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Full Laplace spectrum of distance spheres in symmetric spaces of rank one

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Abstract

We use Lie-theoretic methods to explicitly compute the full spectrum of the Laplace–Beltrami operator on homogeneous spheres which occur as geodesic distance spheres in (compact or non-compact) symmetric spaces of rank one, and provide a single unified formula for all cases. As an application, we find all resonant radii for distance spheres in the compact case, that is, radii where there is bifurcation of embedded constant mean curvature spheres, and show that distance spheres are stable and locally rigid in the non-compact case.

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1 | INTRODUCTION

The family of (simply connected) symmetric spaces of rank one consists of spheres and projective spaces $\mathbb{S}^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$, $\mathbb{C}\mathrm{a}P^2$, together with their non-compact duals, the hyperbolic spaces H^n, $\mathbb{C}H^n$, $\mathbb{H}H^n$, $\mathbb{C}\mathrm{a}H^2$. As Riemannian manifolds, these are *two-point homogeneous spaces*, that is, any two pairs of points at the same distance can be mapped to one another by an isometry. In particular, their distance spheres

$$S(r) = \{x \in M : \operatorname{dist}(x_0, x) = r\},$$

are homogeneous spaces themselves, and two distance spheres are isometric if and only if they have the same radius, regardless of their centers. These homogeneous spheres are the main object of study in this paper; in which we shall use Lie theory to explicitly compute their entire Laplace spectrum, and determine their stability (or lack thereof) as constant mean curvature (CMC) hypersurfaces.

While distance spheres $S(r)$ in $\mathbb{S}^n$ and H^n have constant curvature, that is, are isometric to round spheres, just like in $\mathbb{R}^n$, this is no longer the case in projective and hyperbolic spaces. Geometrically, $S(r) \subset M$ are obtained by rescaling the unit round metric in the vertical direction(s) of the corresponding Hopf bundle by $t > 0$:

$$\begin{aligned}
&\mathbb{S}^1_t \longrightarrow \big(\mathbb{S}^{2n+1}, \mathbf{g}(t)\big) \longrightarrow \mathbb{C}P^n, && \text{if } M = \mathbb{C}P^{n+1} \text{ or } \mathbb{C}H^{n+1};\\
&\mathbb{S}^3_t \longrightarrow \big(\mathbb{S}^{4n+3}, \mathbf{h}(t)\big) \longrightarrow \mathbb{H}P^n, && \text{if } M = \mathbb{H}P^{n+1} \text{ or } \mathbb{H}H^{n+1};\\
&\mathbb{S}^7_t \longrightarrow \big(\mathbb{S}^{15}, \mathbf{k}(t)\big) \longrightarrow \mathbb{S}^8_{1/2}, && \text{if } M = \mathbb{C}\mathrm{a}P^2 \text{ or } \mathbb{C}\mathrm{a}H^2;
\end{aligned} \tag{1.1}$$

where $\mathbb{S}^\ell_t$ denotes the ℓ-dimensional sphere of constant curvature $\sec = 1/t^2$, and then globally rescaling all directions by $\alpha > 0$. With the convention (used throughout this paper) that the above projective and hyperbolic spaces with their canonical metrics have sectional curvatures $1 \leqslant \sec_M \leqslant 4$ and $-4 \leqslant \sec_M \leqslant -1$, respectively, the values of t and α for $S(r) \subset M$ are related to its geodesic radius r as follows:

$$\begin{aligned}
&t = \cos r \quad \text{and} \quad \alpha = \sin r, \quad 0 < r < \pi/2, \quad \text{if } M \text{ is a projective space;}\\
&t = \cosh r \quad \text{and} \quad \alpha = \sinh r, \quad r > 0, \quad \text{if } M \text{ is a hyperbolic space.}
\end{aligned} \tag{1.2}$$

Note that, with these conventions, the above projective spaces have diameter $\pi/2$. Of course, all $S(r)$ become asymptotically round as $r \searrow 0$, that is, they converge (up to homothety by α) to the *unit* round metric, which corresponds to $t = 1$ in each of the families $\mathbf{g}(t)$, $\mathbf{h}(t)$, and $\mathbf{k}(t)$. Furthermore, only the metrics with either $t < 1$ or $t > 1$ appear (up to homotheties) as distance spheres $S(r) \subset M$, according to whether M is projective or hyperbolic.

It is convenient to refer to the Riemannian submersions (1.1) collectively as

$$\mathbb{S}^{2d-1}_t \longrightarrow \big(\mathbb{S}^{N-1}, \mathrm{g}(t)\big) \longrightarrow \mathbb{K}P^n, \tag{1.3}$$

where $\mathbb{K} \in \{\mathbb{C}, \mathbb{H}, \mathbb{C}\mathrm{a}\}$, $d = \dim_{\mathbb{C}} \mathbb{K} \in \{1, 2, 4\}$, $n \geqslant 1$, and $N = 2d(n+1) = \dim M$ is the (real) dimension of the ambient space $\mathbb{K}P^{n+1}$ or $\mathbb{K}H^{n+1}$. Recall that if $\mathbb{K} = \mathbb{C}\mathrm{a}$, that is, $d = 4$, only $n = 1$ is possible due to the non-associativity of Cayley numbers [2, 19], and (1.3) is *not* a homogeneous fibration [14, 16].

Since the fibers of (1.3) are totally geodesic, the projection map "commutes" the Laplace–Beltrami operators of total space and base. In particular, lifting a Laplace eigenfunction of $\mathbb{K}P^n$ produces a Laplace eigenfunction of $(\mathbb{S}^{N-1}, \mathrm{g}(t))$, with the same eigenvalue. Such eigenvalues are called *basic*, and are independent of t. Although it has been known for a long time that all eigenvalues are sums of basic eigenvalues with certain Laplace eigenvalues of the fiber [5, 6], determining exactly which sums of eigenvalues from $\mathbb{K}P^n$ and $\mathbb{S}^{2d-1}$ indeed appear in the spectrum of the total space can be somewhat impractical. We circumvent this with an alternative Lie-theoretic approach based on [21], recently used in [7, 20] and expanded in Section 2 below, which yields our first main result.

Theorem A. *The spectrum of the Laplace–Beltrami operator on the homogeneous sphere* $(\mathbb{S}^{N-1}, \mathrm{g}(t))$, $N = 2d(n+1)$, *as in (1.3), consists of the eigenvalues*

$$\lambda^{(p,q)}(t) = 4p(p+q+d(n+1)-1) + 2dnq + q(q+2d-2)\tfrac{1}{t^2}, \quad p, q \in \mathbb{N}_0, \tag{1.4}$$

which are basic if $q = 0$, *and have multiplicity*

$$m_{p,q} = \frac{2p+q+d(n+1)-1}{d(n+1)-1} \frac{\binom{p+q+d(n+1)-2}{p+q}\binom{p+dn-1}{p}}{\binom{p+q+d-1}{p+q}} \chi(d,q), \tag{1.5}$$

where $\chi(d,q) = (1+\frac{q}{d-1})\frac{\Gamma(q+2d-2)}{\Gamma(q+1)\Gamma(2d-2)}$. *If different pairs* (p,q) *give the same value* $\lambda^{(p,q)}(t)$, *the multiplicity of that eigenvalue is the sum of all the corresponding* $m_{p,q}$.

We take the convention that $\chi(d,q)$ is extended by continuity to its removable singularity at $d = 1$, that is,

$$\chi(1,q) = \lim_{d\to 1}\left(1+\frac{q}{d-1}\right)\frac{\Gamma(q+2d-2)}{\Gamma(q+1)\Gamma(2d-2)} = \begin{cases} 1 & \text{if } q = 0 \\ 2 & \text{if } q \geqslant 1, \end{cases}$$

since $\Gamma(z)$ has a simple pole at $z = 0$ of residue 1, and $\Gamma(a) = (a-1)!$ for all $a \in \mathbb{N}$. Moreover, if $d \geqslant 2$, note that $\chi(d,q) = (1+\frac{q}{d-1})\binom{q+2d-3}{q}$ for all $q \in \mathbb{N}_0$. As usual, we agree that $\binom{a}{b} = 0$ if $a < b$. Despite the convenient unified formulae (1.4) and (1.5), the proof of Theorem A is done analyzing each case $\mathbb{K} \in \{\mathbb{C}, \mathbb{H}, \mathbb{C}\mathrm{a}\}$ separately, and corresponding formulae can be found in Section 6.

Note that setting $t = 1$ in (1.4), the eigenvalues $\lambda^{(p,q)}(1) = k(k+N-2)$, $k \in \mathbb{N}_0$, of the unit round sphere $\mathbb{S}^{N-1}$ are recovered, with $k = 2p+q$. Moreover, (1.5) and combinatorial identities show that its multiplicity $\binom{k+N-1}{N-1} - \binom{k+N-3}{N-1}$ is equal to the sum of $m_{p,q}$ over all $p, q \in \mathbb{N}_0$ satisfying $2p + q = k$. Similarly, setting $q = 0$, one recovers the eigenvalues $\lambda^{(p,0)}(t) = 4p(p+d(n+1)-1)$, $p \in \mathbb{N}_0$, of the projective space $\mathbb{K}P^n$ and the corresponding multiplicities.

