagnosis errors is not being considered in this work.

Let $\{(X_i,\Theta_i): i\geq 0\}$ be a stochastic process, where X_i is the random variable indicating nonconformity of the i-th item produced in a cycle and Θ_i denotes the unobservable process fraction defective when the i-th item is about to be produced. The marginal process $\{\Theta_i: i\geq 0\}$ is assumed to behave like a Markov chain with state space $E = \{0,\theta\}$ and transition matrix

$$M = \begin{bmatrix} 1-\rho & \rho \\ 0 & 1 \end{bmatrix},$$

where $\theta\in[0,1]$ and $\rho\in[0,1]$ are both unknown. It is also assumed that $\Theta_0 = 0$, $X_0 = 0$, $(X_0, X_1,\ldots, X_n)$ are conditionally independent given $(\Theta_0, \Theta_1, \ldots, \Theta_n)$ and that

$$P\{X_i = 1|\Theta_0 = \theta_0, \Theta_1 = \theta_1,\ldots, \Theta_n = \theta_n\} = P\{X_i = 1|\Theta_i = \theta_i\} = \theta_i,\ 0\leq i \leq n,\ n\geq 0.$$

Under this model, the transience of M reflects the impossibility of restoring the fraction defective θ to 0, at the end of a cycle, unless process adjustment is undertaken. Also, the unobservable random waiting time for process shift,

$$\eta = \inf\{ i\geq 1: \Theta_i=\theta\},$$

is easily seen to have a geometric distribution with unknown parameter ρ.

Recalling that under the Taguchi's monitoring strategy an item is to be inspected after every m units produced, the observable random number of inspections carried out until a defective item is found,

$$\tau = \inf\{\, k\geq 1 : X_{km}=1\},$$

or, equivalently, the random vector $(X_m,\ldots, X_{\tau m})$, constitutes the statistical data in a production cycle that carries relevant information on $(\theta,\rho)\in[0,1]^2$. Thus, in order to estimate the unknown, non observable, process parameter vector (θ,ρ), on the basis of the information that $\tau=k$, for some $k\geq 1$, one must obtain the likelihood function

$$L_k(\theta,\rho) = P\{\, \tau = k \mid (\theta,\rho)\},\quad (\theta,\rho)\in[0,1]^2.$$

As shown in Nayebpour and Woodall (1993),

$$P\{\tau = k \mid (\theta,\rho)\} = \theta(1-(1-\rho)^m)\sum_{j=0}^{k-1}(1-\rho)^{mj}(1-\theta)^{k-1-j},\ k\geq 1.$$

It is worth noting that the above statistical model is nonidentifiable in the sense that at least two different points in the parameter space give rise to the same sampling distribution, that is, there exists (θ_1,ρ_1) and (θ_2,ρ_2) in $[0,1]^2$ such that

$$P\{\tau=k \mid (\theta_1,\rho_1)\} = P\{\tau=k \mid (\theta_2,\rho_2)\},\ \forall\, k\geq 1.$$

More precisely, it is not hard to see that

$$L_k(\theta,\rho) = L_k(1-(1-\rho)^m,\ 1-(1-\theta)^{1/m}),\ \forall\, k\geq 1.$$

In situations of nonidentifiability, it is not infrequent that methods of estimation, such as maximum likelihood, least squares, moments and minimum variance, deliver unsatisfactory estimates. Under this circumstance, the estimation problem is not even well-defined in the classical statistics framework. Some authors further advocate that if this is the case, Bayesian methodology should be preferred. Deeper discussions of this issue can be found in Kadane (1974) and Neath and Samaniego (1997). In our setting, the maximum likelihood estimator may, at first sight, be revealed nonunique. By taking into account this mathematical feature of the model, the maximum likelihood estimation and the Bayesian operation for the process parameter vector, (θ, ρ), are the object of the next two sections.

