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Ricci pinched compact hypersurfaces in spheres

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Abstract. We investigate the topology of the compact hypersurfaces in round spheres focusing on those with the Ricci curvature satisfying an appropriate bound determined solely by the mean curvature of the submanifold. In this paper, the application of the Bochner technique yields more robust results compared to those presented in [2] for submanifolds that lay in any codimension.

Compact submanifolds in the unit sphere $f: M^n \to \mathbb{S}^{n+p}$, $n \geq 4$, with any codimension p have been investigated in [2] under a pinching condition on the Ricci curvature that depends solely on the norm of the mean curvature vector field. In this paper, the special case of codimension $p = 1$ is considered. Many examples of hypersurfaces meeting the pinching condition have been given in [2]. In this paper, we are able to obtain more robust results than in [2] because the Bochner technique applies to hypersurfaces in opposition to the case of higher codimension.

Let $f: M^n \to \mathbb{S}^{n+1}$, $n \geq 4$, be an isometric immersion of an n-dimensional compact manifold into the unit sphere. Given an integer k that satisfies $2 \leq k \leq n/2$, we denote

$$b(n,k,H) = \frac{n(k-1)}{k} + \frac{n(k-1)H}{2k^2}\left(nH + \sqrt{n^2H^2 + 4k(n-k)}\right)$$

being H the length of the (normalized) mean curvature vector field of f. Throughout the paper, we assume that at any point of M^n the (not normalized) Ricci curvature satisfies the pinching condition

$$\mathrm{Ric}_M \geq b(n,k,H). \tag{*}$$

We say that $(*)$ is *satisfied with equality* at $x \in M^n$ if the inequality at that point is not strict, that is, there exists a unit vector $X \in T_xM$ such that $\mathrm{Ric}_M(X) = b(n,k,H)$. If it happens otherwise, we say that $(*)$ is *strict* at $x \in M^n$.

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Recall that the generalized Clifford torus is the standard embedding of $\mathbb{T}^n_p(r) = \mathbb{S}^p(r) \times \mathbb{S}^{n-p}(\sqrt{1-r^2})$, $2 \le p \le n-2$, into the unit sphere $\mathbb{S}^{n+1}$, where $\mathbb{S}^p(r)$ denotes the p-dimensional sphere of radius $r < 1$. Computations given in [7] yield that $(*)$ is satisfied if $p = k$ and $(k-1)/(n-2) \le r^2 \le k/n$, and that this happens with equality.

Theorem 1. *Let $f\colon M^n \to \mathbb{S}^{n+1}$, $n \ge 4$, be an isometric immersion of a compact manifold. Assume that f satisfies the pinching condition $(*)$ for some $k \ge 2$ where $k < n/2$ if n is even and $k < (n-1)/2$ if n is odd. Then M^n is simply connected, hence orientable, and one of the following cases occurs:*
(i) The homology groups satisfy

$$H_i(M^n;\mathbb{Z}) = H_{n-i}(M^n;\mathbb{Z}) = 0 \ \textit{for all} \ 1 \le i \le k$$

and $H_{n-k-1}(M^n;\mathbb{Z}) = \mathbb{Z}^{\beta_{k+1}(M)}$, where $\beta_{k+1}(M)$ denotes the $(k+1)$-th Betti number of M^n. This is necessarily the case if $(*)$ is strict at some point.
(ii) The homology groups satisfy

$$H_i(M^n;\mathbb{Z}) = H_{n-i}(M^n;\mathbb{Z}) = 0 \ \textit{for all} \ 1 \le i \le k-1,$$

$H_k(M^n;\mathbb{Z}) \neq 0$ is finite and $H_{n-k}(M^n;\mathbb{Z}) = 0$. For $k = 2$, we also assume that $H \neq 0$ at all points. Then, at any point

$$\lambda(n,k,H) = \frac{1}{2k}\left(nH + \sqrt{n^2H^2 + 4k(n-k)}\right)$$

is a principal curvature whose multiplicity ℓ satisfies $k \le \ell \le n-k-1$. Moreover, equality holds in $(*)$ on the principal distribution $T_\lambda = \ker(A - \lambda I)$ where A denotes the shape operator of f.
(iii) $M^n = \mathbb{T}^n_k(r)$ with $(k-1)/(n-2) \le r^2 \le k/n$ and f is the standard embedding in $\mathbb{S}^{n+1}$.

Hypersurfaces obtained in part (ii) admit the parametrization given by Theorem 8 below on connected components of an open dense subset of M^n.