Several partial descriptions of the spectra in Theorem A appear in the literature, for example, [4, 8, 23, 24]; in particular, the *first* (non-zero) eigenvalue was computed in [8], see also [7]. However, to the best of our knowledge, the *full* Laplace spectrum cannot be directly extracted from these earlier results.

Using Theorem A and (1.2), the full spectrum of the Laplace–Beltrami operator on any distance sphere $S(r)$ in a rank one symmetric space M can be easily computed, since $\Delta_{\alpha \mathrm{g}} = \frac{1}{\alpha}\Delta_{\mathrm{g}}$. Although the lowest dimensional cases are excluded as $n \geqslant 1$ in (1.1), these are trivial since $\mathbb{C}P^1 \cong \mathbb{S}^2(\frac{1}{2})$ and $\mathbb{H}P^1 \cong \mathbb{S}^4(\frac{1}{2})$ are isometric to round spheres with $\sec_M = 4$; and $\mathbb{C}H^1 \cong H^2(\frac{1}{2})$ and $\mathbb{H}H^1 \cong H^4(\frac{1}{2})$ are isometric to real hyperbolic spaces with $\sec_M = -4$, so distance spheres $S(r) \subset M$ in any of these spaces are just round spheres.

The spectrum of distance spheres is closely related to the local ambient geometry, for example, it detects whether a harmonic space is locally symmetric [1]. One of its global consequences is explored in our second main result, concerning the existence of other embedded CMC spheres near distance spheres. More precisely, a distance sphere $S(r_*) \subset M$ is *resonant* if there exists a sequence r_j of radii converging to r_* and a sequence $\Sigma_j \subset M$ of embedded spheres converging to $S(r_*)$, with CMC $H(\Sigma_j) = H(S(r_j))$, which are not congruent to $S(r_j)$. Note that $S(r_*)$ is *non-resonant* if and only if, up to ambient isometries, $S(r)$ are locally the only embedded CMC spheres with their mean curvature if r is sufficiently close to r_*. Recall that a hypersurface $\Sigma \subset M$ has CMC H if and only if it is a stationary point for the functional $\mathrm{Area}(\Sigma) + H\ \mathrm{Vol}(\Sigma)$, where $\mathrm{Area}(\Sigma)$ is the $(N-1)$-volume of Σ and $\mathrm{Vol}(\Sigma)$ is the N-volume of the region enclosed by Σ in M, and Σ is *stable* if it is locally a minimum.

Theorem B. *The distance spheres $S(r)$ in the projective spaces $\mathbb{C}P^{n+1}$, $\mathbb{H}P^{n+1}$, $n \geqslant 1$, and $\mathbb{C}\mathrm{a}P^2$ are resonant if and only if $r = r_p$ for some $p \in \mathbb{N}$, where*

$$r_p := \arctan \sqrt{\frac{4p(p-1) + N(2p-1) + 1}{2d-1}},$$

$d = \dim_{\mathbb{C}} \mathbb{K} \in \{1, 2, 4\}$ per $\mathbb{K} \in \{\mathbb{C}, \mathbb{H}, \mathbb{C}\mathrm{a}\}$, and $N = \dim \mathbb{K}P^{n+1} = 2d(n+1)$. On the other hand, for all $r > 0$, the distance spheres $S(r)$ in the hyperbolic spaces $\mathbb{C}H^{n+1}$, $\mathbb{H}H^{n+1}$, $n \geqslant 1$, and $\mathbb{C}\mathrm{a}H^2$ are stable and non-resonant.

The existence of infinitely many resonant distance spheres in $\mathbb{C}P^{n+1}$ and $\mathbb{H}P^{n+1}$ with radii accumulating at $\pi/2$ had been established in [9]. Nevertheless, the coarser equivariant spectral methods used there do not allow one to explicitly determine which radii $0 < r < \pi/2$ are resonant, nor to handle the case of $\mathbb{C}\mathrm{a}P^2$, since (1.3) is *not* a homogeneous fibration if $\mathbb{K} = \mathbb{C}\mathrm{a}$. Moreover, it was known that $S(r) \subset \mathbb{K}P^{n+1}$ is stable if and only if $0 < r < r_1 = \arctan \sqrt{\frac{N+1}{2d-1}}$, see [4, Theorems 1.3 and 1.4], and that $S(r) \subset \mathbb{K}H^{n+1}$ are stable for all $r > 0$, see [22, Theorem 2].

The path leading from Theorem A to Theorem B is that the stability operator (or *Jacobi operator*) for a CMC hypersurface $\Sigma \subset M$ is $J_\Sigma = \Delta_\Sigma - (\mathrm{Ric}(\vec{n}_\Sigma) + \|A_\Sigma\|^2)$; hence, its spectrum is a shift of the Laplace spectrum of Σ by a curvature term, which is constant if Σ is a distance sphere $S(r)$ as above. Stability of $S(r)$ is equivalent to non-negativity of the first eigenvalue of $J_{S(r)}$, while resonance of $S(r_*)$ is detected by eigenvalues of $J_{S(r)}$ crossing zero at $r = r_*$, see Section 7 for details.

This paper is organized as follows. Section 2 describes a Lie-theoretic approach tailored to compute the Laplace spectrum on the total space of Riemannian submersions such as (1.1). The outcome for the first families in (1.1) is given in Sections 3 and 4, respectively, while Section 5 deals with the third case. These results are unified in Section 6, with the proof of Theorem A. The applications regarding resonance and rigidity are discussed in Section 7, where Theorem B is proven.

2 | COMPUTING THE LAPLACE SPECTRUM OF A HOMOGENEOUS SPACE

2.1 | Basic setting

Let $\mathsf{H} \subset \mathsf{K} \subset \mathsf{G}$ be compact Lie groups, with Lie algebras $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$. Fix a bi-invariant metric on G, that is, an $\mathrm{Ad}(\mathsf{G})$-invariant inner product $\langle \cdot, \cdot \rangle_0$ on $\mathfrak{g}$. For instance, a natural choice on most matrix

Lie groups is

$$\langle X, Y\rangle_0 = -\tfrac{1}{2}\operatorname{Re}(\operatorname{tr}(XY)). \tag{2.1}$$

Let $\mathfrak{p}$ and $\mathfrak{q}$ be the $\langle\cdot,\cdot\rangle_0$-orthogonal complements of $\mathfrak{h}$ in $\mathfrak{k}$, and $\mathfrak{k}$ in $\mathfrak{g}$, so that $\mathfrak{k} = \mathfrak{h}\oplus\mathfrak{p}$ and $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{q} = \mathfrak{h}\oplus(\mathfrak{p}\oplus\mathfrak{q})$ are Cartan decompositions. In particular, the H-action on $\mathfrak{p}\oplus\mathfrak{q}$ via the adjoint representation of G is identified with the isotropy representation of G/H. Note that although $\mathfrak{p}$ and $\mathfrak{q}$ are subrepresentations, they need not be irreducible. Consider the family of $\mathrm{Ad}(\mathsf{H})$-invariant inner products

$$\langle\cdot,\cdot\rangle_{(r,s)} = \frac{1}{r^2}\,\langle\cdot,\cdot\rangle_0\big|_{\mathfrak{p}} + \frac{1}{s^2}\,\langle\cdot,\cdot\rangle_0\big|_{\mathfrak{q}},\quad r,s>0, \tag{2.2}$$

on $\mathfrak{p}\oplus\mathfrak{q}$, which induces a corresponding family of G-invariant metrics $\mathrm{g}_{(r,s)}$ on G/H.

Up to homotheties, this is the *canonical variation* of the Riemannian submersion

$$\mathsf{K}/\mathsf{H} \longrightarrow \mathsf{G}/\mathsf{H} \longrightarrow \mathsf{G}/\mathsf{K}, \tag{2.3}$$

where all spaces are endowed with normal homogeneous metrics induced by $\langle\cdot,\cdot\rangle_0$. In geometric terms, $\mathrm{g}_{(r,s)}$ is obtained by rescaling the vertical and horizontal directions of (2.3) by $1/r$ and $1/s$, respectively. If $\mathfrak{p}$ and $\mathfrak{q}$ are irreducible and non-equivalent as H-modules, then any G-invariant metric on G/H is isometric to some $\mathrm{g}_{(r,s)}$.

2.2 | The Lie-theoretic method

In this section, we describe the Lie-theoretic procedure to compute the Laplace–Beltrami spectrum of $(\mathsf{G}/\mathsf{H}, \mathrm{g}_{(r,s)})$, which relies on knowledge of representation branching rules involving G, K, and H. The discussion below is based on [21] and our earlier work [7, §2], and provides a computationally efficient alternative to more classical methods in [5, 6].

