

A matrix method for quasinormal modes: Kerr and Kerr–Sen black holes

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In this paper, a matrix method is employed to study the scalar quasinormal modes of Kerr as well as Kerr–Sen black holes. Discretization is applied to transfer the scalar perturbation equation into a matrix form eigenvalue problem, where the resulting radial and angular equations are derived by the method of separation of variables. The eigenvalues, quasinormal frequencies ω and angular quantum numbers λ , are then obtained by numerically solving the resultant homogeneous matrix equation. This work shows that the present approach is an accurate, as well as efficient method for investigating quasinormal modes.

Keywords: Kerr black hole; Kerr–Sen black hole; quasinormal modes; Taylor series; Teukolsky equation.

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1. Introduction

Black holes constitute an intriguing topic in astrophysics and theoretical physics, where the gravitational force is so strong that nothing can escape from inside of its event horizon. The study of the properties of black holes might lead to insightful perspectives on quantum gravity. The observation of many astronomical phenomena, as, for instance, gravitational lensing, become more accessible when associated

to very compact stellar objects such as black holes. Among others, one of the most important tools in the study of black holes is the analysis of its quasinormal mode (QNM) oscillations, which describe the late time dynamics of black holes or black hole binaries, and therefore provide valuable information on the inherent properties of the black hole spacetime, as well as its stability. Recently, such signal was observed in the LIGO's first direct detection of gravitational wave.^{1,2}

Generally, the QNM problem can be reformulated in terms of a Schrödinger-type equation. Due to mathematical difficulties, an exact analytic solution is not always attainable. Therefore, semi-analytical approximate and numerical methods have been proposed to calculate the quasinormal frequency (QNF),³⁻⁷ for example, the Pöschl–Teller potential method,⁸ continued fractions method (CFM),^{9,10} the Horowitz–Hubeny method (HH) for anti-de Sitter spacetime,¹¹ the WKB approximation,¹²⁻¹⁴ the finite difference method¹⁵⁻²⁰ and the asymptotic iteration method²¹⁻²⁴ among others.²⁵⁻²⁷

2. A Matrix Approach for Quasinormal Modes

In this paper, we make use of a matrix method²⁸ to calculate the scalar QNF's for rotating Kerr and Kerr–Sen black hole spacetimes. By using the method of separation of variables, the radial and angular parts of the linearized perturbation equation of the scalar fields are given by^{29,30}

$$(1 - u^2) \frac{d}{du} \left[(1 - u^2) \frac{dS_{lm}(u)}{du} \right] - [m^2 - (1 - u^2)(\lambda + a^2 \omega^2 u^2)] S_{lm}(u) = 0, \quad (1)$$

$$\frac{\Delta(r)}{r^2} \frac{d}{dr} \left[\frac{\Delta(r)}{r^2} \frac{dR_{lm}(r)}{dr} \right] + \frac{\Delta(r)}{r^6} \left[r^2 V(r) + 2\Delta(r) - r \frac{d\Delta(r)}{dr} \right] R_{lm}(r) = 0,$$

where

$$V(r) = \frac{1}{\Delta(r)} \{ [a^2 + (2b + r)r]^2 \omega^2 - 4Mam\omega r + a^2 m^2 \} - (a^2 \omega^2 + \lambda), \quad (2)$$

$$\Delta(r) = r(r + 2b) - 2Mr + a^2.$$

Here, $a \in [0, \frac{1}{2}]$ gives the angular momentum per unit mass. When $b \neq 0$, it is the Kerr–Sen black hole case, which reduces to the Kerr black hole spacetime at $b = 0$. For the case of Kerr black hole, in order to compare our results with those from the continuous fraction method, the mass of the black hole is taken to be $M = 1/2$. On the other hand, for the case of Kerr–Sen black hole, the mass $M = (2b + r_0 + r_i)/2$ and angular momentum $a = \sqrt{r_i r_0}$ can be expressed in terms of the event horizon r_0 and the inner horizon r_i . m represents the magnetic quantum number and $u = \cos \theta \in [-1, 1]$. It is noted that we have $\lambda = L(L + 1)$ in the Schwarzschild limit with $a = 0$, where L is the angular quantum number. By making use of the boundary