Let $f\colon M^n \to \mathbb{S}^{n+1}$, $n \ge 3$, be a hypersurface oriented by the unit normal vector field ξ and let A denote the associated shape operator. Let λ be a principal curvature of constant multiplicity ℓ with $2 \le \ell < n$ and let $T_\lambda = \ker(A - \lambda I)$ be the corresponding integrable principal distribution. The associated *focal map* $f_\lambda\colon M^n \to \mathbb{S}^{n+1}$ is given by

$$f_\lambda = \cos\sigma f + \sin\sigma \xi \quad \text{where} \quad \lambda = \cot\sigma.$$

Let L be the space of leaves M^n/T_λ. It is a standard fact (cf. Theorem 3.1 in [1]) that if M^n is complete, then the focal map factors through an immersion $g\colon L \to \mathbb{S}^{n+1}$ of the $(n-\ell)$-dimensional manifold $L^{n-\ell}$. The submanifold g is called the *focal submanifold* associated to λ.

The cases when $k = n/2$ if n is even and $k = (n-1)/2$ if n is odd have been considered in [2] for arbitrary codimension. In fact, for the first case we have from there the following result reiterated here for the sake of completeness.

Theorem 2. ([2]) *Let $f\colon M^n \to \mathbb{S}^{n+1}$, $n \geq 4$, be an isometric immersion of a compact manifold of even dimension. Assume that*

$$Ric_M \geq (n-2)\big(1 + H^2 + H\sqrt{1+H^2}\big) \tag{1}$$

holds at any point of M^n. Then one of the following cases occurs:
(i) M^n is homeomorphic to $\mathbb{S}^n$ and this is necessarily the case if at some point of M^n the inequality (1) is strict.
(ii) The submanifold is the minimal generalized Clifford torus $\mathbb{T}^n_{n/2}(1/\sqrt{2})$.

For hypersurfaces of odd dimension and $k = (n-1)/2$ the following result is quite stronger compared to the one following from [2].

Theorem 3. *Let $f\colon M^n \to \mathbb{S}^{n+1}$, $n \geq 5$, be an isometric immersion of a compact manifold of odd dimension. Assume that it holds that*

$$Ric_M \geq \frac{n(n-3)}{n-1}\left(1 + \frac{H}{n-1}\big(nH + \sqrt{n^2H^2 + n^2 - 1}\big)\right) \tag{2}$$

at any point of M^n. Then one of the following cases occurs:
(i) M^n is homeomorphic to $\mathbb{S}^n$ and this is necessarily the case if at some point of M^n the inequality (2) is strict.
(ii) The homology groups $H_i(M^n;\mathbb{Z})$, $1 \leq i \leq n-1$, vanish with the exception of $H_{(n-1)/2}(M^n;\mathbb{Z}) = \mathbb{Z}_q$ for some $q > 1$. For $n = 5$ let also assume that $H \neq 0$ at any point of M^n. Then $n = 4r + 3$ and $\lambda = \lambda(n,k,H)$ is a principal curvature with multiplicity $k = (n-1)/2$ at any point of M^n and equality holds in (2) on the principal distribution T_λ. Moreover, M^n is diffeomorphic to the unit normal sphere bundle of the corresponding focal submanifold $g\colon L^{k+1} \to \mathbb{S}^{n+1}$, being L^{k+1} homeomorphic to $\mathbb{S}^{k+1}$.
(iii) $M^n = \mathbb{T}^n_{(n-1)/2}(r)$ with $(n-3)/2(n-2) \leq r^2 \leq (n-1)/2n$ and f is the standard embedding in $\mathbb{S}^{n+1}$.

From Theorem 8 given below it follows that the hypersurfaces in part (ii) admit a global parametrization.

The following is a direct consequence of the preceding result.

Corollary 4. *Let $f\colon M^n \to \mathbb{S}^{n+1}$, $n \geq 5$, be an isometric immersion of a compact manifold of odd dimension that satisfies (2). If $H_{(n-1)/2}(M^n,\mathbb{Z})$ is torsion free or if $n = 4r+1$ for $r \geq 2$ then one of the following cases occurs:*
(i) M^n is homeomorphic to $\mathbb{S}^n$ and this is necessarily the case if at some point of M^n the inequality (2) is strict.
(ii) $M^n = \mathbb{T}^n_{(n-1)/2}(r)$ with $(n-3)/2(n-2) \leq r^2 \leq (n-1)/2n$ and f is the standard embedding in $\mathbb{S}^{n+1}$.

1. The Bochner operator

We start with some algebraic preliminaries inspired by Savo [5]. Let V be a real n-dimensional vector space of dimension $n \geq 3$, equipped with a positive definite inner product $\langle\, ,\, \rangle$. We denote by $\mathrm{End}(V)$ the set of self-adjoint endomorphisms of V and by $\Lambda^p V^*$, $1 \leq p \leq n$, the $\binom{n}{p}$-dimensional real vector space defined as the p-th exterior power of the dual vector space $V^* = \mathrm{Hom}(V, \mathbb{R})$ of V. Let $A^{[p]} \in \mathrm{End}(\Lambda^p V^*)$ be given by

$$A^{[p]}\omega(v_1, \ldots, v_p) = \sum_{i=1}^{p} \omega(v_1, \ldots, Av_i, \ldots, v_p),$$