Notation 2.1. Given a compact Lie group J, let $\widehat{\mathsf{J}}$ be its unitary dual, that is, the set of equivalence classes of irreducible unitary representations of J. We shall consider elements of $\widehat{\mathsf{J}}$ as representations (π, V_π), that is, homomorphisms $\pi\colon \mathsf{J}\to\mathsf{GL}(V_\pi)$. Given J-representations (σ, V_σ) and (τ, V_τ), set $[\sigma:\tau] := \dim\operatorname{Hom}_{\mathsf{J}}(V_\sigma, V_\tau)$. Note that if σ is irreducible, then $[\sigma:\tau]$ is the multiplicity of σ in the decomposition of τ in irreducible components. In particular, $[\sigma:\tau]>0$ for only finitely many $\sigma\in\widehat{\mathsf{J}}$.

Definition 2.2. The set of *spherical representations* associated to (G,H) is

$$\widehat{\mathsf{G}}_{\mathsf{H}} := \big\{(\pi, V_\pi)\in\widehat{\mathsf{G}} : V_\pi^{\mathsf{H}}\neq 0\big\} = \big\{\pi\in\widehat{\mathsf{G}} : [1_{\mathsf{H}} : \pi|_{\mathsf{H}}]>0\big\},$$

where V_π^{H} is the subspace of V_π given by H-invariant elements, and 1_{H} is the trivial representation of H. The *Casimir element of* $\mathfrak{g}$ with respect to $\langle\cdot,\cdot\rangle_0$ is the element $\mathrm{Cas}_{\mathfrak{g},\langle\cdot,\cdot\rangle_0} := X_1^2+\cdots+X_{\dim\mathfrak{g}}^2$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$, where $\{X_1,\dots,X_{\dim\mathfrak{g}}\}$ is any $\langle\cdot,\cdot\rangle_0$-orthonormal basis of $\mathfrak{g}$. Since we fixed an $\mathrm{Ad}(\mathsf{G})$-invariant inner product $\langle\cdot,\cdot\rangle_0$, we denote $\mathrm{Cas}_{\mathfrak{g},\langle\cdot,\cdot\rangle_0}$ by $\mathrm{Cas}_{\mathfrak{g}}$, and similarly for the Casimir elements $\mathrm{Cas}_{\mathfrak{k}}$ and $\mathrm{Cas}_{\mathfrak{h}}$ of $(\mathfrak{k}, \langle\cdot,\cdot\rangle_0|_{\mathfrak{k}})$ and $(\mathfrak{h}, \langle\cdot,\cdot\rangle_0|_{\mathfrak{h}})$.

Notation 2.3. Let (φ, V_φ) be a unitary representation of a compact Lie group J. We shall also denote by φ the induced representations of the Lie algebra $\mathfrak{j}$ of J, of its complexification $\mathfrak{j}_{\mathbb{C}} := \mathfrak{j} \otimes_{\mathbb{R}} \mathbb{C}$, and of its universal enveloping algebra $\mathcal{U}(\mathfrak{j}_{\mathbb{C}})$. Since $\varphi(a) \colon V_\varphi \to V_\varphi$ is unitary for all $a \in \mathsf{J}$, it follows that $\varphi(X)$ is skew-symmetric for all $X \in \mathfrak{g}$, and, consequently, $\varphi(-X^2)$ is self-adjoint for all $X \in \mathfrak{g}$.

For $\pi \in \widehat{\mathsf{G}}$, the operator $\pi(\operatorname{Cas}_{\mathfrak{g}}) \colon V_\pi \to V_\pi$ commutes with $\pi(a)$ for all $a \in \mathsf{G}$. Thus, by Schur's lemma, $\pi(-\operatorname{Cas}_{\mathfrak{g}}) = \lambda^\pi \operatorname{Id}_{V_\pi}$ for some $\lambda^\pi > 0$. Analogously, $\tau(-\operatorname{Cas}_{\mathfrak{k}}) = \lambda^\tau \operatorname{Id}_{V_\tau}$ and $\sigma(-\operatorname{Cas}_{\mathfrak{h}}) = \lambda^\sigma \operatorname{Id}_{V_\sigma}$ for $\tau \in \widehat{\mathsf{K}}$ and $\sigma \in \widehat{\mathsf{H}}$. The constants λ^π and λ^τ can be computed explicitly using Lie-theoretic objects, see Subsection 2.3.

Theorem 2.4. *The spectrum of the Laplace–Beltrami operator on the homogeneous space* $(\mathsf{G}/\mathsf{H}, \mathrm{g}_{(r,s)})$ *consists of the eigenvalues*

$$\lambda^{\pi,\tau}(r,s) = (r^2 - s^2)\,\lambda^\tau + s^2\lambda^\pi, \tag{2.4}$$

where $(\pi,\tau) \in \widehat{\mathsf{G}}_{\mathsf{H}} \times \widehat{\mathsf{K}}$ *is such that* $[\tau : \pi|_{\mathsf{K}}] > 0$, *with multiplicity*

$$m_{\pi,\tau} = [1_{\mathsf{H}} : \tau|_{\mathsf{H}}][\tau : \pi|_{\mathsf{K}}] \dim V_\pi.$$

Moreover, (2.4) is basic for the Riemannian submersion $(\mathsf{G}/\mathsf{H}, \mathrm{g}_{(r,s)}) \to \mathsf{G}/\mathsf{K}$ *if* $\pi \in \widehat{\mathsf{G}}_{\mathsf{K}}$ *and* $\tau = 1_{\mathsf{K}}$, *in which case* $\lambda^{\pi,\tau}(r,s) = s^2\lambda^\pi$ *and* $m_{\pi,\tau} = [1_{\mathsf{K}} : \pi|_{\mathsf{K}}] \dim V_\pi$.

Proof. Let $\{X_1, \dots, X_{\dim \mathfrak{p}}\}$ and $\{Y_1, \dots, Y_{\dim \mathfrak{q}}\}$ be $\langle\cdot,\cdot\rangle_0$-orthonormal bases of $\mathfrak{p}$ and $\mathfrak{q}$, respectively, and note that $\{rX_1, \dots, rX_{\dim \mathfrak{p}}, sY_1, \dots, sY_{\dim \mathfrak{q}}\}$ is an orthonormal basis of $\mathfrak{p} \oplus \mathfrak{q}$ with respect to $\langle\cdot,\cdot\rangle_{(r,s)}$. Set $C_{\mathfrak{p}} = X_1^2 + \dots + X_{\dim \mathfrak{p}}^2$, $C_{\mathfrak{q}} = Y_1^2 + \dots + Y_{\dim \mathfrak{q}}^2$, and $C_{(r,s)} = t^2\, C_{\mathfrak{p}} + s^2\, C_{\mathfrak{q}}$. According to [7, Proposition 2.2], the spectrum of the Laplace–Beltrami operator on $(\mathsf{G}/\mathsf{H}, \mathrm{g}_{(r,s)})$ is the union of eigenvalues of $\pi(-C_{(r,s)})|_{V_\pi^{\mathsf{H}}}$, where $\pi \in \widehat{\mathsf{G}}_{\mathsf{H}}$, each with multiplicity $\dim V_\pi$.

We need to show that (2.4) appears in the spectrum of $\pi(-C_{(r,s)})|_{V_\pi^{\mathsf{H}}}$ with multiplicity $[1_{\mathsf{H}} : \tau|_{\mathsf{H}}][\tau : \pi|_{\mathsf{K}}]$ for any $\tau \in \widehat{\mathsf{K}}$ satisfying $[\tau : \pi|_{\mathsf{K}}] > 0$, and that these eigenvalues exhaust the spectrum. For $v \in V_\pi^{\mathsf{H}}$, we have that

$$\begin{aligned}
\pi(-C_{(r,s)}) \cdot v &= r^2\, \pi(-C_{\mathfrak{p}}) \cdot v + s^2\, \pi(-C_{\mathfrak{q}}) \cdot v \\
&= (r^2 - s^2)\, \pi(-C_{\mathfrak{p}}) \cdot v + s^2\, \pi(-C_{\mathfrak{p}} - C_{\mathfrak{q}}) \cdot v \\
&= (r^2 - s^2)\, \pi(-\operatorname{Cas}_{\mathfrak{h}} - C_{\mathfrak{p}}) \cdot v + s^2\, \pi(-\operatorname{Cas}_{\mathfrak{h}} - C_{\mathfrak{p}} - C_{\mathfrak{q}}) \cdot v \\
&= (r^2 - s^2)\, \pi(-\operatorname{Cas}_{\mathfrak{k}}) \cdot v + s^2\, \pi(-\operatorname{Cas}_{\mathfrak{g}}) \cdot v.
\end{aligned} \tag{2.5}$$

The third equality follows from $\pi(\operatorname{Cas}_{\mathfrak{h}}) \cdot v = 0$, since v is H-invariant. While clearly $\pi(-\operatorname{Cas}_{\mathfrak{g}}) \cdot v = \lambda^\pi v$, the computation of the term $\pi(-\operatorname{Cas}_{\mathfrak{k}}) \cdot v$ is more involved.