conditions, one defines

$$R_{lm}(r) = \begin{cases} e^{i\omega r} (r - r_0)^{-i \frac{2Mr_0\omega - am}{r_0 - r_i}} r^i \left[(r_0 + r_i)\omega + \frac{2Mr_0\omega - am}{r_0 - r_i} \right] \Psi_R(r), & \text{non-extreme} \\ & \text{case,} \\ e^{i\omega r + i \frac{2M\omega - m}{r - r_0} r_0} (r - r_0)^{2im - 4iM\omega} r^{2i[(2M + r_0)\omega - m]} \Psi_R(r), & \text{extreme} \\ & \text{case,} \end{cases}$$

$$S_{lm}(u) = e^{a\omega u} (1 + u)^{\frac{|m|}{2}} (1 - u)^{\frac{|m|}{2}} \Phi_S(u), \quad (3)$$

with $r_0 = M - b + \sqrt{(M - b)^2 - a^2}$ and $r_i = M - b - \sqrt{(M - b)^2 - a^2}$ being the event horizon and the inner horizon of the black hole (for the Kerr black hole with $M = 1/2$, we simply have $r_0 = \frac{1}{2} + \sqrt{\frac{1}{4} - a^2}$ and $r_i = \frac{1}{2} - \sqrt{\frac{1}{4} - a^2}$), respectively, so that the wave functions $\Psi_R(r)$ and $\Phi_S(u)$ are both finite at the boundaries.

Equation (1) is a specific form of the Teukolsky equation.³¹ It is noted that in our calculations we do not assume the value of λ by using its Schwarzschild limit, which might lead to inaccurate results. In what follows, we apply the matrix method to evaluate the QNF ω as well as λ .

First, we discretize an unknown wave function $f(x)$ at N points $\{f(x_i)\}$ with $i = 1, 2, \dots, N$. Once convergence is assured, one can expand the function to all the points x_i in the vicinity of a by using the Taylor series,

$$f(x_i) = f(a) + (x_i - a)f'(a) + \frac{1}{2}(x_i - a)^2 f''(a) + \frac{1}{3!}(x_i - a)^3 f'''(a) + \dots \quad (4)$$

Now, repeatedly carrying out the above expansions around all the points x_k with $k = 1, 2, \dots, i - 1, i + 1, \dots, N$, one obtains^{28,32}

$$\Delta F = MD, \quad (5)$$

where

$$\Delta F = (f(x_1) - f(x_k), f(x_2) - f(x_k), \dots, f(x_{k-1}) - f(x_k), f(x_{k+1}) - f(x_k), \dots, f(x_N) - f(x_k))^T$$

$$M = \begin{pmatrix} x_1 - x_k & \frac{(x_1 - x_k)^2}{2} & \dots & \frac{(x_1 - x_k)^i}{i!} & \dots & \frac{(x_1 - x_k)^{N-1}}{(N-1)!} \\ x_2 - x_k & \frac{(x_2 - x_k)^2}{2} & \dots & \frac{(x_2 - x_k)^i}{i!} & \dots & \frac{(x_2 - x_k)^{N-1}}{(N-1)!} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ x_{k-1} - x_k & \frac{(x_{k-1} - x_k)^2}{2} & \dots & \frac{(x_{k-1} - x_k)^i}{i!} & \dots & \frac{(x_{k-1} - x_k)^{N-1}}{(N-1)!} \\ x_{k+1} - x_k & \frac{(x_{k+1} - x_k)^2}{2} & \dots & \frac{(x_{k+1} - x_k)^i}{i!} & \dots & \frac{(x_{k+1} - x_k)^{N-1}}{(N-1)!} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ x_N - x_k & \frac{(x_N - x_k)^2}{2} & \dots & \frac{(x_N - x_k)^i}{i!} & \dots & \frac{(x_N - x_k)^{N-1}}{(N-1)!} \end{pmatrix}, \quad (6)$$

$$D = (f'(x_k), f''(x_k), \dots, f^{(i)}(x_k), \dots, f^{(N-1)}(x_k))^T. \quad (7)$$

Using Cramer's rule, we express the derivatives in D as

$$\begin{aligned} f'(x_k) &= \det(M_1) / \det(M), \\ f''(x_k) &= \det(M_2) / \det(M), \end{aligned} \quad (8)$$

where M_i is the matrix formed by replacing the i th column of M by the column vector ΔF . Therefore, when applying to Eq. (1), it can be written in terms of a matrix equation by employing the above relation. Consequently, ω and λ in Eq. (1) can be evaluated by solving the resultant nonlinear eigenvalue problem.