where $\omega \in \Lambda^p V^*$ and $v_1, \ldots, v_p \in V$. Then associated to $A \in \mathrm{End}(V)$ there is the endomorphism $T_A^{[p]} \in \mathrm{End}(\Lambda^p V^*)$ defined by

$$T_A^{[p]} = (\mathrm{tr}A)A^{[p]} - A^{[p]} \circ A^{[p]}$$

which is self-adjoint with respect to the natural inner product $\langle\, ,\, \rangle$ in $\Lambda^p V^*$.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of V and let $\{\theta_1, \ldots, \theta_n\}$ be the dual basis. For every integer p let $\mathcal{I}_p$ be the set of p-multi-indices

$$\mathcal{I}_p = \left\{ I = (i_1, \ldots, i_p) \colon 1 \leq i_1 < \cdots < i_p \leq n \right\}.$$

For each $I = (i_1, \ldots, i_p) \in \mathcal{I}_p$ set $\theta_I = \theta_{i_1} \wedge \cdots \wedge \theta_{i_p}$ and $e_I = (e_{i_1}, \ldots, e_{i_p})$. For any $I, J \in \mathcal{I}_p$ we have

$$\theta_I(e_J) = \begin{cases} 1 \text{ if } I = J \\ 0 \text{ if otherwise.} \end{cases}$$

Since $\{\theta_I \colon I \in \mathcal{I}_p\}$ is an orthonormal basis of $\Lambda^p V^*$, given $\omega \in \Lambda^p V^*$ we have $\omega = \sum_{I \in \mathcal{I}_p} a_I \theta_I$ where $a_I = \omega(e_I)$.

Lemma 5. *If $A \in \mathrm{End}(V)$, then for any multi-index $I \in \mathcal{I}_p$, with $1 \leq p \leq n$, we have*

$$T_A^{[p]}\theta_I = \Big(trA \sum_{i \in \mathbf{I}} \langle Ae_i, e_i \rangle - \big(\sum_{i \in \mathbf{I}} \langle Ae_i, e_i \rangle \big)^2 \Big) \theta_I,$$

where $\mathbf{I} = \{i_1, \ldots, i_p\}$ and $\{e_1, \ldots, e_n\}$ is an orthonormal basis of V that diagonalizes A.

Proof. We have that

$$A^{[p]}\theta_I = \sum_{J \in \mathcal{I}_p} \left(A^{[p]}\theta_I \right)(e_J)\theta_J. \tag{3}$$

Then we compute $\left(A^{[p]}\theta_I\right)(e_J)$ for any $I, J \in \mathcal{I}_p$. If $I = (i_1, \dots, i_p)$, $1 \le i_1 < \cdots < i_p \le n$ and $J = (j_1, \dots, j_p)$, $1 \le j_1 < \cdots < j_p \le n$, then

$$\begin{aligned}\left(A^{[p]}\theta_I\right)(e_J) &= \theta_I(Ae_{j_1}, \dots, e_{j_p}) + \cdots + \theta_I(e_{j_1}, \dots, Ae_{j_p}) \\ &= \sum_{s=1, s\notin \mathbf{J}\setminus\{j_1\}}^{n} \langle Ae_{j_1}, e_s\rangle \theta_I(e_s, e_{j_2}, \dots, e_{j_p}) \\ &\quad + \sum_{s=1, s\notin \mathbf{J}\setminus\{j_2\}}^{n} \langle Ae_{j_2}, e_s\rangle \theta_I(e_{j_1}, e_s, e_{j_3}, \dots, e_{j_p}) \\ &\quad + \cdots + \sum_{s=1, s\notin \mathbf{J}\setminus\{j_p\}}^{n} \langle Ae_{j_p}, e_s\rangle \theta_I(e_{j_1}, \dots, e_{j_{p-1}}, e_s).\end{aligned}$$

Hence

$$\left(A^{[p]}\theta_I\right)(e_J) = \sum_{j\in\mathbf{J}} \langle Ae_j, e_j\rangle \theta_I(e_J),$$

and thus

$$\left(A^{[p]}\theta_I\right)(e_J) = \begin{cases} \sum_{i\in\mathbf{I}} \langle Ae_i, e_i\rangle & \text{if } J = I \\ 0 & \text{if otherwise.} \end{cases}$$

Then (3) yields

$$A^{[p]}\theta_I = \sum_{i\in\mathbf{I}} \langle Ae_i, e_i\rangle \theta_I$$

for any $I \in \mathcal{I}_p$. The proof now follows from the definition of $T_A^{[p]}$. □

Let M^n be an orientable Riemannian manifold of dimension n. For each integer $0 \le p \le n$, the Hodge-Laplace operator acting on differential p-forms is defined by

$$\Delta = d\delta + \delta d : \Omega^p(M^n) \to \Omega^p(M^n),$$

where d and δ are the differential and the co-differential operators, respectively. For $p = 0$ the Hodge-Laplace operator is just the Laplace-Beltrami operator acting on 0-forms, that is, scalar functions.