Consider the decomposition

$$V_\pi = \bigoplus_{\tau \in \widehat{\mathsf{K}},\, [\tau : \pi|_{\mathsf{K}}] > 0} V_\pi(\tau),$$

where the subspace $V_\pi(\tau)$ is given by the sum of all K-submodules of V_π equivalent to τ. As a K-module, $V_\pi(\tau)$ is equivalent to $[\tau : \pi|_{\mathsf{K}}]$ copies of τ. Since $V_\pi(\tau)$ is obviously invariant under the action of H, we conclude that

$$V_\pi^{\mathsf{H}} = \bigoplus_{\tau\in\widehat{\mathsf{K}},\,[\tau:\pi|_{\mathsf{K}}]>0} V_\pi(\tau)^{\mathsf{H}}.$$

For $v \in V_\pi(\tau)^{\mathsf{H}}$, it follows that $\pi(-\operatorname{Cas}_{\mathfrak{k}}) \cdot v = \lambda^\tau v$, and, consequently, from (2.5),

$$\pi(-C_{(r,s)}) \cdot v = \left((r^2 - s^2)\,\lambda^\tau + s^2\lambda^\pi\right) v = \lambda^{\pi,\tau}(r,s)\, v.$$

Moreover, these eigenvalues exhaust the spectrum of $\pi(-C_{(r,s)})|_{V_\pi^{\mathsf{H}}}$, since

$$\dim V_\pi^{\mathsf{H}} = \sum_{\tau\in\widehat{\mathsf{K}},\,[\tau:\pi|_\pi]>0} \dim V_\pi(\tau)^{\mathsf{H}} = [\tau : \pi|_{\mathsf{K}}]\dim V_\tau^{\mathsf{H}} = [1_{\mathsf{H}} : \tau|_{\mathsf{H}}][\tau : \pi|_{\mathsf{K}}].$$

Finally, by definition, (2.4) is basic for the submersion $(\mathsf{G}/\mathsf{H}, \mathrm{g}_{(r,s)}) \to \mathsf{G}/\mathsf{K}$ if the associated eigenfunctions are constant along the fibers K/H. In this case, they descend to eigenfunctions of the Laplace–Beltrami operator on the base G/K. Applying [7, Proposition 2.2] to G/K, this corresponds to $\pi \in \widehat{\mathsf{G}}_{\mathsf{K}}$ and $\tau = 1_{\mathsf{K}}$. □

Following the method described in Theorem 2.4, the ingredients needed to explicitly determine the Laplace spectrum of the homogeneous space $(\mathsf{G}/\mathsf{H}, \mathrm{g}_{(r,s)})$ are:

(i) the set $\widehat{\mathsf{G}}_{\mathsf{H}}$ of spherical representations associated to (G, H);
(ii) the integers $[1_{\mathsf{H}} : \tau|_{\mathsf{H}}]$ and $[\tau : \pi|_{\mathsf{K}}]$ for $\tau \in \widehat{\mathsf{K}}$ satisfying $[\tau : \pi|_{\mathsf{K}}] > 0$;
(iii) the coefficients λ^π and λ^τ for all $\pi \in \widehat{\mathsf{G}}_{\mathsf{H}}, \tau \in \widehat{\mathsf{K}}$ with $[1_{\mathsf{H}} : \tau|_{\mathsf{H}}][\tau : \pi|_{\mathsf{K}}] > 0$.

All the above are Lie-theoretic in nature. While the first is known in many cases, the second depends on branching rules that are typically rather intricate, making this the most difficult part of the computation, see Subsection 2.4. Fortunately, the scalars λ^π and λ^τ are easily computed using Freudenthal's formula, as follows.

2.3 | Freudenthal's formula

Fix a maximal torus T in G such that $\mathsf{T} \cap \mathsf{K}$ and $\mathsf{T} \cap \mathsf{H}$ are maximal tori in K and H, respectively. Then $\mathfrak{t}_{\mathbb{C}} = \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, and we denote by $\Phi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ its root system. If $\mathfrak{g}$ is not semisimple (but necessarily reductive), then $\Phi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ is the root system associated to the semisimple part $[\mathfrak{g}, \mathfrak{g}]$ with respect to $[\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{t}$. Fix an order on $i\mathfrak{t}^*$ inducing a positive root system $\Phi^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. By the highest weight theorem (see, for example, [17, Theorem 9.4, 9.5] or [18, Theorem 5.110]), irreducible G-representations correspond to elements in the set $P^+(\mathsf{G})$ of dominant G-integral weights. Analogous objects are defined for K and H, provided that the orders are compatible. For $\Lambda \in P^+(\mathsf{G})$, we denote by π_Λ the unique (up to equivalences) irreducible representation of G with the highest weight Λ. Analogously, for $\mu \in P^+(\mathsf{K})$ and $\nu \in P^+(\mathsf{H})$, we denote by τ_μ and σ_ν the representations with highest weight μ and ν, respectively.

Freudenthal's formula (see [25, Lemma 5.6.4] or [17, Proposition 10.6]) applied to $\Lambda \in P^+(\mathsf{G})$ and $\mu \in P^+(\mathsf{K})$ gives, respectively,

$$\lambda^{\pi_\Lambda} = \langle \Lambda, \Lambda + 2\rho_{\mathfrak{g}} \rangle_0, \quad \text{and} \quad \lambda^{\tau_\mu} = \langle \mu, \mu + 2\rho_{\mathfrak{k}} \rangle_0, \tag{2.6}$$

where $\rho_{\mathfrak{g}}$ and $\rho_{\mathfrak{k}}$ are half the sum of positive roots in $\Phi^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ and $\Phi^+(\mathfrak{k}_{\mathbb{C}}, (\mathfrak{t} \cap \mathfrak{k})_{\mathbb{C}})$, respectively, and $\langle \cdot, \cdot \rangle_0$ is the Hermitian extension of $\langle \cdot, \cdot \rangle_0|_{\mathfrak{t}}$ to $\mathfrak{t}^*_{\mathbb{C}}$.

2.4 | Product group

The branching problem needed to compute ingredient (ii) above has an important simplification if $\mathsf{K} = \mathsf{HL} \simeq \mathsf{H} \times \mathsf{L}$, where L is a closed subgroup of G that commutes with H. In this case, the submersion (2.3) becomes

$$\mathsf{L} \longrightarrow \mathsf{G}/\mathsf{H} \longrightarrow \mathsf{G}/(\mathsf{H} \times \mathsf{L}).$$

It is well known that every irreducible K-representation is of the form $\sigma \otimes \phi$ for some $\sigma \in \widehat{\mathsf{H}}$ and $\phi \in \widehat{\mathsf{L}}$. Since $(\sigma \otimes \phi)|_{\mathsf{H}} = \sigma$, any $\tau \in \widehat{\mathsf{K}}$ contributing to $\operatorname{Spec}(\mathsf{G}/\mathsf{H}, \mathrm{g}_{(r,s)})$ in Theorem 2.4 must be of the form $\tau = 1_{\mathsf{H}} \otimes \phi$, and also, $[1_{\mathsf{H}} : \tau|_{\mathsf{H}}] = 1$. Moreover,

$$[\tau : \pi|_{\mathsf{K}}] = [1_{\mathsf{H}} \otimes \phi : \pi|_{\mathsf{K}}] = \dim_{\mathsf{H}\times\mathsf{L}}(V_{1_{\mathsf{H}}} \otimes V_\phi, V_\pi) = \dim_{\mathsf{L}}(V_\phi, V_\pi^{\mathsf{H}}) =: [\phi : V_\pi^{\mathsf{H}}].$$

In other words, since H and L commute, the L-action leaves V_π^{H} invariant, and $[1_{\mathsf{H}} \otimes \phi : \pi|_{\mathsf{K}}]$ is the multiplicity of ϕ in the decomposition of V_π^{H} as an L-module. Furthermore, by Freudenthal's formula, for any $\eta \in P^+(\mathsf{L})$,

$$\lambda^{1_{\mathsf{H}}\otimes\phi_\eta} = \langle \mu_{1_{\mathsf{H}}\otimes\phi_\eta}, \mu_{1_{\mathsf{H}}\otimes\phi_\eta} + 2\rho_{\mathfrak{h}\oplus\mathfrak{l}} \rangle_0 = \langle \eta, \eta + 2\rho_{\mathfrak{l}} \rangle_0, \tag{2.7}$$

where $\mathfrak{l}$ is the Lie algebra of L, and, as before, $\rho_{\mathfrak{l}} = \frac{1}{2}\sum_{\alpha\in\Phi^+(\mathfrak{l}_{\mathbb{C}}, (\mathfrak{t}\cap\mathfrak{l})_{\mathbb{C}})} \alpha$. Therefore, we may restate Theorem 2.4 in this case as follows.