To obtain the desired boundary conditions for the eigenvalue problem, we further introduce the coordinate transformation from r and u to x and z , and transform the wave functions from Ψ_R and Φ_S to χ_R and χ_S as

$$\begin{aligned} \chi_R &= x(x-1)\Psi_R, \\ \chi_S &= z(z-1)\Phi_S, \\ x &= 1 - \frac{r_0}{r}, \\ z &= \frac{1+u}{2}, \end{aligned} \quad (9)$$

where the variable $x \in [0, 1]$ and the wave functions vanish on the boundary. The master field equations in terms of χ_R and χ_S are thus obtained by substituting Eqs. (2), (3) and (9) into Eq. (1),

$$\begin{aligned} \tau_R(x)\chi_R''(x) + \lambda_R(x)\chi_R'(x) + s_R(x)\chi_R(x) &= 0, \\ \tau_S(z)\chi_S''(z) + \lambda_S(z)\chi_S'(z) + s_S(z)\chi_S(z) &= 0, \end{aligned} \quad (10)$$

with the boundary conditions

$$\chi_R(0) = \chi_R(1) = \chi_S(0) = \chi_S(1) = 0. \quad (11)$$

By taking into account Eq. (8), Eq. (10) can be converted into the matrix form

$$\bar{\mathcal{M}}_R \mathcal{F}_R = 0 \quad \text{and} \quad \bar{\mathcal{M}}_S \mathcal{F}_S = 0, \quad (12)$$

where $\mathcal{F}_R = (\chi_R(x_1), \chi_R(x_2), \dots, \chi_R(x_N))^T$ and $\mathcal{F}_S = (\chi_S(z_1), \chi_S(z_2), \dots, \chi_S(z_N))^T$, while $\bar{\mathcal{M}}_R$ and $\bar{\mathcal{M}}_S$ are two matrices with eigenvalues ω and λ to be determined. The boundary conditions can be implemented by replacing the first and the last row of the matrices $\bar{\mathcal{M}}_R$ and $\bar{\mathcal{M}}_S$ with Eq. (11), or equivalently, one redefines the matrices

$$(\mathcal{M}_R)_{k,i} = \begin{cases} \delta_{k,i}, & k = 1 \text{ or } N, \\ (\bar{\mathcal{M}}_R)_{k,i}, & k = 2, 3, \dots, N-1, \end{cases}$$

and

$$(\mathcal{M}_S)_{k,i} = \begin{cases} \delta_{k,i}, & k = 1 \text{ or } N, \\ (\bar{\mathcal{M}}_S)_{k,i}, & k = 2, 3, \dots, N-1, \end{cases} \quad (13)$$

and replaces Eq. (12) by

$$\mathcal{M}_R \mathcal{F}_R = 0 \quad \text{and} \quad \mathcal{M}_S \mathcal{F}_S = 0. \quad (14)$$

The homogeneous equation (14) implies that the eigenvalues λ and ω satisfy the relation

$$\det(\mathcal{M}_R) = 0 \quad \text{and} \quad \det(\mathcal{M}_S) = 0. \quad (15)$$

It is obvious that the precision of the present method depends on the order of the expansion, N . Therefore, in principle, higher accuracy can be achieved by using larger N .

3. Results and Discussions

In Fig. 1, the real and imaginary parts of ω and λ of a massless scalar field for the Kerr–Sen black hole are given as a function of b by using the proposed matrix method. The calculated curves are for different values of the inner horizon r_i with $r_0 = 1$, $m = 0$ and $L = 2$. As b increases from 0 to 1.5, it is found that the real part of the QNF decreases for about 50%, while the imaginary part of the QNF increases more sensitively and approaches zero. Since b measures the deviation of