A key element in our methodology revolves around the Bochner technique, rooted in the Bochner-Weitzenböck formula. It states that the Laplacian of every p-form $\omega \in \Omega^p(M^n)$ on a manifold M^n is given by

$$\Delta\omega = \nabla^*\nabla\omega + \mathcal{B}^{[p]}\omega, \tag{4}$$

where $\nabla^*\nabla$ is the so called rough Laplacian or connection Laplacian and $\mathcal{B}^{[p]}\colon \Omega^p(M^n) \to \Omega^p(M^n)$ is a certain symmetric endomorphism of the bundle of p-forms called the *Bochner operator*.

Proposition 6. *Let $f\colon M^n \to \mathbb{S}^{n+1}$, $n \geq 4$, be an isometric immersion of a compact oriented manifold satisfying the inequality* $(*)$ *for an integer* $2 \leq k \leq n/2$. *Then the Bochner operator* $\mathcal{B}^{[k]}$ *is nonnegative.*

Proof. Let $\{e_1, \ldots, e_n\}$ be a local orthonormal frame of the tangent bundle that diagonalizes the shape operator A and let $\{\theta_1, \ldots, \theta_n\}$ be the dual frame. Set $\theta_I = \theta_{i_1} \wedge \cdots \wedge \theta_{i_n}$ for any $I = (i_1, \ldots, i_k) \in \mathcal{I}_k$. Then for any k-form $\omega = \sum_{I \in \mathcal{I}_k} a_I \theta_I$ it follows from Theorem 1 in [5] that

$$\langle \mathcal{B}^{[k]} \omega, \omega \rangle = k(n-k)\|\omega\|^2 + \langle T_A^{[p]} \omega, \omega \rangle.$$

Using Lemma 5, the above becomes

$$\begin{aligned}\langle \mathcal{B}^{[k]} \omega, \omega \rangle &= k(n-k)\|\omega\|^2 \\ &\quad + \sum_{I \in \mathcal{I}_k} a_I^2 \Big(\operatorname{tr} A \sum_{i \in \mathbf{I}} \langle A e_i, e_i \rangle - \Big(\sum_{i \in \mathbf{I}} \langle A e_i, e_i \rangle \Big)^2 \Big).\end{aligned}$$

Hence, we have from the Cauchy-Schwarz inequality that

$$\langle \mathcal{B}^{[k]} \omega, \omega \rangle \geq k(n-k)\|\omega\|^2 + \sum_{I \in \mathcal{I}_k} a_I^2 \sum_{i \in \mathbf{I}} \big(\operatorname{tr} A \langle A e_i, e_i \rangle - k \|A e_i\|^2 \big).$$

Using that $\operatorname{Ric}_M(X) = n - 1 + \operatorname{tr} A \langle AX, X \rangle - \|AX\|^2$, the above is written as

$$\begin{aligned}\langle \mathcal{B}^{[k]} \omega, \omega \rangle &\geq k(n-k)\|\omega\|^2 \\ &\quad + \sum_{I \in \mathcal{I}_k} a_I^2 \sum_{i \in \mathbf{I}} \big(k \operatorname{Ric}_M(e_i) - k(n-1) - (k-1) \operatorname{tr} A \langle A e_i, e_i \rangle \big).\end{aligned}$$

From Lemma 6 in [2] we have that $\operatorname{tr} A \langle A e_i, e_i \rangle \leq n H \lambda(n, k, H)$. Then using the assumption on the Ricci curvature we obtain

$$\langle \mathcal{B}^{[k]} \omega, \omega \rangle \geq \|\omega\|^2 k \big(n - k + k b(n, k, H) - k(n-1) - n(k-1) H \lambda(n, k, H) \big),$$

and since $k b(n, k, H) = n(k-1)(1 + H \lambda(n, k, H))$ then the right-hand-side vanishes. □

2. A parametrization

In this section, our goal is to provide a parametrization for the hypersurfaces in spheres with a principal curvature of constant multiplicity at least two. This result of independent interest in submanifold theory will be applied in one of the forthcoming proofs.

Let $g\colon L^{n-\ell} \to \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$, $2 \leq \ell \leq n-2$, be an isometric immersion into the unit sphere with unit normal bundle $\Lambda = \{(x, w) \in N_g L \colon \|w\| = 1\}$. Then the projection $\Pi\colon \Lambda \to L^{n-\ell}$ given by $\Pi(x, w) = x$ is a submersion whose vertical distribution is $\mathcal{V} = \ker \Pi_*$. Let the gradient of $\tau \in C^\infty(L)$ with $0 < \tau < \pi/2$

satisfy $\|\nabla\tau\| < 1$ at any point of $L^{n-\ell}$. Finally, let $\Psi\colon \Lambda \to \mathbb{S}^{n+1}$ be the map given by

$$\Psi(x, w) = \exp_{g(x)} \delta(x, w), \tag{5}$$

where $\delta(x, w) = -\tau(x)(g_*\nabla\tau(x) + \sqrt{1 - \|\nabla\tau(x)\|^2}\, w)$ and exp stands for the exponential map of $\mathbb{S}^{n+1}$. For simplicity, we make use of the same notation for the corresponding map when composing Ψ with the inclusion of $\mathbb{S}^{n+1}$ into $\mathbb{R}^{n+2}$. Hence, we may write

$$\Psi(x, w) = \cos\tau(x)g(x) - \sin\tau(x)\big(g_*\nabla\tau(x) + \sqrt{1 - \|\nabla\tau(x)\|^2}\, w\big).$$