Corollary 2.5. *If* $\mathsf{K} = \mathsf{HL}$ *as above, then the spectrum of the Laplace–Beltrami operator on the homogeneous space* $(\mathsf{G}/\mathsf{H}, \mathrm{g}_{(r,s)})$ *consists of the eigenvalues*

$$\lambda^{\pi, 1_{\mathsf{H}}\otimes\phi_\eta}(r,s) = (r^2 - s^2)\,\langle \eta, \eta + 2\rho_{\mathfrak{l}} \rangle_0 + s^2 \lambda^\pi, \tag{2.8}$$

where $(\pi, \phi_\eta) \in \widehat{\mathsf{G}}_{\mathsf{H}} \times \widehat{\mathsf{L}}$ *is such that* $[\phi_\eta : V_\pi^{\mathsf{H}}] > 0$*, with multiplicity*

$$m_{\pi,\phi_\eta} = [\phi_\eta : V_\pi^{\mathsf{H}}] \dim V_\pi.$$

Moreover, (2.8) is basic for the Riemannian submersion $(\mathsf{G}/\mathsf{H}, \mathrm{g}_{(r,s)}) \to \mathsf{G}/\mathsf{K}$ *if* $\eta = 0$*, in which case* $\lambda^{\pi, 1_{\mathsf{H}}\otimes\phi_\eta}(r,s) = s^2 \lambda^\pi$ *and* $m_{\pi,\phi_\eta} = \dim V_\pi$.

3 | EIGENVALUES OF THE LAPLACIAN ON $\mathbb{S}^{2n+1}$

In this section, we determine the full Laplace spectrum of the homogeneous spheres $(\mathbb{S}^{2n+1}, \mathbf{g}(t))$, $n \geqslant 1$, as in (1.1). Although there are several partial results in the literature, for example, this is done for all odd n in [7, Theorem 3.9, Remark 3.10], we include a complete argument below to illustrate the method in Section 2.

A homogeneous metric on $\mathbb{S}^{2n+1}$ is $\mathsf{SU}(n+1)$-invariant if and only if it is $\mathsf{U}(n+1)$-invariant. Although the $\mathsf{U}(n+1)$-action on $\mathbb{S}^{2n+1}$ is not effective, we shall use it since it simplifies some computations. Throughout this section, we set:

$$\begin{aligned} \mathsf{G} &= \mathsf{U}(n+1), & \mathsf{L} &= \left\{ \begin{pmatrix} I & 0 \\ 0 & z \end{pmatrix} : z \in \mathsf{U}(1) \right\}, \\ \mathsf{H} &= \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} : A \in \mathsf{U}(n) \right\}, & \mathsf{K} &= \left\{ \begin{pmatrix} A & 0 \\ 0 & z \end{pmatrix} : A \in \mathsf{U}(n),\, z \in \mathsf{U}(1) \right\}. \end{aligned} \tag{3.1}$$

Clearly, $\mathsf{H} \simeq \mathsf{U}(n)$, $\mathsf{L} \simeq \mathsf{U}(1)$, and $\mathsf{K} = \mathsf{HL} \simeq \mathsf{U}(n)\mathsf{U}(1)$, as in Subsection 2.4, and it is well known that $\mathsf{G}/\mathsf{H} \cong \mathbb{S}^{2n+1}$ and $\mathsf{G}/\mathsf{K} \cong \mathbb{C}P^n$. It is easy to check that

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{i}\theta \end{pmatrix} : \theta \in \mathbb{R} \right\}, \qquad \mathfrak{q} = \left\{ \begin{pmatrix} 0 & v \\ -v^* & 0 \end{pmatrix} : v \in \mathbb{C}^n \right\} \tag{3.2}$$

satisfy $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{p}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q} = \mathfrak{h} \oplus (\mathfrak{p} \oplus \mathfrak{q})$. Moreover, as subrepresentations of the isotropy representation of H, $\mathfrak{p}$ is trivial and $\mathfrak{q}$ is the standard representation.

Consider the G-invariant metrics $\mathrm{g}_{(r,s)}$ on G/H as in Subsection 2.1. Since $\mathfrak{p}$ and $\mathfrak{q}$ are irreducible and non-equivalent, every G-invariant metric on G/H is isometric to some $\mathrm{g}_{(r,s)}$; for example, for all $t > 0$, the metric $\mathbf{g}(t)$ in (1.1) is isometric to $\mathrm{g}_{(\frac{1}{t\sqrt{2}},1)}$.

Proposition 3.1. *For all $n \geqslant 1$, the spectrum of the Laplace–Beltrami operator on $(\mathbb{S}^{2n+1}, \mathrm{g}_{(r,s)})$ consists of the eigenvalues*

$$\lambda^{(p,q)}(r,s) = (4p(p+q+n) + 2nq)s^2 + 2q^2r^2, \quad p, q \in \mathbb{N}_0, \tag{3.3}$$

which are basic if $q = 0$, and have multiplicity

$$m_{p,q} = (2 - \delta_{q0}) \frac{2p+q+n}{n} \binom{p+q+n-1}{p+q} \binom{p+n-1}{p}. \tag{3.4}$$

Proof. Since the groups (3.1) satisfy $\mathsf{K} = \mathsf{HL}$, we may apply Corollary 2.5. Fix the maximal torus of G given by $\mathsf{T} = \{\mathrm{diag}(e^{\mathrm{i}\theta_1}, \dots, e^{\mathrm{i}\theta_{n+1}}) : \theta_1, \dots, \theta_{n+1} \in \mathbb{R}\}$. Note that $\mathsf{T} \cap \mathsf{K}$, $\mathsf{T} \cap \mathsf{H}$, and $\mathsf{T} \cap \mathsf{L}$ are maximal tori in K, H, and L respectively. The Lie algebra $\mathfrak{t}$ and its complexification $\mathfrak{t}_{\mathbb{C}}$ consist of elements $Y = \mathrm{diag}(\mathrm{i}\theta_1, \dots, \mathrm{i}\theta_{n+1})$, where θ_j are in $\mathbb{R}$ and $\mathbb{C}$, respectively. Define $\varepsilon_j : \mathfrak{t}_{\mathbb{C}}^* \to \mathbb{C}$ as $\varepsilon_j(Y) = \mathrm{i}\theta_j$, for Y as above, and note that $\{\frac{1}{\sqrt{2}}\varepsilon_1, \dots, \frac{1}{\sqrt{2}}\varepsilon_{n+1}\}$ is a $\langle \cdot, \cdot \rangle_0$-orthonormal basis of $\mathfrak{t}_{\mathbb{C}}^*$.

With the standard order, we have $\Phi^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\varepsilon_i - \varepsilon_j : 1 \leqslant i < j \leqslant n+1\}$, so half the sum of positive roots is $\rho_{\mathfrak{g}} = \sum_{j=1}^{n+1} \frac{n+2-j}{2} \varepsilon_j$, and the set of dominant integral weights is $P^+(\mathsf{G}) = \{\sum_{j=1}^{n+1} a_j \varepsilon_j \in \bigoplus_{j=1}^{n+1} \mathbb{Z}\varepsilon_j : a_1 \geqslant a_2 \geqslant \cdots \geqslant a_{n+1}\}$.

The classical branching rule from G to H (see, for example, [18, Theorem 9.14]) states that, if $\Lambda = \sum_{i=1}^{n+1} a_i\varepsilon_i \in P^+(\mathsf{G})$ and $\nu = \sum_{i=1}^{n} b_i\varepsilon_i \in P^+(\mathsf{H})$, then $[\sigma_\nu : \pi|_\Lambda] > 0$ if and only if $a_1 \geqslant b_1 \geqslant a_2 \geqslant \cdots \geqslant a_n \geqslant b_n \geqslant a_{n+1}$; in which case $[\sigma_\nu : \pi|_\Lambda] = 1$. We conclude that $\pi_\Lambda \in \widehat{\mathsf{G}}_{\mathsf{H}}$, that is, $\dim V_\pi^{\mathsf{H}} = [1_{\mathsf{H}} : \pi|_\Lambda] > 0$, if and only if $a_i = 0$ for all $2 \leqslant i \leqslant n$ and $a_1 \geqslant 0 \geqslant a_{n+1}$. Therefore, the set of spherical representations is:

$$\widehat{\mathsf{G}}_{\mathsf{H}} = \{\pi_{k,l} := \pi_{l\varepsilon_1 - k\varepsilon_{n+1}} : k, l \in \mathbb{N}_0\}.$$

We henceforth abbreviate $V_{k,l} := V_{\pi_{k,l}}$. It is a simple matter to check that

$$\dim V_{k,l} = \frac{k+l+n}{n}\binom{k+n-1}{k}\binom{l+n-1}{l}, \tag{3.5}$$

by the Weyl dimension formula, see, for example, [18, Theorem 5.84].