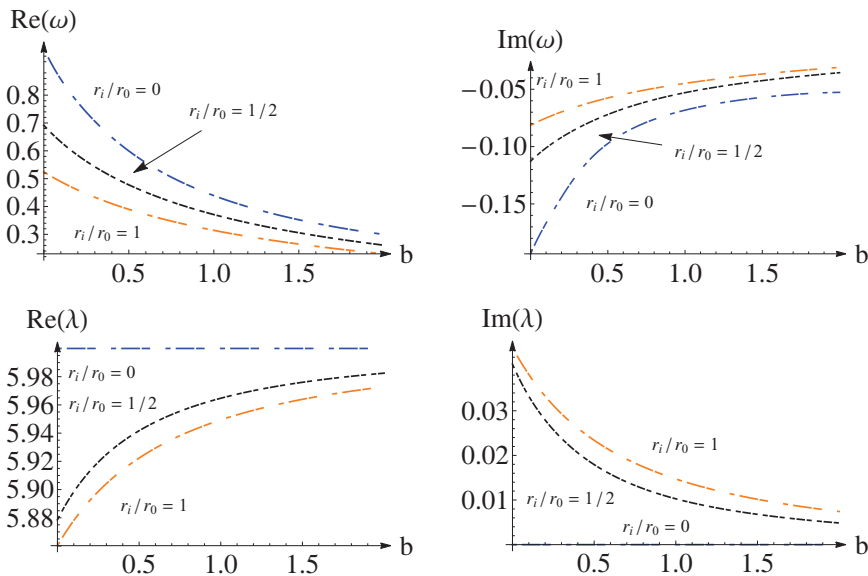


Fig. 1. The calculated ω and λ of massless scalar field for the Kerr–Sen black hole with $m = 0$, $r_0 = 1$ and $L = 2$.

Table 1. The calculated ω and λ of massless scalar field for the Kerr black hole ($b = 0$) obtained by the present matrix method (MM) and by the continued fractions method (CFM). The CFM was carried out up to the 130th order and used as the results are for the states with the principal quantum number $n = 0$ and the mass of black hole $M = 1/2$. The relative errors are shown in the last column, the first value and the second value are the relative errors of real and imaginary parts of the QNF, respectively. The symbol $\mathcal{O}$ is used to indicate that the relative error is less than 10^{-5} .