Observe that $\Psi(\Lambda)$ for constant τ is the boundary of the geodesic tube of radius τ given by $\big\{\exp_{g(x)}(-\theta w)\colon 0 \le \theta \le \tau,\ (x, w) \in \Lambda\big\}$.

Proposition 7. *Let $M^n \subset \Lambda$ be the open subset of points where the map Ψ is regular. Then the following assertions hold:*
(i) We have that $(x, w) \in M^n$ if and only if the self adjoint endomorphism of T_xL given by

$$\begin{aligned} P(x, w)Y &= \cos\tau(x)(Y - \langle Y, \nabla\tau(x)\rangle\nabla\tau(x)) - \sin\tau(x)\mathrm{Hess}\,\tau(x)Y \\ &\quad + \sin\tau(x)\sqrt{1 - \|\nabla\tau(x)\|^2} A^g_w Y \end{aligned}$$

is nonsingular, where A^g_w is the shape operator of g.
(ii) The Gauss map $\eta\colon M^n \to \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ of Ψ is given by

$$\eta(x, w) = \sin\tau(x)g(x) + \cos\tau(x)\big(g_*\nabla\tau(x) + \sqrt{1 - \|\nabla\tau(x)\|^2}\, w\big).$$

(iii) The shape operator A of $\Psi|_M$ has $\cot\tau$ as a principal curvature with corresponding eigenspace $\mathcal{V}$.

Proof. At $(x, w) \in M^n$ for $V \in T_{(x,w)}M$ let $c\colon (-\varepsilon, \varepsilon) \to M^n$ be a curve of the form $c(t) = (\gamma(t), w(t))$ so that $c(0) = (x, w)$ and $V = c'(0) = (Z, w'(0))$. A straightforward computation gives

$$\begin{aligned} \Psi_*(x, w)V &= g_*P(x, w)Z - \sin\tau(x)\alpha_g(Z, \nabla\tau(x)) \\ &- \cos\tau(x)\langle Z, \nabla\tau(x)\rangle\sqrt{1 - \|\nabla\tau(x)\|^2}w + \sin\tau(x)\frac{\langle \mathrm{Hess}(\tau)(x)Z, \nabla\tau(x)\rangle}{\sqrt{1 - \|\nabla\tau(x)\|^2}}w \\ &- \sin\tau(x)\sqrt{1 - \|\nabla\tau(x)\|^2}\frac{\nabla^\perp w}{dt}(0), \end{aligned} \tag{6}$$

where $\alpha_g\colon TL \times TL \to N_gL$ is the second fundamental form of g, and then part (i) follows.

Since $\eta(x, w)$ is a unit vector tangent to $\mathbb{S}^{n+1}$ at $\Psi(x, w)$ we obtain from (6) that $\langle\Psi_*(x, w)V, \eta(x, w)\rangle = 0$, and this proves part (ii).

A straightforward computation yields

$$
\begin{aligned}
&\eta_*(x,w)V\\
&= \sin\tau(x)g_*(Z-\langle Z,\nabla\tau(x)\rangle\nabla\tau(x))\\
&\quad+\cos\tau(x)g_*(\mathrm{Hess}(\tau)Z-\sqrt{1-\|\nabla\tau(x)\|^2}A_wZ))\\
&\quad+\cos\tau(x)\big(\alpha_g(Z,\nabla\tau(x))+\sqrt{1-\|\nabla\tau(x)\|^2}\frac{\nabla^\perp w}{dt}(0))\\
&\quad-\sin\tau(x)\langle Z,\nabla\tau(x)\rangle\sqrt{1-\|\nabla\tau(x)\|^2}w-\cos\tau(x)\frac{\langle\mathrm{Hess}(\tau)Z,\nabla\tau(x)\rangle}{\sqrt{1-\|\nabla\tau(x)\|^2}}w.
\end{aligned}
$$

If $V\in T_{(x,w)}M$ is a vertical vector the above gives

$$\eta_*(x,w)V=\cos\tau(x)\sqrt{1-\|\nabla\tau(x)\|^2}\frac{\nabla^\perp w}{dt}(0).$$

On the other hand, it follows from (6) that

$$\Psi_*(x,w)V=-\sin\tau(x)\sqrt{1-\|\nabla\tau(x)\|^2}\frac{\nabla^\perp w}{dt}(0).$$

Hence each vertical vector $V\in T_{(x,w)}M$ is a principal vector with $\cot\tau(x)$ as the corresponding principal curvature.