3. Maximum Likelihood Estimation

For ease of notation, the Maximum Likelihood (ML) estimation of the parameter vector (θ,α), instead of (θ,ρ), where $\alpha = (1-\rho)^m$, will be obtained. This is clearly of no consequence because of the invariance property of ML estimators. The problem thus consists of determining $(\theta,\alpha)^\wedge: N \to [0,1]^2$, $N = \{1, 2, 3,\ldots\}$, such that

$$L_k((\theta,\alpha)^\wedge(k)) = \max\{L_k(\theta,\alpha) : (\theta,\alpha) \in [0,1]^2\},\ \forall\ k \geq 1,$$

where

$$L_k(\theta,\alpha) = \theta(1-\alpha)\sum_{j=0}^{k-1}\alpha^j(1-\theta)^{k-1-j},\ (\theta,\alpha)\in[0,1]^2. \qquad (3.1)$$

For $k = 1$, the objective function reduces to $L_1(\theta,\alpha)=\theta(1-\alpha)$, which clearly attains its maximum at the point (1,0). For general $k>1$, the solution is not immediate, requiring some standard mathematical manipulation.

Lemma 3.1. In order that $(\theta_0,\alpha_0)\in(0,1)^2$ be an extreme point of $L_k(\theta,\alpha)$, defined in (3.1), it is necessary and sufficient that

$$\alpha_0 = (k\theta_0-1)\,[(k+1)\theta_0-1]^{-1}. \tag{3.2}$$

Proof.

As $L_k(\theta,\alpha)$ possesses infinitely many continuous partial derivatives, (θ_0,α_0) is an extreme point of $L_k(\theta,\alpha)$ if, and only if,

$$\frac{\partial L(\theta,\alpha)}{\partial\theta} = \frac{\partial L(\theta,\alpha)}{\partial\alpha} = 0. \tag{3.3}$$

Since from (3.1)

$$\frac{\partial L(\theta,\alpha)}{\partial\theta} = \sum_{j=0}^{k-1}\alpha^j(1-\alpha)[(1-\theta)^{k-1-j} - \theta(k-1-j)(1-\theta)^{k-2-j}]$$

and

$$\frac{\partial L(\theta,\alpha)}{\partial\alpha} = \sum_{j=0}^{k-1}\theta(1-\theta)^{k-1-j}[j\alpha^{j-1}(1-\alpha) - \alpha^j],$$

(3.3) holds if, and only if,

$$\sum_{j=0}^{k-1} j\alpha^j(1-\theta)^{k-j-1} = \frac{k\theta-1}{\theta^2(1-\alpha)}L_k(\theta,\alpha) \tag{3.4}$$

and

$$\sum_{j=0}^{k-1} j\alpha^j(1-\theta)^{k-j-1} = \frac{\alpha}{\theta(1-\alpha)^2}L_k(\theta,\alpha). \tag{3.5}$$

From (3.4) and (3.5) the result follows easily.

Since

$$L_k(\theta,\alpha) = L_k(1-\alpha,1-\theta),\ \forall k\geq 1 \text{ and } (\theta,\alpha)\in(0,1)^2,$$

it follows from lemma 3.1 that the problem of determining $(\theta_0,\alpha_0)\in(0,1)^2$ that maximizes $L_k(\theta,\alpha)$ amounts to determining $\theta_0\in(1/k,\ 2/(k+1)]$ that maximizes

$$L_k(\theta) = \frac{\theta^2}{(k+1)\theta - 1} \sum_{j=0}^{k-1} \left[\frac{(k\theta - 1)}{(k+1)\theta - 1}\right]^j (1-\theta)^{k-1-j}. \qquad (3.6)$$

For that, the next lemma is a crucial step.

Lemma 3.2. $L_k(\theta)$ defined in (3.6) is monotone increasing in the interval $(1/k, 2/(k+1))$.

Proof.

For $\theta \in (1/k,\ 2/(k+1))$, $L_k(\theta)$ may be rewritten as

$$L_k(\theta) = \frac{\theta^2}{(k+1)\theta - 1} \cdot \frac{(1-\theta)^k - \left(\frac{k\theta - 1}{(k+1)\theta - 1}\right)^k}{(1-\theta) - \left(\frac{k\theta - 1}{(k+1)\theta - 1}\right)}$$

$$= \frac{\theta^2}{[(k+1)\theta - 1](1-\theta) - (k\theta - 1)} \cdot \{(1-\theta)^k - \left(\frac{k\theta - 1}{(k+1)\theta - 1}\right)^k\} \qquad (3.7)$$

.