Note that $\mathsf{L} \simeq \mathsf{U}(1)$ is abelian and $(\mathfrak{l} \cap \mathfrak{t})_{\mathbb{C}}^* = \mathbb{C}\varepsilon_{n+1}$. Thus, its root system is empty, that is, $\rho_{\mathfrak{l}} = 0$, and every L-integral weight is dominant, that is, $P^+(\mathsf{L}) = \{\phi_m := \phi_{m\varepsilon_{n+1}} : m \in \mathbb{Z}\}$. It is well known that $V_{k,l}^{\mathsf{H}} \simeq \phi_{l-k}$ as L-modules; more precisely, $\pi_{k,l}((\begin{smallmatrix} I_n & 0 \\ 0 & z \end{smallmatrix})) \cdot v = z^{l-k}\, v$ for $v \in V_{k,l}^{\mathsf{H}}$ and $z \in \mathsf{U}(1)$, see, for example, [15, Theorem 8.1.2].

From Corollary 2.5, the eigenvalues of $(\mathbb{S}^{2n+1}, \mathrm{g}_{(r,s)})$ are $\lambda^{\pi_{k,l}, 1_{\mathsf{H}} \otimes \phi_{l-k}}(r,s)$ for all $k, l \in \mathbb{N}_0$, with multiplicity $\dim V_{k,l}$. Moreover, by (2.6) and (2.7), we have that

$$\begin{aligned} \lambda^{1_{\mathsf{H}} \otimes \phi_m} &= \langle m\varepsilon_{n+1}, m\varepsilon_{n+1} \rangle_0 = 2m^2, \\ \lambda^{\pi_{k,l}} &= \langle l\varepsilon_1 - k\varepsilon_{n+1} + 2\rho_{\mathfrak{g}}, l\varepsilon_1 - k\varepsilon_{n+1} \rangle_0 = 2l(n+l) + 2k(n+k). \end{aligned}$$

We conclude from (2.8) that the corresponding eigenvalue is

$$\begin{aligned} \lambda^{\pi_{k,l}, 1_{\mathsf{H}} \otimes \phi_{l-k}}(r,s) &= (r^2 - s^2)\,\lambda^{1_{\mathsf{H}} \otimes \phi_{k-l}} + s^2 \lambda^{\pi_{k,l}} \\ &= 2(k-l)^2(r^2 - s^2) + (2l(n+l) + 2k(n+k))s^2 \\ &= (4kl + 2n(k+l))s^2 + 2(k-l)^2 r^2. \end{aligned} \tag{3.6}$$

For convenience of notation, let us reindex $(k,l) \in \mathbb{N}_0^2$ as $(p,q) \in \mathbb{N}_0^2$,

$$p := \min\{k,l\}, \qquad q := \max\{k,l\} - \min\{k,l\} = |k-l|. \tag{3.7}$$

Since $kl = p(p+q)$, $k + l = 2p + q$, and $(k-l)^2 = q^2$, (3.6) is equal to (3.3). Moreover, $q = 0$ if and only if $k = l$, which is equivalent to $\phi_{l-k} = 1_{\mathsf{L}}$, proving the claim regarding basic eigenvalues.

We conclude by determining the contribution $m_{p,q}$ to the multiplicity of the eigenvalue $\lambda^{(p,q)}(r,s) \in \operatorname{Spec}(\mathbb{S}^{2n+1}, \mathrm{g}_{(r,s)})$. On the one hand, if $q = 0$, the only solution to (3.7) is $(k,l) = (p,p)$, so this contribution is $m_{p,0} = \dim V_{p,p}$. On the other hand, if $q > 0$, then there are *two* solutions to (3.7), namely, $(k,l) = (p+q, p)$ and $(k,l) = (p, p+q)$, yielding a contribution of $m_{p,q} = \dim V_{p+q,p} + \dim V_{p,p+q} = 2\dim V_{p+q,p}$. Therefore, (3.4) now follows from (3.5). □

4 | EIGENVALUES OF THE LAPLACIAN ON $\mathbb{S}^{4n+3}$

This short section gives the full Laplace spectrum of the homogeneous spheres $(\mathbb{S}^{4n+3}, \mathbf{h}(t))$, $n \geqslant 1$, based on [7]. Following the same notation as above, we set

$$\begin{aligned} &\mathsf{G} = \mathsf{Sp}(n+1), && \mathsf{L} = \left\{ \begin{pmatrix} I & 0 \\ 0 & z \end{pmatrix} : z \in \mathsf{Sp}(1) \right\}, \\ &\mathsf{H} = \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} : A \in \mathsf{Sp}(n) \right\}, && \mathsf{K} = \left\{ \begin{pmatrix} A & 0 \\ 0 & z \end{pmatrix} : A \in \mathsf{Sp}(n),\ z \in \mathsf{Sp}(1) \right\}. \end{aligned} \tag{4.1}$$

Clearly, $\mathsf{H} \simeq \mathsf{Sp}(n)$, $\mathsf{L} \simeq \mathsf{Sp}(1)$, and $\mathsf{K} = \mathsf{HL} \simeq \mathsf{Sp}(n)\mathsf{Sp}(1)$, as in Subsection 2.4, and it is well known that $\mathsf{G}/\mathsf{H} \cong \mathbb{S}^{4n+3}$ and $\mathsf{G}/\mathsf{K} \cong \mathbb{H}P^n$. It is easy to check that

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} : a \in \operatorname{Im} \mathbb{H} \right\}, \quad \mathfrak{q} = \left\{ \begin{pmatrix} 0 & v \\ -v^* & 0 \end{pmatrix} : v \in \mathbb{H}^n \right\}. \tag{4.2}$$

They satisfy $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{p}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q} = \mathfrak{h} \oplus (\mathfrak{p} \oplus \mathfrak{q})$. As subrepresentations of the isotropy representation of H, $\mathfrak{p}$ is equivalent to three copies of the trivial representation, and $\mathfrak{q}$ is the standard representation.

Consider again the G-invariant metrics $\mathrm{g}_{(r,s)}$ on G/H, as in Subsection 2.1. This is a 2-parameter subfamily of the 4-parameter family of G-invariant metrics on G/H, see, for example, [7, Section 3.2]. For all $t > 0$, the metric $\mathbf{h}(t)$ in (1.1) is isometric to $\mathrm{g}_{(\frac{1}{t},1)}$.

Proposition 4.1. *For all $n \geqslant 1$, the spectrum of the Laplace–Beltrami operator on $(\mathbb{S}^{4n+3}, \mathrm{g}_{(r,s)})$ consists of eigenvalues*

$$\lambda^{(p,q)}(r,s) = (4p(p+q+2n+1) + 4qn)s^2 + q(q+2)r^2, \quad p, q \in \mathbb{N}_0, \tag{4.3}$$

which are basic if $q = 0$, and have multiplicity

$$m_{p,q} = \frac{(2p+q+2n+1)(q+1)^2}{(2n+1)(p+q+1)} \binom{p+q+2n}{p+q} \binom{p+2n-1}{p}. \tag{4.4}$$

Proof. This follows from Corollary 2.5, analogously to Proposition 3.1, using the appropriate branching law. Alternatively, it follows from [7, Lemma 3.2, Remark 3.3] replacing (p,q) with $(p+q,p)$, and setting $a = b = c = r/\sqrt{2}$, which implies that $\nu_j^{(q)}(a,b,c) = \frac{1}{2} r^2 q(q+2)$ for all $1 \leqslant j \leqslant q+1$. Accordingly, the multiplicity (4.4) is $q+1$ times that in [7, (3.11)], since $\nu_j^{(q)}$ does not depend on j in this case. $\square$

5 | EIGENVALUES OF THE LAPLACIAN ON $\mathbb{S}^{15}$

In this section, we determine the full Laplace spectrum of $(\mathbb{S}^{15}, \mathbf{k}(t))$. The Lie groups $\mathsf{H} \subset \mathsf{K} \subset \mathsf{G}$ in this case do not follow the pattern (3.1) and (4.1) of the previous sections. Namely, the inclusion $\mathsf{H} \subset \mathsf{K}$ is *not* given by a block embedding, and there is *no* Lie subgroup $\mathsf{L} \subset \mathsf{K}$ such that $\mathsf{K} = \mathsf{HL}$. In particular, Corollary 2.5 no longer applies.

Let $\mathsf{G} = \mathsf{Spin}(9)$, and identify its Lie algebra $\mathfrak{g} = \mathfrak{spin}(9)$ with $\mathfrak{so}(9)$ in the standard way. Let K be the subgroup of G isomorphic to $\mathsf{Spin}(8)$ with Lie algebra

$$\mathfrak{k} = \{\operatorname{diag}(X, 0) \in \mathfrak{g} : X \in \mathfrak{so}(8)\} \simeq \mathfrak{so}(8).$$

Clearly, $\mathsf{G}/\mathsf{K} \cong \mathbb{S}^8$. In order to define the subgroup $\mathsf{H} \subset \mathsf{K}$, which is isomorphic to $\mathsf{Spin}(7)$, but whose Lie algebra $\mathfrak{h} \subset \mathfrak{k}$ is not a block inclusion as $\mathfrak{k} \subset \mathfrak{g}$ above, we follow an approach tailored to apply the branching law of Baldoni–Silva [3, §6].