(L, m, a)	MM with $N = 18$	CFM with $N = 130$	Relative error
{0, 0, 0}	$\omega = 0.221393 - 0.209814i$	$\omega = 0.22091 - 0.209791i$	{0.22%, 0.01%}
	$\lambda = 0$	$\lambda = 0$	{ $\mathcal{O}$, $\mathcal{O}$ }
{0, 0, 0.2}	$\omega = 0.223892 - 0.206538i$	$\omega = 0.223398 - 0.206506i$	{0.21%, 0.02%}
	$\lambda = -0.0000993953 + 0.00123315i$	$\lambda = -0.0000966273 + 0.00123025i$	{2.87%, 0.24%}
{0, 0, 0.4}	$\omega = 0.229763 - 0.1914i$	$\omega = 0.229074 - 0.191404i$	{0.30%, $\mathcal{O}$ }
	$\lambda = -0.000858889 + 0.00469192i$	$\lambda = -0.00084195 + 0.00467792i$	{2.01%, 0.30%}
{1, 0, 0}	$\omega = 0.585879 - 0.19532i$	$\omega = 0.585819 - 0.195523i$	{0.01%, -0.10%}
	$\lambda = 2$	$\lambda = 2$	{ $\mathcal{O}$, $\mathcal{O}$ }
{1, 0, 0.2}	$\omega = 0.592162 - 0.192525i$	$\omega = 0.592155 - 0.192526i$	{ $\mathcal{O}$, $\mathcal{O}$ }
	$\lambda = 1.99247 + 0.00547387i$	$\lambda = 1.99247 + 0.00547382i$	{ $\mathcal{O}$, $\mathcal{O}$ }
{1, 0, 0.4}	$\omega = 0.613394 - 0.18013i$	$\omega = 0.613388 - 0.180136i$	{ $\mathcal{O}$, $\mathcal{O}$ }
	$\lambda = 1.96698 + 0.0212409i$	$\lambda = 1.96698 + 0.0212413i$	{ $\mathcal{O}$, $\mathcal{O}$ }
{2, 0, 0}	$\omega = 0.967288 - 0.193518i$	$\omega = 0.967284 - 0.193532i$	{ $\mathcal{O}$, $\mathcal{O}$ }
	$\lambda = 6$	$\lambda = 6$	{ $\mathcal{O}$, $\mathcal{O}$ }
{2, 0, 0.2}	$\omega = 0.97772 - 0.19093i$	$\omega = 0.97772 - 0.19093i$	{ $\mathcal{O}$, $\mathcal{O}$ }
	$\lambda = 5.98075 + 0.00781144i$	$\lambda = 5.98075 + 0.00781144i$	{ $\mathcal{O}$, $\mathcal{O}$ }
{2, 0, 0.4}	$\omega = 1.01425 - 0.179338i$	$\omega = 1.01425 - 0.179339i$	{ $\mathcal{O}$, $\mathcal{O}$ }
	$\lambda = 5.91671 + 0.0302983i$	$\lambda = 5.91671 + 0.0302984i$	{ $\mathcal{O}$, $\mathcal{O}$ }
{1, 1, 0}	$\omega = 0.585879 - 0.19532i$	$\omega = 0.585872 - 0.19532i$	{ $\mathcal{O}$, $\mathcal{O}$ }
	$\lambda = 2$	$\lambda = 2$	{ $\mathcal{O}$, $\mathcal{O}$ }
{1, 1, 0.2}	$\omega = 0.663138 - 0.191587i$	$\omega = 0.663133 - 0.191583i$	{ $\mathcal{O}$, $\mathcal{O}$ }
	$\lambda = 1.99677 + 0.00203427i$	$\lambda = 1.99677 + 0.00203422i$	{ $\mathcal{O}$, $\mathcal{O}$ }
{1, 1, 0.4}	$\omega = 0.806547 - 0.166271i$	$\omega = 0.806545 - 0.166265i$	{ $\mathcal{O}$, $\mathcal{O}$ }
	$\lambda = 1.98003 + 0.008622i$	$\lambda = 1.98003 + 0.00862165i$	{ $\mathcal{O}$, $\mathcal{O}$ }
{2, 1, 0}	$\omega = 0.967288 - 0.193518i$	$\omega = 0.967288 - 0.193518i$	{ $\mathcal{O}$, $\mathcal{O}$ }
	$\lambda = 6$	$\lambda = 6$	{ $\mathcal{O}$, $\mathcal{O}$ }
{2, 1, 0.2}	$\omega = 1.04428 - 0.190295i$	$\omega = 1.04428 - 0.190295i$	{ $\mathcal{O}$, $\mathcal{O}$ }
	$\lambda = 5.98192 + 0.00681854i$	$\lambda = 5.98192 + 0.00681853i$	{ $\mathcal{O}$, $\mathcal{O}$ }
{2, 1, 0.4}	$\omega = 1.18404 - 0.170251i$	$\omega = 1.18404 - 0.170251i$	{ $\mathcal{O}$, $\mathcal{O}$ }
	$\lambda = 5.90568 + 0.027756i$	$\lambda = 5.9056 + 0.0277559i$	{ $\mathcal{O}$, $\mathcal{O}$ }
{2, 2, 0}	$\omega = 0.967288 - 0.193518i$	$\omega = 0.967288 - 0.193518i$	{ $\mathcal{O}$, $\mathcal{O}$ }
	$\lambda = 6$	$\lambda = 6$	{ $\mathcal{O}$, $\mathcal{O}$ }
{2, 2, 0.2}	$\omega = 1.11929 - 0.189863i$	$\omega = 1.11929 - 0.189862i$	{ $\mathcal{O}$, $\mathcal{O}$ }
	$\lambda = 5.99304 + 0.00243193i$	$\lambda = 5.99304 + 0.00243193i$	{ $\mathcal{O}$, $\mathcal{O}$ }
{2, 2, 0.4}	$\omega = 1.41365 - 0.163041i$	$\omega = 1.41365 - 0.163041i$	{ $\mathcal{O}$, $\mathcal{O}$ }
	$\lambda = 5.95475 + 0.0106276i$	$\lambda = 5.95475 + 0.0106275i$	{ $\mathcal{O}$, $\mathcal{O}$ }