Furthermore, straightforward calculations give:

$$\frac{dL_k(\theta)}{d\theta} = u(\theta)\{\frac{(1-\theta)^{k-1}[(k+1)\theta - 1]^{k+1}}{(k\theta - 1)^{k-1}} - 1\},$$

where

$$u(\theta) = \frac{[k(k+1)\theta^2 - 2(k+1)\theta + 2](k\theta - 1)^{k-1}}{[2-(k+1)\theta]^2[(k+1)\theta - 1]^{k+1}}.$$

Since

$$k(k+1)\theta^2 - 2(k+1)\theta + 2 > 0,\ \forall \theta \in (1/k,\ 2/(k+1)),$$

it follows that $\frac{dL_k(\theta)}{d\theta} > 0$ if, and only if,

$$v(\theta) = \frac{(1-\theta)^{k-1}[(k+1)\theta - 1]^{k+1}}{[k\theta - 1]^{k-1}} - 1 > 0, \ \forall\theta \in (1/k,\ 2/(k+1)). \tag{3.8}$$

In order to see that (3.8) holds observe first that $\lim_{\theta \to (1/k)+} v(\theta) = \infty$ and $\lim_{\theta \to (2/(k+1))-} v(\theta) = 0$. In addition, after standard calculations one gets

$$\frac{dv(\theta)}{d\theta} = \frac{-k(k\theta-1)^{k-2}(1-\theta)^{k-2}[(k+1)\theta-1]^{k}[(k+1)\theta-2]^2}{(k\theta-1)^{2k-2}} < 0, \ \forall\theta \in (1/k,\ 2/(k+1)).$$

Therefore, $v(\theta)$ is strictly decreasing in (1/k, 2/(k+1)) and $\inf\{v(\theta): \theta \in (1/k,\ 2/(k+1))\} = 0$. Consequently, $v(\theta) > 0$, $\forall\theta \in (1/k,\ 2/(k+1))$, and the proof is complete.

Theorem 3.1. The ML estimator of the process parameter vector (θ,ρ), on the basis of the information carried by τ, the observable random number of inspections carried out until a defective item is found, is given by

$$(\theta,\rho)\hat{}(\tau) = \left(\frac{2}{\tau+1}, 1 - \left[\frac{\tau-1}{\tau+1}\right]^{1/m}\right). \tag{3.9}$$

Proof.

Lemma 3.2 implies that $L_k(\theta)$ reaches its maximum in the interior of the parameter space at the point $(\theta_0, \alpha_0) = \left(\frac{2}{k+1}, \frac{k-1}{k+1}\right)$. In order to evaluate the maximum of $L_k(\theta,\alpha)$ on $[0,1]^2$, the entire parameter space, it suffices to evaluate the maximum of $L_k(\theta,\alpha)$ on its boundary and compare it with $L_k(\frac{2}{k+1},\frac{k-1}{k+1})$.

Note that $L_k(\theta,\alpha)$ vanishes in the line segments $\theta = 0$ and $\alpha = 1$. Along the line segments $\theta = 1$ and $\alpha = 0$, $L_k(\theta, \alpha)$ attains its maximum value of $\frac{(k-1)^{k-1}}{k^k}$ at the points (1, (k-1)/k) and (1/k, 0), respectively.

Since

$$\frac{(k-1)^{k-1}}{k^k} = \frac{k(k-1)^{k-1}}{(k+1)^{k+1}} \cdot \left[\frac{k+1}{k}\right]^{k+1}$$

and the nonnegative function on $[1,\infty)$, defined by $f(x) = \left[\frac{x+1}{x}\right]^{x+1}$, is easily seen to be decreasing, it follows that $\max\{\left[\frac{k+1}{k}\right]^{k+1} : k \geq 1\} = f(1) = 4$, and

$$\frac{(k-1)^{k-1}}{k^k} < \frac{4k(k-1)^{k-1}}{(k+1)^{k+1}} = L_k(\frac{2}{k+1},\frac{k-1}{k+1}).$$

Consequently the ML estimator of (θ,α), on the basis of the information carried by τ=k, k≥1, is given by (θ,α)^(k)=(2/(k+1) , (k-1)/(k+1)) and the result follows from the invariance property of the ML estimators.

It should be mentioned that in the case of available information from n independent production cycles, that is, when the random numbers of inspections carried out until a defective item is found, $\tau_1,\ldots,\tau_n$, can be observed on n independent cycles, the ML estimator of the process parameter vector (θ,ρ) can be easily shown to be

$$(\theta,\rho)\hat{}(\tau_1,\ldots,\tau_n) = (\frac{2}{\bar{\tau}+1}, 1-\left[\frac{\bar{\tau}-1}{\bar{\tau}+1}\right]^{1/m}), \qquad (3.9)$$

where $\bar{\tau} = (\tau_1 + \tau_2 + \ldots + \tau_n)/n$. The proof is completely analogous with the one just presented.

