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Local Banach-space dichotomies and ergodic spaces

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Abstract. We prove a local version of Gowers' Ramsey-type theorem [Ann. of Math. 156 (2002)], as well as local versions both of the Banach space first dichotomy (the "unconditional/HI" dichotomy) of Gowers [Ann. of Math. 156 (2002)] and of the third dichotomy (the "minimal/tight" dichotomy) due to Ferenczi–Rosendal [J. Funct. Anal. 257 (2009)]. This means that we obtain versions of these dichotomies restricted to certain families of subspaces called D-families, of which several concrete examples are given. As a main example, non-Hilbertian spaces form D-families; therefore versions of the above properties for non-Hilbertian spaces appear in new Banach space dichotomies. As a consequence we obtain new information on the number of subspaces of non-Hilbertian Banach spaces, making some progress towards the "ergodic" conjecture of Ferenczi–Rosendal and towards a question of Johnson.

Keywords. Ergodic Banach spaces, Ramsey theory, Banach-space dichotomies, non-Hilbertian spaces, minimal Banach spaces, hereditarily indecomposable Banach spaces

1. Introduction and background

In this paper, we will only consider real Banach spaces; however, all of our results transpose to the complex case. Unless otherwise specified, when writing about a *Banach space* (or simply a *space*), we shall mean an infinite-dimensional Banach space, by *subspace*, we shall mean an infinite-dimensional, closed vector subspace, and by *direct sum*, we shall mean a topological direct sum. By *operator*, we shall always mean a bounded linear

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operator. By *Hilbertian space* we mean a space which is linearly isomorphic (but not necessarily isometric) to a Hilbert space. For all other unexplained notation, see the end of this introduction.

1.1. Ergodic Banach spaces

A Banach space is said to be *homogeneous* if it is isomorphic to all of its (closed, infinite-dimensional) subspaces. A famous problem due to Banach, and known as the *homogeneous space problem*, asked whether, up to isomorphism, ℓ_2 is the only homogeneous Banach space. The answer turned out to be positive; this problem was eventually solved in the 1990's by a combination of results by Gowers–Maurey [26], Komorowski–Tomczak-Jaegermann [35], and Gowers [25].

The homogeneous space characterization of the Hilbert space shows that, as soon as a separable Banach space X is non-Hilbertian, it should have at least two non-isomorphic subspaces. Thus, the following general question was asked by Godefroy:

Question 1.1 (Godefroy). *How many different subspaces, up to isomorphism, can a separable, non-Hilbertian Banach space have?*

This question seems to be very difficult in general, although good lower bounds for several particular classes of spaces are now known. A seemingly simplest particular case of Godefroy's question was formulated by Johnson:

Question 1.2 (Johnson). *Does there exist a separable Banach space having exactly two different subspaces, up to isomorphism?*

Even this question is still open. More generally, it is not known whether there exists a separable, non-Hilbertian Banach space with at most countably many different subspaces, up to isomorphism; anticipating our considerations below, let us note that a positive answer to Conjecture 1.9 would imply that such a space does not exist. In the rest of this paper, a separable Banach space having exactly two different subspaces, up to isomorphism, will be called a *Johnson space*.

It turns out that the right setting to study Godefroy's question is the theory of the *classification of definable equivalence relations*. This theory studies equivalence relations E on non-empty standard Borel spaces X which, when seen as subsets of X^2, have a sufficiently low descriptive complexity (in general, Borel or analytic). Recall that a *Polish space* is a separable and completely metrizable topological space. A *standard Borel space* is a set X equipped with a σ-algebra $\mathcal{B}$ such that $\mathcal{B}$ is the Borel σ-algebra associated to some Polish topology on X. When X is a standard Borel space, the X^n's, for $n \geqslant 1$, will always be endowed with the product σ-algebras; this makes them standard Borel spaces as well. A subset A of a standard Borel space $(X, \mathcal{B})$ is said to be *Borel* if it is an element of $\mathcal{B}$, *analytic* if it is the projection of a Borel subset of X^2, and *coanalytic* if its complement is analytic. A *Borel mapping* between two standard Borel spaces is a mapping for which the preimage of every Borel set is Borel, and an *isomorphism* is a Borel bijection (then automatically its inverse is Borel). It is a classical fact in descriptive set

theory that all uncountable standard Borel spaces are isomorphic, and that a Borel subset of a standard Borel space is itself a standard Borel space when equipped with the induced σ-algebra. For the proofs of all the aforementioned facts, see [34]. The central notion of the theory is *Borel-reducibility*.

Definition 1.3. Let X, Y be non-empty standard Borel spaces, and E, F be equivalence relations on X and Y respectively.

- We say that E *Borel-reduces* to F, denoted by $(X, E) \leqslant_B (Y, F)$ (or simply $E \leqslant_B F$), if there is a Borel mapping $f : X \to Y$ (called a *reduction*) such that for all $x, y \in X$, we have $x \mathrel{E} y \Leftrightarrow f(x) \mathrel{F} f(y)$.
- We say that E and F are *Borel-equivalent*, denoted by $E \equiv_B F$, if $E \leqslant_B F$ and $F \leqslant_B E$.
- We denote by $E <_B F$ the fact that $E \leqslant_B F$ and $E \not\equiv_B F$.

The Borel-reducibility relation defines a hierarchy of complexities on the class of all equivalence relations on standard Borel spaces, the complexity classes being the equivalence classes of $\equiv_B$.

Observe that a reduction f from (X, E) to (Y, F) induces a one-to-one mapping $X/E \to Y/F$, and in particular, if $E \leqslant_B F$, then $|X/E| \leqslant |Y/F|$. Thus, complexity classes can be seen as Borel cardinalities: studying the complexity of an equivalence relation gives us at least as much information as counting its classes. If E is analytic and has at most countably many classes, then E is actually Borel and $E \leqslant_B F \Leftrightarrow |X/E| \leqslant |Y/F|$. Thus, for such an E, the complexity of E and the number of its classes agree. However, for relations with uncountably many classes, it turns out that the complexity of the relation gives strictly more information than the number of its classes. The classification of relations with exactly continuum-many classes is extremely complex and is actually the main focus of the theory.

We now define a particular equivalence relation that will be important in the rest of this paper. Denote by Δ the *Cantor space*, that is, $\{0, 1\}^{\mathbb{N}}$ with the product topology and the associated standard Borel structure.

Definition 1.4. The equivalence relation $\mathbf{E}_0$ on Δ is defined as follows: two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are $\mathbf{E}_0$-*equivalent* if and only if $x_n = y_n$ eventually.

It can easily be shown that $(\Delta, =) <_B (\Delta, \mathbf{E}_0)$; in particular, $\mathbf{E}_0$ and equality on the Cantor space are examples of two inequivalent equivalence relations both having continuum-many classes. It follows from important dichotomies by Silver [56] and by Harrington–Kechris–Louveau [28] that the following family of equivalence relations:

$$(1, =) <_B (2, =) <_B (3, =) <_B \cdots <_B (\mathbb{N}, =) <_B (\Delta, =) <_B (\Delta, \mathbf{E}_0)$$

is an exhaustive initial segment of the whole hierarchy of Borel equivalence relations, in the sense that every Borel equivalence relation E is either Borel-equivalent to some element of this hierarchy, or is strictly above $\mathbf{E}_0$. Note that this is not true anymore when E is

only supposed to be analytic (an analytic equivalence relation E which is strictly above $(\mathbb{N}, =)$ and incomparable to $(\mathbb{R}, =)$ is constructed in [56]). For a complete presentation of the theory of classification of definable equivalence relations, see [33]; note for example that $\mathbf{E_0}$ is still quite low in the whole hierarchy.

One of the main applications of this theory is the study of the complexity of classification problems in mathematics. When one wants to classify a class $\mathscr{C}$ of objects up to isomorphism, it is often possible to equip $\mathscr{C}$ with a natural Borel structure for which the isomorphism relation is, in general, analytic. Knowing the complexity of this isomorphism relation gives an indication on the difficulty of the associated classification problem. For instance, such a class of structures can be classified by real invariants if and only if the isomorphism relation on this class is reducible to $(\mathbb{R}, =)$ (or equivalently, to $(\Delta, =)$). Conversely, if $(\Delta, \mathbf{E_0})$ is reducible to the isomorphism relation on this class, this implies that the associated classification problem is quite complex.

One can in particular study the classification problem for closed vector subspaces of a given separable Banach space X. To do this, we first need to put a standard Borel structure on $\mathrm{Sub}(X)$; this was first done by Bossard [9]. We refer to his paper for more details and proofs. The set $\mathrm{Sub}(X)$ is endowed with the *Effros Borel structure*, that is, the σ-algebra generated by sets of the form $\{Y \in \mathrm{Sub}(X) \mid Y \cap U \neq \emptyset\}$, where U ranges over all open subsets of X. This makes it a standard Borel space, on which the isomorphism relation is analytic. It is clear from the definition that this Borel structure on $\mathrm{Sub}(X)$ only depends on the isomorphic structure of X; in particular, if $T : X \to Y$ is an isomorphism between two separable Banach spaces, then T induces a Borel isomorphism between $\mathrm{Sub}(X)$ and $\mathrm{Sub}(Y)$. It is also easy to see that if Y is a subspace of X, then the Effros Borel structure on $\mathrm{Sub}(Y)$ coincides with the trace on $\mathrm{Sub}(Y)$ of the Effros Borel structure on $\mathrm{Sub}(X)$.

We also record the following lemma, which will be useful in applications. Here, $\mathcal{P}(\mathbb{N})$ is identified with the Cantor space, and if $(x_i)_{i\in I}$ is a family of elements of a Banach space X, we will let $[x_i \mid i \in I] = \overline{\mathrm{span}(x_i \mid i \in I)}$.

Lemma 1.5. *Let X be a separable Banach space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of X. Then the mapping $j : \mathcal{P}(\mathbb{N}) \to \mathrm{Sub}(X)$ defined by $j(A) = [x_n \mid n \in A]$ is Borel.*

Proof. Let U be an open subset of X; it is enough to prove that

$$\mathcal{V} := \{A \in \mathcal{P}(N) \mid j(A) \cap U \neq \emptyset\}$$

is an open subset of $\mathcal{P}(\mathbb{N})$. Let $A \in \mathcal{V}$. Then since U is open, U contains a finite linear combination of the x_n's, $n \in A$, such there is a finite $s \subseteq A$ with $[x_n \mid n \in s] \cap U \neq \emptyset$. In particular, the open neighborhood $\{B \in \mathcal{P}(\mathbb{N}) \mid s \subseteq B\}$ of A is entirely contained in $\mathcal{V}$. ∎

Let us mention that the Effros Borel structure can also be used to study the isomorphism relation on the class of *all* finite- and infinite-dimensional separable Banach spaces. Indeed, using the fact that the separable Banach space $\mathcal{C}(\Delta)$ is isometrically universal for this class, we can identify the class of all finite- and infinite-dimensional separable Banach spaces with $\mathrm{Sub}(\mathcal{C}(\Delta))$. Using this coding, it has been shown by Ferenczi, Louveau

and Rosendal [19] that the isomorphism relation on the class of all finite- and infinite-dimensional separable Banach spaces is *analytic-complete*, that is, it is maximum for $\leqslant_B$ among all analytic equivalence relations on standard Borel spaces. This gives a formal proof of the heuristic fact that there is no reasonable classification of separable Banach spaces, up to isomorphism.

We can also simply study the complexity of the isomorphism relation on $\mathrm{Sub}(X)$ for any separable Banach space X; this complexity gives strictly more information than the number of different subspaces of X, up to isomorphism, *including the finite-dimensional ones*. So Godefroy's question can be generalized by asking, for spaces X with infinitely many different subspaces up to isomorphism, what is the complexity of the isomorphism relation of $\mathrm{Sub}(X)$. In their investigation of this question, Ferenczi and Rosendal [20] defined the following class of separable Banach spaces:

Definition 1.6. A separable Banach space X is said to be *ergodic* if $\mathbf{E}_0$ is Borel-reducible to the isomorphism relation on $\mathrm{Sub}(X)$.

In particular, ergodic Banach spaces have continuum-many pairwise non-isomorphic subpaces, and their subspaces cannot be classified by real numbers, up to isomorphism. Immediate consequences of this definition are that ℓ_2 is non-ergodic, that a subspace of a non-ergodic space is itself non-ergodic, and that the notion of ergodicity is invariant under isomorphism. Ergodic Banach spaces are quite complex and on the contrary, non-ergodic spaces are expected to be regular in some sense. Ferenczi and Rosendal have shown several regularity properties for non-ergodic spaces. For instance:

Theorem 1.7 (Ferenczi–Rosendal, [21, 50]). *Let X be a non-ergodic Banach space with an unconditional basis. Then X is isomorphic to $X \oplus Y$ for every subspace Y spanned by a (finite or infinite) subsequence of the basis. In particular, X is isomorphic to its square and to its hyperplanes.*

Theorem 1.8 (Ferenczi–Rosendal, [20]). *Let X be a non-ergodic separable Banach space. Then X has a subspace Y with an unconditional basis such that Y is isomorphic to $Y \oplus Z$ for every block-subspace Z of Y.*

All these results led them to the following conjecture:

Conjecture 1.9 (Ferenczi–Rosendal). *Every separable non-Hilbertian Banach space is ergodic.*

This conjecture is still open. We quote below some of the most relevant partial results supporting the conjecture.

Definition 1.10. A Banach space X is said to be *minimal* if it embeds isomorphically into all of its subspaces.

The notion of minimality was based on the classical examples of the ℓ_p's, $1 \leqslant p < \infty$, and c_0 (and their subspaces). Later on, the dual of Tsirelson's space, and then Schlumprecht's arbitrarily distortable space, were added to the list; see [11, 54, 60], as well as [10] for variants of Schlumprecht's example.

Theorem 1.11 (Ferenczi, [15]). *Every non-ergodic separable Banach space contains a minimal subspace.*

It is a consequence of Kwapień's theorem [36] that a space is Hilbertian if and only if there exists a constant K such all of its finite-dimensional subspaces are K-isomorphic to a Euclidean space. This property may be relaxed as follows:

Definition 1.12. A Banach space X is said to be *asymptotically Hilbertian* if there exists a constant K such that for every $n \in \mathbb{N}$, there exists a finite-codimensional subspace Y of X all of whose n-dimensional subspaces are K-isomorphic to ℓ_2^n.

Theorem 1.13 (Anisca, [3]). *Every asymptotically Hilbertian, non-Hilbertian separable Banach space is ergodic.*

A generalization of the last result will be proved in this paper (see Theorem 5.25), using a different method than Anisca's.

Theorem 1.14 (Cuellar Carrera, [12]). *Every non-ergodic separable Banach space has type p and cotype q for all $p < 2 < q$.*

This last result is particularly significant since it shows that counterexamples to Conjecture 1.9 should be geometrically very close to being Hilbertian. In particular, the ℓ_p's, $1 \leqslant p \neq 2 < \infty$, and c_0 are ergodic (this had already been shown by Ferenczi and Galego [17] for the ℓ_p's, $1 \leqslant p < 2$, and c_0). A consequence of this, combined with James' theorem, is that non-ergodic spaces having an unconditional basis should be reflexive.

We refer to the survey [23] as well as to Ferenczi's Thèse d'Habilitation [16] for more details. These references list in particular better estimates on the complexity of the isomorphism relation between subspaces for several classical Banach spaces.

On the way to possible answers to Johnson's Question 1.2 and Ferenczi–Rosendal's Conjecture 1.9 we identify two weaker conjectures to be studied in the present paper.

Conjecture 1.15. *Every Johnson space has an unconditional basis.*

Conjecture 1.16. *Every non-ergodic non-Hilbertian separable Banach space contains a non-Hilbertian subspace having an unconditional basis.*

Conjectures 1.15 and 1.16 are important because they allow us to reduce Johnson's and Ferenczi–Rosendal's conjectures to the case of spaces having an unconditional basis, for which, as we saw above, we already know many properties. We have not been able to solve these conjectures, but we make significant progress on them in Section 6.

1.2. Gowers' classification program

In order to motivate our forthcoming definitions, we first present the main steps of the solution of the homogeneous space problem. We start with a definition.

Definition 1.17 (Gowers–Maurey, [26]). A Banach space X is *hereditarily indecomposable* (HI) if it contains no direct sum of two subspaces.

HI spaces exist: they were first built by Gowers and Maurey [26], as a solution to the unconditional basic sequence problem; they were the first spaces known to contain no subspace with an unconditional basis. Independently of the existence of HI spaces, the combination of the following three results solves positively the homogeneous space problem:

Theorem 1.18 (Gowers–Maurey, [26]). *An HI space is isomorphic to no proper subspace of itself.*

Theorem 1.19 (Komorowski–Tomczak-Jaegermann, [35]). *Every Banach space contains either a subspace without unconditional basis, or an isomorphic copy of* ℓ_2.

Theorem 1.20 (Gowers' first dichotomy, [25]). *Every Banach space contains either a subspace with an unconditional basis, or an HI subspace.*

We refer to [35] for a more precise statement of Theorem 1.19. Gowers' first dichotomy is especially important, since it allows one to restrict the homogeneous space problem to two special cases, the case of spaces with an unconditional basis and the case of HI spaces. In both of these radically opposite cases, we have at our disposed specific tools allowing us to solve the problem more efficiently. Based on this remark, Gowers [25] suggested a classification program for separable Banach spaces "up to subspace". The goal is to build a list of classes of separable Banach spaces, as fine as possible, satisfying the following requirements:

(1) the classes are hereditary: if X belongs to a class $\mathcal{C}$ then all subspaces of X also belong to $\mathcal{C}$ (or, in the case of classes defined by properties of bases, all block-subspaces of X belong to $\mathcal{C}$);

(2) the classes are pairwise disjoint;

(3) knowing that a space belongs to a class gives much information about the structure of this space;

(4) every Banach space contains a subspace belonging to one of these classes.

Such a list is in general called a *Gowers list*. The most difficult property to prove is in general (4); Gowers' first dichotomy proves this property for the two classes of spaces with an unconditional basis and HI spaces, thus showing that these two classes form a Gowers list. In the same paper [25], Gowers suggests that this list could be refined by proving new dichotomies in the same spirit, and himself proves a second dichotomy. Three other dichotomies were then proved by Ferenczi and Rosendal [22], leading to a Gowers list with six classes (all of which are now known to be non-empty) and 19 possible subclasses.

All of these dichotomies draw a border between "regular" spaces (spaces sharing many properties with classical spaces such as the ℓ_p's, $1 \leqslant p < \infty$, or c_0), and a class of "pathological" or "exotic" spaces. These dichotomies are often important in studying the complexity of the isomorphism relation between subspaces of a space X; when X is on the "pathological" side, we expect this relation to be rather complex. We present below the

most important of the dichotomies by Ferenczi and Rosendal (called "third dichotomy" in [22]), which will be particularly relevant in this paper.

Definition 1.21 (Ferenczi–Rosendal). (1) Let $(e_n)_{n\in\mathbb{N}}$ be a basis of some Banach space. A Banach space X is *tight in* the basis (e_n) if there is an infinite sequence of non-empty intervals $I_0 < I_1 < \cdots$ of integers such that for every infinite $A \subseteq \mathbb{N}$, we have $X \not\sqsubseteq [e_n \,|n \notin \bigcup_{i\in A} I_i\,.]$.

(2) A basis $(e_n)_{n\in\mathbb{N}}$ is said to be *tight* if every Banach space is tight in it. A Banach space X is *tight* if it has a tight basis.

For reflexive Banach spaces, it is known that all bases are tight if one of them is [22, Corollary 3.5]. Note that there is a more intuitive characterization of tightness [18]: X is tight in (e_n) exactly when the set of $A \subseteq \mathbb{N}$ such that X embeds into $[e_n \mid n \in A]$ is meager (in the natural topology on $\mathcal{P}(\mathbb{N})$ obtained by identifying it with the Cantor space). However, the definition with the intervals I_i is more operational, allowing for example to distinguish forms of tightness according to the dependence between X and the sequence (I_i).

Theorem 1.22 (Ferenczi–Rosendal). *Every Banach space has either a minimal subspace, or a tight subspace.*

This dichotomy will be referred as *the minimal/tight dichotomy*. Here, the "regular" class is the class of minimal spaces, and the "pathological" class is the class of tight spaces: these spaces are isomorphic to very few of their own subspaces. An example of a tight space is Tsirelson's space [22]. The minimal/tight dichotomy is a generalization of Theorem 1.11 (which itself improved the main result of [43]): indeed, it can be shown quite easily that tight spaces are ergodic, which, combined with the dichotomy, shows that non-ergodic separable spaces should have a minimal subspace.

Ferenczi–Rosendal's definition of tightness is restricted to Schauder bases. This was not a relevant loss of generality for Theorem 1.22. For our local versions of this dichotomy, however, it will be important to extend the notion to FDD's. To give a concrete example of our need to use FDD's, note that one may force a space to be non-Hilbertian just by imposing restrictions on the summands of an FDD, without any condition on the way they "add up"; this would of course not be possible with bases. The definition is straightforward, and properties of tight bases extend without difficulty to tight FDD's:

Definition 1.23 (Tight FDD's). (1) Let $(F_n)_{n\in\mathbb{N}}$ be an FDD of some Banach space. A Banach space X is *tight in* (F_n) if there is an infinite sequence of non-empty intervals $I_0 < I_1 < \cdots$ of integers such that for every infinite $A \subseteq \mathbb{N}$, we have $X \not\sqsubseteq [F_n \mid n \notin \bigcup_{i\in A} I_i]$.

(2) An FDD $(F_n)_{n\in\mathbb{N}}$ is said to be *tight* if every Banach space is tight in it.

It is clear from the definition that if a space is spanned by a tight FDD, then it has a tight subspace.

1.3. Local Ramsey theory

Dichotomies such as Gowers' or Ferenczi–Rosendal's present drawbacks if one wants to deal with problems related to ergodicity. Indeed, ℓ_2 always belongs to the "regular" class defined by those dichotomies, which makes them useless for spaces containing an isomorphic copy of ℓ_2. Typically, if a space is ℓ_2-saturated, but non-Hilbertian, then these dichotomies do not provide information on the structure of the space itself.

For this reason, it would be interesting to have dichotomies similar to Gowers' or Ferenczi–Rosendal's, but which avoid ℓ_2, that is, dichotomies of the form "every non-Hilbertian Banach space X contains a non-Hilbertian subspace either in $\mathcal{R}$, or in $\mathcal{P}$", where $\mathcal{R}$ is a class of "regular" spaces, and $\mathcal{P}$ is a class of "pathological" spaces. Proving such dichotomies is the main goal of this paper.

Gowers' and Ferenczi–Rosendal's dichotomies are proved using combinatorial methods, and especially Ramsey theory. Here, for an infinite $M \subseteq \mathbb{N}$, we denote by $[M]^\infty$ the set of infinite subsets of M; we see $[\mathbb{N}]^\infty$ as a subset of the Cantor space, endowed with the induced topology.

Theorem 1.24 (Silver, [55]). *Let $\mathcal{X} \subseteq [\mathbb{N}]^\infty$ be analytic. Then there exists an infinite $M \subseteq \mathbb{N}$ such that either $[M]^\infty \subseteq \mathcal{X}$, or $[M]^\infty \subseteq \mathcal{X}^c$.*

A topological proof of Silver's theorem was obtained by E. Ellentuck [14]. Similar topologies to those introduced by Ellentuck will be considered in Section 3.1.

The proofs of both Gowers' dichotomies in [25] are based on a version of Theorem 1.24 in the context of Banach spaces, known as *Gowers' Ramsey-type theorem for Banach spaces*. Here, $\mathbb{N}$ is replaced with a separable Banach space X, the set $\mathcal{X}$ becomes a set of normalized sequences in X, and the monochromatic set M becomes a subspace of X. In this context, a result exactly similar to Theorem 1.24 does not hold, and the conclusion has to be weakened, using a game-theoretic framework. The exact statement of Gowers' Ramsey-type theorem is a bit technical and will be given in Section 2 (Theorem 2.10); a more comprehensive presentation of this theory can be found in [7, Part B, Chapter IV]. The proofs of the dichotomies of Ferenczi and Rosendal in [22] use either Gowers' Ramsey-type theorem, or similar methods based on Ramsey theory and games.

If one wants to prove Banach-space dichotomies where the outcome space lies in some prescribed family of subspaces (for instance, non-Hilbertian subspaces), one needs adapted Ramsey-theoretic results. Fortunately, such results exist in classical Ramsey theory; they form a topic usually called *local Ramsey theory*. Here, the word *local* refers to the fact that we want to find a monochromatic subset *locally*, that is, in a prescribed family of subsets. We present below the local version of Silver's Theorem 1.24, due to Mathias [40]. A complete presentation of local Ramsey theory can be found in [59, Chapter 7].

Definition 1.25. (1) A *coideal* on $\mathbb{N}$ is a non-empty subset $\mathcal{H} \subseteq [\mathbb{N}]^\infty$ such that, for all $A, B \in \mathcal{P}(\mathbb{N})$,

(a) if $A \in \mathcal{H}$ and $A \subseteq B$, then $B \in \mathcal{H}$;

(b) if $A \cup B \in \mathcal{H}$, then either $A \in \mathcal{H}$ or $B \in \mathcal{H}$.

(2) The coideal $\mathcal{H}$ is said to be P^+ if for every decreasing sequence $(A_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{H}$, there exists $A_\infty \in \mathcal{H}$ such that for every $n \in \mathbb{N}$, $A_\infty \subseteq^* A_n$ (meaning here that $A_\infty \setminus A_n$ is finite).

(3) The coideal $\mathcal{H}$ is said to be *selective* if for every decreasing sequence $(A_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{H}$, there exists $A_\infty \in \mathcal{H}$ such that for every $n \in \mathbb{N}$, $A_\infty \setminus [\![0, n]\!] \subseteq A_n$.

Theorem 1.26 (Mathias, [40]). *Let $\mathcal{H}$ be a selective coideal on $\mathbb{N}$, and let $\mathcal{X} \subseteq [\mathbb{N}]^\infty$ be analytic. Then there exists $M \in \mathcal{H}$ such that either $[M]^\infty \subseteq \mathcal{X}$, or $[M]^\infty \subseteq \mathcal{X}^c$.*

A local Ramsey theory in Banach spaces has already been developed by Smythe [57]. He proves an analogue of Gowers' Ramsey-type theorem where the outcome space is ensured to lie in some prescribed family $\mathcal{H}$ of subspaces of the space X in which we work. The conditions on the family $\mathcal{H}$ are similar to those in the definition of a selective coideal. However, in the context of Banach spaces, these conditions become quite restrictive and it is not clear that they are met by "natural" families in a Banach-space-theoretic sense. Smythe's theory seems to be more adapted to dealing with problems of genericity, as illustrated in [57].

In this paper, we shall prove a local version of Gowers' Ramsey-type theorem for families $\mathcal{H}$ satisfying weaker conditions, which are closer to the definition of P^+-coideals (Theorem 4.1). This theorem has a weaker conclusion than Smythe's theorem; however, the range of families $\mathcal{H}$ to which it applies is much broader and includes "natural" families in a Banach-space-theoretic sense, for instance the family of non-Hilbertian subspaces of a given space. These families, called *D-families*, will be defined and studied in Section 3. In order to motivate their definition, we state a sufficient condition for being a P^+-coideal which is well-known to set-theoreticians. This fact is folklore; it is, for instance, an easy consequence of [42, Lemma 1.2].

Lemma 1.27. *Let $\mathcal{H}$ be a coideal on $\mathbb{N}$. If $\mathcal{H}$ is G_δ when seen as a subset of the Cantor space, then $\mathcal{H}$ is P^+.*

1.4. Organization of the paper

After the introductory Section 1, Section 2 is still mainly a background section, presenting the formalism of Gowers spaces, as well as their approximate versions, developed by de Rancourt [13] as a generalization of Gowers Ramsey-type theory in Banach spaces, and necessary to prove local dichotomies.