It remains to prove that there are no other eigenvectors associated to $\cot\tau$. We have to show that any solution of

$$\eta_*(x,w)V=-\cot\tau\Psi_*(x,w)V$$

with $V=(Z,w'(0))$ as above satisfies $Z=0$. Taking the normal component to g, we obtain that

$$(\eta_*(x,w)V)_{N_{g(x)}L}=-\cot\tau(\Psi_*(x,w)V)_{N_{g(x)}L}.$$

Then a straightforward computation gives that $\langle Z,\nabla\tau\rangle=0$. Now taking the tangent component, we have

$$(\eta_*(x,w)V)_{g_*T_xL}=-\cot\tau(\Psi_*(x,w)V)_{g_*T_xL}$$

and conclude that $Z=0$, as we wished. □

In the sequel, let $f\colon M^n\to\mathbb{S}^{n+1}$, $n\geq 4$, be an orientable hypersurface in the unit sphere with Gauss map η and associated shape operator A. Let $\lambda>0$ be a principal curvature of constant multiplicity $2\leq\ell\leq n-2$ and corresponding principal curvature distribution $T_\lambda=\ker(A-\lambda I)$. It is a standard fact that T_λ is integrable and that the leaves are umbilical submanifolds in $\mathbb{S}^{n+1}$ along which λ is constant. In addition, it is well-known that the leaves are complete if M^n is complete.

The associated focal map $h\colon M^n\to\mathbb{S}^{n+1}\subset\mathbb{R}^{n+2}$ to λ is defined as

$$h=\exp_f(\sigma\eta)=\cos\sigma f+\sin\sigma\eta,\tag{7}$$

where $\sigma \in C^\infty(M)$, $0 < \sigma < \pi/2$, is given by $\cot\sigma = \lambda$. Let $V \subset M^n$ be an open saturated subset (i.e., V is a union of maximal leaves of T_λ) and L the quotient space of leaves of V. Then L is Hausdorff, and hence is an $(n-\ell)$-dimensional manifold, if either V is the saturation of some cross section to the foliation or all leaves through points of V are complete. In the following result we assume that $L^{n-\ell}$ is a manifold and let $\pi : V \to L^{n-\ell}$ be the projection. Hence the focal map factors through an immersion $g\colon L^{n-\ell} \to \mathbb{S}^{n+1}$, that is, $h = g \circ \pi$, which is called the *focal submanifold* associated to λ. If M^n is compact, it follows from the Corollary given in page 18 of [3] that also $L^{n-\ell}$ is a compact manifold.

Theorem 8. *Let $f\colon M^n \to \mathbb{S}^{n+1}$, $n \geq 4$, be an orientable hypersurface with a principal curvature λ of constant multiplicity ℓ with $2 \leq \ell \leq n-2$. Then $\tau \in C^\infty(L)$, $0 < \tau < \pi/2$, given by $\sigma = \tau \circ \pi$ satisfies $\|\nabla\tau\| < 1$ at any point of $L^{n-\ell}$. Let Λ be the unit normal bundle of the focal submanifold g. Then there is a local diffeomorphism $j\colon V \to \Lambda$ such that $f|_V = \Psi \circ j$ where $\Psi\colon \Lambda \to \mathbb{S}^{n+1}$ is the map given by (5). If M^n is compact then $L^{n-\ell}$ is compact, $j\colon M^n \to \Lambda$ is a covering map and Ψ is a global parametrization.*

Conversely, let $g\colon L^{n-\ell} \to \mathbb{S}^{n+1}$, $2 \leq \ell \leq n-2$, be a connected submanifold and let $M^n \subset \Lambda$ be the open subset of points of the unit normal bundle of g where the map Ψ given by (5) is regular. Let $\tau \in C^\infty(L)$, $0 < \tau < \pi/2$, be such that $\|\nabla\tau\| < 1$ at any point of $L^{n-\ell}$. Then $\Psi\colon M^n \to \mathbb{S}^{n+1}$ has a principal curvature $\lambda = \cot\tau \circ \pi$ of constant multiplicity ℓ.