Notice that, independently of the initial inspection interval m, the ML estimator of (θ,ρ) derived in theorem 3.1 takes values on the triangle $\theta \geq \rho$ of the parameter space, except only for m=1, in which case $\theta\hat{}=\rho\hat{}$. Consequently, this estimator is not suitable in situations in which θ is known beforehand to be strictly smaller than ρ. Also, the information $\tau=1$ yields an estimate which outlines a process that produces exclusively defective items, namely, $\theta\hat{}(1)=\rho\hat{}(1)=1$.

In a sense, these shortcomings of the ML estimator of (θ,ρ) result from the fact that the likelihood function is the only mathematical device that brings knowledge or information about the process parameter vector. For this reason, the Bayesian approach to the estimation of (θ,ρ) is explored in the next section. Under this approach two mathematical devices bring knowledge about the parameter vector: the likelihood function and the prior distribution which describes an expert's opinion about it. A complete exposition of the Bayesian methodology can be found in Bernardo and Smith (1994).

4. The Bayesian Approach

In this section, the well-known Bayesian paradigm for uncertainty updating (knowledge acquirement) of unknown quantities is adopted by considering jointly independent beta densities as a prior distribution for (θ, ρ). More precisely, the prior joint probability density function, $f(\theta, \rho)$, for (θ, ρ), is given by

$$f(\theta,\rho)=\frac{\Gamma(a_1+b_1)}{\Gamma(a_1)\Gamma(b_1)}\theta^{a_1-1}(1-\theta)^{b_1-1}\frac{\Gamma(a_2+b_2)}{\Gamma(a_2)\Gamma(b_2)}\rho^{a_2-1}(1-\rho)^{b_2-1},\ (\theta,\rho)\in[0,1]^2,$$

where $\Gamma(.)$ denotes the gamma function and a_1, $b_1 > 0$ and a_2, $b_2 > 0$ are the parameters of the prior independent marginal beta distributions for θ and ρ, respectively. Besides making the Bayesian operation mathematically tractable, the assumption of independence is also reasonable in a wide variety of production environments. Also, the beta choice covers mathematical transcriptions of a broad spectrum of opinions about (θ, ρ).

From Bayes' Theorem, the density $f(\theta,\rho\,|\,\tau=k)$, of the posterior distribution of (θ,ρ), given the data $\tau = k$, satisfies

$$f(\theta,\rho\,|\,\tau=k)=\frac{L_k(\theta,\rho)f(\theta,\rho)}{\int_{[0,1]}\int_{[0,1]}L_k(\theta,\rho)f(\theta,\rho)d\theta d\rho}$$

$$\propto \sum_{j=0}^{k-1}\theta^{a_1}(1-\theta)^{k-1-j+b_1-1}\rho^{a_2-1}(1-\rho)^{mj+b_2-1}(1-(1-\rho)^m),\ (\theta,\rho)\in[0,1]^2.$$

Consequently, for $(\theta,\rho)\in[0,1]^2$,

$$f(\theta,\rho\,|\,\tau=k)=\frac{1}{C}\sum_{j=0}^{k-1}\theta^{a_1}(1-\theta)^{b_1+k-j-2}\rho^{a_2-1}(1-\rho)^{b_2+mj-1}(1-(1-\rho)^m),$$

where $C = C(m,k,a_1,b_1,a_2,b_2)$ is given by

$$\sum_{j=0}^{k-1}\frac{\Gamma(a_1+1)\Gamma(b_1+k-1-j)}{\Gamma(a_1+b_1+k-j)}\left\{\frac{\Gamma(a_2)\Gamma(b_2+mj)}{\Gamma(a_2+b_2+mj)}-\frac{\Gamma(a_2)\Gamma(b_2+m(j+1))}{\Gamma(a_2+b_2+m(j+1))}\right\}.$$

From the joint posterior distribution for (θ,ρ) above, one can easily obtain the posterior marginal densities. Specifically, the posterior marginal density of θ is given by

$$f(\theta\mid\tau=k)=\frac{1}{C}\sum_{j=0}^{k-1}\left\{\frac{\Gamma(a_2)\Gamma(b_2+mj)}{\Gamma(a_2+b_2+mj)}-\frac{\Gamma(a_2)\Gamma(b_2+m(j+1))}{\Gamma(a_2+b_2+m(j+1))}\right\}\theta^{a_1}(1-\theta)^{b_1+k-j-2},$$