In Section 3 we define and study the notion of D-family (Definition 3.2). By analogy to Lemma 1.27, a set of subspaces of a Banach space X will be called a D-family if it is closed under finite-dimensional modifications and is G_δ for a certain rather fine topology on the set of subspaces of X. This will ensure on the one hand that such families have a diagonalization property similar to the P^+-property, and on the other hand that they have a good behavior relative to FDD's, so that "local" Ramsey theorems, i.e. restricted to subspaces in the D-family, may be hoped for. The concrete examples of D-families are

associated to the important notion of degree d (Definition 3.15) allows us to formalize quantitative estimates relating the finite-dimensional subspaces F of a space X, and X itself, by assigning to them a positive real number $d(X, F)$. A subspace Y of X is d-small (Definition 3.16), when the degrees $d(Y, F)$ are uniformly bounded for $F \subseteq Y$, and d-large otherwise; the conditions on the definition of degree imply that the family of d-large subspaces of X is a D-family (Proposition 3.25). Several classical properties of Banach spaces are equivalent to being d-small for a well-chosen degree d, for instance being Hilbertian, having a certain fixed type or cotype, or having Gordon–Lewis local unconditional structure [24], Examples 3.17.

In Section 4, we concentrate on the formalism of approximate Gowers spaces to prove our local version of Gowers' Ramsey-type theorem (Theorem 4.1) for analytic games. Then we deduce from it a local version of Gowers' first dichotomy (Theorem 4.4). This "first dichotomy" in the case of a D-family induced by a degree d may be stated as follows:

Theorem 1.28 (see Theorem 4.5). *Let X be a d-large Banach space. Then X has a d-large subspace Y such that either*

(1) *Y is spanned by a UFDD, or*

(2) *Y contains no direct sum of two d-large subspaces.*

The first alternative is stronger than containing an unconditional basic sequence, and the second one, a "pathological" property, is weaker than the HI property.

In Section 5, we will then prove a local version of the minimal/tight dichotomy (Theorem 5.5). In the case of a degree d, this dichotomy may be stated as follows:

Theorem 1.29 (see Theorem 5.6). *Let X be a d-large Banach space. Then X has a d-large subspace Y such that either*

(1) *Y isomorphically embeds into all of its d-large subspaces, or*

(2) *Y is spanned by an FDD in which every d-large Banach space is tight.*

The property satisfied by Y in the first alternative will be called *d-minimality*. Note that the word "minimal" here refers to a minimal element, among d-large spaces, for the relation of embedding between subspaces. So a d-minimal space should not be thought of as small in this context; it is a d-large space. The proof of Theorem 1.29 is more delicate than for the first local dichotomy; it is inspired by a proof by Rosendal of a variant of the classical minimal/tight dichotomy [52] and relies on the formalism of Gowers spaces. Quite importantly towards the questions of Godefroy and Johnson, we prove that the relation between tightness and ergodicity still holds in the local version. Our precise result, in the case of a degree d, is the following:

Theorem 1.30 (see Theorem 5.16). *Let X be a d-large Banach space spanned by an FDD in which every d-large Banach space is tight. Then X is ergodic.*

Consequently, a d-large and non-ergodic separable space must contain a d-minimal subspace (Corollary 5.17), and the study of d-minimal spaces turns out to be quite relevant. We end the section with additional observations about the d-minimality property, and some consequences. For example, we generalize the result by Anisca that non-Hilbertian spaces which are asymptotically Hilbertian must be ergodic (Theorem 1.13) to the case of d-large spaces which are "asymptotically d-small" (Theorem 5.25).

Finally, in Section 6, we consider the Hilbertian degree $d_2(F)$, defined as the Banach–Mazur distance of F to the Euclidean space of the same dimension, and for which the class of d-small spaces is exactly the class of Hilbertian spaces. In this case the two dichotomies immediately translate as (to avoid confusion let us stress that each of (1) and (2) states a dichotomy, but (1) versus (2) is not):

Theorem 1.31. *Every non-Hilbertian Banach space contains a non-Hilbertian subspace which:*

(1) *either is spanned by a UFDD, or does not contain any direct sum of non-Hilbertian subspaces,*

(2) *either isomorphically embeds into all of its non-Hilbertian subspaces, or has an FDD in which every non-Hilbertian space is tight.*

We therefore give some applications of the theory developed in the previous sections for the study of ergodicity and Johnson's question, applying these new dichotomies using only non-Hilbertian subspaces. We reproduce two of our results below as an illustration:

Theorem 1.32 (see Corollary 6.16). *Let X be a Johnson space. Then X has a Schauder basis; moreover, X has an unconditional basis if and only if it is isomorphic to its square.*

Theorem 1.33 (see Theorems 6.5 and 6.23). *Let X be a separable, non-Hilbertian, non-ergodic Banach space. Then X has a non-Hilbertian subspace Y which isomorphically embeds into all of its non-Hilbertian subspaces, and which moreover satisfies one of the following two properties:*

(1) *Y has an unconditional basis;*

(2) *Y contains no direct sum of two non-Hilbertian subspaces.*

We moreover conjecture that the second alternative in Theorem 1.33 cannot actually happen (Conjecture 6.26). We end the section by identifying non-trivial examples of spaces which do not contain direct sums of non-Hilbertian subspaces, Examples 6.21 and 6.22, and giving a list of open problems.

1.5. Definitions and notation

This subsection lists the main classical definitions and notation that will be needed in this work. We denote by $\mathbb{N}$ the set of non-negative integers, and by $\mathbb{R}_+$ the set of non-negative real numbers. We denote by Ban^{∞} the class of all (infinite-dimensional) Banach spaces, by $\mathrm{Ban}^{<\infty}$ the class of finite-dimensional normed spaces, and we let $\mathrm{Ban} =$

$\mathrm{Ban}^{\infty} \cup \mathrm{Ban}^{<\infty}$. Given a Banach space X, we denote by $\mathrm{Sub}^{\infty}(X)$ the set of (infinite-dimensional, closed) subspaces of X, by $\mathrm{Sub}^{<\infty}(X)$ the set of finite-dimensional subspaces of X, and we let $\mathrm{Sub}(X) = \mathrm{Sub}^{<\infty}(X) \cup \mathrm{Sub}^{\infty}(X)$. For $Y, Z \in \mathrm{Sub}(X)$, we will say that Y is *almost contained* in Z, and write $Y \subseteq^* Z$, when a finite-codimensional subspace of Y is contained in Z.

When writing about a Banach space X, we will in general assume that it comes with a fixed norm, which we will usually denote by $\|\cdot\|$. The unit sphere of X for this norm will be denoted by S_X, and if necessary we will denote by $\delta_{\|\cdot\|}$ the distance induced by this norm. For $x \in X$ and $r \geqslant 0$, we denote by $\mathrm{B}(x, r)$ the open ball centered at x with radius r (which is just the empty set when $r = 0$).

Given two finite- or infinite-dimensional Banach spaces X and Y, the space of continuous linear operators from X to Y will be denoted by $\mathscr{L}(X, Y)$, or simply by $\mathscr{L}(X)$ when $X = Y$. It will be equipped with the operator norm coming from the norms of X and Y, and this norm will also be usually denoted by $\|\cdot\|$. For $C \geqslant 1$, a *C-isomorphism* between X and Y is an isomorphism $T : X \to Y$ such that $\|T\| \cdot \|T^{-1}\| \leqslant C$. The *Banach–Mazur distance* between X and Y, denoted by $d_{BM}(X, Y)$ is the infimum of the $C \geqslant 1$ such that there exists a C-isomorphism between X and Y (if X and Y are not isomorphic, then $d_{BM}(X, Y) = \infty$). A space will be called *Hilbertian* if it is at finite Banach–Mazur distance to a Hilbert space, and *ℓ_2-saturated* if every subspace of X has a Hilbertian subspace. A *C-embedding* from X into Y is an embedding which is a C-isomorphism onto its image. We write $X \sqsubseteq Y$ if X isomorphically embeds into Y, and $X \sqsubseteq_C Y$ if X C-embeds into Y.

Two families $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ of elements of a Banach space X are said to be *C-equivalent*, for $C \geqslant 1$, if for every family $(a_i)_{i\in I}$ of real numbers with finite support, we have

$$\frac{1}{C}\Big\|\sum_{i\in I} a_i y_i\Big\| \leqslant \Big\|\sum_{i\in I} a_i x_i\Big\| \leqslant C\Big\|\sum_{i\in I} a_i y_i\Big\|.$$

In this case, there is a unique C^2-isomorphism $T : \overline{\mathrm{span}(x_i \mid i \in I)} \to \overline{\mathrm{span}(y_i \mid i \in I)}$ such that for every i, we have $T(x_i) = y_i$. The families (x_i) and (y_i) are simply said to be *equivalent* if they are C-equivalent for some $C \geqslant 1$.

In this paper, we will often use the notion of *finite-dimensional decomposition* (FDD). Recall that an FDD of a space X is a sequence $(F_n)_{n\in\mathbb{N}}$ of non-zero finite-dimensional subspaces of X such that every $x \in X$ can be written in a unique way as $\sum_{n=0}^{\infty} x_n$, where $x_n \in F_n$ for all $n \in \mathbb{N}$. In this case there exists a constant C such that for all $x \in X$ and all $n \in \mathbb{N}$, we have $\|\sum_{i<n} x_i\| \leqslant C\|x\|$. The smallest such C is called the *constant* of the FDD (F_n). A sequence of finite-dimensional subspaces which is an FDD of the closed subspace it generates will simply be called an *FDD*, without more specification. An *unconditional finite-dimensional decomposition* (UFDD) is an FDD $(F_n)_{n\in\mathbb{N}}$ such that for every sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in F_n$ for all n, if $\sum_{n=0}^{\infty} x_n$ converges, then this convergence is unconditional. In this case, there is a constant K such that for all such sequences (x_n), and for every sequence of signs $(\varepsilon_n)_{n\in\mathbb{N}} \in \{-1, 1\}^{\mathbb{N}}$, we have

$\|\sum_{i<n} x_n\| \leqslant K\|\sum_{i<n} \varepsilon_n x_n\|$. The smallest such K is called the *unconditional constant* of the UFDD (F_n).

Fix an FDD $(F_n)_{n\in\mathbb{N}}$ of a space X. For $x = \sum_{n=0}^{\infty} x_n \in X$, the *support* of x is $\mathrm{supp}(x) = \{n \in \mathbb{N} \mid x_n \neq 0\}$. For $A \subseteq X$, we let $\mathrm{supp}(A) = \bigcup_{x\in A} \mathrm{supp}(x)$. A *blocking* of (F_n) is a sequence $(G_n)_{n\in\mathbb{N}}$ of finite-dimensional subspaces of X for which there exists a partition of $\mathbb{N}$ into non-empty successive intervals $I_0 < I_1 < \cdots$ such that for every n, $G_n = \bigoplus_{i\in I_n} F_i$ (here, for two non-empty sets A and B of integers, we write $A < B$ if $\forall i \in A\ \forall j \in B\ i < j$). A *block-FDD* of (F_n) is a sequence $(G_n)_{n\in\mathbb{N}}$ of non-zero finite-dimensional subspaces of X such that $\mathrm{supp}(G_0) < \mathrm{supp}(G_1) < \cdots$. A blocking is a particular case of a block-FDD. A block-FDD of (F_n) is itself an FDD, and its constant is less than or equal to the constant of (F_n); moreover, if (F_n) is a UFDD, then a block-FDD of (F_n) is also a UFDD, and its unconditional constant is less than or equal to that of (F_n). A *block-sequence* of (F_n) is a sequence $(x_n)_{n\in\mathbb{N}}$ of vectors of X such that $(\mathbb{R}x_n)_{n\in\mathbb{N}}$ is a block-FDD of (F_n). Such a sequence is a basic sequence, with constant less than or equal to the constant of the FDD (F_n).

If $(F_i)_{i\in I}$ is a family of finite-dimensional subspaces of a Banach space X, we will let $[F_i \mid i \in I] = \overline{\sum_{i\in I} F_i}$. This notation will often (but not only) be used in the case where (F_i) is a (finite or infinite) subsequence of an FDD.

For $C \geqslant 1$, a *C-bounded minimal system* in a Banach space X is a family $(x_i)_{i\in I}$ of non-zero elements of X such that for every family $(a_i)_{i\in I}$ of real numbers with finite support and for every $i_0 \in I$, we have $\|a_{i_0}x_{i_0}\| \leqslant C\|\sum_{i\in I} a_i x_i\|$. Every separable Banach space contains a countable bounded minimal system whose closed span is the whole space; several more precise results by Terenzi show that such a system can be chosen to have properties that are very close to those of a Schauder basis (see for example [58]). A normalized, 1-bounded minimal system is called an *Auerbach system*; by Auerbach's lemma [27, Theorem 1.16], every finite-dimensional normed space has an Auerbach basis (that is, a basis which is an Auerbach system). A basic sequence with constant $\leqslant C$ is a $2C$-bounded minimal system. But there are other interesting examples. For instance, let $(F_n)_{n\in\mathbb{N}}$ be an FDD with constant C. Let, for $n \in \mathbb{N}$, $d_n = \sum_{m<n} \dim(F_m)$, and let $(e_i)_{d_n\leqslant i<d_{n+1}}$ be an Auerbach basis of F_n. Then the sequence $(e_i)_{i\in\mathbb{N}}$ is a $2C$-bounded minimal system.

Given two families $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ that are K-equivalent, if (x_i) is a C-bounded minimal system, then (y_i) is a CK^2-bounded minimal system. We will also often use the following small perturbation principle for bounded minimal systems:

Lemma 1.34. *For every $C \geqslant 1$ and every $\varepsilon > 0$, there exists $\delta > 0$ with the following property: if $(x_i)_{i\in I}$ is a C-bounded minimal system in a Banach space X and $(y_i)_{i\in I}$ is a family of elements of X, and if*

$$\sum_{i\in I} \frac{\|x_i - y_i\|}{\|x_i\|} \leqslant \delta,$$

then (x_i) and (y_i) are $(1+\varepsilon)$-equivalent.

The proof is routine. This is a classical result for basic sequences (see for example [1, Theorem 1.3.9]), and the proof is exactly the same for bounded minimal systems.

2. Gowers spaces

In this section, we present the formalism of Gowers spaces. This formalism will be our main tool to prove dichotomies. It has been developed by de Rancourt [13] as a generalization of Gowers' Ramsey-type theory in Banach spaces developed in [25]. The proofs of all the results presented in this section can be found in [13].

2.1. Gowers spaces

For a set Π, denote by $\Pi^{<\mathbb{N}}$ the set of all finite sequences of elements of Π. A sequence of length n will usually be denoted by $s = (s_0, \dots, s_{n-1})$, and the unique sequence of length 0 will be denoted by $\emptyset$. Let $\mathrm{Seq}(\Pi) = \Pi^{<\mathbb{N}} \setminus \{\emptyset\}$. For $s \in \Pi^{<\mathbb{N}}$ and $x \in \Pi$, the concatenation of s and x will be denoted by $s^\frown x$.

Definition 2.1. A *Gowers space* is a quintuple $\mathcal{G} = (\mathbb{P}, \Pi, \leqslant, \leqslant^*, \lhd)$, where $\mathbb{P}$ is a non-empty set (the set of *subspaces*), Π is an at most countable non-empty set (the set of *points*), $\leqslant$ and $\leqslant^*$ are two quasiorders on $\mathbb{P}$ (i.e. reflexive and transitive binary relations), and $\lhd \subseteq \mathrm{Seq}(\Pi) \times \mathbb{P}$ is a binary relation, satisfying the following properties:

(1) for all $p, q \in \mathbb{P}$, if $p \leqslant q$, then $p \leqslant^* q$;

(2) for all $p, q \in \mathbb{P}$, if $p \leqslant^* q$, then there exists $r \in \mathbb{P}$ such that $r \leqslant p$, $r \leqslant q$ and $p \leqslant^* r$;

(3) for every $\leqslant$-decreasing sequence $(p_i)_{i\in\mathbb{N}}$ of elements of $\mathbb{P}$, there exists $p^* \in \mathbb{P}$ such that for all $i \in \mathbb{N}$, $p^* \leqslant^* p_i$;

(4) for every $p \in \mathbb{P}$ and $s \in \Pi^{<\mathbb{N}}$, there exists $x \in \Pi$ such that $s^\frown x \lhd p$;

(5) for all $s \in \mathrm{Seq}(\Pi)$ and $p, q \in \mathbb{P}$, if $s \lhd p$ and $p \leqslant q$, then $s \lhd q$.

We say that $p, q \in \mathbb{P}$ are *compatible* if there exists $r \in \mathbb{P}$ such that $r \leqslant p$ and $r \leqslant q$. For brevity, we will often write $p \lessapprox q$ when $p \leqslant q$ and $q \leqslant^* p$.

The prototypical example of a Gowers space is the following. Let K be an at most countable field. The *Rosendal space* over K is $\mathcal{R}_K = (\mathbb{P}, \Pi, \subseteq, \subseteq^*, \lhd)$, where

- Π is a countably infinite-dimensional K-vector space;
- $\mathbb{P}$ is the set of all infinite-dimensional subspaces of Π;
- $\subseteq$ is inclusion between subspaces;
- $\subseteq^*$ is almost inclusion, defined by $Y \subseteq^* Z$ iff Z contains a finite-codimensional subspace of Y;
- $(x_0, \dots, x_n) \lhd Y$ iff $x_n \in Y$.

Here, $Z \lessapprox Y$ iff Z is a finite-codimensional subspace of Y, and Y and Z are compatible iff $Y \cap Z$ is infinite-dimensional.

In the case of the Rosendal space, whether $s \lhd p$ actually only depends on p and on the last term of s. This is the case in most usual examples of Gowers spaces; spaces satisfying this property will be called *forgetful Gowers spaces*. In these spaces, we will allow ourselves to view $\lhd$ as a binary relation on $\Pi \times \mathbb{P}$. However, in the proof of Theorem 5.5, we will use a Gowers space which is not forgetful.

In the rest of this subsection, we fix a Gowers space $\mathcal{G} = (\mathbb{P}, \Pi, \leqslant, \leqslant^*, \lhd)$. To every $p \in \mathbb{P}$, we associate the following four games:

Definition 2.2. Let $p \in \mathbb{P}$.

(1) *Gowers' game below* p, denoted by G_p, is defined in the following way:

I	p_0		p_1	$\dots$
II		x_0		x_1 $\dots$

where the x_i's are elements of Π, and the p_i's are elements of $\mathbb{P}$. The rules are the following:

- for **I**: for all $i \in \mathbb{N}$, $p_i \leqslant p$;
- for **II**: for all $i \in \mathbb{N}$, $(x_0, \dots, x_i) \lhd p_i$.

The outcome of the game is the sequence $(x_i)_{i\in\mathbb{N}} \in \Pi^{\mathbb{N}}$.

(2) The *asymptotic game below* p, denoted by F_p, is defined in the same way as G_p, except that this time we moreover require that $p_i \lessapprox p$.

(3) The *adversarial Gowers games* below p, denoted by A_p and B_p, are obtained by mixing Gowers' game and the asymptotic game. The game A_p is defined in the following way:

I		x_0, q_0		x_1, q_1	$\dots$
II	p_0		y_0, p_1		y_1, p_2 $\dots$

where the x_i's and the y_i's are elements of Π, and the p_i's and the q_i's are elements of $\mathbb{P}$. The rules are the following:

- for **I**: for all $i \in \mathbb{N}$, $(x_0, \dots, x_i) \lhd p_i$ and $q_i \lessapprox p$;
- for **II**: for all $i \in \mathbb{N}$, $(y_0 \dots, y_i) \lhd q_i$ and $p_i \leqslant p$.

The outcome of the game is the pair of sequences $((x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}) \in (\Pi^{\mathbb{N}})^2$.

(4) The game B_p is defined in the same way as A_p, except that this time we require $p_i \lessapprox p$, whereas we only require $q_i \leqslant p$.

In this paper, when dealing with games, we shall use a convention introduced by Rosendal: we associate an *outcome* to the game, and define a winning condition in terms of the outcome belonging or not to a determined set. For example, saying that player **II** has a strategy to reach a set $\mathcal{X} \subseteq \Pi^{\mathbb{N}}$ in the game G_p means that she has a winning strategy in the game whose rules are those of G_p and whose winning condition is that the outcome belongs to $\mathcal{X}$.

We endow Π with the discrete topology and $\Pi^{\mathbb{N}}$ with the product topology. The two main results about Gowers spaces, proved by de Rancourt [13], are the following:

Theorem 2.3 (Abstract Rosendal theorem, [13]). *Let $\mathcal{X} \subseteq \Pi^{\mathbb{N}}$ be analytic, and let $p \in \mathbb{P}$. Then there exists $q \leqslant p$ such that either*

- *player **I** has a strategy to reach $\mathcal{X}^c$ in F_q, or*
- *player **II** has a strategy to reach $\mathcal{X}$ in G_q.*

Theorem 2.4 (Adversarial Ramsey principle, [13]). *Let $\mathcal{X} \subseteq (\Pi^{\mathbb{N}})^2$ be Borel, and let $p \in \mathbb{P}$. Then there exists $q \leqslant p$ such that either*

- *player **I** has a strategy to reach $\mathcal{X}$ in A_q, or*
- *player **II** has a strategy to reach $\mathcal{X}^c$ in B_q.*

Remark 2.5. The definition of the games A_p and B_p we give here is slightly different from the original definition in [13]. This is to save notation in the rest of the paper, and in particular in the proof of Theorem 5.5, which will be quite technical. The version of Theorem 2.4 we state above is thus slightly weaker than the original one.

Theorem 2.3 has been stated and proved by Rosendal [51] in the special case of the Rosendal space, as a discrete version of Gowers' Ramsey-type theorem in Banach spaces. Theorem 2.4 has been proved by Rosendal [53] for $\mathbf{\Sigma}^3_0$ and $\mathbf{\Pi}^3_0$ subsets, in the case of the Rosendal space; he also conjectured the result for Borel sets, which has been proved by de Rancourt [13].

2.2. Approximate Gowers spaces

Approximate Gowers spaces are a version of Gowers spaces where the set of points is not anymore a countable set, but a Polish metric space. This formalism is more convenient to obtain approximate Ramsey-type theorems in Banach spaces, for example.

In this section and in the rest of this paper, we use the following notation: if (Π, δ) is a metric space, if $\mathcal{X} \subseteq \Pi^{\mathbb{N}}$ and if $\Delta = (\Delta_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers, then we let $(\mathcal{X})_\Delta = \{(x_n)_{n\in\mathbb{N}} \in \Pi^{\mathbb{N}} \mid \exists (y_n)_{n\in\mathbb{N}} \in \mathcal{X}\ \forall n \in \mathbb{N}\ \delta(x_n, y_n) \leqslant \Delta_n\}$.

Definition 2.6. An *approximate Gowers space* is a sextuple $\mathcal{G} = (\mathbb{P}, \Pi, \delta, \leqslant, \leqslant^*, \lhd)$, where $\mathbb{P}$ is a non-empty set, Π is a non-empty Polish space, δ is a compatible distance on Π, $\leqslant$ and $\leqslant^*$ are two quasiorders on $\mathbb{P}$, and $\lhd \subseteq \Pi \times \mathbb{P}$ is a binary relation, satisfying axioms (1)–(3) as in the definition of a Gowers space and moreover the following two axioms:

(4) for every $p \in \mathbb{P}$, there exists $x \in \Pi$ such that $x \lhd p$;

(5) for all $x \in \Pi$ and $p, q \in \mathbb{P}$, if $x \lhd p$ and $p \leqslant q$, then $x \lhd q$.

The relation $\lessapprox$ and the compatibility relation on $\mathbb{P}$ are defined in the same way as for a Gowers space.

With this definition, approximate Gowers spaces are always forgetful, that is, the relation $\lhd$ is defined as a subset of $\Pi \times \mathbb{P}$ and not as a subset of $\mathrm{Seq}(\Pi) \times \mathbb{P}$ (this technical restriction seems to be needed to prove approximate versions of Theorems 2.3 and 2.4). In all cases we will encounter in this paper, $\lhd$ will actually be the membership relation.

The prototypical example of an approximate Gowers space is the following. Let X be a separable Banach space. The *canonical approximate Gowers space over* X is $\mathcal{G}_E = (\mathbb{P}, S_X, \delta_{\|\cdot\|}, \subseteq, \subseteq^*, \in)$, where

- $\mathbb{P}$ is the set of all (infinite-dimensional) subspaces of X;
- S_X is the unit sphere of X;
- $\delta_{\|\cdot\|}$ is the distance on S_X induced by the norm of X;
- $\subseteq$ is inclusion between subspaces;
- $\subseteq^*$ is almost inclusion between subspaces, as defined in Section 1.5;
- $\in$ is the membership relation between points and subspaces.

Here, $Z \lessapprox Y$ iff Z is a finite-codimensional subspace of Y, and Y and Z are compatible iff $Y \cap Z$ is infinite-dimensional.

In the context of approximate Gowers spaces, de Rancourt [13] proved an approximate version of Theorem 2.4, but we will not use it in this paper. However, we will introduce an approximate version of Theorem 2.3. In the rest of this subsection, we fix an approximate Gowers space $\mathcal{G} = (\mathbb{P}, \Pi, \delta, \leqslant, \leqslant^*, \lhd)$. In this space, Gowers' game G_p is defined in the same way as in Gowers spaces (Definition 2.2), apart from the fact that the rule $(x_0, \dots, x_i) \lhd p_i$ is obviously replaced with $x_i \lhd p_i$. We will also define a strengthening of the asymptotic game. Recall that a subset of Π is said to be *relatively compact* if its closure in Π is compact. In what follows, for $K \subseteq \Pi$ and $p \in \mathbb{P}$, we abusively write $K \lhd p$ to mean that the set $\{x \in K \mid x \lhd p\}$ is dense in K.

Definition 2.7. A *system of relatively compact sets* for the approximate Gowers space $\mathcal{G}$ is a set $\mathcal{K}$ of relatively compact subsets of Π, equipped with an associative binary operation $\oplus$, satisfying the following property: for all $p \in \mathbb{P}$ and $K, L \in \mathcal{K}$, if $K \lhd p$ and $L \lhd p$, then $K \oplus L \lhd p$.

If $(\mathcal{K}, \oplus)$ is a system of relatively compact sets for $\mathcal{G}$ and if $(K_n)_{n\in\mathbb{N}}$ is a sequence of elements of $\mathcal{K}$, then

- for $A \subseteq \mathbb{N}$ finite, denote by $\bigoplus_{n\in A} K_n$ the sum $K_{n_1} \oplus \cdots \oplus K_{n_k}$, where $n_1, \dots, n_k$ are the elements of A taken in increasing order;
- a *block-sequence* of (K_n) is, by definition, a sequence $(x_i)_{i\in\mathbb{N}} \in \Pi^{\mathbb{N}}$ for which there exists an increasing sequence $A_0 < A_1 < A_2 < \cdots$ of non-empty sets of integers such that for every $i \in \mathbb{N}$, we have $x_i \in \bigoplus_{n\in A_i} K_n$.

Denote by $\mathrm{bs}((K_n)_{n\in\mathbb{N}})$ the set of all block-sequences of (K_n).

In the canonical approximate Gowers space $\mathcal{G}_X$ over a separable Banach space X, we can define a natural system of relatively compact sets, $(\mathcal{K}_X, \oplus_X)$, as follows: the elements of $\mathcal{K}_X$ are the unit spheres of finite-dimensional subspaces of X and the operation $\oplus_X$

on $\mathcal{K}_X$ is defined by $S_F \oplus_X S_G = S_{F+G}$. Observe that, given an FDD $(F_n)_{n\in\mathbb{N}}$ of a subspace of X, the block-sequences of $(S_{F_n})_{n\in\mathbb{N}}$ in the sense given by the latter definition are exactly the normalized block-sequences of (F_n) in the Banach-theoretic sense.