Proof. Since the converse follows from Proposition 7 we only argue for the direct part of the statement. Let $V \subset M^n$ be such that the quotient $L^{n-\ell}$ is a manifold. Along $g\colon L^{n-\ell} \to \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ the vector $\eta(x)$ decomposes as

$$\eta(x) = \langle g(\pi(x)), \eta(x)\rangle g(\pi(x)) + g_*(\pi(x))Z + \delta$$

for $Z \in T_{\pi(x)}L$ and $\delta \in N_gL(\pi(x))$. Then that $\langle g(x), \eta(x)\rangle = \sin\tau(\pi(x))$ follows from (7). Also

$$\langle \eta(x), g_*(\pi(x))\tilde{X}\rangle = \langle Z, \tilde{X}\rangle$$

for any $X \in T_xV$ and $\tilde{X} = \pi_*(x)X \in T_{\pi(x)}L$. Moreover, since

$$\begin{aligned}\langle \eta(x), g_*(\pi(x))\tilde{X}\rangle = \langle \eta(x), h_*(x)X\rangle &= \cos\sigma(x)\langle\nabla\sigma, X\rangle \\ &= \cos\tau(\pi(x))\langle\nabla\tau(\pi(x)), \tilde{X}\rangle\end{aligned}$$

then $Z = \cos\tau(\pi(x))\nabla\tau(\pi(x))$. It follows that

$$\|\delta\| = \cos\tau(\pi(x))\sqrt{1 - \|\nabla\tau(\pi(x))\|^2}.$$

We argue that $\|\nabla\tau\| < 1$ at any point of $L^{n-\ell}$ and, in particular, we have that $\|\delta\| > 0$. On the contrary, suppose that at some point $\|\nabla\tau\| = 1$ and let $Y \in TM$ be such that $\pi_*Y = \nabla\tau$. Then $\eta = \sin\tau\circ\pi\; g + \cos\tau\circ\pi\; g_*\pi_*Y$. It follows that $Y(\sigma) = 1$ and that

$$(\sin\tau\circ\pi)(g_*\pi_*Y) = (\cos\tau\circ\pi)g\circ\pi - f.$$

On the other hand, we have from (7) that

$$\begin{aligned}(\sin\tau\circ\pi)(g_*\pi_*Y) &= \sin\sigma h_*Y\\ &= \sin\sigma[f_*(\cos\sigma Y-\sin\sigma AY)+Y(\sigma)(-\sin\sigma f+\cos\sigma\eta)]\end{aligned}$$

and thus $AY=\cot\sigma Y$, which is a contradiction. From (7) there is a unit vector field $\delta_1\in\Gamma(N_gL)$ such that

$$f=(\cos\tau\circ\pi)g\circ\pi-(\sin\tau\circ\pi)\big((g_*\nabla\tau)\circ\pi+\sqrt{1-\|(\nabla\tau)\circ\pi\|^2}\delta_1\big).$$

Define $j\colon V\to\Lambda$ by $j(x)=(\pi(x),\delta_1(\pi(x))$. Then the previous equation yields $f(x)=\Psi\circ j(x)$. Hence j is a local diffeomorphism and Ψ is regular on $j(V)$. If M^n is compact we take $V=M^n$, and then L^{n-k} is a compact manifold, j is a covering map and $\Psi(\Lambda)=f(M)$. □

3. The proofs

In this section, the proofs of the results stated in the introduction are given.

Proof of Theorem 1. From Theorem 1 in [2] we have that M^n is simply connected and therefore orientable. Moreover, either we are in case (i) or we have that

$$H_i(M^n;\mathbb{Z})=0=H_{n-i}(M^n;\mathbb{Z})\ \text{ for all }\ 1\le i\le k-1,$$

$H_k(M^n;\mathbb{Z})\neq 0$, $H_{n-k}(M^n;\mathbb{Z})=\mathbb{Z}^{\beta_k(M)}$ and $\lambda(n,k,H)$ is a principal curvature of f with multiplicity at least k at any point of M^n.

We need to distinguish two cases:

Case I. Suppose that $\beta_k(M)=0$. Poincaré duality gives $\beta_{n-k}(M)=0$, and therefore

$$H_k(M^n;\mathbb{Z})=\mathrm{Tor}(H_k(M^n;\mathbb{Z}))\ \text{ and }\ H_{n-k}(M^n;\mathbb{Z})=\mathrm{Tor}(H_{n-k}(M^n;\mathbb{Z})).$$

Since $H_{k-1}(M^n;\mathbb{Z})=0$ we obtain from the universal coefficient theorem for cohomology (cf. Corollary 4 in [6, p. 244]) that

$$\mathrm{Tor}(H^k(M^n;\mathbb{Z}))=\mathrm{Tor}(H_{k-1}(M^n;\mathbb{Z}))=0.$$

Poincaré duality yields

$$\mathrm{Tor}(H_{n-k}(M^n;\mathbb{Z}))=\mathrm{Tor}(H^k(M^n;\mathbb{Z}))=0$$

and hence $H_{n-k}(M^n;\mathbb{Z})=0$. Thus we are in part (ii) since the remaining of the statement follows from the aforementioned Theorem 1 in [2].