$\theta\in[0,1]$, that is, given $\tau=k$, θ is distributed according to a mixture of k beta densities with parameters a_1+1 and $b_1+k-1-j$, $j=0,\ldots,k-1$, weighted, respectively, by

$$p_j=\frac{1}{C}\frac{\Gamma(a_1+1)\Gamma(b_1+k-1-j)}{\Gamma(a_1+b_1+k-j)}\left\{\frac{\Gamma(a_2)\Gamma(b_2+mj)}{\Gamma(a_2+b_2+mj)}-\frac{\Gamma(a_2)\Gamma(b_2+m(j+1))}{\Gamma(a_2+b_2+m(j+1))}\right\}.$$

By an abuse of notation, this can be written as

$$\theta\mid\tau=k\sim\sum_{j=0}^{k-1}p_j\text{Beta}(a_1+1,b_1+k-j-1).$$

Also, the posterior marginal density of ρ is a difference of mixtures of beta densities. By another abuse of notation,

$$\rho\mid\tau=k\sim\sum_{j=0}^{k-1}u_j\text{Beta}(a_2,b_2+mj)-\sum_{j=0}^{k-1}v_j\text{Beta}(a_2,b_2+m(j+1)),$$

where

$$u_j=\frac{1}{C}\frac{\Gamma(a_1+1)\Gamma(b_1+k-j-1)}{\Gamma(a_1+b_1+k-j)}\frac{\Gamma(a_2)\Gamma(b_2+mj)}{\Gamma(a_2+b_2+mj)}$$

and

$$v_j=\frac{1}{C}\frac{\Gamma(a_1+1)\Gamma(b_1+k-j-1)}{\Gamma(a_1+b_1+k-j)}\frac{\Gamma(a_2)\Gamma(b_2+m(j+1))}{\Gamma(a_2+b_2+m(j+1))}$$

The above results show that the *a priori* stochastic independence between θ and ρ is not preserved *a posteriori*, except when $\tau=1$. Also, since the resultant marginal posterior densities do not look mathematically inspiring, summaries in terms of posterior means, medians or modes, may become appealing. However, despite their "mathematical looks", the joint posterior distribution of (θ,ρ) and its corresponding marginals are precisely the ultimate inferential results one can achieve concerning (θ,ρ). Under Bayesian conditioning they combine an expert's prior knowledge about (θ,ρ) with the whole information on it carried by the data through the likelihood function.

It should also be emphasized that other prior distributions having proper subsets of the parameter space, $[0,1]^2$, such as the triangle $\theta<\rho$ or the line segment $\theta=\rho$, as their supports can be treated similarly, coherently yielding posterior distributions supported by the same proper subsets of the parameter space $[0,1]^2$. In this manner, a broader range of opinions concerning (θ,ρ) may be contemplated.

When the information from n independent production cycles is available, that is, when the random numbers of inspections carried out until a defective item is found, $\tau_1,\ldots,\tau_n$, can be observed on n independent cycles, the Bayesian operation still applies, regardless of whether the n resulting observations are made known altogether or sequentially, that is, one at a time.

Also, the Bayesian approach to inference is immune against nonidentifiability. According to Lindley (1971), "In passing it might be noted that unidentifiability causes no real difficulty in the Bayesian approach".

Finally, it should be noticed that under the Bayesian paradigm the uncertainty updating of η, the waiting time for the process shift, is also feasible. In fact, for $i \geq 1$, one has *a priori*

$$\begin{aligned} P\{\eta=i\} &= \int_{[0,1]}\int_{[0,1]} P\{\eta=i \mid (\theta,\rho)\}\, f(\theta,\rho)\, d\theta d\rho \\ &= \int_{[0,1]}\int_{[0,1]} (1-\rho)^{i-1} \rho\, f(\theta,\rho) d\theta d\rho \\ &= \int_{[0,1]} \Gamma(a_2+b_2)[\Gamma(a_2)\Gamma(b_2)]^{-1} \rho^{a2+1-1} (1-\rho)^{b2+i-1-1} d\rho. \end{aligned}$$