Definition 2.8. Let $(\mathcal{K}, \oplus)$ be a system of relatively compact sets for $\mathcal{G}$, and $p \in \mathbb{P}$. The *strong asymptotic game below* p, denoted by SF_p, is defined as follows:

I	p_0		p_1	$\dots$
II		K_0		K_1 $\dots$

where the K_n's are elements of $\mathcal{K}$, and the p_n's are elements of $\mathbb{P}$. The rules are the following:

- for **I**: for all $n \in \mathbb{N}$, $p_n \lessapprox p$;
- for **II**: for all $n \in \mathbb{N}$, $K_n \lhd p_n$.

The outcome of the game is the sequence $(K_n)_{n\in\mathbb{N}} \in \mathcal{K}^{\mathbb{N}}$.

We endow $\Pi^{\mathbb{N}}$ with the product topology. The following result, proved by de Rancourt [13], is the approximate version of Theorem 2.3.

Theorem 2.9 (Abstract Gowers theorem, [13]). *Let $(\mathcal{K}, \oplus)$ be a system of relatively compact sets for $\mathcal{G}$. Let $\mathcal{X} \subseteq \Pi^{\mathbb{N}}$ be analytic, let $p \in \mathbb{P}$ and let Δ be a sequence of positive real numbers. Then there exists $q \leqslant p$ such that either*

- *player **I** has a strategy in SF_q to build a sequence $(K_n)_{n\in\mathbb{N}}$ such that* $\mathrm{bs}((K_n)_{n\in\mathbb{N}}) \subseteq \mathcal{X}^c$, *or*
- *player **II** has a strategy in G_q to reach $(\mathcal{X})_\Delta$.*

From this abstract result, we can easily recover the original Ramsey-type theorem proved by Gowers [25], and used to deduce his first dichotomy (Theorem 1.20) along with another dichotomy:

Theorem 2.10 (Gowers). *Let X be a separable Banach space, $\mathcal{X} \subseteq (S_X)^{\mathbb{N}}$ be analytic and Δ be a sequence of positive real numbers. Then there exists a subspace Y of X such that either*

- *Y has a basis $(y_n)_{n\in\mathbb{N}}$ such that all normalized block-sequences of (y_n) belong to $\mathcal{X}^c$, or*
- *player **II** has a strategy in G_Y to reach $(\mathcal{X})_\Delta$.*

In the statement of this theorem, G_Y denotes the Gowers game relative to the canonical approximate Gowers space $\mathcal{G}_X$. The original statement proved by Gowers is a bit different in its formulation, but both are easily seen to be equivalent. As an illustration of the formalism of approximate Gowers spaces, we now prove Theorem 2.10.

Proof of Theorem 2.10. Work in the canonical approximate Gowers space $\mathcal{G}_X$, with the system of relatively compact sets $(\mathcal{K}_X, \oplus_X)$ defined above. Apply Theorem 2.9 to $\mathcal{X}$, $p = X$, and Δ. Then either we get a subspace $Y \subseteq X$ such that player **II** has a strategy

in G_Y to reach $(\mathcal{X})_\Delta$, and we are done, or we get a subspace $Y \subseteq X$ such that player **I** has a strategy τ in SF_Y to build a sequence $(K_n)_{n\in\mathbb{N}}$ with $\mathrm{bs}((K_n)_{n\in\mathbb{N}}) \subseteq \mathcal{X}^c$. We can assume that the strategy τ is such that for every run of the game SF_Y,

I	Y_0		Y_1	$\dots$
II		S_{F_0}		S_{F_1} $\dots$

played according to τ, the natural projection $[F_i \mid i < n] \oplus Y_n \to [F_i \mid i < n]$ has norm at most 2. Now consider any run of the game where **I** plays according to τ and **II** plays unit spheres of subspaces of dimension 1: $S_{\mathbb{R}y_0}, S_{\mathbb{R}y_1}, \dots$. Then by construction, $(y_n)_{n\in\mathbb{N}}$ is a basic sequence with constant at most 2, and because **I** played according to τ, all normalized block-sequences of (y_n) belong to $\mathcal{X}^c$. ∎

The main goal of the next section is to investigate conditions on families $\mathcal{H}$ of subspaces of X for which a local version of Theorem 2.10 can be proved, that is, a version of Theorem 2.10 where we can ensure that the subspace Y given by the theorem is in $\mathcal{H}$. Such a result will be proved in Section 4.

3. D-families: definition and examples

In this section, we introduce the notion of a D-family: these families will be those for which we will be able to prove local Banach-space dichotomies. The "D" in the name of D-families refers both to the possibility of proving such dichotomies, and to the fundamental property that one can diagonalize among such families (see Lemma 3.5 below). We will then give sufficient conditions for being a D-family, and examples.

3.1. Definition and first properties

As seen in the previous section, the main ingredient in proving dichotomies of a Ramsey-theoretic nature in a given family of subspaces is the possibility of diagonalizing among elements of this family. Inspired by Lemma 1.27, we will define D-families as families of subspaces that are G_δ for a certain topology. This will ensure, on the one hand, that a diagonalization property similar to that in the definition of a P^+-coideal will be satisfied by these families, and on the other hand that they have a good behavior relative to FDD's.

Fix a Banach space X. For $F \in \mathrm{Sub}^{<\infty}(X)$ and $Y \in \mathrm{Sub}(X)$ such that $F \subseteq Y$, let $[F, Y] := \{Z \in \mathrm{Sub}(X) \mid F \subseteq Z \subseteq Y\}$; and for $\varepsilon > 0$, let $[F, Y]_\varepsilon^X$ be the set of $Z \in \mathrm{Sub}(X)$ for which there exist $Z' \in [F, Y]$ and an isomorphism $T : Z' \to Z$ with $\|T - \mathrm{Id}_{Z'}\| < \varepsilon$ (the latter set will be simply denoted by $[F, Y]_\varepsilon$ when there is no ambiguity about the ambient space X). To avoid any misunderstanding, we stress that the notation $[F, Y]$ should not be confused with the one used to denote the closed linear span of a sequence of vectors or of finite-dimensional subspaces.

Lemma 3.1. *The sets $[F, Y]_\varepsilon$, for $\varepsilon > 0$, $F \in \mathrm{Sub}^{<\infty}(X)$ and $Y \in \mathrm{Sub}(X)$ such that $F \subseteq Y$, form a basis for a topology on* $\mathrm{Sub}(X)$. *Given $Y \in \mathrm{Sub}(X)$, a basis of neighborhoods of Y for this topology is given by the $[F, Y]_\varepsilon$'s for $\varepsilon > 0$ and $F \subseteq Y$.*

Proof. What we have to show is that given $\varepsilon_i > 0$, $Y_i \in \mathrm{Sub}(X)$, and $F_i \in \mathrm{Sub}^{<\infty}(X)$ such that $F_i \subseteq Y_i$ for $1 \leqslant i \leqslant n$, and given $Z \in \bigcap_{i=1}^n [F_i, Y_i]_{\varepsilon_i}$, there exist $\varepsilon > 0$ and a finite-dimensional subspace $F \subseteq Z$ such that $[F, Z]_\varepsilon \subseteq \bigcap_{i=1}^n [F_i, Y_i]_{\varepsilon_i}$. For each i, fix $Z_i \in [F_i, Y_i]$ and an isomorphism $T_i : Z_i \to Z$ such that $\|T_i - \mathrm{Id}_{Z_i}\| < \varepsilon_i$. Fix $\varepsilon > 0$ such that for every i, $\|T_i - \mathrm{Id}_{Z_i}\| + \varepsilon(1 + \varepsilon_i) < \varepsilon_i$, and let $F = \sum_{i=1}^n T_i(F_i)$. Then we have $F \subseteq Z$. Now let $W \in [F, Z]_\varepsilon$, and fix $1 \leqslant i \leqslant n$; we show that $W \in [F_i, Y_i]_{\varepsilon_i}$. Fix $W' \in [F, Z]$ and an isomorphism $T : W' \to W$ such that $\|T - \mathrm{Id}_{W'}\| < \varepsilon$. Then the map $T \circ (T_i)_{\restriction T_i^{-1}(W')}$ is an isomorphism from $T_i^{-1}(W')$ to W, and $T_i^{-1}(W') \in [F_i, Y_i]$. Moreover,

$$\begin{aligned}\|T \circ (T_i)_{\restriction T_i^{-1}(W')} - \mathrm{Id}_{T_i^{-1}(W')}\| &\leqslant \|(T - \mathrm{Id}_{W'}) \circ (T_i)_{\restriction T_i^{-1}(W')}\| + \|T_i - \mathrm{Id}_{Z_i}\| \\ &\leqslant \varepsilon(1 + \varepsilon_i) + \|T_i - \mathrm{Id}_{Z_i}\| < \varepsilon_i,\end{aligned}$$

concluding the proof. ∎

The topology on $\mathrm{Sub}(X)$ defined in Lemma 3.1 will be called the *Ellentuck topology*. It does not depend on the choice of the equivalent norm on X. This name was given because of the similarity between this topology and other topologies that arise in the context of Ramsey spaces, and that are also called *Ellentuck*. See [59] for more details.

Definition 3.2. A *D-family* of subspaces of X is a family $\mathcal{H} \subseteq \mathrm{Sub}^{\infty}(X)$ satisfying the following two properties:

(1) $\mathcal{H}$ is stable under finite-dimensional modifications, i.e. for every $Y \in \mathrm{Sub}^{\infty}(X)$ and every $F \in \mathrm{Sub}^{<\infty}(X)$, we have $Y \in \mathcal{H}$ if and only if $Y + F \in \mathcal{H}$;

(2) $\mathcal{H}$, seen as a subset of $\mathrm{Sub}(X)$, is G_δ for the Ellentuck topology.

We now prove a few properties of D-families. In what follows, we fix a D-family $\mathcal{H}$ of subspaces of X.

Definition 3.3. Let $Y \in \mathrm{Sub}(X)$. The *restriction* of $\mathcal{H}$ to Y is the set $\mathcal{H}_{\restriction Y} = \mathcal{H} \cap \mathrm{Sub}(Y)$.

Lemma 3.4. *Let $Y \in \mathrm{Sub}^{\infty}(X)$. The Ellentuck topology on $\mathrm{Sub}(Y)$ coincides with the topology induced on $\mathrm{Sub}(Y)$ by the Ellentuck topology on $\mathrm{Sub}(X)$. In particular, $\mathcal{H}_{\restriction Y}$ is a D-family of subspaces of Y.*

Proof. Observe that for every $\varepsilon > 0$, every $Z \in \mathrm{Sub}(Y)$ and every finite-dimensional subspace $F \subseteq Z$, we have $[F, Z]_\varepsilon^Y = [F, Z]_\varepsilon^X \cap \mathrm{Sub}(Y)$. The left-hand side of this equality is the general form of a basic neighborhood of Z in the Ellentuck topology on $\mathrm{Sub}(Y)$, and the right-hand side is the general form of a basic neighborhood of Z in the topology induced on $\mathrm{Sub}(Y)$ by the Ellentuck topology on $\mathrm{Sub}(X)$. Thus, these topologies coincide.

Therefore, since $\mathcal{H}$ is G_δ for the Ellentuck topology on $\mathrm{Sub}(X)$, its intersection with $\mathrm{Sub}(Y)$ is G_δ for the Ellentuck topology on $\mathrm{Sub}(Y)$, proving the second part. ∎

Lemma 3.5. *Let $(Y_n)_{n\in\mathbb{N}}$ be a decreasing family of elements of $\mathcal{H}$. Then there exists $Y_\infty \in \mathcal{H}$ such that for every $n \in \mathbb{N}$, $Y_\infty \subseteq^* Y_n$.*

Proof. Let $(\mathcal{U}_n)_{n\in\mathbb{N}}$ be a decreasing family of Ellentuck-open subsets of $\mathrm{Sub}(X)$ such that $\mathcal{H} = \bigcap_{n\in\mathbb{N}} \mathcal{U}_n$. We define inductively an increasing sequence $(F_n)_{n\in\mathbb{N}}$ of finite-dimensional subspaces of X in the following way. Let $F_0 = \{0\}$. The space F_n being defined, by axiom (1) in the definition of a D-family, the subspace $Y_n + F_n$ is in $\mathcal{H}$, so in $\mathcal{U}_n$; thus there exists a finite-dimensional subspace $F_{n+1} \subseteq Y_n + F_n$ such that $[F_{n+1}, Y_n + F_n] \subseteq \mathcal{U}_n$. We can even assume that $F_n \subseteq F_{n+1}$ and $\dim(F_{n+1}) \geqslant n+1$. This completes the construction.

Now let $Y_\infty = \overline{\bigcup_{n\in\mathbb{N}} F_n}$. By construction, for every $n \in \mathbb{N}$ we have $Y_\infty \subseteq F_n + Y_n$, so $Y_\infty \subseteq^* Y_n$. This also implies that $Y_\infty \in [F_{n+1}, Y_n + F_n] \subseteq \mathcal{U}_n$, so finally $Y_\infty \in \mathcal{H}$. ∎

Corollary 3.6. *$\mathcal{G}_\mathcal{H} = (\mathcal{H}, S_X, \delta_{\|\cdot\|}, \subseteq, \subseteq^*, \in)$ is an approximate Gowers space.*

Definition 3.7. An *$\mathcal{H}$-good FDD* is an FDD $(F_n)_{n\in\mathbb{N}}$ of a subspace of X such that for every infinite $A \subseteq \mathbb{N}$, the subspace $[F_n \mid n \in A]$ is in $\mathcal{H}$.

This terminology is motivated by the fact that we want to prove dichotomies where the outcome space is in the family $\mathcal{H}$ (for instance, non-Hilbertian). In our dichotomies, good FDD's will play a similar role to the basis $(y_n)_{n\in\mathbb{N}}$ in the statement of Theorem 2.10.

Lemma 3.8. *Let $(F_n)_{n\in\mathbb{N}}$ be an FDD of a subspace of X. Suppose that $[F_n \mid n \in \mathbb{N}] \in \mathcal{H}$. Then there exists a blocking $(G_n)_{n\in\mathbb{N}}$ of (F_n) which is $\mathcal{H}$-good.*

Proof. Let $(\mathcal{U}_n)_{n\in\mathbb{N}}$ be a decreasing family of Ellentuck-open subsets of $\mathrm{Sub}(X)$ such that $\mathcal{H} = \bigcap_{n\in\mathbb{N}} \mathcal{U}_n$. For every $k \in \mathbb{N}$, let $Y_k = [F_l \mid l \geqslant k]$. We build (G_n) by induction as follows. Suppose that the G_m's have been built for $m < n$, and let $k_n = \max \mathrm{supp}(G_{n-1}) + 1$ if $n \geqslant 1$, $k_n = 0$ otherwise. By axiom (1) in the definition of a D-family, for every $A \subseteq \{0, \dots, n-1\}$, we have $[G_m \mid m \in A] \oplus Y_{k_n} \in \mathcal{U}_n$, so there exists a finite-dimensional subspace $K_n^A \subseteq [G_m \mid m \in A] \oplus Y_{k_n}$ and $\varepsilon_n^A > 0$ such that $[K_n^A,\ [G_m \mid m \in A] \oplus Y_{k_n}]_{\varepsilon_n^A} \subseteq \mathcal{U}_n$. We can assume that $K_n^A = [G_m \mid m \in A] \oplus H_n^A$ for some finite-dimensional subspace $H_n^A \subseteq Y_{k_n}$. Now let H_n be the finite-dimensional subspace of Y_{k_n} generated by all the H_n^A's, $A \subseteq \{0, \dots, n-1\}$, and let $\varepsilon_n = \min\{\varepsilon_n^A \mid A \subseteq \{0, \dots, n-1\}\}$. For every $A \subseteq \{0, \dots, n-1\}$, we have

$$\big[[G_m \mid m \in A] \oplus H_n,\ [G_m \mid m \in A] \oplus Y_{k_n}\big]_{\varepsilon_n} \subseteq \mathcal{U}_n.$$

Now consider an isomorphism $T_n : Y_0 \to Y_0$ such that

- T_n is equal to the identity on $[F_k \mid k < k_n]$;
- $T_n(Y_{k_n}) = Y_{k_n}$;
- $T_n(H_n) \subseteq [F_k \mid k_n \leqslant k < k_{n+1}]$ for some $k_{n+1} > k_n$;
- $\|T_n - \mathrm{Id}_{Y_0}\| < \varepsilon_n$.

We let $G_n = [F_k \mid k_n \leqslant k < k_{n+1}]$, and this completes the construction.

It is clear that (G_n) is a blocking of (F_n). We show that it is $\mathcal{H}$-good. Let $A \subseteq \mathbb{N}$ be infinite and $n \in A$; it is enough to show that $[G_m \mid m \in A] \in \mathcal{U}_n$. We know that T_n fixes $[G_m \mid m \in A,\ m < n]$, and we have $T_n(H_n) \subseteq G_n$, thus $(T_n)^{-1}([G_m \mid m \in A])$ contains

$[G_m \mid m \in A,\ m < n] \oplus H_n$. Moreover, $(T_n)^{-1}$ stabilizes Y_{k_n}, which contains the G_m's for $m \geqslant n$, so $(T_n)^{-1}([G_m \mid m \in A])$ is contained in $[G_m \mid m \in A,\ m < n] \oplus Y_{k_n}$. Hence,

$$(T_n)^{-1}([G_m \mid m \in A]) \in \big[[G_m \mid m \in A,\ m < n] \oplus H_n,\ [G_m \mid m \in A,\ m < n] \oplus Y_{k_n}\big],$$

and since $\|T_n - \mathrm{Id}_{Y_0}\| < \varepsilon_n$, we finally get

$$[G_m \mid m \in A] \in \big[[G_m \mid m \in A,\ m < n] \oplus H_n,\ [G_m \mid m \in A,\ m < n] \oplus Y_{k_n}\big]_{\varepsilon_n} \subseteq \mathcal{U}_n,$$

as desired. ∎

Lemma 3.9. *For every $Y \in \mathcal{H}$ and every $\varepsilon > 0$, there exists a subspace of Y having an $\mathcal{H}$-good FDD $(F_n)_{n\in\mathbb{N}}$ with constant at most $1 + \varepsilon$.*

Proof. By Lemma 3.8, it is enough to build an FDD (F_n) in such a way that $[F_n \mid n \in \mathbb{N}] \in \mathcal{H}$; passing to a blocking, we can turn it into an $\mathcal{H}$-good FDD having the same constant. Let $(\mathcal{U}_n)_{n\in\mathbb{N}}$ be a decreasing family of Ellentuck-open subsets of $\mathrm{Sub}(X)$ such that $\mathcal{H} = \bigcap_{n\in\mathbb{N}} \mathcal{U}_n$. We build the FDD $(F_n)_{n\in\mathbb{N}}$ by induction on n. Suppose that $F_0, \dots, F_{n-1}$ have been built. Let Y_n be a finite-codimensional subspace of Y with $Y_n \cap [F_i \mid i < n] = \{0\}$ and such that the natural projection from $[F_i \mid i < n] \oplus Y_n$ onto $[F_i \mid i < n]$ has norm at most $1 + \varepsilon$. If $n \geqslant 1$, we can even assume that $Y_n \subseteq Y_{n-1}$. We have $[F_i \mid i < n] \oplus Y_n \in \mathcal{U}_n$, so we can find a finite-dimensional subspace $F_n \subseteq Y_n$ such that $[[F_i \mid i \leqslant n],\ [F_i \mid i < n] \oplus Y_n] \subseteq \mathcal{U}_n$. This completes the construction.

By construction, for every $n \in \mathbb{N}$, we have $[F_i \mid i \geqslant n] \subseteq Y_n$, so the natural projection from $[F_i \mid i < n] \oplus [F_i \mid i \geqslant n]$ onto $[F_i \mid i < n]$ has norm at most $1 + \varepsilon$. This shows that (F_n) is an FDD with constant at most $1 + \varepsilon$. Moreover, for every $n \in \mathbb{N}$, we have $[F_i \mid i \in \mathbb{N}] \in [[F_i \mid i \leqslant n], [F_i \mid i < n] \oplus Y_n] \subseteq \mathcal{U}_n$, so $[F_i \mid i \in \mathbb{N}] \in \mathcal{H}$. ∎

The next lemma is an $\mathcal{H}$-good version of Bessaga–Pełczyński's selection principle.

Lemma 3.10. *Let Y be a subspace of X having an FDD $(F_n)_{n\in\mathbb{N}}$, and let $U \in \mathcal{H}$ be such that $U \subseteq Y$. Then there exists a subspace Z of Y spanned by an $\mathcal{H}$-good block-FDD $(G_n)_{n\in\mathbb{N}}$ of (F_n) such that Z isomorphically embeds into U.*

Lemma 3.10 was stated in this form for greater clarity, but for several applications in this paper, we will need a more general and more precise version of it, stated and proven below. It can be seen as an amalgamation of Lemmas 3.10 and 3.5.

Lemma 3.11. *Let $(Y_k)_{k\in\mathbb{N}}$ be a family of subspaces of X such that for every $k \in \mathbb{N}$, Y_k has an FDD $(F_n^k)_{n\in\mathbb{N}}$. Assume that for every $k \in \mathbb{N}$, $(F_n^{k+1})_{n\in\mathbb{N}}$ is a block-FDD of $(F_n^k)_{n\in\mathbb{N}}$. Let $(U_k)_{k\in\mathbb{N}}$ be a decreasing family of elements of $\mathcal{H}$ and let $(\Delta_k)_{k\in\mathbb{N}}$ be a sequence of positive real numbers. Assume that for every $k \in \mathbb{N}$, we have $U_k \subseteq Y_k$.*

Then there exists a subspace $Z \subseteq X$ generated by an $\mathcal{H}$-good FDD $(G_k)_{k\in\mathbb{N}}$ such that for every $k \in \mathbb{N}$, $(G_l)_{l\geqslant k}$ is a block-FDD of $(F_n^k)_{n\in\mathbb{N}}$; and there exists an isomorphic embedding $T : Z \to X$ such that for every $k \in \mathbb{N}$, we have $T(G_k) \subseteq U_k$ and $\|(T - \mathrm{Id}_Z)_{\restriction [G_l \mid l\geqslant k]}\| \leqslant \Delta_k$, and the FDD $(T(G_k))_{k\in\mathbb{N}}$ of $T(Z)$ is $\mathcal{H}$-good.

Moreover, if we are given, for every $k \in \mathbb{N}$, a subset $D_k \subseteq S_X$ such that, for every finite $A \subseteq \mathbb{N}$, $D_k \cap [F_n^k \mid n \in A]$ is dense in the unit sphere of $[F_n^k \mid n \in A]$, then we can ensure that for every $k \in \mathbb{N}$, the space G_k has a basis made up of elements of D_k.

Proof. Without loss of generality, we can assume that $\Delta_0 \leqslant 1/2$ and $\Delta_{k+1} \leqslant \Delta_k/2$ for all $k \in \mathbb{N}$. Let $(\mathcal{U}_k)_{k\in\mathbb{N}}$ be a decreasing family of Ellentuck-open subsets of $\mathrm{Sub}(X)$ such that $\mathcal{H} = \bigcap_{k\in\mathbb{N}} \mathcal{U}_k$. Let C be the constant of the FDD (F_n^0). We build by induction on k an FDD (G_k), along with an FDD $(H_k)_{k\in\mathbb{N}}$ of a subspace of X, such that for all $k \in \mathbb{N}$, $\dim(G_k) = \dim(H_k)$. We also build, at the same time, a sequence of isomorphisms $T_k : G_k \to H_k$; the embedding T will be defined as the unique bounded linear mapping on Z extending all the T_k's.

Suppose that the G_l's, the H_l's and the T_l's are built for $l < k$. Let $r_k \in \mathbb{N}$ be defined as follows: if $k = 0$, then $r_k = 0$, and otherwise r_k is such that $\mathrm{supp}(G_{k-1}) < \mathrm{supp}(F_{r_k}^k)$, where the supports are taken with respect to the FDD $(F_n^{k-1})_{n\in\mathbb{N}}$. Let $Y_k' = [F_n^k \mid n \geqslant r_k]$, and let U_k' be a finite-codimensional subspace of U_k such that $U_k' \subseteq Y_k'$ and $U_k' \cap [H_l \mid l < k] = \{0\}$; if $k \geqslant 1$, we moreover suppose that $U_k' \subseteq U_{k-1}'$. For every $A \subseteq \{0, \dots, k-1\}$, the subspaces $[G_l \mid l \in A] \oplus U_k'$ and $[H_l \mid l \in A] \oplus U_k'$ are in $\mathcal{H}$, so as in the proof of Lemma 3.8, we can find $\varepsilon_k > 0$ and a non-zero finite-dimensional subspace $H_k \subseteq U_k'$ such that for all A, both basic open sets

$$\big[[G_l \mid l \in A] \oplus H_k, [G_l \mid l \in A] \oplus U_k'\big]_{\varepsilon_k}$$

and

$$\big[[H_l \mid l \in A] \oplus H_k, [H_l \mid l \in A] \oplus U_k'\big]_{\varepsilon_k}$$

are contained in $\mathcal{U}_k$. We can even assume that for all $l < k$, $\varepsilon_k \leqslant \varepsilon_l/2^{k-l}$. Since $H_k \subseteq Y_k' = [F_n^k \mid n \geqslant r_k]$, we can find a finite-dimensional subspace $G_k \subseteq Y_k'$ having finite support relative to the FDD $(F_n^k)_{n\geqslant r_k}$ of Y_k', and a linear mapping $T_k : G_k \to H_k$ such that $\|T_k - \mathrm{Id}_{G_k}\| \leqslant \frac{\Delta_k}{4C}$ and $\|T_k^{-1} - \mathrm{Id}_{H_k}\| \leqslant \frac{\varepsilon_k}{24C(C+1)}$. We can even ensure that G_k has a basis made up of elements of D_k. This finishes the induction.

As desired, for every $k \in \mathbb{N}$, $(G_l)_{l\geqslant k}$ is a block-FDD of $(F_n^k)_{n\in\mathbb{N}}$. In particular, $(G_k)_{k\in\mathbb{N}}$ is a block-FDD of (F_n^0) and hence has constant at most C. Let $Z = [G_k \mid k \in \mathbb{N}]$ and $\widetilde{Z} = \bigoplus_{k\in\mathbb{N}} G_k$, a dense vector subspace of Z. Define $\widetilde{T} : \widetilde{Z} \to X$ as the unique linear mapping extending all the T_k's on their domains. For every eventually null sequence $(x_l)_{l\in\mathbb{N}}$ with $x_l \in G_l$ for all $l \in \mathbb{N}$ and for every $k \in \mathbb{N}$, we have

$$\begin{aligned}\Big\|(\widetilde{T} - \mathrm{Id}_{\widetilde{Z}})\Big(\sum_{l=k}^{\infty} x_l\Big)\Big\| &\leqslant \sum_{l=k}^{\infty} \|(T_l - \mathrm{Id}_{G_l})(x_l)\| \leqslant \sum_{l=k}^{\infty} \frac{\Delta_l}{4C}\|x_l\| \leqslant \sum_{l=k}^{\infty} \frac{\Delta_k}{2^{l-k+2}C}\|x_l\| \\ &\leqslant \sum_{l=k}^{\infty} \frac{\Delta_k}{2^{l-k+2}C} \cdot 2C\Big\|\sum_{l=k}^{\infty} x_l\Big\| \leqslant \Delta_k\Big\|\sum_{l=k}^{\infty} x_l\Big\|.\end{aligned}$$

This shows that $\|(\widetilde{T} - \mathrm{Id}_{\widetilde{Z}})_{\restriction \bigoplus_{l\geqslant k} G_l}\| \leqslant \Delta_k$. In particular, $\widetilde{T}$ is a bounded operator on $\widetilde{Z}$, so it extends to a bounded operator $T : Z \to X$ still satisfying $\|(T - \mathrm{Id}_Z)_{\restriction [G_l \mid l\geqslant k]}\| \leqslant \Delta_k$

for every $k \in \mathbb{N}$. In particular, since $\Delta_0 \leqslant 1/2$, the latter inequality shows that T is an isomorphic embedding, with $\|T\| \leqslant 3/2$ and $\|T^{-1}\| \leqslant 2$. In particular, $(H_k)_{k\in\mathbb{N}}$ is an FDD of a subspace of X, with constant at most $3C$.