Let $\mathsf{F} \cong \mathsf{F}_4^{-20}$ be the simply connected Lie group associated to the real simple Lie algebra $\mathfrak{f}$ of type FII. The maximal compact subgroup of F (which is unique up to conjugation) is isomorphic to G, and $\mathsf{F}/\mathsf{G} \cong \mathbb{C}\mathrm{a}H^2$. Fix the maximal torus $\mathsf{T} \subset \mathsf{G} \subset \mathsf{F}$ with Lie algebra

$$\mathfrak{t} = \left\{ \operatorname{diag}\left(\begin{pmatrix} 0 & \mathrm{i}\theta_1 \\ -\mathrm{i}\theta_1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & \mathrm{i}\theta_4 \\ -\mathrm{i}\theta_4 & 0 \end{pmatrix}, 1 \right) \in \mathfrak{g} : \theta_1, \dots, \theta_4 \in \mathbb{R} \right\}. \tag{5.1}$$

Its complexification $\mathfrak{t}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$, with elements as in (5.1) where $\theta_1, \dots, \theta_4 \in \mathbb{C}$. The functionals $\varepsilon_j : \mathfrak{t}_{\mathbb{C}}^* \to \mathbb{C}$ that map such an element to θ_j form a $\mathbb{C}$-basis of $\mathfrak{t}_{\mathbb{C}}^*$. Fix an order on $\mathrm{i}\mathfrak{t}$ such that the corresponding positive root systems of $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{f}_{\mathbb{C}}$ with respect to $\mathfrak{t}_{\mathbb{C}}$ are

$$\Phi^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\varepsilon_i : 1 \leqslant i \leqslant 4\} \cup \{\varepsilon_i \pm \varepsilon_j : 1 \leqslant i < j \leqslant 4\},$$
$$\Phi^+(\mathfrak{f}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \Phi^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) \cup \left\{ \tfrac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4) \right\}.$$

Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{g}$ on $\mathfrak{f}$ with respect to the Killing form of $\mathfrak{f}$, so that $\mathfrak{f} = \mathfrak{g} \oplus \mathfrak{m}$ is a Cartan decomposition. Set $\alpha = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4) \in \Phi^+(\mathfrak{f}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, and choose root vectors $X_\alpha \in (\mathfrak{f}_{\mathbb{C}})_\alpha$ and $X_{-\alpha} \in (\mathfrak{f}_{\mathbb{C}})_{-\alpha}$ satisfying $[X_\alpha, X_{-\alpha}] \in \mathfrak{p}$. Then $\mathfrak{a} := \mathbb{R}(X_{\alpha_4} + X_{-\alpha_4})$ is a maximal abelian subalgebra of $\mathfrak{p}$. Finally, define H as the centralizer of $\mathfrak{a}$ in K, that is,

$$\mathsf{H} = \{k \in \mathsf{K} : \operatorname{Ad}(k) \cdot \mathfrak{a} = 0\}.$$

It can be checked that its Lie algebra $\mathfrak{h} = \{X \in \mathfrak{k} : [X, \mathfrak{a}] = 0\}$ is isomorphic to $\mathfrak{so}(7)$, and $\mathsf{H} \simeq \mathsf{Spin}(7)$, $\mathsf{G}/\mathsf{H} \cong \mathbb{S}^{15}$, and $\mathsf{K}/\mathsf{H} \cong \mathbb{S}^7$.

Consider the $\langle \cdot, \cdot \rangle_0$-orthogonal complements $\mathfrak{p}$ and $\mathfrak{q}$, and G-invariant metrics $\mathrm{g}_{(r,s)}$, as in Subsection 2.1. As subrepresentations of the isotropy representation of H, $\mathfrak{p}$ is the standard representation, and $\mathfrak{q}$ is the spin representation. Since they are irreducible and non-equivalent, every G-invariant metric on G/H is isometric to some $\mathrm{g}_{(r,s)}$; for example, for all $t > 0$, the metric $\mathbf{k}(t)$ in (1.1) is isometric to $\mathrm{g}_{(\frac{1}{t}, 2)}$.

Proposition 5.1. *The spectrum of the Laplace–Beltrami operator on* $(\mathbb{S}^{15}, \mathrm{g}_{(r,s)})$ *consists of eigenvalues*

$$\lambda^{(p,q)}(r,s) = \big(p^2 + p(q+7) + 2q\big)s^2 + q(q+6)r^2, \quad p, q \in \mathbb{N}_0, \tag{5.2}$$

which are basic if $q = 0$, *and have multiplicity*

$$m_{p,q} = \frac{2p+q+7}{7}\left(1 + \frac{q}{3}\right) \frac{\binom{p+q+6}{p+q}\binom{p+3}{p}\binom{q+5}{q}}{\binom{p+q+3}{p+q}}. \tag{5.3}$$

Proof. In order to apply Theorem 2.4, we first state the branching law from G to H in order to determine $\widehat{\mathsf{G}}_{\mathsf{H}}$ and the integers $[1_{\mathsf{H}} : \tau|_{\mathsf{H}}]$ and $[\tau : \pi|_{\mathsf{K}}]$ for $\tau \in \widehat{\mathsf{K}}$ satisfying $[\tau : \pi|_{\mathsf{K}}] > 0$. Note that $\mathsf{T} \cap \mathsf{K}$ is a maximal torus in K, and the corresponding positive root system is $\Phi^+(\mathfrak{k}_{\mathbb{C}}, (\mathfrak{t} \cap \mathfrak{k})_{\mathbb{C}}) = \{\varepsilon_i \pm \varepsilon_j : 1 \leqslant i < j \leqslant 4\}$ with simple roots $\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4, \varepsilon_3 + \varepsilon_4$. We have that

$$P^+(\mathsf{G}) = \left\{ \sum_{i=1}^{4} a_i\varepsilon_i : \begin{array}{l} a_1 \geqslant a_2 \geqslant a_3 \geqslant a_4 \geqslant 0, \\ 2a_i \in \mathbb{Z},\ a_i - a_j \in \mathbb{Z} \text{ for } 1 \leqslant i, j \leqslant 4 \end{array} \right\},$$

$$P^+(\mathsf{K}) = \left\{ \sum_{i=1}^{4} a_i\varepsilon_i : \begin{array}{l} a_1 \geqslant a_2 \geqslant a_3 \geqslant |a_4|, \\ 2a_i \in \mathbb{Z},\ a_i - a_j \in \mathbb{Z} \text{ for } 1 \leqslant i, j \leqslant 4 \end{array} \right\}.$$

The fundamental weights of $\Phi^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ are $\omega_1 = \varepsilon_1$, $\omega_2 = \varepsilon_1 + \varepsilon_2$, $\omega_3 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$, and $\omega_4 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)$, and they satisfy $P^+(\mathsf{G}) = \bigoplus_{i=1}^{4} \mathbb{N}_0\omega_i$.

We define $\varphi : \mathfrak{t}_{\mathbb{C}}^* \to \mathfrak{t}_{\mathbb{C}}^*$ to be the linear map determined by

$$\begin{aligned} \varphi(\varepsilon_1) &= \tfrac{1}{2}(+\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4), \\ \varphi(\varepsilon_2) &= \tfrac{1}{2}(+\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \varepsilon_4), \\ \varphi(\varepsilon_3) &= \tfrac{1}{2}(+\varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \varepsilon_4), \\ \varphi(\varepsilon_4) &= \tfrac{1}{2}(-\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4). \end{aligned}$$

One can check that $\varphi^2 = \mathrm{Id}$. Since φ permutes the simple roots of $\Phi^+(\mathfrak{k}_{\mathbb{C}}, (\mathfrak{t} \cap \mathfrak{k})_{\mathbb{C}})$, namely, $\varphi(\varepsilon_1 - \varepsilon_2) = \varepsilon_3 - \varepsilon_4$, $\varphi(\varepsilon_2 - \varepsilon_3) = \varepsilon_2 - \varepsilon_3$, $\varphi(\varepsilon_3 - \varepsilon_4) = \varepsilon_1 - \varepsilon_2$, and $\varphi(\varepsilon_3 + \varepsilon_4) = \varepsilon_3 + \varepsilon_4$, we have that φ is an automorphism of $\Phi^+(\mathfrak{k}_{\mathbb{C}}, (\mathfrak{t} \cap \mathfrak{k})_{\mathbb{C}})$, which extends to an automorphism of $\mathfrak{k}_{\mathbb{C}}$ that we denote again by φ. It turns out (see [3, p. 248]) that $\varphi(\mathfrak{h})$ is a copy of $\mathfrak{so}(7)$ embedded in $\mathfrak{k} \simeq \mathfrak{so}(8)$. More precisely, $\varphi(\mathfrak{h}) = \{\operatorname{diag}(X, 0, 0) \in \mathfrak{g} : X \in \mathfrak{so}(7)\}$, so the simple roots are: $\varepsilon_1 - \varepsilon_2 = \varphi(\varepsilon_3 - \varepsilon_4)$, $\varepsilon_2 - \varepsilon_3 = \varphi(\varepsilon_2 - \varepsilon_3)$, $\varepsilon_3 = \varphi(\frac{1}{2}(+\varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \varepsilon_4))$, and $\Phi^+(\varphi(\mathfrak{h})_{\mathbb{C}}, (\mathfrak{t} \cap \varphi(\mathfrak{h}))_{\mathbb{C}}) = \{\varepsilon_i : 1 \leqslant i \leqslant 3\} \cup \{\varepsilon_i \pm \varepsilon_j : 1 \leqslant i < j \leqslant 3\}$, hence

$$P^+(\mathsf{H}') = \left\{ \sum_{i=1}^{3} c_i\varepsilon_i : \begin{array}{l} c_1 \geqslant c_2 \geqslant c_3 \geqslant 0, \\ 2c_i \in \mathbb{Z},\ c_i - c_j \in \mathbb{Z} \text{ for } 1 \leqslant i, j \leqslant 3 \end{array} \right\},$$

where H' denotes the only connected Lie subgroup of K with Lie algebra $\varphi(\mathfrak{h})$.