Thus

$$P\{\eta=i\} = \{\Gamma(a_2+b_2)[\Gamma(a_2)\Gamma(b_2)]^{-1}\} \; \{[\Gamma(a_2+1)\Gamma(b_2+i-1)]\,[\Gamma(a_2+b_2+i)]^{-1}\}\, I_{\{1,2,\ldots\}}(i),$$

where $I_A(\cdot)$ denotes the indicator function of the set A. Posterior to the data $\tau=k$,

$$\begin{aligned} P\{\eta=i \mid \tau=k\} &= \frac{P\{\eta=i, \tau=k\}}{P\{\tau=k\}} \\ &= \frac{\int_{[0,1]}\int_{[0,1]} P\{\eta=i, \tau=k \mid (\theta,\rho)\} f(\theta,\rho) d\theta d\rho}{\int_{[0,1]}\int_{[0,1]} P\{\tau=k \mid (\theta,\rho)\} f(\theta,\rho) d\theta d\rho} \\ &= \frac{\int_{[0,1]}\int_{[0,1]} (1-\rho)^{i-1}\rho(1-\theta)^{k-(]i/m[-1)-1}\theta f(\theta,\rho) d\theta d\rho}{\int_{[0,1]}\int_{[0,1]} P\{\tau=k \mid (\theta,\rho)\} f(\theta,\rho) d\theta d\rho}, \end{aligned}$$

for $i/m \leq k$, where $]x[$ denotes the smallest integer bigger than or equal to x.

Standard arithmetic then yields

$$P\{\eta=i \mid \tau=k\} = \frac{\dfrac{\Gamma(a_1+b_1)}{\Gamma(a_1)\Gamma(b_1)} \cdot \dfrac{\Gamma(a_2+b_2)}{\Gamma(a_2)\Gamma(b_2)} \cdot \dfrac{\Gamma(a_1+1)\Gamma(b_1+k-]i/m[)}{\Gamma(a_1+b_1+k-]i/m[+1)} \cdot \dfrac{\Gamma(a_2+1)\Gamma(b_2+i-1)}{\Gamma(a_2+b_2+i)}}{\int_{[0,1]}\int_{[0,1]} \sum_{j=0}^{k-1} \theta[1-(1-\rho)^m](1-\rho)^{mj}(1-\theta)^{k-1-j} f(\theta,\rho) d\theta d\rho} I_{\{1,2,\ldots,km\}}(i)$$

$$= \frac{\frac{\Gamma(b_1 + k - \lceil i/m \rceil)}{\Gamma(a_1 + b_1 + k - \lceil i/m \rceil + 1)} \cdot \frac{\Gamma(a_2 + 1)\Gamma(b_2 + i - 1)}{\Gamma(a_2 + b_2 + i)}}{\sum_{j=0}^{k-1} \frac{\Gamma(a_2)\Gamma(b_1 + k - j - 1)}{\Gamma(a_1 + b_1 + k - j)} \cdot \{\frac{\Gamma(b_2 + mj)}{\Gamma(a_2 + b_2 + mj)} - \frac{\Gamma(b_2 + m(j+1))}{\Gamma(a_2 + b_2 + m(j+1))}\}} I_{(1,2,\ldots,km)}(i) \,.$$

The above development concerns, in particular, situations in which a recall of produced items must be conducted in order to repair the existing defective ones. It is worth mentioning that as the model proposed by Nayebpour and Woodall (1993) does not include η as a parameter, no ML estimator for it can be found.

5. Conclusions

The assessment of the optimal inspection interval under Taguchi's on-line quality monitoring procedure for attributes depends on the unknown value of a specific parameter vector besides appropriate cost factors. The derivation of estimators for this parameter vector is then focused in this work for the particular statistical model proposed by Nayebpour and Woodall (1993), a variant of Taguchi's original setting that incorporates a geometric distribution for the waiting time for the process shift. Both the maximum likelihood (ML) estimator and the Bayesian operation are developed on the basis of the number of inspections to be carried out until a defective item is found. The ML estimator is shown to be sensible in general, although it yields unsatisfactory estimates in certain situations. The Bayesian solution applies to a broader range of production processes since it considers an expert's knowledge of the parameter vector in question. The uncertainty updating of the waiting time for the process shift is also shown to be feasible under the Bayesian paradigm. Nevertheless, a full Bayesian decision-theoretic approach to the problem of choosing the optimal inspection interval, without reference to Renewal Theory, has not been dealt with and will be the subject of forthcoming papers.

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