It remains to show that the FDD's (G_k) and (H_k) are $\mathcal{H}$-good. For (H_k), the proof is similar to that of Lemma 3.8: given $A \subseteq \mathbb{N}$ infinite and $k \in A$, we have

$$[H_l \mid l \in A,\, l \geqslant k] \subseteq U'_k,$$

so

$$[H_l \mid l \in A] \in \big[[H_l \mid l \in A,\, l < k] \oplus H_k, [H_l \mid l \in A,\, l < k] \oplus U'_k\big],$$

and by construction, the set on the right-hand side is contained in $\mathcal{U}_k$, as desired.

For (G_k), we need one more estimate. For all k, let $K_k = [G_l \mid l < k]$, $V_k = [H_l \mid l \geqslant k]$, and $W_k = K_k \oplus V_k$. Define $S_k : W_k \to Z$ as the unique operator which is equal to the identity on K_k, and to T^{-1} on V_k. For all $l \geqslant k$, we have

$$\|T_l^{-1} - \mathrm{Id}_{H_l}\| \leqslant \frac{\varepsilon_l}{24C(C+1)} \leqslant \frac{\varepsilon_k}{2^{l-k} \cdot 24C(C+1)}.$$

Thus, knowing that $(H_l)_{l\geqslant k}$ is an FDD of V_k with constant at most $3C$, and using the same proof as for T, we can show that

$$\|(T^{-1})_{\restriction V_k} - \mathrm{Id}_{V_k}\| \leqslant \sum_{l\geqslant k} \frac{\varepsilon_k}{2^{l-k} \cdot 24C(C+1)} \cdot 6C = \frac{\varepsilon_k}{2(C+1)}.$$

Now recall that, by construction, $\mathrm{supp}(K_k) < \mathrm{supp}(Y'_k)$, where the supports are taken with respect to the FDD (F_n^0), and moreover $V_k \subseteq Y'_k$. Since the FDD (F_n^0) has constant C, the natural projection $W_k \to V_k$ has norm at most $C+1$. Since $S_k - \mathrm{Id}_{W_k}$ is the composition of this projection and $(T^{-1})_{\restriction V_k} - \mathrm{Id}_{V_k}$, we deduce that $\|S_k - \mathrm{Id}_{W_k}\| \leqslant \varepsilon_k/2$.

We are now ready to prove that the FDD (G_k) is $\mathcal{H}$-good. Let A be an infinite subset of $\mathbb{N}$, and let $k \in A$; we want to prove that $[G_l \mid l \in A] \in \mathcal{U}_k$. We have

$$[G_l \mid l \in A] = S_k\big([G_l \mid l \in A, l < k] \oplus H_k \oplus [H_l \mid l \in A,\, l > k]\big),$$

and

$$[G_l \mid l \in A, l < k] \oplus H_k \oplus [H_l \mid l \in A,\, l > k] \\ \in \big[[G_l \mid l \in A, l < k] \oplus H_k,\ [G_l \mid l \in A,\, l < k] \oplus U'_k\big],$$

so using the fact that $\|S_k - \mathrm{Id}_{W_k}\| < \varepsilon_k$, we get

$$[G_l \mid l \in A] \in \big[[G_l \mid l \in A,\, l < k] \oplus H_k,\ [G_l \mid l \in A,\, l < k] \oplus U'_k\big]_{\varepsilon_k}.$$

By construction, the latter basic open set is contained in $\mathcal{U}_k$, as desired. ∎

In the rest of this section, we introduce sufficient conditions for being a D-family, which will be convenient for applications.

3.2. Wijsman and slice topologies

Let X be a Banach space with a fixed norm $\|\cdot\|$. For N an equivalent norm on X, for $Y \in \mathrm{Sub}(X)$ and for $x \in X$, we denote by $N_{X/Y}(x)$ the norm of the class of x in the quotient X/Y, when this quotient is equipped with the corresponding quotient norm. Thus, we have an injective mapping

$$\varphi_N : \mathrm{Sub}(X) \to \mathbb{R}^X, \quad Y \mapsto N_{X/Y}.$$

The *Wijsman topology* associated to N on $\mathrm{Sub}(X)$ is the topology obtained by pulling back through φ_N the product topology on $\mathbb{R}^X$. For this topology, a net (Y_λ) of elements of $\mathrm{Sub}(X)$ converges to $Y \in \mathrm{Sub}(X)$ if and only if for every $x \in X$, $N_{X/Y_\lambda}(x) \to N_{X/Y}(x)$. In general, this topology depends on the choice of the equivalent norm N (see [8, Section 2.4]).

The *slice topology* on $\mathrm{Sub}(X)$ is the topology generated by sets of the form $\{Y \in \mathrm{Sub}(X) \mid Y \cap U \neq \emptyset\}$ and $\{Y \in \mathrm{Sub}(X) \mid \delta_{\|\cdot\|}(Y, C) > 0\}$, where U ranges over open subsets of X, C ranges over non-empty bounded closed convex subsets of X, and $\delta_{\|\cdot\|}(Y, C) = \inf_{x\in C,\, y\in Y} \|x - y\|$. The name *slice topology* comes from the fact that in the previous definition, it is actually enough to make C range over slices of closed balls, that is, non-empty sets of the form $\{x \in X \mid \|x\| \leqslant r,\ x^*(x) \geqslant a\}$, where $r > 0$, $x^* \in X^*$, and $a \in \mathbb{R}$ (see [8, Lemma 2.4.4]). It is easy to see that the slice topology on $\mathrm{Sub}(X)$ only depends on the isomorphic structure of X, but not on the norm.

The main properties of the Wijsman and the slice topologies can be found in [8]. We gather some useful ones below.

Theorem 3.12 (see [8]). (1) *If X is separable, then all the Wijsman topologies on* $\mathrm{Sub}(X)$ *associated to equivalent norms are Polish.*

(2) *If X is separable and has separable dual, then the slice topology on* $\mathrm{Sub}(X)$ *is Polish.*

(3) *The slice topology on* $\mathrm{Sub}(X)$ *is the coarsest topology refining all the Wijsman topologies associated to equivalent norms on X.*

(4) (Hess' theorem) *If X is separable, then the Borel σ-algebra associated with any Wijsman topology on* $\mathrm{Sub}(X)$ *coincides with the Effros Borel structure on this set.*

(5) *If X is separable and has separable dual, then the Borel σ-algebra associated with the slice topology on* $\mathrm{Sub}(X)$ *coincides with the Effros Borel structure on this set.*

These topologies are easier to manipulate than the Ellentuck topology. However, we have the following result:

Proposition 3.13. *The Ellentuck topology on* $\mathrm{Sub}(X)$ *is finer than the slice topology. In particular, it is finer than all the Wijsman topologies associated to equivalent norms.*

Proof. Fix a non-empty open subset U of X. We will show that $\mathcal{U} = \{Y \in \mathrm{Sub}(X) \mid Y \cap U \neq \emptyset\}$ is Ellentuck-open. For this, consider $Y \in \mathcal{U}$. We fix $x_0 \in U \cap Y$ and $\varepsilon > 0$ such that $\mathrm{B}(x_0, \varepsilon\|x_0\|) \subseteq U$; we will show that $[\mathbb{R}x_0, Y]_\varepsilon \subseteq \mathcal{U}$. Let $Z \in [\mathbb{R}x_0, Y]_\varepsilon$, and fix $Z' \in [\mathbb{R}x_0, Y]$ and an isomorphism $T : Z' \to Z$ with $\|T - \mathrm{Id}_{Z'}\| < \varepsilon$. If $x_0 \neq 0$, then

$\|T(x_0) - x_0\| < \varepsilon\|x_0\|$, so by the choice of ε, we have $T(x_0) \in U \cap Z$; this remains true if $x_0 = 0$. This shows that $Z \in \mathcal{U}$.

Now fix a non-empty bounded closed convex subset C of X. We will show that $\mathcal{V} = \{Y \in \mathrm{Sub}(X) \mid \delta_{\|\cdot\|}(Y, C) > 0\}$ is Ellentuck-open. For this, we fix $Y \in \mathcal{V}$ and we show that $[\{0\}, Y]_\varepsilon \subseteq \mathcal{V}$ for small enough ε. More precisely, let $\eta = \delta_{\|\cdot\|}(Y, C)$ and let $R \geqslant 1$ be such that $C \subseteq \mathrm{B}(0, R - \eta)$. Then we take $\varepsilon = \frac{\eta}{4R}$ (so in particular $\varepsilon \leqslant 1/2$). Let $Z \in [\{0\}, Y]_\varepsilon$, and fix $Z' \in [\{0\}, Y]$ and an isomorphism $T : Z' \to Z$ with $\|T - \mathrm{Id}_{Z'}\| < \varepsilon$. For any $x \in Z'$, it is enough to show that $\delta_{\|\cdot\|}(T(x), C) \geqslant \eta/2$. If $\|x\| > 2R$, then $\|T(x)\| \geqslant \|x\| - \|T(x) - x\| \geqslant \|x\|/2 > R$, so $\delta_{\|\cdot\|}(T(x), C) \geqslant \eta$. If $\|x\| \leqslant 2R$, then $\|T(x) - x\| \leqslant \eta/2$. And since $x \in Y$, we have $\delta_{\|\cdot\|}(x, C) \geqslant \eta$, so $\delta_{\|\cdot\|}(T(x), C) \geqslant \eta/2$. ∎

Corollary 3.14. *Consider $\mathcal{H} \subseteq \mathrm{Sub}^\infty(X)$ satisfying the following two properties:*

(1) *for every $Y \in \mathrm{Sub}^\infty(X)$ and every $F \in \mathrm{Sub}^{<\infty}(X)$, we have $Y \in \mathcal{H}$ if and only if $Y + F \in \mathcal{H}$;*

(2) *$\mathcal{H}$, seen as a subset of $\mathrm{Sub}(X)$, is G_δ for one of the Wijsman topologies, or for the slice topology.*

Then $\mathcal{H}$ is a D-family of subspaces of X.

3.3. Degrees

Degrees will be our main way of defining D-families throughout this paper. A degree allows one to define a notion of largeness on the class of all Banach spaces, and this notion gives rise to a D-family when restricted to the set of subspaces of some fixed Banach space.

We define an *approximation pair* as a pair (X, F) where X is a (finite- or infinite-dimensional) Banach space, and F is a finite-dimensional subspace of X. We denote by AP the class of approximation pairs. If $(X, F), (Y, G) \in \mathrm{AP}$, a *morphism* from (X, F) to (Y, G) is a pair $\varphi = (S, T)$, where $S : G \to F$ and $T : X \to Y$ are operators that make the following diagram commute:

$$\begin{array}{ccc} F & \xhookrightarrow{\iota} & X \\ {\scriptstyle S}\uparrow & & \downarrow{\scriptstyle T} \\ G & \xhookrightarrow{\iota} & Y \end{array}$$

where the ι's stand for inclusions. The *norm* of the morphism φ is defined as $\|\varphi\| = \|S\| \cdot \|T\|$ if $G \neq \{0\}$, and $\|\varphi\| = 1$ if $G = \{0\}$.

Definition 3.15. A *degree* is a mapping $d : \mathrm{AP} \to \mathbb{R}_+$ for which there exists $K_d : [1, \infty) \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

- K_d is non-decreasing in each variable;
- for all $t \in \mathbb{R}_+$, $\lim_{s \to 1} K_d(s, t) = t$;

and for every $(X, F), (Y, G) \in \mathrm{AP}$ and every morphism $\varphi : (X, F) \to (Y, G)$, we have

$$d(Y, G) \leqslant K_d(\|\varphi\|, d(X, F)).$$

Definition 3.16. Given a degree d, we say that a Banach space X is *d-small* if $\sup_{F \in \mathrm{Sub}^{<\infty}(X)} d(X, F) < \infty$, while X is *d-large* otherwise.

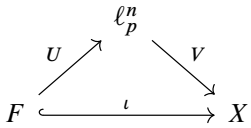
For most degrees we will consider, the value of $d(X, F)$ will actually only depend on F. Degrees satisfying this property will be called *internal degrees*, and $d(X, F)$ will simply be denoted by $d(F)$. To verify that $d : \mathrm{Ban}^{<\infty} \to \mathbb{R}_+$ is an internal degree, it is enough to find $K_d : [1, \infty) \times \mathbb{R}_+ \to \mathbb{R}_+$ as above such that for every embedding $S : G \to F$ between two finite-dimensional spaces, we have $d(G) \leqslant K_d(\|S\| \cdot \|S^{-1}\|, d(F))$ (where S^{-1} is defined on $S(G)$, and with the convention that $\|S\| \cdot \|S^{-1}\| = 1$ when $G = \{0\}$).

Examples 3.17. (1) Let $d(F) = \dim(F)$. Then d is an internal degree, witnessed by $K_d(s, t) = t$. A space is d-small if and only if it is finite-dimensional.

(2) Let $d(F) = d_{BM}(F, \ell_2^{\dim(F)})$. Then d is an internal degree, witnessed by $K_d(s, t) = st$. A space is d-small if and only if it is Hilbertian (a consequence of Kwapień's theorem [36]).

(3) Fix $1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty$. Let $d(F)$ be the type p constant (resp. the cotype q constant) of F. Then d is an internal degree, witnessed by $K_d(s, t) = st$. A space is d-small if and only if it has type p (resp. cotype q). If X is d-small, then $\sup_{F \in \mathrm{Sub}^{<\infty}(X)} d(F)$ is the type p constant (resp. the cotype q constant) of X.

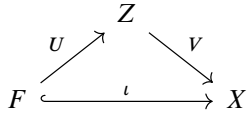
(4) Fix $1 \leqslant p \leqslant \infty$. For $(X, F) \in \mathrm{AP}$, define $d(X, F)$ as the infimum of the M's for which the canonical inclusion of F into X M-factorizes through some ℓ_p^n, meaning there exist $n \in \mathbb{N}$ and operators $U : F \to \ell_p^n$ and $V : \ell_p^n \to X$ with $\|U\| \cdot \|V\| = M$, making the following diagram commute:



Then d is a degree, witnessed by $K_d(s, t) = st$. By [38, Theorem 4.3] (and the classical fact that ℓ_2^n's are uniformly complemented in L_p, $1 < p < \infty$),

- if $1 < p < \infty$, a space is d-small if and only if it is either an $\mathcal{L}_p$-space, or a Hilbertian space;
- if $p = 1$ or $p = \infty$, a space is d-small if and only if it is an $\mathcal{L}_p$-space.

(5) For $(X, F) \in \mathrm{AP}$, define $d(X, F)$ as the infimum of the M's for which there exist a space Z with a 1-unconditional basis and operators $U : F \to Z$ and $V : Z \to X$ with $\|U\| \cdot \|V\| = M$, making the following diagram commute:



Then d is a degree, witnessed by $K_d(s,t) = st$. A space is d-small if and only if it has Gordon–Lewis local unconditional structure (GL-lust) [24]. If X is d-small, then $\sup_{F \in \mathrm{Sub}^{<\infty}(X)} d(X,F)$ is the GL-lust constant of X.

(6) For $(X,F) \in \mathrm{AP}$, define $d(X,F)$ as the infimum of the K's such that F is K-complemented in X. Then d is not a degree. To see this, pick $F = G \subset X \subset Y$ in such a way that F is 1-complemented in X but not n-complemented in Y, so that $d(X,F) = 1$ and $d(Y,G) \geqslant n$; and take $S = \mathrm{Id}_F$, T the canonical inclusion of X into Y. This may be surprising, in view of the fact that a space is Hilbertian if and only if there exists $K \geq 1$ such that all its finite-dimensional subspaces are K-complemented in it (see [1, Theorem 12.42]), i.e., the Hilbertian spaces would be characterized as the d-small spaces if d were a degree.

Remark 3.18. The notion of degree is closely related to Pietsch's classical theory of operator ideals [47]. Recall [47, Chapter 6] that a *quasi-norm* on an operator ideal $\mathfrak{A}$ is a mapping $A : \mathfrak{A} \to \mathbb{R}_+$ such that

(1) $A(\mathrm{Id}_X) = 1$ whenever X is 1-dimensional,

(2) there exists a constant $K \geqslant 1$ such that $A(S+T) \leqslant K(A(S) + A(T))$ whenever S, T belong to $\mathfrak{A}$ and $S + T$ makes sense,

(3) $A(U \circ T \circ S) \leqslant \|U\| \cdot A(T) \cdot \|S\|$ whenever T belongs to $\mathfrak{A}$ and $U \circ T \circ S$ makes sense.

Hence, to every quasi-normed operator ideal $(\mathfrak{A}, A)$, we can associate a degree d_A defined by $d_A(X,F) = A(\iota_{F,X})$ for every approximation pair (X,F), where $\iota_{F,X}$ denotes the inclusion map of F into X. It is interesting to observe that all examples of degrees given in Examples 3.17 come in this fashion:

(1) The dimension is the degree associated to the ideal of *nuclear operators* (see [47, §6.3.1]); this is a consequence of Auerbach's lemma.

(2) The internal degree $d(F) = d_{BM}(F, \ell_2^{\dim(F)})$ is the degree associated to the ideal of *Hilbert operators* (see [47, §6.6.1]).

(3) The type p and cotype q constants are the degrees associated to the ideals of *type p* and *cotype q operators*, normed with the type p and cotype q constants, respectively (see [48, Section 3.a]).

(4) The degree defined in Examples 3.17 (4) is the degree associated to the ideal of *discretely p-factorable* operators (see [47, §19.3.11]).

(5) The degree defined in Examples 3.17 (5) is the degree associated to the ideal of *σ-nuclear operators* (see [47, §23.2.1]); this is a consequence of [47, Theorem 23.2.5].

We warn the reader that given a quasi-normed operator ideal $(\mathfrak{A}, A)$, the class of d_A-small spaces does not in general coincide with the space ideal associated to $\mathfrak{A}$ (see [47, §2.1.2]). This is so for instance for examples (4) and (5) above.

We will not further develop the link between degrees and quasi-normed operator ideals in this paper, as this would be of limited practical use. Indeed, exhibiting a degree adapted

to a given situation is in general much easier than exhibiting a quasi-normed operator ideal, as shown by the examples above.

We can also define a notion of asymptotic smallness:

Definition 3.19. Let d be a degree. A Banach space X is said to be *asymptotically d-small* if there exists a constant K such that for every $n \in \mathbb{N}$, there exists a finite-codimensional subspace $X_n \subseteq X$ such that every n-dimensional subspace $F \subseteq X_n$ satisfies $d(X, F) \leqslant K$.

When $d(X, F) = d_{BM}(F, \ell_2^{\dim(F)})$, asymptotically d-small Banach spaces are exactly asymptotically Hilbertian Banach spaces (see Definition 1.12).

If d is an internal degree, then a subspace of a d-small space is itself d-small, and a subspace of an asymptotically d-small space is itself asymptotically d-small. This is not true in general; for example, $L_p([0, 1])$ is an $\mathcal{L}_p$-space, and for $1 \leqslant p \neq 2 \leqslant \infty$, the only non-Hilbertian subspaces of $L_p([0, 1])$ which are $\mathcal{L}_p$ are the complemented ones [38]. Similarly, the property of having Gordon–Lewis local unconditional structure is not stable under passing to subspaces; consider ℓ_p spaces for $1 \leqslant p \neq 2 < \infty$, a consequence of, e.g., [35].

Remark 3.20. A useful property of degrees is that for $F \subseteq G \subseteq Y \subseteq X$, where the spaces F and G are finite-dimensional and X and Y are arbitrary, we have $d(X, F) \leqslant d(Y, G)$. To see this, just consider the morphism $(\mathrm{Id}_F, \mathrm{Id}_Y)$ from (Y, G) to (X, F).

In the rest of this subsection, we fix a degree d.

Lemma 3.21. *For every $n \in \mathbb{N}$, there exists a constant $C_d(n)$ such that for every $(X, F) \in$ AP with $\dim(F) = n$, we have $d(X, F) \leqslant C_d(n)$. In particular, every finite-dimensional space is d-small.*

Proof. Let $(X, F) \in$ AP with $\dim(F) = n$. If $n \geq 1$, then let $T : \ell_1^n \to F$ be an n-isomorphism. Then (T^{-1}, T) is a morphism from (ℓ_1^n, ℓ_1^n) to (X, F), with norm at most n. So, letting $C_d(n) = K_d(n, d(\ell_1^n, \ell_1^n))$, it follows that $d(X, F) \leqslant C_d(n)$. The proof of the case $n = 0$ is similar, replacing ℓ_1^n by $\{0\}$. ∎

Lemma 3.22. *The properties of being d-small, d-large, and asymptotically d-small are invariant under isomorphism.*

Proof. Let X and Y be Banach spaces, and $T : X \to Y$ be an isomorphism. First suppose that X is d-small, and let $K = \sup_{F \in \mathrm{Sub}^{<\infty}(X)} d(X, F)$. Let $G \in \mathrm{Sub}^{<\infty}(Y)$. Then $((T^{-1})_{\restriction G}, T)$ is a morphism from $(X, T^{-1}(G))$ to the pair (Y, G), so

$$d(Y, G) \leqslant K_d\big(\|(T^{-1})_{\restriction G}\| \cdot \|T\|, d(X, T^{-1}(G))\big) \leqslant K_d(\|T^{-1}\| \cdot \|T\|, K).$$

This bound does not depend on G, so Y is d-small.

Now suppose that X is asymptotically d-small and fix a constant K witnessing it. We will show that Y is asymptotically d-small, witnessed by the constant $L =$

$K_d(\|T^{-1}\| \cdot \|T\|, K)$. Let $n \in \mathbb{N}$. There exists a finite-codimensional subspace $X_n \subseteq X$ such that for every n-dimensional subspace $F \subseteq X_n$, we have $d(X, F) \leqslant K$. Let $Y_n = T(X_n)$, and let $G \subseteq Y_n$ be an n-dimensional subspace. Then $((T^{-1})_{\restriction G}, T)$ is a morphism from $(X, T^{-1}(G))$ to (Y, G), so $d(Y, G) \leqslant K_d(\|T^{-1}\| \cdot \|T\|, d(X, T^{-1}(G)))$. Since $T^{-1}(G) \subseteq X_n$, we have $d(X, T^{-1}(G)) \leqslant K$, so $d(Y, G) \leqslant L$, as desired. ∎

Lemma 3.23. (1) *A complemented subspace of a d-small space is d-small.*

(2) *A complemented subspace of an asymptotically d-small space is asymptotically d-small.*

Proof. Fix a Banach space X, a complemented subspace Y of X, and a projection $P : X \to Y$.

(1) Suppose X is d-small, and let $K = \sup_{F \in \mathrm{Sub}^{<\infty}(X)} d(X, F)$. Let $F \subseteq Y$ be a finite-dimensional subspace. Then (Id_F, P) is a morphism from (X, F) to (Y, F), so we have $d(Y, F) \leqslant K_d(\|P\|, d(X, F)) \leqslant K_d(\|P\|, K)$. Hence Y is d-small.

(2) Suppose X is asymptotically d-small, witnessed by a constant K. We will show that Y is asymptotically d-small, witnessed by the constant $K_d(\|P\|, K)$. Let $n \in \mathbb{N}$, and fix a finite-codimensional subspace $X_n \subseteq X$ such that for every n-dimensional subspace $F \subseteq X_n$, we have $d(X, F) \leqslant K$. Let $Y_n = X_n \cap Y$. For an n-dimensional subspace $F \subseteq Y_n$, (Id_F, P) is a morphism from (X, F) to (Y, F), so we have $d(Y, F) \leqslant K_d(\|P\|, d(X, F)) \leqslant K_d(\|P\|, K)$, as desired. ∎

Lemma 3.24. *Let X be a Banach space, and Y be a finite-codimensional subspace of X. Then:*

(1) *X is d-small iff Y is d-small;*

(2) *X is asymptotically d-small iff Y is asymptotically d-small.*

Proof. Since Y is complemented in X, we know by Lemma 3.23 that if X is d-small (resp. asymptotically d-small), then so is Y. So in both cases, we just have one direction to show.

(1) Suppose that Y is d-small, and let $K = \sup_{G \in \mathrm{Sub}^{<\infty}(Y)} d(Y, G)$. By Lemma 3.21, we can suppose that X is infinite-dimensional. We denote by k the codimension of Y in X. Recall that by [21, Lemma 3], every k-codimensional subspace of X is $A(k)$-isomorphic to Y, where the constant $A(k)$ only depends on k.

Let $F \subseteq X$ be finite-dimensional; we want to bound $d(X, F)$. Find a subspace $Z \subseteq X$ with codimension k containing F. Let $T : Z \to Y$ be an $A(k)$-isomorphism. We have $d(Y, T(F)) \leqslant K$. Moreover, $(T_{\restriction F}, T^{-1})$ is a morphism from $(Y, T(F))$ to (X, F), so $d(X, F) \leqslant K_d(\|T\| \cdot \|T^{-1}\|, K) \leqslant K_d(A(k), K)$, as desired.

(2) Suppose that Y is asymptotically d-small. Then there exist a constant K and finite-codimensional subspaces $Y_n \subseteq Y$ for all n such that for every n-dimensional subspace $F \subseteq Y_n$, we have $d(Y, F) \leqslant K$. In particular, for such an F, we also have $d(X, F) \leqslant K$, showing that X is asymptotically d-small. ∎

Proposition 3.25. *Let X be a Banach space, and $\mathcal{H}$ be the set of subspaces of X that are d-large. Then $\mathcal{H}$ is a D-family.*

Proof. Lemma 3.21 shows that $\mathcal{H}$ contains only infinite-dimensional subspaces. The stability of $\mathcal{H}$ under finite-dimensional modifications comes from Lemma 3.24. Now we need to prove that $\mathcal{H}$ is Ellentuck-G_δ. For every $n \in \mathbb{N}$, let $\mathcal{U}_n = \{Y \in \mathrm{Sub}(X) \mid \exists F \in \mathrm{Sub}^{<\infty}(Y)\ d(Y,F) > n\}$, so that $\mathcal{H} = \bigcap_{n\in\mathbb{N}} \mathcal{U}_n$. We will show that all the $\mathcal{U}_n$'s are open.

Fix $n \in \mathbb{N}$ and $Y \in \mathcal{U}_n$. Let $F \in \mathrm{Sub}^{<\infty}(Y)$ be such that $d(Y,F) > n$. We know that $\lim_{s\to 1} K_d(s,n) = n$, so there exists $\varepsilon \in (0,1)$ such that $K_d(\frac{1+\varepsilon}{1-\varepsilon}, n) < d(Y,F)$. We will show that $[F,Y]_\varepsilon \subseteq \mathcal{U}_n$. Let $Z \in [F,Y]_\varepsilon$, and let $Z' \in [F,Y]$ and $T : Z' \to Z$ be an isomorphism such that $\|T - \mathrm{Id}_{Z'}\| < \varepsilon$. Then $\|T\| \leqslant 1+\varepsilon$ and $\|T^{-1}\| \leqslant \frac{1}{1-\varepsilon}$. So $(T_{\restriction F}, T^{-1})$ is a morphism of norm at most $\frac{1+\varepsilon}{1-\varepsilon}$ from $(Z, T(F))$ to (Y,F). Thus, $d(Y,F) \leqslant K_d(\frac{1+\varepsilon}{1-\varepsilon}, d(Z,T(F)))$. If we had $d(Z,T(F)) \leqslant n$, we would have $d(Y,F) \leqslant K_d(\frac{1+\varepsilon}{1-\varepsilon}, n)$, contradicting the choice of ε. So $d(Z,T(F)) > n$, witnessing $Z \in \mathcal{U}_n$. ∎

Corollary 3.26. *Given a sequence $(d_n)_{n\in\mathbb{N}}$ of degrees and a space X, the family of subspaces of X that are large for all the d_n's is a D-family, and in the same way, for fixed $N \in \mathbb{N}$, the family of subspaces of X that are large for at least one d_n, $n \leqslant N$, is also a D-family.*

Proof. Since the class of G_δ subsets of a topological space is closed under countable intersections and under finite unions, this is a consequence of Proposition 3.25. ∎

For instance, for $2 < q_0 \leqslant \infty$ fixed, the family of subspaces of X that do not have any cotype $q < q_0$ is a D-family.