Case II. Suppose that $\beta_k(M)>0$. Then M^n carries a nontrivial harmonic k-form ω. Taking the scalar product with ω on both sides of (4) gives

$$\|\nabla\omega\|^2+\langle\mathcal{B}^{[p]}\omega,\omega\rangle+\frac{1}{2}\Delta\|\omega\|^2=\langle\Delta\omega,\omega\rangle$$

which is given in [5] and in [4] as Lemma 3.4. Proposition 6 yields that the Bochner operator $\mathcal{B}^{[k]}$ is nonnegative, and since $\Delta\omega=0$ then $\Delta\|\omega\|^2\le 0$. From the maximum principle it follows that $\|\omega\|^2$ is a positive constant. Hence ω is parallel. Thus M^n supports a nontrivial parallel k-form. Then Theorem 4 in [5] gives that $f(M)$ is the torus $\mathbb{T}_k^n(r)$, and from [7] we have that the mean curvature is $H = (k - nr^2)/nr\sqrt{1-r^2}$ with $r^2 \leq k/n$. Since the Ricci curvature of $\mathbb{T}_p^n(r)$ in the principal directions attains the values $(k-1)/r^2$ and $(n-k-1)/(1-r^2)$, then the condition $(*)$ yields that $r^2 \geq (k-1)/(n-2)$. □

Proof of Theorem 3. By Theorem 1 in [2] we have that either

$$H_i(M^n; \mathbb{Z}) = H_{n-i}(M^n; \mathbb{Z}) = 0 \text{ for all } 1 \leq i \leq k,$$

or

$$H_i(M^n; \mathbb{Z}) = H_{n-i}(M^n; \mathbb{Z}) = 0 \text{ for all } 1 \leq i \leq k-1,$$

$H_k(M^n; \mathbb{Z}) \neq 0$ and $H_{k+1}(M^n; \mathbb{Z}) = \mathbb{Z}^{\beta_k(M)}$.

In the first case, the manifold M^n is a homology sphere. Theorem 1 in [2] also yields that M^n is simply connected. Thus in this case the Hurewicz homomorphisms between the homotopy and homology groups are isomorphisms, and hence M^n is a homotopy sphere. Since the generalized Poincaré conjecture holds due to the work of Smale and Freedman then M^n is homeomorphic to $\mathbb{S}^n$.

Hereafter we deal with the second case. Part (ii) of Theorem 1 in [2] yields that $\lambda = \lambda(n, k, H)$ is a principal curvature with multiplicity k at any point of M^n. By Theorem 3 in [2] we have that either the homology of M^n is isomorphic to the one of $\mathbb{S}^k \times \mathbb{S}^{k+1}$, or the homology of M^n is $H_k(M^n, \mathbb{Z}) = \mathbb{Z}_q$ for some $q > 1$, satisfies $H_0(M^n, \mathbb{Z}) = H_n(M^n, \mathbb{Z}) = \mathbb{Z}$ and is trivial in all other cases. By Proposition 7 in [2] we have that M^n is diffeomorphic to the total space of a sphere bundle $\mathbb{S}^k \hookrightarrow \mathsf{E} \xrightarrow{p} L^{k+1}$ over the quotient manifold $L^{k+1} = M^n/T_\lambda$. Moreover, by Theorem 3 in [2] we know that L^{k+1} is homeomorphic to $\mathbb{S}^{k+1}$.

We need to distinguish two cases:

Case I. Suppose that $H_k(M^n, \mathbb{Z}) = \mathbb{Z}_q$ for some $q > 1$. It follows from part (ii) of Theorem 3 in [2] that $n = 4r + 3$. Since $\lambda = \lambda(n, k, H)$ is a principal curvature with multiplicity k at any point of M^n, then Theorem 8 gives that f is globally a composition $f = \Psi \circ j$, where $j\colon M^n \to \Lambda$ is a covering map and Λ is the unit normal bundle of the compact associated focal submanifold $g\colon L^{k+1} \to \mathbb{S}^{n+1}$. It is clear that the compact hypersurface $\Psi\colon \Lambda \to \mathbb{S}^{n+1}$ satisfies $(*)$. Theorem 1 in [2] yields that the manifolds M^n and Λ are simply connected and therefore $j\colon M^n \to \Lambda$ is a diffeomorphism.

Case II. Suppose that the homology of M^n is isomorphic to the homology of $\mathbb{S}^k \times \mathbb{S}^{k+1}$. Hence $\beta_k(M) = 1$. Then M^n carries a nontrivial harmonic k-form. Proposition 6 implies that the Bochner operator $\mathcal{B}^{[k]}$ is nonnegative. Since $H^k(M^n; \mathbb{R}) \neq 0$ then by Proposition 6 every harmonic k-form is parallel. Thus the manifold M^n supports a nontrivial parallel k-form. We have from Theorem 4 in [5] that $f(M)$ is the torus $\mathbb{T}_k^n(r)$ whose mean curvature is $H = (k - nr^2)/nr\sqrt{1-r^2}$ with $r^2 \leq k/n$. Since the Ricci curvature of $\mathbb{T}_k^n(r)$ in the principal directions attains the values $(k-1)/r^2$ and $k/(1-r^2)$, then condition $(*)$ yields $r^2 \geq (k-1)/(n-2)$. □

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Declarations

Conflict of Interest The authors declare that they have no conflict of interest to this work.

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