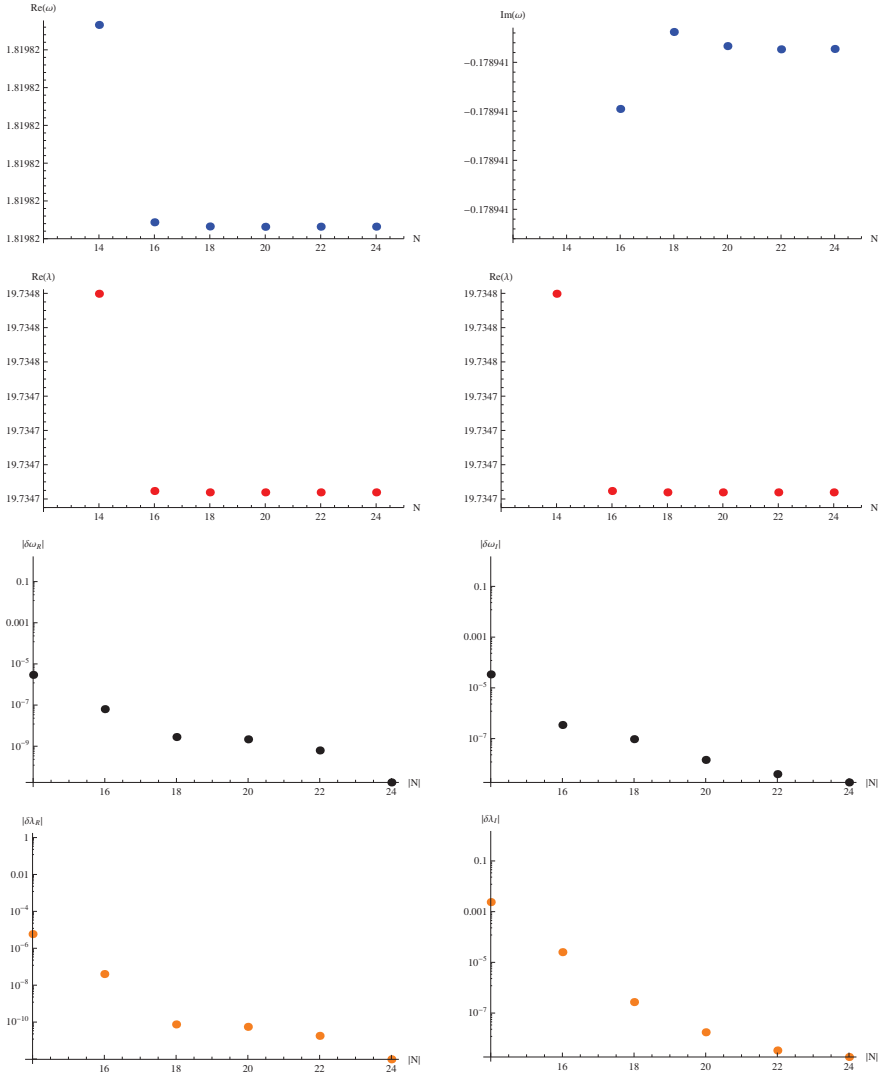


Fig. 2. The calculated values and their relative error of ω and λ of massless scalar field for the Kerr black hole with $b = 0$, $M = 1/2$, $n = 0$, $m = 1$, $a = 0.4$ and $L = 4$. The results are presented as a function of the number of interpolation points N . The relative error is defined by $\delta A \equiv \frac{|A_c - A_e|}{|A_e|}$, where A_c is the calculated result for quantity A by the proposed matrix method, and A_e is that by the continued fraction method (up to 130th order).

the theory from general relativity, if the QNF is not sensitive to the magnitude of b , it might imply certain indeterminacy in the black hole spacetime parametrization corresponding to the observed QNF. Therefore, as pointed out recently,³³ such calculations might be relevant to the future LIGO and VIRGO data.

In order to study the precision of the proposed method, in Table 1, we show the calculated ω and λ for the Kerr black hole by the present matrix method,

and they are compared to those obtained by the CFM. The present method is carried out by considering $N = 18$ interpolation points. For comparison purpose, the calculations by the continued fraction method are carried out up to the 130th order, and therefore the results can be viewed as exact. The fourth column of the table gives the relative error of the present method. It can be inferred from Table 1 that the deviations from the exact results are reasonably small, especially for the $L \geq 1$ cases. By only considering a modest amount of terms in the discretization, the relative error is less than 10^{-5} in most cases.

In addition, we show in Fig. 2 the calculated ω and λ , as well as their relative errors, as a function of the number of interpolation points N . The calculations are carried out with $b = 0$, $M = 1/2$, $n = 0$, $m = 1$, $a = 0.4$ and $L = 4$. It is found that the relative error becomes smaller than 10^{-5} once $N \geq 16$, and it continues to decrease with increasing N . This convergence behavior indicates that the proposed method is accurate as well as stable.

Concerning the efficiency of the present method, since a major part of the method involves the solution of nonlinear algebraic equations, it may take advantage of advanced algebraic equation solvers, such as *Mathematica* and *Matlab*. In general, we expect the proposed method to have the same level of efficiency in comparison with several other methods based on nonlinear algebraic equations, such as the continued fraction method, the HH method, etc. However, a distinct feature of the matrix method is that a big chunk of analytic calculations associated with Eq. (8) is independent of any specific metric, once N is fixed. In this regard, the method is flexible for dealing with different black hole background. Moreover, the evaluation of Eq. (8) can be carried out beforehand, which in turn saves the computation time. Therefore, we come to the conclusion that the present method can be used as an efficient tool for the study of QNM. We plan to apply the proposed method further to study more sophisticated black hole spacetimes in future investigations.

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