If d is a degree and X a Banach space, the D-family defined in Proposition 3.25 will be denoted by $\mathcal{H}_d^X$, or by $\mathcal{H}_d$ when there is no ambiguity. An $\mathcal{H}_d$-good FDD will simply be called *d-good*. In the case of families defined by a degree, we have a useful strengthening of the notion of good FDD's:

Definition 3.27. An FDD $(F_n)_{n\in\mathbb{N}}$ of a Banach space X is *d-better* if $d(X, F_n) \xrightarrow[n\to\infty]{} \infty$.

This implies that (F_n) is a d-good FDD. Indeed, if $A \subseteq \mathbb{N}$ is infinite, then for every $n \in A$, we have $d([F_m \mid m \in A], F_n) \geqslant d(X, F_n)$. Below we prove a weak converse to this; it can be seen as the d-better version of Lemma 3.8.

Lemma 3.28. *Let $(F_n)_{n\in\mathbb{N}}$ be an FDD of a d-large Banach space X. Then there exists a blocking $(G_n)_{n\in\mathbb{N}}$ of (F_n) which is a d-better FDD.*

Proof. Let C be the constant of the FDD (F_n). For each $k \in \mathbb{N}$, let $X_k = [F_l \mid l \geqslant k]$ and let $P_k : X \to X_k$ the projection associated to the FDD. We build (G_n) by induction as follows. Suppose that the G_m's have been built for $m < n$, and let $k_n = \max \mathrm{supp}(G_{n-1}) + 1$ if $n \geqslant 1$, and $k_n = 0$ otherwise. The space X_{k_n} is d-large, so there exists $H_n \in \mathrm{Sub}^{<\infty}(X_{k_n})$ such that $d(X_{k_n}, H_n) \geqslant n$. Now consider an isomorphism $T_n : X_{k_n} \to X_{k_n}$

such that $\|T_n\| \cdot \|T_n^{-1}\| \leqslant 2$ and $T_n(H_n) \subseteq [F_k \mid k_n \leqslant k < k_{n+1}]$ for some k_{n+1}. Let $G_n = [F_k \mid k_n \leqslant k < k_{n+1}]$. This finishes the construction of (G_n).

To prove that (G_n) is d-better, fix n and consider the morphism $((T_n)_{\restriction H_n}, T_n^{-1} \circ P_{k_n})$ from (X, G_n) to (X_{k_n}, H_n). It has norm at most $2(1 + C)$, so

$$n \leqslant d(X_{k_n}, H_n) \leqslant K_d(2(1 + C), d(X, G_n)).$$

In particular, for every constant K, as long as $n > K_d(2(1 + C), K)$, we have $d(X, G_n) > K$. This shows that $d(X, G_n) \xrightarrow[n\to\infty]{} \infty$. ∎

As an illustration, note that if $d(F)$ is the dimension of F, then any FDD is d-good, while a d-better FDD is an FDD where the dimensions of the summands tend to infinity.

4. The first dichotomy

In this section, we generalize Gowers' Ramsey-type Theorem 2.10 to D-families. As an application, we prove our first dichotomy (Theorem 4.4), a local version of Gowers' first dichotomy (Theorem 1.20).

4.1. A local version of Gowers' Ramsey-type theorem

In this subsection, we fix a Banach space X, and a D-family $\mathcal{H}$ of subspaces of X. We work in the approximate Gowers space $\mathcal{G}_{\mathcal{H}} = (\mathcal{H}, S_X, \delta_{\|\cdot\|}, \subseteq, \subseteq^*, \in)$ defined in the last section (see Corollary 3.6). Each time we will mention Gowers' game or the asymptotic game, we will be referring to the games relative to this space. Note that Gowers' game relative to this space is in general different from the original game defined by Gowers. For $Y \in \mathcal{H}$, the game G_Y has the following form:

I	Y_0		Y_1	$\dots$
II		y_0		y_1 $\dots$

where the y_n's are elements of S_X, and the Y_n's are elements of $\mathcal{H}$, with the constraint that for all $n \in \mathbb{N}$, $Y_n \subseteq Y$ and $y_n \in Y_n$. The outcome is, as usual, the sequence $(y_n)_{n\in\mathbb{N}} \in (S_X)^{\mathbb{N}}$.

Our local version of Gowers' Theorem 2.10 is the following:

Theorem 4.1. *Let $\mathcal{X} \subseteq (S_X)^{\mathbb{N}}$ be analytic, let $Y \in \mathcal{H}$, let Δ be a sequence of positive real numbers and let $\varepsilon > 0$. Then there exists $Z \in \mathcal{H}_{\restriction Y}$ such that either*

- *Z has an $\mathcal{H}$-good FDD $(G_n)_{n\in\mathbb{N}}$, with constant at most $1 + \varepsilon$, and all of whose normalized block-sequences belong to $\mathcal{X}^c$, or*
- *player **II** has a strategy in G_Z to reach $(\mathcal{X})_\Delta$.*

Moreover,

- *if $\mathcal{H} = \mathcal{H}_d$ for some degree d, then we can even assume that the FDD (G_n) is d-better;*

- *if Y comes with a fixed FDD $(F_n)_{n\in\mathbb{N}}$, then we can also assume that (G_n) is a block-FDD of (F_n).*

Gowers' Theorem 2.10 is just the special case of the last theorem when $\mathcal{H} = \mathrm{Sub}^\infty(X)$ (which is a D-family, the family of d-large subspaces of X for the internal degree $d(F) = \dim(F)$).

Proof of Theorem 4.1. We start with the general case; the "moreover" part will be dealt with separately at the end of the proof. We proceed as in the proof of Theorem 2.10: we apply the abstract Gowers Theorem 2.9 to the approximate Gowers space $\mathcal{G}_\mathcal{H}$, endowed with the system of relatively compact sets $(\mathcal{K}_X, \oplus_X)$ defined in Section 2.2 (recall that the elements of $\mathcal{K}_X$ are the unit spheres of finite-dimensional subspaces of X, and that $S_F \oplus_X S_G = S_{F+G}$). In case we get $Z \in \mathcal{H}_{\restriction Y}$ such that player **II** has a strategy in G_Z to reach $(\mathcal{X})_\Delta$, we are done. So we now suppose that there exists $U \in \mathcal{H}_{\restriction Y}$ such that player **I** has a strategy τ in SF_U to build a sequence $(K_n)_{n\in\mathbb{N}}$ with $\mathrm{bs}((K_n)_{n\in\mathbb{N}}) \subseteq \mathcal{X}^c$. We can assume that the strategy τ is such that for every run of the game SF_U,

I	U_0		U_1	$\dots$
II		S_{G_0}		S_{G_1} $\dots$

played according to τ, we have $[G_i \mid i < n] \cap U_n = \{0\}$ and the natural projection $[G_i \mid i < n] \oplus U_n \to [G_i \mid i < n]$ has norm at most $1 + \varepsilon$. By Lemma 3.8, it is enough to build an FDD $(G_n)_{n\in\mathbb{N}}$ of a subspace of U, with constant at most $1 + \varepsilon$, such that $[G_n \mid n \in \mathbb{N}] \in \mathcal{H}$, and all of whose normalized block-sequences belong to $\mathcal{X}^c$.

Let $(\mathcal{U}_n)_{n\in\mathbb{N}}$ be a decreasing sequence of Ellentuck-open subsets of $\mathrm{Sub}(X)$ such that $\bigcap_{n\in\mathbb{N}} \mathcal{U}_n = \mathcal{H}$. We describe a run $(U_0, S_{G_0}, U_1, S_{G_1}, \dots)$ of the game SF_U where **I** plays according to τ, by describing the moves of **II**. Suppose that $U_0, S_{G_0}, \dots, U_{n-1}, S_{G_{n-1}}$ have just been played. According to the strategy τ, player **I** plays U_n, a finite-codimensional subspace of U. Since $U \in \mathcal{H}$, we have $[G_i \mid i < n] \oplus U_n \in \mathcal{H} \subseteq \mathcal{U}_n$. So we can find a finite-dimensional subspace $G_n \subseteq U_n$ such that $[[G_i \mid i \leqslant n], [G_i \mid i < n] \oplus U_n] \subseteq \mathcal{U}_n$. We make **II** play S_{G_n}, finishing the construction.

Exactly as in the proof of Lemma 3.9, we can show that $(G_n)_{n\in\mathbb{N}}$ is an FDD of a subspace $Z \in \mathcal{H}_{\restriction U}$, with constant at most $1 + \varepsilon$. Since the game SF_U has been played according to τ, we have $\mathrm{bs}((S_{G_n})_{n\in\mathbb{N}}) \subseteq \mathcal{X}^c$, which exactly means that every normalized block-sequence of the FDD (G_n) is in $\mathcal{X}^c$.

If $\mathcal{H} = \mathcal{H}_d$ for some degree d, then by Lemma 3.28, we can replace the FDD (G_n) with one of its blockings which is d-better, and this blocking will still satisfy the conclusion of the theorem.

We now prove the refinement of the theorem in the case where Y has a fixed FDD $(F_n)_{n\in\mathbb{N}}$. Without loss of generality, we assume that the sequence $\Delta = (\Delta_n)_{n\in\mathbb{N}}$ is decreasing, $\Delta_0 < 1$, and $(1 + \varepsilon/2)\frac{1+\Delta_0}{1-\Delta_0} \leqslant 1 + \varepsilon$. The general case applied to $\mathcal{X}' = (\mathcal{X})_{\Delta/2}$ (which is still analytic) and to the sequence $\Delta' = \Delta/2$ gives a $U \in \mathcal{H}_{\restriction Y}$ such that either player **II** has a strategy in G_U to reach $(\mathcal{X}')_{\Delta'}$, or U has an $\mathcal{H}$-good FDD $(K_n)_{n\in\mathbb{N}}$ with constant at most $1 + \varepsilon/2$ all of whose normalized block-sequences belong to $(\mathcal{X}')^c$. In the first case, we are done, since $(\mathcal{X}')_{\Delta'} \subseteq \mathcal{X}_\Delta$. In the second case, we apply

Lemma 3.11 to $Y_k = Y$, $(F_n^k)_{n\in\mathbb{N}} = (F_n)_{n\in\mathbb{N}}$, $U_k = [K_n \mid n \geqslant k]$ for every $k \in \mathbb{N}$. This gives us a $Z \in \mathcal{H}_{\restriction Y}$ spanned by an $\mathcal{H}$-good block-FDD $(G_n)_{n\in\mathbb{N}}$ of (F_n), and an isomorphic embedding $T : Z \to U$ such that for every $n \in \mathbb{N}$, $\|(T - \mathrm{Id}_Z)_{\restriction [G_k | k\geqslant n]}\| < \Delta'_n/2$ and $T(G_n) \subseteq U_n$. Modifying T if necessary, we can even assume that for every n, $T(G_n)$ has finite support on the FDD (K_n) (of course, doing a modification does not necessarily preserve the FDD $(T(G_n))_{n\in\mathbb{N}}$ being $\mathcal{H}$-good, but this fact will not matter in this proof). Since $T(G_n) \subseteq U_n$ for every n, we have $\lim_{n\to\infty} \min \mathrm{supp}(G_n) = \infty$ (the supports being taken with respect to the FDD (K_n)). Thus, extracting a subsequence if necessary, we can assume that $(T(G_n))$ is a block-FDD of (K_n).

We now prove that the FDD (G_n) is as desired. Recall that $\|T - \mathrm{Id}_Z\| \leqslant \Delta_0$, so $\|T\| \leqslant 1 + \Delta_0$ and $\|T^{-1}\| \leqslant \frac{1}{1-\Delta_0}$. Since $(T(G_n))$ is a block-FDD of (K_n), which has constant at most $1 + \varepsilon/2$, we deduce that $(T(G_n))$ also has constant at most $1 + \varepsilon/2$. Hence, (G_n) has constant at most $(1 + \varepsilon/2)\frac{1+\Delta_0}{1-\Delta_0} \leqslant 1 + \varepsilon$, as desired. Now let $(x_i)_{i\in\mathbb{N}}$ be a normalized block-sequence of (G_n) ; we prove that $(x_i) \in \mathcal{X}^c$. For every i, $x_i \in [G_n \mid n \geqslant i]$ so $\|T(x_i) - x_i\| \leqslant \Delta'_i/2$. Hence, letting $y_i = \frac{T(x_i)}{\|T(x_i)\|}$, we have $\|x_i - y_i\| \leqslant \Delta'_i$. Observe that $(y_i)_{i\in\mathbb{N}}$ is a normalized block-sequence of $(T(G_n))$, so of (K_n); hence, it is in $(\mathcal{X}')^c$. Thus, (x_i) is in $((\mathcal{X}')^c)_{\Delta'}$, which is contained in $\mathcal{X}^c$. This finishes the proof. ∎

Remark 4.2. The essential difference between Theorem 4.1 and Smythe's local version of Gowers' Ramsey-type theorem proved in [57] is that, in Smythe's theorem, the original Gowers game appears: player **I** is allowed to play whatever subspace he wants, not only elements of $\mathcal{H}$. The cost is that the conditions on the family $\mathcal{H}$ in Smythe's theorem are much more restrictive than in ours. Thus, it is not clear at all that Smythe's theorem could apply to the families we shall consider (for instance, the family of non-Hilbertian subspaces of a given Banach space).

4.2. The dichotomy

We now want to prove a local version of Gowers' first dichotomy (Theorem 1.20), that is, a similar dichotomy where we moreover ensure that the resulting subspace will be in a fixed D-family. To do this, we will need local versions of the two possible conclusions. In particular, we will need a weakening of the notion of HI space.

Definition 4.3. Let X be a Banach space and let $\mathcal{H}$ be a D-family of subspaces of X. We say that a subspace of X is *$\mathcal{H}$-decomposable* if it is equal to the direct sum of two elements of $\mathcal{H}$. The space X is *hereditarily $\mathcal{H}$-indecomposable*, or *$\mathcal{H}$-HI*, if $X \in \mathcal{H}$ and X contains no $\mathcal{H}$-decomposable subspace.

If d is a degree, we call a space X *hereditarily d-indecomposable*, or *d-HI*, if it is hereditarily $\mathcal{H}_d^X$-indecomposable; in other words, if it is d-large and no subspace of it is a direct sum of two d-large subspaces.

In the case where $\mathcal{H} = \mathrm{Sub}^\infty(X)$, i.e. where d is the dimension, we recover the notion of HI spaces.

Theorem 4.4 (The first dichotomy). *Let X be a Banach space, and let $\mathcal{H}$ be a D-family of subspaces of X, containing X. Then there exists $Y \in \mathcal{H}$ such that either*

- *Y has an $\mathcal{H}$-good UFDD, or*
- *Y is hereditarily $\mathcal{H}_{\restriction Y}$-indecomposable.*

Moreover, if X comes with a fixed FDD, then in the first case, the UFDD of Y can be taken as a block-FDD of the FDD of X.

This is a true dichotomy in the sense that both classes it defines are, in some way, hereditary with respect to $\mathcal{H}$ (every block-FDD of a UFDD is a UFDD, and if Y is hereditarily $\mathcal{H}_{\restriction Y}$-indecomposable, then every subspace $Z \in \mathcal{H}_{\restriction Y}$ is hereditarily $\mathcal{H}_{\restriction Z}$-indecomposable); and these classes are obviously disjoint, since a subspace Y with an $\mathcal{H}$-good UFDD has continuum-many decompositions as a direct sum of elements of $\mathcal{H}$.

We spell out the version of the dichotomy when the D-family is induced by a degree d, taking into account Lemma 3.28.

Theorem 4.5 (The first dichotomy for degrees). *Let X be a Banach space, and let d be a degree such that X is d-large. Then there exists a d-large subspace $Y \subseteq X$ which either has a d-better UFDD, or is hereditarily d-indecomposable.*

Proof of Theorem 4.4. We fix a sequence $\Delta = (\Delta_i)_{i\in\mathbb{N}}$ of positive real numbers satisfying the following property: for every normalized basic sequence $(x_i)_{i\in\mathbb{N}}$ in X with constant at most 2, and for every normalized sequence $(y_i)_{i\in\mathbb{N}}$ in X such that $\|x_i - y_i\| \leqslant \Delta_i$ for all $i \in \mathbb{N}$, the sequences (x_i) and (y_i) are 2-equivalent. Let $\mathcal{X}$ be the set of sequences $(x_i)_{i\in\mathbb{N}} \in (S_X)^{\mathbb{N}}$ satisfying the following property: for every $N \in \mathbb{N}$, there exists an eventually null sequence $(a_i)_{i\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ such that $\| \sum_{i \text{ even}} a_i x_i \| > N \| \sum_{i\in\mathbb{N}} a_i x_i \|$. The set $\mathcal{X}$ is a G_δ subset of $(S_X)^{\mathbb{N}}$. We apply Theorem 4.1 to X, to the set $\mathcal{X}$, to the sequence Δ, and to $\varepsilon = 1$.

First case: There exists $Y \in \mathcal{H}$ with an $\mathcal{H}$-good FDD $(F_n)_{n\in\mathbb{N}}$ such that no normalized block-sequence of (F_n) belongs to $\mathcal{X}$. Moreover, if X comes with a fixed FDD, then (F_n) is a block-FDD of the FDD of X.

We then show that (F_n) is a UFDD. For every n, let $y_n \in F_n$, and let $A \subseteq \mathbb{N}$ be infinite and coinfinite, and suppose that $\sum_{n\in\mathbb{N}} y_n$ converges; we will show that $\sum_{n\in A} y_n$ converges. Without loss of generality, we can assume that $0 \in A$. Consider a sequence $0 = n_0 < n_1 < n_2 < \cdots$ of integers such that $A = \bigcup_{i \text{ even}} [\![n_i, n_{i+1} - 1]\!]$. For every $i \in \mathbb{N}$, consider $x_i \in [F_n \mid n_i \leqslant n < n_{i+1}]$ with $\|x_i\| = 1$ and $a_i \in \mathbb{R}$ such that $\sum_{n_i \leqslant n < n_{i+1}} y_n = a_i x_i$. Then $(x_i)_{i\in\mathbb{N}}$ is a normalized block-sequence of (F_n), so it does not belong to $\mathcal{X}$. Hence, there exists $N \in \mathbb{N}$ such that for every $k \leqslant l$,

$$\Big\| \sum_{\substack{k\leqslant i<l \\ i \text{ even}}} a_i x_i \Big\| \leqslant N \Big\| \sum_{k\leqslant i<l} a_i x_i \Big\|.$$

We show that $\sum_{n\in A} y_n$ converges using Cauchy's criterion. Fix $\varepsilon > 0$; there is $n_\varepsilon \in \mathbb{N}$ such that for every $q \geqslant p \geqslant n_\varepsilon$, we have $\| \sum_{p\leqslant n<q} y_n \| \leqslant \varepsilon$; we can moreover assume

that n_ε is one of the n_k's. Fix $q \geqslant p \geqslant n_\varepsilon$. Fixing k and l such that $n_{k-1} \leqslant p < n_k$ and $n_l \leqslant q < n_{l+1}$, we have

$$\begin{aligned}\Big\| \sum_{\substack{p \leqslant n < q \\ n \in A}} y_n \Big\| &\leqslant \Big\| \sum_{p \leqslant n < n_k} y_n \Big\| + \Big\| \sum_{\substack{n_k \leqslant n < n_l \\ n \in A}} y_n \Big\| + \Big\| \sum_{n_l \leqslant n < q} y_n \Big\| \\ &\leqslant 2\varepsilon + \Big\| \sum_{\substack{k \leqslant i < l \\ i \text{ even}}} a_i x_i \Big\| \leqslant 2\varepsilon + N \Big\| \sum_{k \leqslant i < l} a_i x_i \Big\| \\ &= 2\varepsilon + N \Big\| \sum_{n_k \leqslant n < n_l} y_n \Big\| \leqslant (N+2)\varepsilon,\end{aligned}$$

settling this first case.

*Second case: There exists $Y \in \mathcal{H}$ such that player **II** has a strategy in G_Y to reach $(\mathcal{X})_\Delta$.*

We then show that Y is hereditarily $\mathcal{H}_{\restriction Y}$-indecomposable. Fix $U, V \in \mathcal{H}_{\restriction Y}$ and $N \in \mathbb{N}$. It is enough to construct $u \in U$ and $v \in V$ such that $\|u\| > \frac{N}{4}\|u+v\|$. Consider a run of the game G_Y:

I	$Z_0 \cap U$		$Z_1 \cap V$		$Z_2 \cap U$		$Z_3 \cap V$	$\dots$
II		z_0		z_1		z_2		z_3	$\dots$

where **II** plays using a strategy to reach $(\mathcal{X})_\Delta$, and where **I** plays as follows:

- if n is even, **I** plays $Z_n \cap U$ where Z_n is a finite-codimensional subspace of Y such that the natural projection $[z_i \mid i < n] \oplus Z_n \to [z_i \mid i < n]$ has norm at most 2, and $Z_n \subseteq Z_{n-1}$ for $n \geqslant 1$;
- if n is odd, **I** plays $Z_n \cap V$ for Z_n as above.

At the end of the game, player **II** will have built a normalized basic sequence $(z_n)_{n\in\mathbb{N}}$ with constant at most 2, which is in $(\mathcal{X})_\Delta$, and such that $z_n \in U$ for n even, while $z_n \in V$ for n odd.

Now choose a sequence $(z'_n)_{n\in\mathbb{N}} \in \mathcal{X}$ such that for every $n \in \mathbb{N}$, $\|z'_n - z_n\| \leqslant \Delta_n$. Choose $(a_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ eventually null such that $\|\sum_{n \text{ even}} a_n z'_n\| > N \|\sum_{n\in\mathbb{N}} a_n z'_n\|$. By the choice of Δ, the sequences (z_n) and (z'_n) are 2-equivalent, so we have

$$\Big\| \sum_{n \text{ even}} a_n z_n \Big\| \geqslant \frac{1}{2} \Big\| \sum_{n \text{ even}} a_n z'_n \Big\| > \frac{N}{2} \Big\| \sum_{n\in\mathbb{N}} a_n z'_n \Big\| \geqslant \frac{N}{4} \Big\| \sum_{n\in\mathbb{N}} a_n z_n \Big\|.$$

Thus, $u = \sum_{n \text{ even}} a_n z_n$ and $v = \sum_{n \text{ odd}} a_n z_n$ satisfy the desired property. ∎

5. The second dichotomy

In this section, we prove our second dichotomy, a local version of Ferenczi–Rosendal's dichotomy between minimal and tight spaces (Theorem 1.22). We also discuss some consequences, in particular concerning ergodicity.

5.1. The statement of the dichotomy

In this subsection, we state our second dichotomy. Just as for the first one, we first need to provide appropriate local versions of the notions of minimality and tightness. In the whole section, we fix a Banach space X, a D-family $\mathcal{H}$ of subspaces of X, and a degree d.

Definition 5.1. We say that X is *$\mathcal{H}$-minimal* if $X \in \mathcal{H}$ and X isomorphically embeds into every element of $\mathcal{H}$. If $\mathcal{H}$ is induced by the degree d, then we say that X is *d-minimal*.

So X is d-minimal if it is d-large and embeds into any of its d-large subspaces – again note that d-minimality is not a notion of "smallness". In particular, if d is the dimension (or equivalently $\mathcal{H} = \mathrm{Sub}^{\infty}(X)$), we recover the usual notion of minimality. Also observe that if X is $\mathcal{H}$-minimal, then it is separable; this is for instance a consequence of Lemma 3.9.

Definition 5.2. Let $(F_n)_{n\in\mathbb{N}}$ be an FDD of a subspace of X.

(1) The FDD (F_n) is said to be *$\mathcal{H}$-tight* if every $Y \in \mathcal{H}$ is tight in (F_n).

(2) The space X is said to be *$\mathcal{H}$-tight* if $X \in \mathcal{H}$ and X has an $\mathcal{H}$-tight FDD.

If $\mathcal{H}$ is induced by the degree d, then we say that the FDD (F_n) is *d-tight*, and the space X is *d-tight*.

So X is d-tight if it is d-large and has an FDD in which every d-large Banach space is tight. When $d(F)$ is the dimension of F, we recover the usual notion of tight FDD.

Note that $\mathcal{H}$-minimality is a hereditary notion in the sense that if X is $\mathcal{H}$-minimal, then every $Y \in \mathcal{H}$ is $\mathcal{H}_{\restriction Y}$-minimal. The notion of $\mathcal{H}$-tightness is also hereditary in the following sense:

Lemma 5.3. *Let $(F_n)_{n\in\mathbb{N}}$ be an FDD of a subspace of X.*

(1) *If a Banach space Y is tight in (F_n), then it is also tight in all of its block-FDD's.*

(2) *If (F_n) is $\mathcal{H}$-tight, then all of its block-FDD's are $\mathcal{H}$-tight. In particular, if the FDD (F_n) witnesses that X is $\mathcal{H}$-tight, then every $Y \in \mathcal{H}$ generated by a block-FDD of (F_n) is $\mathcal{H}_{\restriction Y}$-tight.*

Proof. We only prove (1), since (2) is an immediate consequence. Let $(G_m)_{m\in\mathbb{N}}$ be a block-FDD of (F_n), and let $I_0 < I_1 < I_2 < \cdots$ be a sequence of non-empty successive intervals witnessing the tightness of Y in (F_n). Observe that every infinite subsequence of (I_i) still witnesses the tightness of Y in (F_n). Thus, without loss of generality, we can assume that for every $m \in \mathbb{N}$, there is at most one $i \in \mathbb{N}$ such that $I_i \cap \mathrm{supp}(G_m) \neq \emptyset$. If there are infinitely many I_i's that intersect no set of the form $\mathrm{supp}(G_m)$, then by tightness, $Y \not\sqsubseteq [G_m \mid m \in \mathbb{N}]$ so Y is tight in (G_m). Otherwise, passing again to a subsequence if necessary, we can assume that for every $i \in \mathbb{N}$, I_i intersects at least one of the $\mathrm{supp}(G_m)$'s. For every $i \in \mathbb{N}$, let $J_i = \{m \in \mathbb{N} \mid I_i \cap \mathrm{supp}(G_m) \neq \emptyset\}$. Then the J_i's are non-empty intervals and $J_0 < J_1 < J_2 < \cdots$; moreover, by construction, for every infinite $A \subseteq \mathbb{N}$

we have $[G_m \mid m \notin \bigcup_{i\in A} J_i] \subseteq [F_n \mid n \notin \bigcup_{i\in A} I_i]$, so $Y \not\subseteq [G_m \mid m \notin \bigcup_{i\in A} J_i]$. This shows that Y is tight in (G_m). ∎

Corollary 5.4. *If X is $\mathcal{H}$-tight (resp. d-tight), then it has an FDD which is $\mathcal{H}$-tight and $\mathcal{H}$-good (resp. d-tight and d-better).*

Proof. In the case of a D-family, starting from any $\mathcal{H}$-tight FDD $(F_n)_{n\in\mathbb{N}}$ of X, we can find a blocking $(G_n)_{n\in\mathbb{N}}$ of this FDD which is $\mathcal{H}$-good, using Lemma 3.8. The FDD (G_n) is still $\mathcal{H}$-tight, by Lemma 5.3. In the case of a degree, the proof is the same, using this time Lemma 3.28 to pass to a better blocking. ∎

Theorem 5.5 (The second dichotomy). *Suppose that $X \in \mathcal{H}$. Then X has a subspace $Y \in \mathcal{H}$ such that either*

- *Y is $\mathcal{H}_{\restriction Y}$-minimal, or*
- *Y is $\mathcal{H}_{\restriction Y}$-tight.*

Moreover, if X comes with a fixed FDD, then in the latter case, the $\mathcal{H}$-tight FDD of Y can be taken as a block-FDD of the FDD of X.

This is a true dichotomy: indeed, as we already saw, the notions of $\mathcal{H}$-minimality and $\mathcal{H}$-tightness are hereditary in a certain sense, and obviously an $\mathcal{H}$-tight space cannot be $\mathcal{H}$-minimal.

It is worth spelling out the version of the second dichotomy for degrees:

Theorem 5.6 (The second dichotomy for degrees). *Suppose that X is d-large. Then X has a d-large subspace Y which is either d-minimal or d-tight.*

In the case where $d(F) = \dim(F)$, we get back Theorem 1.22.

The rest of this section is organized as follows. In Section 5.2, we prove Theorem 5.5. Then, in Section 5.3, we study the properties of $\mathcal{H}$-minimal and $\mathcal{H}$-tight spaces, and we deduce some consequences of Theorem 5.5.

5.2. *The proof of Theorem 5.5*

This proof is inspired by the proof by Rosendal [52] of a variant of the minimal/tight dichotomy. This dichotomy will again be proved using combinatorial methods, but its proof is quite delicate and cannot be done in the formalism of approximate Gowers spaces. We will, instead, use the formalism of Gowers spaces, and work with countable vector spaces instead of Banach spaces.

In the general case, we can reduce to the case where X has an $\mathcal{H}$-good FDD, using Lemma 3.9. If X already comes with a fixed FDD, we can assume that this FDD is $\mathcal{H}$-good, using Lemma 3.8. So, in what follows, we will assume that X comes with a fixed $\mathcal{H}$-good FDD $(E_n)_{n\in\mathbb{N}}$, and we will prove that either X has a subspace Y which is $\mathcal{H}_{\restriction Y}$-minimal, or (E_n) has an $\mathcal{H}$-tight block-FDD.

Let C be the constant of the FDD (E_n). For every $n \in \mathbb{N}$, let $d_n = \sum_{i<n} \dim(E_n)$ and fix a normalized basis $(e_i)_{d_n \leqslant i < d_{n+1}}$ of E_n. Let K be a countable subfield of $\mathbb{R}$ having the following property: for every eventually null sequence $(x_i)_{i\in\mathbb{N}} \in K^{\mathbb{N}}$, we have $\| \sum_{i\in\mathbb{N}} x_i e_i \| \in K$. Such a field can be built in the following way: we fix $K_0 = \mathbb{Q}$, for every $n \in \mathbb{N}$, we let K_{n+1} be the subfield of $\mathbb{R}$ generated by K_n and by all reals of the form $\| \sum_{i\in\mathbb{N}} x_i e_i \|$, where $(x_i)_{i\in\mathbb{N}}$ is an eventually null sequence of elements of K_n, and finally we let $K = \bigcup_{n\in\mathbb{N}} K_n$. In the rest of this subsection, vector spaces over K will be denoted by script letters, and closed $\mathbb{R}$-vector subspaces of E (of finite or infinite dimension) will be denoted by italic letters. Let $\mathscr{X}$ be the K-vector subspace of X generated by all the e_i's. For $\mathscr{Y}$ a (finite- or infinite-dimensional) K-vector subspace of $\mathscr{X}$, we let $\overline{\mathscr{Y}}$ be its closure in X. This is an $\mathbb{R}$-vector subspace of X, and we have $\overline{\mathscr{X}} = X$. Also let $S_{\mathscr{Y}}$ be the set of normalized vectors of $\mathscr{Y}$. Since, for $x \in \mathscr{Y} \setminus \{0\}$, we have $\frac{x}{\|x\|} \in \mathscr{Y}$, we deduce that $S_{\mathscr{Y}}$ is dense in $S_{\overline{\mathscr{Y}}}$.

Lemma 5.7. *Let $\mathscr{Y}$ be a K-vector subspace of $\mathscr{X}$. Then $\overline{\mathscr{Y}}$ is $\mathbb{R}$-finite-dimensional if and only if $\mathscr{Y}$ is K-finite-dimensional, and in this case, their dimensions are equal.*

Proof. Let $(f_0, \dots, f_{k-1})$ be a K-free family in $\mathscr{Y}$. Let $N \in \mathbb{N}$ be such that all the f_l's are in $\operatorname{span}_K(e_0, \dots, e_{N-1})$, and let M be the matrix of the family $(f_1, \dots, f_k)$ in the basis $(e_0, \dots, e_{N-1})$. Then, over the field K, the matrix M has at least one non-zero $k \times k$ minor. But the determinant does not depend on the field, so this is also true over $\mathbb{R}$. Hence, the family $(f_0, \dots, f_{k-1})$ is $\mathbb{R}$-free. We deduce that $\dim_{\mathbb{R}}(\overline{\mathscr{Y}}) \geqslant \dim_K(\mathscr{Y})$.

Conversely, if $(f_0, \dots, f_{k-1})$ is a K-generating family in $\mathscr{Y}$, then it is an $\mathbb{R}$-generating family in $\operatorname{span}_{\mathbb{R}}(\mathscr{Y})$, which is equal to $\overline{\mathscr{Y}}$ since it is finite-dimensional. So $\dim_{\mathbb{R}}(\overline{\mathscr{Y}}) \leqslant \dim_K(\mathscr{Y})$. ∎

All along this subsection, we will use the following notation: if $(\mathscr{U}_i)_{i\in I}$ is a sequence of finite-dimensional vector subspaces of $\mathscr{X}$, we let $[\mathscr{U}_i \mid i \in I]$ be the K-vector subspace of $\mathscr{X}$ spanned by the $\mathscr{U}_i$'s. For every $n \in \mathbb{N}$, we let $\mathscr{E}_n$ be the K-vector subspace of E_n generated by the e_i's for $d_n \leqslant i < d_{n+1}$, and we let $\mathscr{E} = (\mathscr{E}_n)_{n\in\mathbb{N}}$. Obviously we have $\overline{\mathscr{E}_n} = E_n$ and $\mathscr{X} = [\mathscr{E}_n \mid n \in \mathbb{N}]$. For $(\mathscr{F}_n)_{n\in\mathbb{N}}$ a sequence of non-zero finite-dimensional K-vector subspaces of $\mathscr{X}$ whose sum is a direct sum, we define a *block-FDD* of $(\mathscr{F}_n)$ as a sequence $(\mathscr{G}_m)_{m\in\mathbb{N}}$ of non-zero finite-dimensional K-vector subspaces of $\mathscr{X}$ for which there exists a sequence $A_0 < A_1 < \cdots$ of finite sets of integers such that for every m, we have $\mathscr{G}_m \subseteq \bigoplus_{n\in A_m} \mathscr{F}_n$. In what follows, we will only consider block-FDD's of $\mathscr{E}$. A block-FDD $(\mathscr{F}_m)_{m\in\mathbb{N}}$ of $\mathscr{E}$ will often be denoted by $\mathscr{F}$; thus, when we speak about a block-FDD $\mathscr{F}$ without further explanation, it will be supposed that its terms are denoted by $\mathscr{F}_m$, and we will also let $[\mathscr{F}] = [\mathscr{F}_m \mid m \in \mathbb{N}]$. Observe that if $\mathscr{F}$ is a block-FDD of $\mathscr{E}$, then $(\overline{\mathscr{F}_m})_{m\in\mathbb{N}}$ is a block-FDD of (E_n). So we will say that $\mathscr{F}$ is *good* if and only if $(\overline{\mathscr{F}_m})$ is an $\mathcal{H}$-good block-FDD of (E_n). If $\mathscr{F}$ is a block-FDD of $\mathscr{E}$ and $m_0 \in \mathbb{N}$, we will denote by $\mathscr{F}^{(m_0)}$ the block-FDD $(\mathscr{F}_{m+m_0})_{m\in\mathbb{N}}$. If $\mathscr{F}$ is good, then $\mathscr{F}^{(m_0)}$ is also good.

We now define the Gowers space in which we will work. We let $\mathbb{P}$ be the set of good block-FDD's of $\mathscr{E}$. If $\mathscr{F}, \mathscr{G} \in \mathbb{P}$, we let $\mathscr{F} \leqslant \mathscr{G}$ if $\mathscr{F}$ is a block-FDD of $\mathscr{G}$. We let $\mathscr{F} \leqslant^* \mathscr{G}$

if there exists $m \in \mathbb{N}$ such that $\mathscr{F}^{(m)} \leqslant \mathscr{G}$. We let Π be the set of pairs $(\mathscr{U}, x)$ where $\mathscr{U}$ is a non-zero finite-dimensional subspace of $\mathscr{X}$ and x is an element of $\mathscr{X}$. For $\mathscr{F} \in \mathbb{P}$ and a sequence $(\mathscr{U}_0, x_0, \dots, \mathscr{U}_k, x_k) \in \mathrm{Seq}(\Pi)$, we write $(\mathscr{U}_0, x_0, \dots, \mathscr{U}_k, x_k) \lhd \mathscr{F}$ if $\mathscr{U}_k \subseteq [\mathscr{F}]$ and $x_k \in [\mathscr{U}_l \mid l \leqslant k]$.

Lemma 5.8. *$\mathscr{G} = (\mathbb{P}, \Pi, \leqslant, \leqslant^*, \lhd)$ is a Gowers space.*

Proof. The only non-trivial thing to verify is that the diagonalization axiom is satisfied. So let $(\mathscr{F}^k)_{k\in\mathbb{N}}$ be a $\leqslant$-decreasing sequence of elements of $\mathbb{P}$. We apply Lemma 3.11 to $U_k = Y_k = [\mathscr{F}^k]$, $F_n^k = \overline{\mathscr{F}_n^k}$, and $D_k = S_{[\mathscr{F}^k]}$. We get an $\mathcal{H}$-good FDD $(G_n)_{n\in\mathbb{N}}$ of a subspace of X such that for every $k \in \mathbb{N}$, $(G_{n+k})_{n\in\mathbb{N}}$ is a block-FDD of $(F_n^k)_{n\in\mathbb{N}}$, and G_k has a basis in D_k. This last condition shows that G_k can be written as $\overline{\mathscr{G}_k}$, where $\mathscr{G}_k$ is a finite-dimensional subspace of $[\mathscr{F}^k]$. Since $(G_{n+k})_{n\in\mathbb{N}}$ is a block-FDD of $(\overline{\mathscr{F}_n^k})_{n\in\mathbb{N}}$, and since all of the $\mathscr{G}_{n+k}$'s, for $n \in \mathbb{N}$, are vector subspaces of $[\mathscr{F}^k]$, we deduce that $(\mathscr{G}_{n+k})_{n\in\mathbb{N}}$ is a block-FDD of $\mathscr{F}^k$. Thus, $\mathscr{G} = (\mathscr{G}_n)_{n\in\mathbb{N}}$ is in $\mathbb{P}$, and $\mathscr{G} \leqslant^* \mathscr{F}^k$ for every $k \in \mathbb{N}$, as desired. ∎

From now on, we work in the Gowers space $\mathscr{G}$. The asymptotic game $F_{\mathscr{F}}$, Gowers' game $G_{\mathscr{F}}$, or the adversarial Gowers games $A_{\mathscr{F}}$ and $B_{\mathscr{F}}$, will always be considered with respect to this space. To save writing, we will make the following abuse of notation: in a play of $F_{\mathscr{F}}$ or $G_{\mathscr{F}}$ played as follows:

I	$\mathscr{F}^0$		$\mathscr{F}^1$	$\dots$
II		$\mathscr{U}_0,\ x_0$		$\mathscr{U}_1,\ x_1$ $\dots$

the outcome will be considered to be the sequence $(x_0, x_1, \dots)$ (according to the definition given in Section 2, this should be $(\mathscr{U}_0, x_0, \mathscr{U}_1, x_1, \dots)$). Similarly, in a play of $A_{\mathscr{F}}$ or $B_{\mathscr{F}}$ played as follows:

I		$\mathscr{U}_0,\ x_0,\ \mathscr{G}^0$		$\mathscr{U}_1,\ x_1,\ \mathscr{G}^1$		$\dots$
II	$\mathscr{F}^0$		$\mathscr{V}_0,\ y_0,\ \mathscr{F}^1$		$\mathscr{V}_1,\ y_1,\ \mathscr{F}^2$	$\dots$

the outcome will be considered to be the pair of sequences $((x_0, x_1, \dots), (y_0, y_1, \dots))$. Hence, for instance saying that player **II** has a strategy in $B_{\mathscr{F}}$ to produce two equivalent sequences means that player **II** has a strategy to ensure that the sequences $(x_i)_{i\in\mathbb{N}}$ and $(y_i)_{i\in\mathbb{N}}$ produced during the game are equivalent, for the usual notion of equivalence between sequences in a Banach space.

Observe that in this Gowers space, for $\mathscr{F}, \mathscr{G} \in \mathbb{P}$, if $\mathscr{F} \lessapprox \mathscr{G}$, then there exist $m, n \in \mathbb{N}$ such that $\mathscr{F}^{(m)} = \mathscr{G}^{(n)}$.

Lemma 5.9. *There exists $\mathscr{F} \in \mathbb{P}$ having the following property: either player **I** has a strategy in $A_{\mathscr{F}}$ to produce two inequivalent sequences, or player **II** has a strategy in $B_{\mathscr{F}}$ to produce two equivalent sequences.*

Proof. The set of pairs $((x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}) \in (\mathscr{X}^{\mathbb{N}})^2$ that are equivalent is an F_σ subset of $(\mathscr{X}^{\mathbb{N}})^2$ for the product of the discrete topologies on $\mathscr{X}$. Thus, this result is a direct consequence of Theorem 2.4. ∎

We now fix an $\mathscr{F} \in \mathbb{P}$ as given by Lemma 5.9. We say that a sequence $(u_i)_{i\in\mathbb{N}} \in \mathscr{X}^{\mathbb{N}}$ is *$\mathscr{F}$-correct* if there exist $\mathscr{G} \leqslant \mathscr{F}$ and a partition of $\mathbb{N}$ into non-empty successive intervals $I_0 < I_1 < \cdots$ such that for every $m \in \mathbb{N}$, the finite sequence $(u_i)_{i\in I_m}$ is a basis of $\mathscr{G}_m$. The next proposition contains the combinatorial content of Theorem 5.5.

Proposition 5.10. *At least one of the following statements is satisfied:*

(1) *For every $\mathscr{F}$-correct sequence $(u_i)_{i\in\mathbb{N}}$, player **I** has a strategy in $F_{\mathscr{F}}$ to build a sequence $(x_i)_{i\in\mathbb{N}}$ that is not equivalent to (u_i).*

(2) *There exists an $\mathscr{F}$-correct sequence $(u_i)_{i\in\mathbb{N}}$ such that player **II** has a strategy in $G_{\mathscr{F}}$ to build a sequence $(x_i)_{i\in\mathbb{N}}$ that is equivalent to (u_i).*

Proof. We assume that (1) is not satisfied and we prove (2). We fix an $\mathscr{F}$-correct sequence $(u_i)_{i\in\mathbb{N}}$ such that player **I** has no strategy in $F_{\mathscr{F}}$ to build a sequence that is not equivalent to (u_i). By the determinacy of this game (for the fundamentals on the theory of determinacy, see [34, Section 20]), player **II** has a strategy τ in $F_{\mathscr{F}}$ to build a sequence which is equivalent to (u_i). By correctness of the sequence (u_i), we can also fix $\mathscr{G} \leqslant \mathscr{F}$ and a partition of $\mathbb{N}$ into non-empty successive intervals $I_0 < I_1 < \cdots$ such that for every $m \in \mathbb{N}$, $(u_i)_{i\in I_m}$ is a basis of $\mathscr{G}_m$.

*First step: Player **II** has a strategy in $A_{\mathscr{F}}$ to build two equivalent sequences.*

We describe this strategy on a play $(\mathscr{G}, \mathscr{U}_0, x_0, \mathscr{F}^0, \mathscr{V}_0, y_0, \mathscr{G}, \ldots)$ of $A_{\mathscr{F}}$, in which the FDD's played by **II** will always be equal to $\mathscr{G}$. This game will be played at the same time as an auxiliary play $(\mathscr{H}^0, \mathscr{W}_0, z_0, \mathscr{H}^1, \mathscr{W}_1, z_1, \ldots)$ of $F_{\mathscr{F}}$ during which player **II** always plays according to her strategy τ. Actually, the $\mathscr{U}_i$'s played by **I** in $A_{\mathscr{F}}$ will not matter at all in this proof, so we will omit them in the notation; the only thing to observe is that for every $i \in \mathbb{N}$, we will necessarily have $x_i \in [\mathscr{G}]$. At the same time as the games are played, a sequence of integers $0 = k_0 < k_1 < \cdots$ will be constructed. The idea is that turn i of the game $A_{\mathscr{F}}$ will be played at the same time as turns $k_i, k_i + 1, \ldots, k_{i+1} - 1$ of the game $F_{\mathscr{F}}$. Suppose that we are just before turn i of $A_{\mathscr{F}}$, so the x_j's, the $\mathscr{F}^j$'s, the $\mathscr{V}_j$'s, and the y_j's have been defined for all $j < i$. Also suppose that the integers k_j have been defined for all $j \leqslant i$, and that we are just before turn k_i of $F_{\mathscr{F}}$, so the $\mathscr{H}^k$'s, the $\mathscr{W}_k$'s and the z_k's have been played for all $k < k_i$. In the diagram below we represent turn i of $A_{\mathscr{F}}$, and turns $k_i, \ldots, k_{i+1} - 1$ of $F_{\mathscr{F}}$.

$F_{\mathscr{F}}$

I		$\mathscr{F}^i$		$\ldots$		$\mathscr{F}^i$		$\ldots$
II	$\ldots$		$\mathscr{W}_{k_i},\ z_{k_i}$		$\ldots$		$\mathscr{W}_{k_{i+1}-1},\ z_{k_{i+1}-1}$	

$A_{\mathscr{F}}$

I	$\ldots$		$x_i,\ \mathscr{F}^i$				$\ldots$	
II		$\ldots, \mathscr{G}$			$\mathscr{V}_i,\ y_i,\ \ldots$			

We now describe how these turns are played. In $A_{\mathscr{F}}$, the strategy of player **II** will first consist in playing $\mathscr{G}$. Then player **I** answers with a vector $x_i \in [\mathscr{G}]$ and an FDD $\mathscr{F}^i \lessapprox \mathscr{F}$. Thus, x_i can be decomposed in the basis $(u_k)_{k\in\mathbb{N}}$: we can find $k_{i+1} \in \mathbb{N}$ and $(a_i^k)_{k<k_{i+1}} \in K^{k_{i+1}}$ such that $x_i = \sum_{k<k_{i+1}} a_i^k u_k$. Moreover, we can assume that $k_{i+1} > k_i$.

Now, during the $k_{i+1} - k_i$ following turns of the game $F_{\mathscr{F}}$, we will let player **I** play $\mathscr{F}^i$ (so for every $k_i \leqslant k < k_{i+1}$ we have $\mathscr{H}^k = \mathscr{F}^i$). According to the strategy τ, player **II** will answer with $\mathscr{W}_{k_i}$, $z_{k_i}, \dots, \mathscr{W}_{k_{i+1}-1}$, $z_{k_{i+1}-1}$. We now let $\mathscr{V}_i = \mathscr{W}_{k_i} + \cdots + \mathscr{W}_{k_{i+1}-1}$, and $y_i = \sum_{k<k_{i+1}} a_i^k z_k$. Since all the $\mathscr{W}_k$'s, for $k_i \leqslant k < k_{i+1}$ are finite-dimensional subspaces of $[\mathscr{F}^i]$, so is $\mathscr{V}_i$. And since all the z_k's for $k_i \leqslant k < k_{i+1}$ are elements of $\mathscr{W}_0 + \cdots + \mathscr{W}_{k_{i+1}-1} = \mathscr{V}_0 + \cdots + \mathscr{V}_i$, y_i is also an element of $\mathscr{V}_0 + \cdots + \mathscr{V}_i$. So we can let **II** play $\mathscr{V}_i$ and y_i in $A_{\mathscr{F}}$, which finishes the description of the strategy.

The fact that in $F_{\mathscr{F}}$, player **II** always plays according to the strategy τ, ensures that the sequences $(u_k)_{k\in\mathbb{N}}$ and $(z_k)_{k\in\mathbb{N}}$ are equivalent. Observe that the sequence $(x_i)_{i\in\mathbb{N}}$ is built from (u_k) in exactly the same way as $(y_i)_{i\in\mathbb{N}}$ is built from (z_k); this ensures that $(x_i)_{i\in\mathbb{N}}$ and $(y_i)_{i\in\mathbb{N}}$ are equivalent, concluding this step of the proof.

*Second step: Player **II** has a strategy σ in $B_{\mathscr{F}}$ to build two equivalent sequences.*

Indeed, by the first step, **I** has no strategy in $A_{\mathscr{F}}$ to build two inequivalent sequences; so the conclusion immediately follows from the choice of $\mathscr{F}$.

*Third step: Player **II** has a strategy in $G_{\mathscr{F}}$ to build a sequence $(y_i)_{i\in\mathbb{N}}$ that is equivalent to (u_i).*

This is the conclusion of the proof. We describe this strategy on a play of $G_{\mathscr{F}}$ that will be played simultaneously with a play of $B_{\mathscr{F}}$ where **II** will play according to her strategy σ, and a play of $F_{\mathscr{F}}$ where **II** will play according to her strategy τ (for a fixed $i \in \mathbb{N}$, turns i of all of these three games will be played at the same time). The moves of the players during turn i of the games are described in the diagram below.

$F_{\mathscr{F}}$

I	$\mathscr{F}^i$		$\dots$
II	$\dots$	$\mathscr{U}_i,\ x_i$	

$B_{\mathscr{F}}$

I		$\mathscr{U}_i,\ x_i,\ \mathscr{H}^i$	
II	$\dots,\ \mathscr{F}^i$		$\mathscr{V}_i,\ y_i,\ \dots$

$G_{\mathscr{F}}$

I	$\mathscr{H}^i$		$\dots$
II	$\dots$	$\mathscr{V}_i,\ y_i$	

We now describe these moves more precisely. Suppose that in $G_{\mathscr{F}}$, player **I** plays $\mathscr{H}^i$. We look at the move $\mathscr{F}^i$ made by **II** in $B_{\mathscr{F}}$ according to her strategy σ, and we let **I** copy this move in $F_{\mathscr{F}}$. In this game, according to her strategy τ, player **II** will answer with some $\mathscr{U}_i$ and x_i. Now, in $B_{\mathscr{F}}$, we can let **I** answer with $\mathscr{U}_i$, x_i and $\mathscr{H}^i$. In this game, according to her strategy σ, player **II** answers with some $\mathscr{V}_i$ and some y_i. Then the strategy of player **II** in $G_{\mathscr{F}}$ will consist in answering with $\mathscr{V}_i$ and y_i.

Let us verify that this strategy is successful. The outcome of the game $F_{\mathscr{F}}$ is the sequence $(x_i)_{i\in\mathbb{N}}$; the use of the strategy τ by **II** ensures that this sequence is equivalent to (u_i). The outcome of the game $B_{\mathscr{F}}$ is the pair of sequences $((x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}})$; the use by **II** of her strategy σ ensures that these two sequences are equivalent. We deduce that the sequences (u_i) and (y_i) are equivalent, concluding the proof. ∎

Now, for every $m \in \mathbb{N}$, let $F_m = \overline{\mathscr{F}_m}$. The sequence $(F_m)_{m\in\mathbb{N}}$ is an $\mathcal{H}$-good block-FDD of (E_n) and we can let $Y = [F_m \mid m \in \mathbb{N}]$. By Proposition 5.10, Theorem 5.5 will be proved once we have proved the following two lemmas:

Lemma 5.11. *Suppose that there exists an $\mathscr{F}$-correct sequence $(u_i)_{i\in\mathbb{N}}$ such that player **II** has a strategy in $G_{\mathscr{F}}$ to build a sequence $(x_i)_{i\in\mathbb{N}}$ that is equivalent to (u_i). Let $Z = [u_i \mid i \in \mathbb{N}]$. Then Z is $\mathcal{H}_{\restriction Z}$-minimal.*

Lemma 5.12. *Suppose that for every $\mathscr{F}$-correct sequence $(u_i)_{i\in\mathbb{N}}$, player **I** has a strategy in $F_{\mathscr{F}}$ to build a sequence $(x_i)_{i\in\mathbb{N}}$ that is not equivalent to (u_i). Then the FDD $(F_i)_{i\in\mathbb{N}}$ is $\mathcal{H}$-tight.*

We start with the following technical lemma:

Lemma 5.13. *For every $U \in \mathcal{H}_{\restriction Y}$, there exists a $\mathscr{G} \leqslant \mathscr{F}$ such that $\overline{[\mathscr{G}]}$ isomorphically embeds into U.*

Proof. This is a consequence of Lemma 3.11. Indeed, apply it to $Y_k = Y$, to $F_n^k = F_n$, $U_k = U$, and $D_k = S_{[\mathscr{F}]}$. Then Lemma 3.11 gives a subspace $Z \subseteq Y$ generated by an $\mathcal{H}$-good block-FDD $(G_n)_{n\in\mathbb{N}}$ of (F_n) such that Z can be isomorphically embedded into U. Moreover, for every $n \in \mathbb{N}$, G_n has a basis made up of elements of $S_{[\mathscr{F}]}$, so $G_n = \overline{\mathscr{G}_n}$ for some finite-dimensional subspace $\mathscr{G}_n$ of $[\mathscr{F}]$. Hence, $\mathscr{G} = (\mathscr{G}_n)_{n\in\mathbb{N}}$ is a good block-FDD of $\mathscr{F}$, and $\overline{[\mathscr{G}]}$ isomorphically embeds into U, as desired. ∎

Proof of Lemma 5.11. By the definition of correctness, we have $Z \in \mathcal{H}$. We want to prove that Z isomorphically embeds into every element of $\mathcal{H}_{\restriction Z}$; by Lemma 5.13, it is enough to prove that Z isomorphically embeds into $\overline{[\mathscr{G}]}$ for every $\mathscr{G} \leqslant \mathscr{F}$. For this, consider a play of $G_{\mathscr{F}}$ where player **I** always plays $\mathscr{G}$, and **II** plays using her strategy. The outcome will be a sequence $(x_i)_{i\in\mathbb{N}}$ of elements of $\mathscr{G}$ which is equivalent to (u_i). Thus the mapping $u_i \mapsto x_i$ uniquely extends to an isomorphic embedding $Z \to \overline{[\mathscr{G}]}$. ∎

Proof of Lemma 5.12. By Lemma 5.13, it is enough to prove that every subspace of the form $\overline{[\mathscr{G}]}$, for $\mathscr{G} \leqslant \mathscr{F}$, is tight in (F_n). So we fix such a $\mathscr{G}$ and we let $Z = \overline{[\mathscr{G}]}$.

First step: For every $K \geqslant 1$, there exists an infinite sequence $I_0^K < I_1^K < \cdots$ of non-empty intervals of integers such that for every infinite $A \subseteq \mathbb{N}$ with $0 \in A$, we have $Z \not\subseteq_K [F_n \mid n \notin \bigcup_{k\in A} I_k^K]$.

For every $n \in \mathbb{N}$, let $d_n = \sum_{m<n} \dim(\mathscr{G}_m)$, and fix a normalized basis $(u_i)_{d_n \leqslant i < d_{n+1}}$ of $\mathscr{G}_n$ that is also a 2-bounded minimal system (see Section 1.5); this can be done by taking first an Auerbach basis of $\overline{\mathscr{G}_n}$, and then a small perturbation of it. The sequence $(u_i)_{i\in\mathbb{N}}$ we have just built is $\mathscr{F}$-correct and is a $4C$-bounded minimal system. Thus, we can fix a strategy τ for player **I** in $F_{\mathscr{F}}$ to build a sequence $(x_i)_{i\in\mathbb{N}}$ that is not equivalent to (u_i). In the game $F_{\mathscr{F}}$, we will consider that player **II** is allowed to play against the rules, but immediately loses if she does; hence, we can assume that the strategy τ is a mapping defined on the whole set $\Pi^{<\mathbb{N}}$ of finite sequences of elements of Π. For every such sequence s, $\tau(s)$ is an element of $\mathbb{P}$ such that $\tau(s) \lessapprox \mathscr{F}$; hence without loss of generality, we can assume that $\tau(s) = \mathscr{F}^{(\widetilde{\tau}(s))}$ for some $\widetilde{\tau}(s) \in \mathbb{N}$. This defines a mapping $\widetilde{\tau} : \Pi^{<\mathbb{N}} \to \mathbb{N}$.

Let $R = \{x \in X \mid 1 \leqslant \|x\| \leqslant K\}$. Let $\delta > 0$ have the following property: for every $4CK^2$-bounded minimal system $(x_i)_{i\in I}$ and for every family $(y_i)_{i\in I}$ in X, if

$$\sum_{i\in I} \frac{\|x_i - y_i\|}{\|x_i\|} \leqslant \delta,$$

then the families (x_i) and (y_i) are equivalent. For every finite-dimensional subspace $\mathscr{U}$ of $[\mathscr{F}]$ and for every $i \in \mathbb{N}$, we let $\mathfrak{N}_i(\mathscr{U})$ be a finite $(2^{-(i+2)}\delta)$-net in $\mathscr{U} \cap R$. Given $n \in \mathbb{N}$, we say that a sequence $(\mathscr{U}_0, x_0, \ldots, \mathscr{U}_{i-1}, x_{i-1}) \in \Pi^{<\mathbb{N}}$ is *n-small* if it has the following properties:

- there exists a sequence of successive non-empty intervals of integers $J_0 < \cdots < J_{i-1} < n$ such that for every $j < i$, $\mathscr{U}_j = [\mathscr{F}_m \mid m \in J_j]$;
- for every $j < i$, we have $x_j \in \mathfrak{N}_j([\mathscr{U}_k \mid k \leqslant j])$.

For n fixed, there are only finitely many n-small sequences. Hence we can define a sequence $(n_k)_{k\in\mathbb{N}}$ of integers in the following way: let $n_0 = 0$, and for $k \in \mathbb{N}$, choose $n_{k+1} > n_k$ such that for every n_k-small sequence $s \in \Pi^{<\mathbb{N}}$, we have $n_{k+1} \geqslant \widetilde{\tau}(s)$. We now let, for every $k \in \mathbb{N}$, $I_k^K = [\![n_k, n_{k+1} - 1]\!]$, and show that the sequence of intervals $I_0^K < I_1^K < \cdots$ is as desired.

Suppose not. Then there exists an infinite $A \subseteq \mathbb{N}$ with $0 \in A$, and there exists an isomorphic embedding $T : Z \to [F_n \mid n \notin \bigcup_{k\in A} I_k^K]$ such that $\|T^{-1}\| = 1$ and $\|T\| \leqslant K$. In particular, the sequence $(T(u_i))_{i\in\mathbb{N}}$ is K-equivalent to (u_i), so it is a $4CK^2$-bounded minimal system. We also know that, for every $i \in \mathbb{N}$, $1 \leqslant \|T(u_i)\| \leqslant K$. For every $i \in \mathbb{N}$, we fix $y_i \in [\mathscr{F}_n \mid n \notin \bigcup_{k\in A} I_k^K] \cap R$ such that $\|y_i - T(u_i)\| \leqslant 2^{-(i+2)}\delta$. Since A is infinite, we can find $k_{i+1} \in A$ such that $\operatorname{supp}(y_i) < I_{k_{i+1}}^K$ (here, the support is taken with respect to the FDD $\mathscr{F}$). We can also let $k_0 = 0$; hence, we have defined a sequence $(k_i)_{i\in\mathbb{N}}$ of elements of A. We can even assume that for every i, we have $k_{i+1} \geqslant k_i + 2$. We let, for every $i \in \mathbb{N}$, $J_i = [\![n_{k_i+1}, n_{k_{i+1}} - 1]\!]$ and $\mathscr{U}_i = [\mathscr{F}_n \mid n \in J_i]$. Hence, we have a partition of $\mathbb{N}$ into an infinite sequence $I_{k_0}^K < J_0 < I_{k_1}^K < J_1 < \cdots$ of non-empty

successive intervals. Since all the k_i's are in A, we have $[\mathscr{F}_n \mid n \notin \bigcup_{k\in A} I_k^K] \subseteq [\mathscr{F}_n \mid n \in \bigcup_{i\in\mathbb{N}} J_i]$; so all the y_i's are in $[\mathscr{F}_n \mid n \in \bigcup_{i\in\mathbb{N}} J_i]$. Thus, for every $i \in \mathbb{N}$, we have $y_i \in [\mathscr{F}_n \mid n \in \bigcup_{j\leqslant i} J_j] = [\mathscr{U}_j \mid j \leqslant i]$. Hence, we can find $x_i \in \mathfrak{N}_i([\mathscr{U}_j \mid j \leqslant i])$ satisfying $\|x_i - y_i\| \leqslant 2^{-(i+2)}\delta$. In particular, $\|x_i - T(u_i)\| \leqslant 2^{-(i+1)}\delta$. So we have

$$\sum_{i\in\mathbb{N}} \frac{\|x_i - T(u_i)\|}{\|T(u_i)\|} \leqslant \sum_{i\in\mathbb{N}} \|x_i - T(u_i)\| \leqslant \delta,$$

so since $(T(u_i))$ is a $4CK^2$-bounded minimal system, and by the choice of δ, we deduce that the sequences $(T(u_i))$ and (x_i) are equivalent. In particular, the sequences (u_i) and (x_i) are equivalent.

Towards a contradiction, we now prove that (u_i) and (x_i) are *not* equivalent. For this, we first observe that for every $i \in \mathbb{N}$, the sequence $(\mathscr{U}_0, x_0, \dots, \mathscr{U}_{i-1}, x_{i-1})$ is n_{k_i}-small. Thus, letting $p_i = \widetilde{\tau}(\mathscr{U}_0, x_0, \dots, \mathscr{U}_{i-1}, x_{i-1})$, we deduce that $p_i \leqslant n_{k_i+1} = \min J_i$. In particular, $\mathscr{U}_i \subseteq [\mathscr{F}^{(p_i)}]$. Since moreover $x_i \in [\mathscr{U}_j \mid j \leqslant i]$, we deduce that in the following play of $F_{\mathscr{F}}$:

I	$\mathscr{F}^{(p_0)}$		$\mathscr{F}^{(p_1)}$		$\dots$
II		$\mathscr{U}_0,\ x_0$		$\mathscr{U}_1,\ x_1$	$\dots$

player **II** always respects the rules. Since moreover player **I** plays according his strategy τ, we deduce that he wins the game and that the outcome (x_i) is not equivalent to (u_i). This is a contradiction.

Second step: Z is tight in (F_n).

This is the conclusion of the proof. We keep the sequences of intervals $(I_i^K)_{i\in\mathbb{N}}$ built in the previous step. We recall the following classical result: for every $d \in \mathbb{N}$, there exists a constant $c(d) \geqslant 1$ such that for every Banach space U and for any two subspaces $V, W \subseteq U$ both having codimension d, V and W are $c(d)$-isomorphic (see [21, Lemma 3]) – incidentally, an upper bound $c(d) \leq 4d(1+\sqrt{d})^2$ may be obtained, through the fact that any d-codimensional subspace is $(\sqrt{d}+1+\varepsilon)$-complemented for any $\varepsilon > 0$ (a consequence of local reflexivity and the Kadets–Snobar theorem [1, Theorem 12.1.6] and John's result that all d-dimensional spaces are $\sqrt{d}$-isomorphic to ℓ_2^d [1, Theorem 12.1.4].

We build a sequence $I_1 < I_2 < \cdots$ of non-empty successive intervals of integers in the following way. Once all the I_l's for $l < k$ are defined, we can choose I_k such that

- for every positive integer $N \leqslant k$, I_k contains at least one interval of $(I_i^N)_{i\in\mathbb{N}}$;
- $\max I_k \geqslant d_k + \max(\max I_0^{N_k}, \min I_k)$, where $d_k = \dim([F_n \mid n < \min I_k])$ and $N_k = \lceil kc(d_k)\rceil$.

We show that the sequence $(I_k)_{k\geqslant 1}$ witnesses the tightness of Z in (F_n).

Claim 5.14. *For every infinite $A \subseteq \mathbb{N}\setminus\{0\}$ and every $k_0 \in A$, we have*

$$\Big[F_n \;\Big|\; n \notin \bigcup_{k\in A} I_k\Big] \sqsubseteq_{c(d_{k_0})} \Big[F_n \;\Big|\; n \notin I_0^{N_{k_0}} \cup \bigcup_{\substack{k\in A\\ k>k_0}} I_k\Big].$$

Proof. Let $n_0 = \min I_{k_0}$, so that $d_{k_0} = \dim [F_n \mid n < n_0]$. It is enough to prove that

$$[F_n \mid n < n_0] \oplus \Big[F_n \;\Big|\; n \geqslant n_0,\ n \notin \bigcup_{k\in A} I_k \Big] \sqsubseteq_{c(d_{k_0})} \Big[F_n \;\Big|\; n \notin I_0^{N_{k_0}} \cup \bigcup_{\substack{k\in A\\ k>k_0}} I_k \Big].$$

Since $\max I_{k_0} \geqslant d_{k_0} + \max(\max I_0^{N_{k_0}}, \min I_{k_0})$, in particular

$$\dim [F_n \mid \max(\max I_0^{N_{k_0}}, \min I_{k_0}) < n \leqslant \max I_{k_0}] \geqslant d_{k_0}.$$

So we can find a finite-dimensional subspace $H \subseteq [F_n \mid n \in I_{k_0}]$ with $I_0^{N_{k_0}} < \operatorname{supp}(H)$ and $\dim(H) = d_{k_0}$ (here, the supports are taken with respect to the FDD (F_n)). Since $k_0 \in A$, we have

$$H \cap \Big[F_n \;\Big|\; n \geqslant n_0,\ n \notin \bigcup_{k\in A} I_k \Big] = \{0\}.$$

Thus, both subspaces

$$[F_n \mid n < n_0] \oplus \Big[F_n \;\Big|\; n \geqslant n_0,\ n \notin \bigcup_{k\in A} I_k \Big]$$

and

$$H \oplus \Big[F_n \;\Big|\; n \geqslant n_0,\ n \notin \bigcup_{k\in A} I_k \Big]$$

have codimension d_{k_0} in

$$[F_n \mid n < n_0] \oplus H \oplus \Big[F_n \;\Big|\; n \geqslant n_0,\ n \notin \bigcup_{k\in A} I_k \Big],$$

so they are $c(d_{k_0})$-isomorphic. Hence, to conclude the proof, it is enough to see that

$$H \oplus \Big[F_n \;\Big|\; n \geqslant n_0,\ n \notin \bigcup_{k\in A} I_k \Big] \subseteq \Big[F_n \;\Big|\; n \notin I_0^{N_{k_0}} \cup \bigcup_{\substack{k\in A\\ k>k_0}} I_k \Big].$$

The inclusion

$$H \subseteq \Big[F_n \;\Big|\; n \notin I_0^{N_{k_0}} \cup \bigcup_{\substack{k\in A\\ k>k_0}} I_k \Big]$$

is a consequence of the fact that $\operatorname{supp}(H) \subseteq I_{k_0}$ and $I_0^{N_{k_0}} < \operatorname{supp}(H)$. And to prove the inclusion

$$\Big[F_n \;\Big|\; n \geqslant n_0,\ n \notin \bigcup_{k\in A} I_k \Big] \subseteq \Big[F_n \;\Big|\; n \notin I_0^{N_{k_0}} \cup \bigcup_{\substack{k\in A\\ k>k_0}} I_k \Big],$$

it is enough to see that for all $n \geqslant n_0$, if $n \in I_0^{N_{k_0}}$, then $n \in \bigcup_{k\in A} I_k$. This is a consequence of the fact that $n_0 = \min I_{k_0}$ and $\max I_0^{N_{k_0}} \leqslant \max I_{k_0}$. ∎

We now conclude the proof of the lemma. Let $A \subseteq \mathbb{N} \setminus \{0\}$ be infinite and assume, towards a contradiction, that $Z \sqsubseteq [F_n \mid n \notin \bigcup_{k\in A} I_k]$. Then we can choose $k_0 \in A$ such that $Z \sqsubseteq_{k_0} [F_n \mid n \notin \bigcup_{k\in A} I_k]$. Using Claim 5.14 and the fact that $k_0 c(d_{k_0}) \leqslant N_{k_0}$, we get

$$Z \sqsubseteq_{N_{k_0}} \Big[F_n \;\Big|\; n \notin I_0^{N_{k_0}} \cup \bigcup_{\substack{k\in A\\ k>k_0}} I_k\Big].$$

But by construction, $I_0^{N_{k_0}} \cup \bigcup_{\substack{k\in A\\ k>k_0}} I_k$ contains infinitely many intervals of the sequence $(I_i^{N_{k_0}})_{i\in\mathbb{N}}$, including its initial term $I_0^{N_{k_0}}$. This contradicts the definition of $(I_i^{N_{k_0}})_{i\in\mathbb{N}}$. ∎

5.3. $\mathcal{H}$-minimal and $\mathcal{H}$-tight spaces

In this section, we prove several properties of $\mathcal{H}$-minimal and $\mathcal{H}$-tight spaces. We deduce consequences of Theorem 5.5. We start by studying $\mathcal{H}$-tight spaces.

Definition 5.15. We say that the D-family $\mathcal{H}$ is *invariant under isomorphism* if for every $Y, Z \in \mathrm{Sub}^\infty(X)$ such that Y and Z are isomorphic, we have $Y \in \mathcal{H} \Leftrightarrow Z \in \mathcal{H}$.

Theorem 5.16. (1) *If X is $\mathcal{H}$-tight and $\mathcal{H}$ is invariant under isomorphism, then X is ergodic.*

(2) *If X is d-tight, then X is ergodic.*

An important consequence of Theorems 5.5 and 5.16 is the following:

Corollary 5.17. (1) *If $X \in \mathcal{H}$ is non-ergodic and $\mathcal{H}$ is invariant under isomorphism, then there exists $Y \in \mathcal{H}$ which is $\mathcal{H}_{\restriction Y}$-minimal.*

(2) *If X is d-large and non-ergodic, then X has a d-minimal subspace.*

To prove Theorem 5.16, we will use a sufficient condition for the reducibility of $\mathbf{E}_0$ proved by Rosendal [50, Theorem 15]. Let $\mathbf{E}_0'$ be the equivalence relation on $\mathcal{P}(\mathbb{N})$ (identified with the Cantor space) defined as follows: if $A, B \in \mathcal{P}(\mathbb{N})$, we say that $A\mathbf{E}_0'B$ if there exists $n \in \mathbb{N}$ such that $|A \cap [\![0, n]\!]| = |B \cap [\![0, n]\!]|$ and $A \setminus [\![0, n]\!] = B \setminus [\![0, n]\!]$. The result proved by Rosendal is the following:

Proposition 5.18. *Let E be a meager equivalence relation on $\mathcal{P}(\mathbb{N})$ with $\mathbf{E}_0' \subseteq E$. Then $\mathbf{E}_0 \leqslant_B E$.*

To prove Theorem 5.16, we will combine Proposition 5.18 with ideas developed by Ferenczi and Godefroy [18]. They proved that if $(e_i)_{i\in\mathbb{N}}$ is a basis and X a Banach space, then X is tight in (e_i) if and only if the set of $A \subseteq \mathbb{N}$ such that $X \sqsubseteq [e_i \mid i \in A]$ is meager in $\mathcal{P}(\mathbb{N})$. This extends immediately to the case when (e_i) is replaced by an FDD (F_i).

Proof of Theorem 5.16. As usual, we only prove the result for D-families. By Corollary 5.4, we can find an $\mathcal{H}$-good, $\mathcal{H}$-tight FDD $(F_n)_{n\in\mathbb{N}}$ of X. We fix a sequence $(e_i)_{i\in\mathbb{N}}$

of elements of X, and a partition of $\mathbb{N}$ into non-empty successive intervals $J_0 < J_1 < \cdots$ such that for every $n \in \mathbb{N}$, $(e_i)_{i\in J_n}$ is a basis of F_n. For every infinite $A \subseteq \mathbb{N}$, we let $X_A = [e_i \mid i \in A]$, and we define the equivalence relation E on $\mathcal{P}(\mathbb{N})$ by AEB if and only if X_A and X_B are isomorphic. Since the mapping $A \mapsto X_A$ from $\mathcal{P}(\mathbb{N})$ to $\mathrm{Sub}(X)$ is Borel (see Lemma 1.5), it is enough to prove that $\mathbf{E}_0 \leqslant_B E$.

We have $\mathbf{E}_0' \subseteq E$: indeed, if $A\mathbf{E}_0'B$, then fixing $n \in \mathbb{N}$ witnessing it, we have $X_A = X_{A\setminus[\![0,n]\!]} \oplus X_{A\cap[\![0,n]\!]}$ and $X_B = X_{A\setminus[\![0,n]\!]} \oplus X_{B\cap[\![0,n]\!]}$, and $\dim(X_{A\cap[\![0,n]\!]}) = \dim(X_{B\cap[\![0,n]\!]})$ is finite, so X_A and X_B are isomorphic. So, by Proposition 5.18, it is enough to prove that E is meager. Since E is analytic, it has the Baire property, so by Kuratowski–Ulam's theorem [34, Theorem 8.41], it is enough to prove that for every $A \in \mathcal{P}(\mathbb{N})$, the E-equivalence class of A is meager. We distinguish two cases.

First case: $\mathcal{H}_{\restriction X_A} = \emptyset$.

For all $N \in \mathbb{N}$, let $\mathcal{U}_N = \{B \in \mathcal{P}(\mathbb{N}) \mid \exists n \geqslant N\ J_n \subseteq B\}$. This is a dense open subset of $\mathcal{P}(\mathbb{N})$, so $\mathcal{C} := \bigcap_{N\in\mathbb{N}} \mathcal{U}_N$ is comeager in $\mathcal{P}(\mathbb{N})$. For $B \in \mathcal{C}$, the space X_B contains infinitely many F_n's. Since (F_n) is an $\mathcal{H}$-good FDD, this implies that $\mathcal{H}_{\restriction X_B} \neq \emptyset$. Since $\mathcal{H}_{\restriction X_A} = \emptyset$ and $\mathcal{H}$ is invariant under isomorphism, this implies that X_A and X_B are not isomorphic. Hence, the set of $B \in \mathcal{P}(\mathbb{N})$ such that X_B is isomorphic to X_A is meager in $\mathcal{P}(\mathbb{N})$.

Second case: $\mathcal{H}_{\restriction X_A} \neq \emptyset$.

In this case, X_A has a subspace which is tight in (F_n), so X_A itself is tight in (F_n). Let $I_0 < I_1 < \cdots$ be a sequence of intervals witnessing it. For all $k \in \mathbb{N}$, let $K_k = \bigcup_{n\in I_k} J_n$. For every infinite $D \subseteq \mathbb{N}$, we have $X_A \not\subseteq [e_i \mid i \notin \bigcup_{k\in D} K_k]$. For all $N \in \mathbb{N}$, let $\mathcal{U}_N = \{B \in \mathcal{P}(\mathbb{N}) \mid \exists k \geqslant N\ K_k \cap B = \emptyset\}$. This is a dense open subset of $\mathcal{P}(\mathbb{N})$, so $\mathcal{C} := \bigcap_{N\in\mathbb{N}} \mathcal{U}_N$ is comeager in $\mathcal{P}(\mathbb{N})$. If $B \in \mathcal{C}$, then there exists an infinite $D \subseteq \mathbb{N}$ such that $X_B \subseteq [e_i \mid i \notin \bigcup_{k\in D} K_k]$. In particular, X_B cannot be isomorphic to X_A. Hence, the set of $B \in \mathcal{P}(\mathbb{N})$ such that X_B is isomorphic to X_A is meager in $\mathcal{P}(\mathbb{N})$. ∎

We now study the properties of $\mathcal{H}$-minimal spaces.

Definition 5.19. We say that X is *uniformly $\mathcal{H}$-minimal* if $X \in \mathcal{H}$ and there exists a constant C such that X C-embeds into every element of $\mathcal{H}$. We say that X is *uniformly d-minimal* if it is uniformly $\mathcal{H}_d$-minimal.

The statement of the following proposition was improved from a previous version of this paper thanks to an observation of O. Kurka.

Proposition 5.20. (1) *Suppose that the D-family $\mathcal{H}$ is invariant under isomorphisms. If X is $\mathcal{H}$-minimal, then it is uniformly $\mathcal{H}$-minimal.*

(2) *If X is d-minimal, then it is uniformly d-minimal.*

Proof. (2) is a consequence of (1) and the fact that $\mathcal{H}_d$ is invariant under isomorphisms. To prove (1), we start with the following claim:

Claim 5.21. *There exists $Y \in \mathcal{H}$ which is uniformly $\mathcal{H}_{\restriction Y}$-minimal.*

Proof. Let $(\mathcal{U}_n)_{n\in\mathbb{N}}$ be a decreasing sequence of Ellentuck-open subsets of $\mathrm{Sub}(X)$ such that $\mathcal{H} = \bigcap_{n\in\mathbb{N}} \mathcal{U}_n$. The hyperplanes of X are in $\mathcal{H}$, and they are pairwise isomorphic with a uniform constant. Thus, there exists a constant $K \geqslant 1$ such that X K-embeds into all of its hyperplanes. As a consequence, for every $m \in \mathbb{N}$, X K^m-embeds into all of its subspaces of codimension m.

Suppose that (1) is not satisfied. We inductively build a decreasing sequence $(Y_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{H}$ and an increasing sequence $(F_n)_{n\in\mathbb{N}}$ of finite-dimensional subspaces of X in the following way. Let $Y_0 = X$ and $F_0 = \{0\}$. If Y_n and F_n have been defined, then by assumption, Y_n is not uniformly $\mathcal{H}_{\restriction Y_n}$-minimal, so there exists $Y_{n+1} \in \mathcal{H}_{\restriction Y_n}$ such that Y_n does not $(nK^{\dim(F_n)})$-embed into Y_{n+1}. The subspace $Y_{n+1} + F_n$ is also in $\mathcal{H}$, so in $\mathcal{U}_n$; thus, we can choose F_{n+1} such that $F_n \subseteq F_{n+1} \subseteq Y_{n+1} + F_n$ and $[F_{n+1}, Y_{n+1} + F_n] \subseteq \mathcal{U}_n$. This completes the induction.

We now let $Y = \overline{\bigcup_{n\in\mathbb{N}} F_n}$. For every $n \in \mathbb{N}$, we have $Y \subseteq Y_{n+1} + F_n$, so that $Y \in [F_{n+1}, Y_{n+1} + F_n] \subseteq \mathcal{U}_n$; hence, $Y \in \mathcal{H}$. Since X is $\mathcal{H}$-minimal, there exists a C-embedding $T : X \to Y$ for some constant C. For all $n \in \mathbb{N}$, let $X_n = T^{-1}(Y \cap Y_{n+1})$. Recall that $Y \subseteq Y_{n+1} + F_n$; we deduce that X_n has codimension at most $\dim(F_n)$ in X. Hence, X $K^{\dim(F_n)}$-embeds into X_n, so X $(CK^{\dim(F_n)})$-embeds into Y_{n+1}. In particular, Y_n $(CK^{\dim(F_n)})$-embeds into Y_{n+1}. For $n \geqslant C$, this contradicts the definition of Y_{n+1}. ∎

We now prove that X is uniformly $\mathcal{H}$-minimal. Let $Y \in \mathcal{H}$ be uniformly $\mathcal{H}_{\restriction Y}$-minimal, with constant K, given by Claim 5.21. Let C be such that X C-embeds into Y. If Z is an arbitrary element of $\mathcal{H}$, then Z also C-embeds into Y, so the uniformly $\mathcal{H}_{\restriction Y}$-minimal space Y CK-embeds into Z, and therefore X C^2K-embeds into Z. ∎

An interesting consequence of Proposition 5.20 in the case of internal degrees is that if X is d-minimal, then d-large subspaces of X are uniformly d-large, in the following sense:

Lemma 5.22. *Suppose that d is an internal degree, and X is d-minimal. Then there exists a mapping $\Gamma : \mathbb{N} \to \mathbb{R}_+$ with $\lim_{n\to\infty} \Gamma(n) = \infty$ having the following property: for every d-large subspace $Y \subseteq X$ and every $n \in \mathbb{N}$, there exists an n-dimensional subspace $F \subseteq Y$ with $d(F) \geqslant \Gamma(n)$.*

Proof. Recall that if d is an internal degree, writing $d(F)$ for $F \in \mathrm{Ban}^{<\infty}$, we actually mean $d(X, F)$ for any $X \in \mathrm{Ban}$ such that $F \subseteq X$. In particular, if $S : G \to F$ is an isomorphism for $F, G \in \mathrm{Ban}^{<\infty}$, then (S, S^{-1}) is a morphism from (F, F) to (G, G), so we have $d(G) \leqslant K_d(\|S\| \cdot \|S^{-1}\|, d(F))$.

By Proposition 5.20, there exists a constant C such that X C-embeds into all of its d-large subspaces. For all $n \in \mathbb{N}$, let $\gamma(n) = \sup\{d(F) \mid F \in \mathrm{Sub}^{<\infty}(X), \dim(F) = n\}$, which is finite by Lemma 3.21. By Remark 3.20, γ is non-decreasing, and since X is

d-large, it tends to infinity. Now for all $n \in \mathbb{N}$, let

$$\Gamma(n) = \sup\{t \in \mathbb{R}_+ \mid K_d(C,t) \leqslant \gamma(n)/2\},$$

with the convention that $\sup \emptyset = 0$. This defines a mapping $\Gamma : \mathbb{N} \to [0,\infty]$; we will see later that it actually only takes finite values.

We first show that $\lim_{n\to\infty} \Gamma(n) = \infty$. Fix $K \geqslant 0$. There is $n_0 \in \mathbb{N}$ such that $\gamma(n_0) \geqslant 2K_d(C,K)$. Now fix $n \geqslant n_0$. For all $t \leqslant K$, we have $K_d(C,t) \leqslant K_d(C,K) \leqslant \gamma(n_0)/2 \leqslant \gamma(n)/2$, so by definition of $\Gamma(n)$, we have $\Gamma(n) \geqslant K$, as desired.

Now, we fix a d-large subspace Y of X, and $n \in \mathbb{N}$, and we build an n-dimensional subspace $F \subseteq Y$ such that $d(F) \geqslant \Gamma(n)$ (this will in particular show that $\Gamma(n)$ is finite). Let $T : X \to Y$ be a C-embedding. Fix an n-dimensional subspace $G \subseteq X$ with $d(G) > \gamma(n)/2$. Let $F = T(G)$. Then by the remark at the beginning of the proof, $\gamma(n)/2 < d(G) \leqslant K_d(C, d(F))$. In particular, if $t \in \mathbb{R}_+$ is such that $K_d(C,t) \leqslant \gamma(n)/2$, then $t < d(F)$. Thus, $\Gamma(n) \leqslant d(F)$, as desired. ∎

A d-minimal space that is not minimal has to be saturated with d-small subspaces. If d is an internal degree, then for such a space X, Lemma 5.22 is quite surprising: it implies that for subspaces $Y \subseteq X$, either the degrees of finite-dimensional subspaces of Y are bounded, or their maximal value grows quite fast to infinity (at least at the same speed as Γ), but no intermediate growth is possible. This suggests that the structure of finite-dimensional subspaces of such a space X must be rather peculiar. We do not know any example of a d-minimal space that is not minimal, and this last remark makes us think that maybe such spaces do not exist when the degree d is internal.

Question 5.23. *Does there exist an internal degree d such that some infinite-dimensional spaces are d-large, and for which all d-minimal Banach spaces are minimal? Does there exist one for which there exist d-minimal, non-minimal Banach spaces?*

An immediate consequence of Lemma 5.22 is the following:

Corollary 5.24. *If d is an internal degree, then d-minimal spaces cannot be asymptotically d-small.*

As a consequence, we obtain:

Theorem 5.25. *If d is an internal degree, then d-large, asymptotically d-small Banach spaces are ergodic.*

Proof. Suppose X is a d-large, asymptotically d-small Banach space. Then all subspaces of X are asymptotically d-small, so X has no d-minimal subspaces. By Corollary 5.17, X is ergodic. ∎

Theorem 5.25 is a generalization of Anisca's Theorem 1.13, which corresponds to the case of the degree defined by $d_{BM}(F, \ell_2^{\dim(F)})$. This degree is studied in detail in the next section.

6. The Hilbertian degree

In this last section, we study the consequences of all the previous results in the special case of the *Hilbertian degree*, that is, the internal degree defined by $d_{BM}(F, \ell_2^{\dim(F)})$, for which small spaces are exactly Hilbertian spaces, as a consequence of Kwapień's theorem [36]. We shall denote this degree d_2:

$$d_2(F) := d_{BM}(F, \ell_2^{\dim(F)}).$$

For brevity, d_2-better FDD's will sometimes be called *better FDD's*. Let us spell out that a non-Hilbertian space is a d_2-HI space if it contains no direct sum of two non-Hilbertian subspaces, and d_2-minimal if it embeds into all of its non-Hilbertian subspaces ("minimal among non-Hilbertian spaces"). An FDD is d_2-tight if all non-Hilbertian spaces are tight in it. In the case of the Hilbertian degree, our two dichotomies can be summarized as follows:

Theorem 6.1. *Let X be a non-Hilbertian Banach space. Then X has a non-Hilbertian subspace Y satisfying one of the following mutually exclusive properties:*

(1) *Y is d_2-minimal and has a d_2-better UFDD;*

(2) *Y has a d_2-better d_2-tight UFDD;*

(3) *Y is d_2-minimal and d_2-hereditarily indecomposable;*

(4) *Y is d_2-tight and d_2-hereditarily indecomposable.*

It is clear from the definitions that if a Banach space X does not contain any isomorphic copy of ℓ_2, then the d_2-HI property is just the HI property and d_2-minimality is just classical minimality. It is also easy to check that if X is not ℓ_2-saturated, then our two local dichotomies do not provide more information than the original ones.

In the case of ℓ_2-saturated Banach spaces, Theorem 6.1 is more interesting and can be seen as the starting point of a Gowers list for ℓ_2-saturated, non-Hilbertian spaces. It would be interesting to extend this list and to study it more carefully (this could also be done in the case of other degrees). In particular, in the case of ℓ_2-saturated spaces, the only class of those defined by Theorem 6.1 that we know to be non-empty is (2), as will be seen in Corollary 6.11.

Question 6.2. *Which classes of those defined by Theorem 6.1 contain ℓ_2-saturated Banach spaces?*

It would also be interesting to know where the classical ℓ_2-saturated spaces lie in this classification. Perhaps the most iconic example of such a space is James' quasi-reflexive space [29]. Another important one is Kalton–Peck's [32] twisted Hilbert space Z_2. Since Z_2 has a 2-dimensional UFDD which is symmetric and therefore is a good UFDD, the cases (3) and (4) are excluded for subspaces of Z_2. Of course other twisted Hilbert spaces than Z_2 are also relevant. Note that Kalton proved that (non-trivial) twisted Hilbert spaces fail to have an unconditional basis [31]. Another ℓ_2-saturated space of interest could be G. Petsoulas' space [46], whose properties have some similarities to (but are weaker

than) those of d_2-HI spaces. Another example was announced very recently by Argyros, Manoussakis and Motakis and will be commented upon in the subsection on d_2-HI spaces.

Question 6.3. *Does James' space belong to one of the classes defined by Theorem 6.1? If not, in which of those classes can we find subspaces of James' space?*

Question 6.4. *Does Kalton–Peck's space contain a non-Hilbertian d_2-minimal subspace?*

6.1. The property of minimality among non-Hilbertian spaces

In this subsection, we study basic properties of minimality among non-Hilbertian spaces (or d_2-minimality). This property is particularly important in the study of ergodicity, since in the case of the Hilbertian degree, Corollary 5.17 takes the following form:

Theorem 6.5. *Every non-ergodic, non-Hilbertian separable Banach space contains a d_2-minimal subspace.*

In particular, Ferenczi–Rosendal's Conjecture 1.9 reduces to the special case of d_2-minimal spaces.

Concerning their relationship to Johnson's Question 1.2, we can say even more. Indeed, the following result has been proved by Anisca [2] (originally under a finite cotype hypothesis which may be removed due to, e.g., Theorem 1.14).

Theorem 6.6 (Anisca). *A separable Banach space having finitely many different subspaces, up to isomorphism, contains an isomorphic copy of ℓ_2.*

The result of Anisca is based on the construction, in unconditional spaces with finite cotype not containing copies of ℓ_2, and for each n, of a subspace having n-dimensional UFDD's but no UFDD of smaller dimension.

In particular, this applies to Johnson spaces and we get:

Proposition 6.7. *Every Johnson space is d_2-minimal.*

In the rest of this paper, d_2-minimal spaces that are not minimal will be called *non-trivial d_2-minimal*; these spaces are necessarily ℓ_2-saturated. We do not know of any example of a non-trivial d_2-minimal space. If X is such a space, then Lemma 5.22 shows that there is a uniform lower bound on the growth rates of the functions $n \mapsto \sup\{d_{BM}(F, \ell_2^n) \mid F \in \mathrm{Sub}^{<\infty}(Y),\ \dim(F) = n\}$, where Y ranges over non-Hilbertian subspaces of X. This very surprising property suggests that either non-trivial d_2-minimal spaces do not exist, or the structure of their finite-dimensional subspaces is rather peculiar. Note that, however, this uniform growth property holds for the spaces L_p for $2 < p < \infty$, which are not ℓ_2-saturated (nor d_2-minimal) but contain copies of ℓ_2. This is a consequence of the fact that the spaces L_p for $2 < p < \infty$ are finitely representable in all of their non-Hilbertian subspaces (this can be deduced from [45, Proposition 3.1]).

Question 6.8. *Does there exist a non-trivial d_2-minimal space?*

We now study additional properties of d_2-minimal spaces, in particular those related to the existence of basic sequences. In the case of the Hilbertian degree, Corollary 5.24 takes the following form:

Proposition 6.9. *An asymptotically Hilbertian Banach space cannot be d_2-minimal.*

Example 6.10. Let $(p_n)_{n\in\mathbb{N}}$ be a sequence of real numbers greater than 1 and tending to 2, and let $(k_n)_{n\in\mathbb{N}}$ be a sequence of natural numbers tending to ∞ such that $\lim_{n\to\infty} d_{BM}(\ell_{p_n}^{k_n}, \ell_2^{k_n}) = \infty$. Consider the space $X = (\bigoplus_{n\in\mathbb{N}} \ell_{p_n}^{k_n})_{\ell_2}$. It has a better UFDD and is non-Hilbertian, ℓ_2-saturated and asymptotically Hilbertian (this last property follows from [37, Corollary 5]). In particular, it cannot have a d_2-minimal subspace. So by Theorem 5.6, some block-FDD of its UFDD is d_2-tight.

Example 6.10 shows:

Corollary 6.11. *The class of non-Hilbertian, ℓ_2-saturated Banach spaces having a better d_2-tight UFDD is non-empty.*

The property of being asymptotically Hilbertian is closely related to property (H) of Pisier.

Definition 6.12 (Pisier, [49]). A Banach space X is said to have *property* (H) if for every $\lambda \geqslant 1$, there exists a constant $K(\lambda)$ such that for every finite, normalized, λ-unconditional basic sequence $(x_i)_{i<n}$ of elements of X, we have

$$\frac{\sqrt{n}}{K(\lambda)} \leqslant \left\| \sum_{i<n} x_i \right\| \leqslant K(\lambda)\sqrt{n}.$$

Recall that all normalized λ-unconditional basic sequences in Hilbert spaces are λ-equivalent to the canonical basis of ℓ_2 (see for instance [1, Theorem 8.3.5]). A consequence is that every Hilbertian space has property (H). Thus, property (H) is a property of proximity to Hilbertian spaces. The proof of the following result of Johnson (unpublished) can be found in Pisier's paper [49].

Proposition 6.13 (Johnson). *Every space with property* (H) *is asymptotically Hilbertian.*

In particular, d_2-minimal spaces fail property (H). A consequence is the following:

Lemma 6.14. *Let X be a d_2-minimal space. Then there exists $\lambda_0 \geqslant 1$ with the following property: in every non-Hilbertian subspace Y of X, one can find finite-dimensional subspaces F with a normalized λ_0-unconditional basis for which the Banach–Mazur distance $d_{BM}(F, \ell_2^{\dim(F)})$ is arbitrarily large.*

Proof. By Proposition 5.20, X uniformly embeds into all of its non-Hilbertian subspaces. In particular, it is enough to prove the result for $Y = X$. Let λ_0 witness that X fails property (H). Towards a contradiction, suppose there is a constant C such that every finite-dimensional subspace of X with a normalized λ_0-unconditional basis is C-isomorphic to

a Euclidean space. Let F be such a subspace and $(x_i)_{i<n}$ be its unconditional basis. Choose an isomorphism $T : F \to \ell_2^n$ with $\|T\| \leqslant C$ and $\|T^{-1}\| = 1$, and let $y_i = T(x_i)$ and $z_i = y_i/\|y_i\|$ for all $i < n$. Then $(y_i)_{i<n}$ is $C\lambda_0$-unconditional, and so is $(z_i)_{i<n}$. Hence, (z_i) is $C\lambda_0$-equivalent to the canonical basis of ℓ_2^n. Since, for all $i < n$, we have $1 \leqslant \|y_i\| \leqslant C$, and (z_i) is $C\lambda_0$-unconditional, we have, for every sequence $(a_i)_{i<n} \in \mathbb{R}^n$,

$$\frac{1}{C\lambda_0}\Big\|\sum_{i<n} a_i z_i\Big\| \leqslant \Big\|\sum_{i<n} a_i y_i\Big\| \leqslant C^2\lambda_0\Big\|\sum_{i<n} a_i z_i\Big\|,$$

hence (y_i) and (z_i) are $C^2\lambda_0$-equivalent. Moreover, we know that (x_i) and (y_i) are C-equivalent. We deduce that (x_i) is $C^4\lambda_0^2$-equivalent to the canonical basis of ℓ_2^n. In particular, for $K = C^4\lambda_0^2$, we have

$$\frac{\sqrt{n}}{K} \leqslant \Big\|\sum_{i<n} x_i\Big\| \leqslant K\sqrt{n},$$

contradicting the choice of λ_0. ∎

Theorem 6.15. (1) *Every d_2-minimal space has a non-Hilbertian subspace with a Schauder basis.*

(2) *Every d_2-minimal space having an unconditional FDD has a non-Hilbertian subspace with an unconditional basis.*

A consequence of this theorem is that alternative (1) in Theorem 6.1 can be replaced with "Y is d_2-minimal and has an unconditional basis".

Knowing that every non-Hilbertian subspace of a Johnson space is isomorphic to the space itself, another consequence is:

Corollary 6.16. *Every Johnson space has a Schauder basis. Moreover, it has an unconditional basis if and only if it is isomorphic to its square.*

Proof. If it has an unconditional basis then it is isomorphic to its square by Theorem 1.7. Conversely, if it is isomorphic to its square then it is not d_2-HI and by the first local dichotomy (Theorem 4.5), it must have a UFDD. It follows from Theorem 6.15 that the space has an unconditional basis. ∎

A few additional restrictions on the existence of Johnson spaces follow from Corollary 6.16. Every Johnson space is HAPpy (every subspace has the Approximation Property). If a Johnson space X has an unconditional basis then it is reflexive, all of its subspaces have GL-lust and therefore the GL-property, so X has weak cotype 2 [39, Theorem 40]. On the other hand, since X is not weak Hilbert, X cannot have weak type 2 in this case (see [49] for these notions). For non-Hilbertian examples of HAPpy spaces with a symmetric basis (and therefore also non-asymptotically-Hilbertian), see [30].

Theorem 6.15 naturally raises the following two questions (the first one had already been asked by Pełczyński [44]):

Question 6.17. (1) *Does every non-Hilbertian space have a non-Hilbertian subspace with a Schauder basis?*

(2) *Does every non-Hilbertian space with unconditional FDD have a non-Hilbertian subspace with an unconditional basis?*

Proof of Theorem 6.15. (1) Let X be a d_2-minimal space, and fix λ_0 as given by Lemma 6.14 for X. Build an FDD $(F_n)_{n\in\mathbb{N}}$ of a subspace of X, along with a decreasing sequence $(Y_n)_{n\in\mathbb{N}}$ of finite-codimensional subspaces of X, by induction as follows. Let $Y_0 = X$. If the subspace Y_n and all the F_m's, for $m < n$, are built, we can find $F_n \subseteq Y_n$ with a normalized λ_0-unconditional basis such that $d_{BM}(F_n, \ell_2^{\dim(F_n)}) \geqslant n$. We then find a finite-codimensional subspace $Y_{n+1} \subseteq Y_n$ with $Y_{n+1} \cap [F_m \mid m \leqslant n] = \{0\}$ such that the first projection $[F_m \mid m \leqslant n] \oplus Y_{n+1} \to [F_m \mid m \leqslant n]$ has norm at most 2. This finishes the induction.

The sequence $(F_n)_{n\in\mathbb{N}}$ we just built is an FDD of a non-Hilbertian subspace Y of X. It has constant at most 2, and all the F_n's have a basis with constant at most λ_0. Thus, concatenating these bases, we get a basis of Y with constant at most $2 + 4\lambda_0$, as desired.

(2) Let X be a d_2-minimal space with a UFDD $(F_n)_{n\in\mathbb{N}}$. If (F_n) has a normalized block-sequence spanning a non-Hilbertian subspace, then we are done. So from now on, we assume that every normalized block-sequence of (F_n) spans a Hilbertian subspace. Since normalized block-sequences of (F_n) are unconditional, we deduce that all of them are equivalent to the canonical basis of ℓ_2. Our first step is to prove that this holds uniformly.

Claim 6.18. *There exists a constant C satisfying the following property: every normalized block-sequence of (F_n) is C-equivalent to the canonical basis of ℓ_2.*

Proof. We prove the formally weaker, but actually equivalent, statement: there exist $n_0 \in \mathbb{N}$ and a constant C such that every normalized block-sequence of $(F_n)_{n\geqslant n_0}$ is C-equivalent to the canonical basis of ℓ_2. Suppose that this does not hold. Then for every $n_0, N \in \mathbb{N}$ we can find a finite normalized block-sequence $(x_i)_{i<i_0}$ of $(F_n)_{n\geqslant n_0}$ which is not N-equivalent to the canonical basis of $\ell_2^{i_0}$. Applying this for successive values of N, we can build by induction a normalized block-sequence $(x_i)_{i\in\mathbb{N}}$ of (F_n) and a sequence $0 = i_0 < i_1 < i_2 < \cdots$ such that for every $N \in \mathbb{N}$, the sequence $(x_i)_{i_N\leqslant i<i_{N+1}}$ is not N-equivalent to the canonical basis of $\ell_2^{i_{N+1}-i_N}$. In particular, $(x_i)_{i\in\mathbb{N}}$ is not equivalent to the canonical basis of ℓ_2, a contradiction. ∎

We now finish the proof of Theorem 6.15, proceeding as in (1). Fix λ_0 as given by Lemma 6.14 for X. Observe that for every $n_0 \in \mathbb{N}$, we can find a finite-dimensional subspace $G \subseteq [F_n \mid n \geqslant n_0]$, finitely supported on the FDD (F_n), having a normalized $2\lambda_0$-unconditional basis and such that $d_{BM}(G, \ell_2^{\dim(G)})$ is arbitrarily large: indeed, it is enough to take a small perturbation of a (not necessarily finitely supported) finite-dimensional subspace $G' \subseteq [F_n \mid n \geqslant n_0]$ with a normalized λ_0-unconditional basis and large $d_{BM}(G', \ell_2^{\dim(G')})$. Using this remark, we can build a better block-FDD $(G_k)_{k\in\mathbb{N}}$ of (F_n) such that all the G_k's have a $2\lambda_0$-unconditional basis. Let $i_k = \sum_{l<k} \dim(G_l)$

for every $k \in \mathbb{N}$, and denote by $(x_i)_{i_k \leqslant i < i_{k+1}}$ the unconditional basis of G_k. To conclude the proof, it is enough to prove that the sequence $(x_i)_{i\in\mathbb{N}}$ is unconditional.

So let $(a_i)_{i\in\mathbb{N}}$ be a finitely supported sequence of real numbers and $(\varepsilon_i)_{i\in\mathbb{N}}$ be a sequence of signs. For every $k \in \mathbb{N}$, let $b_k, c_k \geqslant 0$ and $y_k, z_k \in S_{G_k}$ be such that $b_k y_k = \sum_{i_k \leqslant i < i_{k+1}} a_i x_i$ and $c_k z_k = \sum_{i_k \leqslant i < i_{k+1}} \varepsilon_i a_i x_i$. In particular, $b_k = \| \sum_{i_k \leqslant i < i_{k+1}} a_i x_i \|$ and $c_k = \| \sum_{i_k \leqslant i < i_{k+1}} \varepsilon_i a_i x_i \|$, so since the sequence $(x_i)_{i_k \leqslant i < i_{k+1}}$ is $2\lambda_0$-unconditional, we have $c_k \leqslant 2\lambda_0 b_k$. Now, since $(y_k)_{k\in\mathbb{N}}$ and $(z_k)_{k\in\mathbb{N}}$ are normalized block-sequences of (F_n), they are C-equivalent to the canonical basis of ℓ_2. Thus, we have

$$\begin{aligned}\Big\| \sum_{i\in\mathbb{N}} \varepsilon_i a_i x_i \Big\| &= \Big\| \sum_{k\in\mathbb{N}} c_k z_k \Big\| \leqslant C \sqrt{\sum_{k\in\mathbb{N}} c_k^2} \\ &\leqslant 2\lambda_0 C \sqrt{\sum_{k\in\mathbb{N}} b_k^2} \leqslant 2\lambda_0 C^2 \Big\| \sum_{i\in\mathbb{N}} a_i x_i \Big\|,\end{aligned}$$

proving that the sequence $(x_i)_{i\in\mathbb{N}}$ is $2\lambda_0 C^2$-unconditional. ∎

6.2. *Properties of d_2-HI spaces*

Recall that d_2-HI spaces are non-Hilbertian Banach spaces that do not contain any direct sum of two non-Hilbertian subspaces. HI spaces are of course d_2-HI. We could only discover two other examples of d_2-HI spaces. Before presenting them, we recall a basic result in operator theory (see [41, Proposition 3.2]). The terminology of the next definition is from [26].

Definition 6.19. An operator $T : X \to Y$ between two Banach spaces is *infinitely singular* if there is no finite-codimensional subspace $X_0 \subseteq X$ such that $T_{\restriction X_0} : X_0 \to T(X_0)$ is an isomorphism.

Proposition 6.20 (Folklore). *An operator $T : X \to Y$ between two Banach spaces is infinitely singular if and only if for every $\varepsilon > 0$, there exists a subspace $X_\varepsilon \subseteq X$ such that $\|T_{\restriction X_\varepsilon}\| \leqslant \varepsilon$.*

Example 6.21. Let Y be an HI space. Then $X = Y \oplus \ell_2$ is d_2-HI. Indeed, denote by $p_Y : X \to Y$ and $p_{\ell_2} : X \to \ell_2$ the two projections. Suppose that two non-Hilbertian subspaces $U, V \subseteq X$ are such that the sum $U + V$ is direct. Then $(p_{\ell_2})_{\restriction U}$ and $(p_{\ell_2})_{\restriction V}$ are infinitely singular, so by Proposition 6.20, we can find subspaces $U' \subseteq U$ and $V' \subseteq V$ on which p_{ℓ_2} has arbitrarily small norm. In particular, U' and V' can be chosen in such a way that $\|(p_{\ell_2})_{\restriction U' \oplus V'}\| \leqslant 1/2$. Thus, p_Y induces an isomorphism between $U' \oplus V'$ and $p_Y(U' \oplus V')$. In particular, the sum $p_Y(U') + p_Y(V')$ is direct, contradicting the fact that Y is HI.

Example 6.22. In [6], Argyros and Raikoftsalis build, for every $1 \leqslant p < \infty$ (resp. for $p = \infty$), a space $\mathfrak{X}_p$ having the following properties: $\mathfrak{X}_p \cong \mathfrak{X}_p \oplus \ell_p$ (resp. $\mathfrak{X}_p \cong \mathfrak{X}_p \oplus c_0$), and for every decomposition as a direct sum $\mathfrak{X}_p = Y \oplus Z$, we have $Y \cong \mathfrak{X}_p$ and $Z \cong \ell_p$ (resp. $Z \cong c_0$), or vice versa. The space $\mathfrak{X}_p$ is built as an *HI Schauder sum* of copies of ℓ_p

(resp. c_0); the construction of such a sum is quite involved and is given in [4, Section 7]. In [6], the following results are proved:

(1) $\mathfrak{X}_p$ does not contain any direct sum of two HI subspaces (see [6, proof of Lemma 1]);

(2) for every subspace $Y \subseteq \mathfrak{X}_p$ not containing any HI subspace, and every $\varepsilon > 0$, there exists a projection P of $\mathfrak{X}_p$ with image isomorphic to ℓ_p (resp. to c_0) such that $\|(\mathrm{Id}_{\mathfrak{X}_p} - P)_{\restriction Y}\| \leqslant \varepsilon$ (see [6, Lemma 3]).

This implies that $\mathfrak{X}_2$ is d_2-HI. Indeed, if two subspaces $Y, Z \subseteq \mathfrak{X}_2$ are such that the sum $Y + Z$ is direct, then by (1), one of them does not contain any HI subspace, for example Y. Choosing a projection P as given by (2) for $\varepsilon = 1/2$, we find that $P_{\restriction Y}$ is an isomorphism onto its image, which is contained in an isomorphic copy of ℓ_2; so Y is Hilbertian.

The interest of d_2-HI spaces in the study of ergodicity, and in particular of our Conjectures 1.15 and 1.16, comes from the following result:

Theorem 6.23. *Let X be a non-ergodic, non-Hilbertian separable Banach space. Then X has a non-Hilbertian subspace Y such that either*

- *Y has an unconditional basis, or*
- *Y is simultaneously d_2-minimal and d_2-HI.*

Proof. By Corollary 5.17, we can assume that X is d_2-minimal. By Theorem 4.5, X has either a subspace with a better UFDD, or a d_2-HI subspace. In the first case, Theorem 6.15 shows that we can find a further non-Hilbertian subspace having an unconditional basis. ∎

It would of course be interesting to remove the second alternative, thus reducing somehow the problem to spaces with unconditional bases. This motivates the following question:

Question 6.24. *Does there exist a non-ergodic Banach space which is simultaneously d_2-minimal and d_2-HI?*

Both examples of d_2-HI spaces given above contain an HI subspace. In particular, they are ergodic, and they cannot be d_2-minimal. Thus, Question 6.24 reduces to the special case of d_2-HI spaces that do not contain any HI subspace. The latter spaces are exactly those d_2-HI spaces that are ℓ_2-saturated. We know of no examples of such spaces.

Question 6.25. *Do there exist ℓ_2-saturated d_2-HI spaces?*

After this article was submitted, Argyros, Manoussakis and Motakis [5] announced that they were able to build an ℓ_2-saturated d_2-HI space, thus giving an affirmative answer to Question 6.25. The construction will be published in a forthcoming paper, and the paper [5] gives the construction of analogues of that space.

We now come back to Question 6.24. We conjecture that the answer to this question is negative, and we actually have the following stronger conjecture:

Conjecture 6.26. *A Banach space cannot be simultaneously d_2-minimal and d_2-HI.*

This conjecture is motivated by the fact that the d_2-HI property is a weakening of the HI property, and it is known that HI spaces have many different subspaces, up to isomorphism. For example, Gowers–Maurey's Theorem 1.18 says that HI spaces cannot be isomorphic to any proper subspace of themselves. This implies, in particular, that they cannot be minimal. It would be tempting to adapt Gowers–Maurey's approach to d_2-HI spaces. Note, however, that in the case of d_2-HI spaces, we cannot hope to have a result as strong as Gowers–Maurey's, since the spaces presented in Examples 6.21 and in 6.22 are both isomorphic to their hyperplanes, and even to their direct sum with ℓ_2. However, we can hope that these spaces cannot be isomorphic to "too deep" subspaces of themselves. This is at least the case for our first example, as shown by the following lemma:

Lemma 6.27. *Let Y be an HI space and let $X = Y \oplus \ell_2$. Then every subspace of X that is isomorphic to X is complemented in X by a (finite- or infinite-dimensional) Hilbertian subspace.*

Proof. Denote by $P_Y : X \to Y$ and $P_{\ell_2} : X \to \ell_2$ the projections. Let $U \subseteq X$ be an isomorphic copy of X; we can write $U = V \oplus W$, where $V \cong Y$ and $W \cong \ell_2$. Suppose that $(P_Y)_{\restriction V}$ is infinitely singular. Then by Proposition 6.20, we can find a subspace $V'' \subseteq V$ on which P_Y has small norm. In particular, P_{ℓ_2} would induce an isomorphism between V'' and a subspace of ℓ_2, a contradiction. Thus, $(P_Y)_{\restriction V}$ is not infinitely singular: we can find a finite-codimensional subspace V' of V such that P_Y induces an isomorphism between V' and $P_Y(V')$.

Observe that $V' \cong P_Y(V')$, and V' and $P_Y(V')$ are subspaces of respectively V and Y that are HI and isomorphic. By Gowers–Maurey's Theorem 1.18, we deduce that the codimension of $P_Y(V')$ in Y is equal to the codimension of V' in V, so is finite. So write $Y = P_Y(V') \oplus F$, where F has finite dimension. We have $P_Y(V') \subseteq V + \ell_2$, so $Y \subseteq V + \ell_2 + F$, so

$$X = V + \ell_2 + F = U + \ell_2 + F.$$

Letting Z be a complement of $U \cap (\ell_2 + F)$ in $\ell_2 + F$, we see that Z is Hilbertian and $X = U \oplus Z$. ∎

Question 6.28. *Let X be d_2-HI and let Y be a subspace of X which is isomorphic to X. Does it follow that Y is complemented by a (finite- or infinite-dimensional) Hilbertian subspace?*

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