We are now in position to state the branching law from G to H established by Baldoni–Silva [3, Theorem 6.3]. For $\Lambda = \sum_{i=1}^{4} a_i\varepsilon_i \in P^+(\mathsf{G})$,

$$\pi_\Lambda|_{\mathsf{H}} = \sum_{\substack{b_1\varepsilon_1 + b_2\varepsilon_2 + b_3\varepsilon_3 + b_4\varepsilon_4 \in P^+(\mathsf{K}): \\ a_1 \geqslant b_1 \geqslant a_2 \geqslant b_2 \geqslant a_3 \geqslant b_3 \geqslant a_4 \geqslant |b_4|, \\ a_1 - b_1 \in \mathbb{Z}, \\ b_1'\varepsilon_1 + \dots + b_4'\varepsilon_4 := \varphi(b_1\varepsilon_1 + \dots + b_4\varepsilon_4),}} \quad \sum_{\substack{\nu := c_1\varepsilon_1 + c_2\varepsilon_2 + c_3\varepsilon_3 \in P^+(\mathsf{H}'): \\ b_1' \geqslant c_1 \geqslant b_2' \geqslant c_2 \geqslant b_3' \geqslant c_3 \geqslant |b_4'|, \\ b_1' - c_1 \in \mathbb{Z}}} \sigma_\nu \circ \varphi. \tag{5.4}$$

We claim that

$$\widehat{\mathsf{G}}_{\mathsf{H}} = \{\pi_{p,q} := \pi_{p\omega_1 + q\omega_4} : p, q \in \mathbb{N}_0\}.$$

Recall that $\omega_1 = \varepsilon_1$ and $\omega_4 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)$. Let $\Lambda = \sum_{i=1}^4 a_i \varepsilon_i \in P^+(\mathsf{G})$. Of course, the trivial H-representation 1_{H} coincides with σ_ν with $\nu = 0$, that is, $c_1 = c_2 = c_3 = 0$. Therefore, if $[1_{\mathsf{H}} : \pi_\Lambda|_{\mathsf{H}}] > 0$, then the coefficients in (5.4) satisfy $b_2' = b_3' = b_4' = 0$ and $b_1' \in \mathbb{N}_0$, which gives $b_1 = b_2 = b_3 = -b_4 = b_1'/2$, and consequently $a_2 = a_3 = a_4 = b_1'/2$ and $a_1 - b_1'/2 \in \mathbb{N}_0$. We conclude that $\Lambda = b_1'\omega_4 + (a_1 - b_1'/2)\omega_1$, as claimed.

Moreover, for $\Lambda = p\omega_1 + q\omega_4$ with $p, q \in \mathbb{N}_0$, we have that $[1_{\mathsf{H}} : \pi_\Lambda|_{\mathsf{H}}] = 1$ (that is, 1_{H} occurs exactly once in $\pi_\Lambda|_{\mathsf{H}}$) since the coefficients b_i, b_i' for $1 \leqslant i \leqslant 4$ in (5.4) are uniquely determined in terms of p and q; indeed, $b_1' = 2b_1 = 2b_2 = 2b_3 = -2b_4 = q$ and $b_2' = b_3' = b_4' = 0$. This implies that there exists only one $\mu \in P^+(\mathsf{K})$ satisfying $[1_{\mathsf{H}} : \tau_\mu|_{\mathsf{H}}] = [\tau_\mu : \pi_\Lambda|_{\mathsf{K}}] = 1$, which is given by $\mu_q := \sum_{i=1}^4 b_i \varepsilon_i = \frac{q}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4)$.

By the above and Theorem 2.4, the eigenvalues of $(\mathbb{S}^{15}, \mathrm{g}_{(r,s)})$ are

$$\lambda^{(p,q)}(r,s) = \lambda^{\pi_{p,q},\tau_q}(r,s) = (r^2 - s^2)\,\lambda^{\tau_q} + s^2 \lambda^{\pi_{p,q}}, \quad p, q \in \mathbb{N}_0,$$

where $\tau_q \in \widehat{\mathsf{K}}$ has the highest weight $\mu_q = \frac{q}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4)$, with multiplicity $m_{p,q} = \dim V_{\pi_{p,q}}$ equal to (5.3) by the Weyl dimension formula, see, for example, [18, Theorem 5.84]. Moreover, $\widehat{\mathsf{G}}_{\mathsf{K}} = \{\pi_{p,0} = \pi_{p\varepsilon_1} : p \in \mathbb{N}_0\}$ by the classical branching law from Spin(9) to Spin(8), so $\lambda^{(p,q)}(r,s)$ is basic if $q = 0$.

The only remaining step is to determine the scalars λ^{τ_q} and $\lambda^{\pi_{p,q}}$. It is easy to check that $\langle \varepsilon_i, \varepsilon_j \rangle_0 = \delta_{ij}$ for all $1 \leqslant i, j \leqslant 4$. Freudenthal's formula (2.6) gives

$$\begin{aligned}
\lambda^{\tau_q} &= \langle \mu_q, \mu_q + 2\rho_{\mathfrak{k}} \rangle \\
&= \left\langle \tfrac{q}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4), \tfrac{q}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4) + \sum_{i=1}^4 (8 - 2i)\varepsilon_i \right\rangle \\
&= \tfrac{q}{2}(\tfrac{q}{2} + 6) + \tfrac{q}{2}(\tfrac{q}{2} + 4) + \tfrac{q}{2}(\tfrac{q}{2} + 2) + \tfrac{q^2}{4} \\
&= q(q + 6).
\end{aligned}$$

Similarly, since $2\rho_{\mathfrak{g}} = \sum_{i=1}^4 (9 - 2i)\varepsilon_i$, we have that

$$\begin{aligned}
\lambda^{\pi_{p,q}} &= \langle p\omega_1 + q\omega_4, p\omega_1 + k\omega_4 + 2\rho_{\mathfrak{g}} \rangle \\
&= (p + \tfrac{q}{2})(p + \tfrac{q}{2} + 7) + \tfrac{q}{2}(\tfrac{q}{2} + 5) + \tfrac{q}{2}(\tfrac{q}{2} + 3) + \tfrac{q}{2}(\tfrac{q}{2} + 1) \\
&= p^2 + p(q + 7) + q^2 + 8q.
\end{aligned}$$

Combining the above, one obtains (5.2), which concludes the proof. □

6 | UNIFIED FORMULAE

In order to prove Theorem A in the Introduction, we collect in Table 1 the Laplace spectra of the homogeneous spheres $(\mathbb{S}^{N-1}, \mathrm{g}(t))$ in (1.3), as computed in Propositions 3.1, 4.1, and 5.1, keeping in mind the isometries relating homogeneous metrics in their geometric description (1.1) with their algebraic description (2.2).

TABLE 1 Eigenvalues of the homogeneous spheres $(\mathbb{S}^{N-1}, \mathrm{g}(t))$ in (1.3), where $N = 2d(n+1)$, $d = \dim_{\mathbb{C}} \mathbb{K} \in \{1,2,4\}$, and $p, q \in \mathbb{N}_0$

$\mathbb{K}$	$(\mathbb{S}^{N-1}, \mathbf{g}(t))$	Parameters (r,s) and Laplace–Beltrami spectrum
$\mathbb{C}$ $d=1$	$(\mathbb{S}^{2n+1}, \mathbf{g}(t))$	$(r,s) = (\frac{1}{t\sqrt{2}}, 1)$ $\lambda^{(p,q)}(t) = 4p(p+q+n) + 2nq + q^2 \frac{1}{t^2}$ $m_{p,q} = (2-\delta_{q0}) \frac{2p+q+n}{n} \binom{p+q+n-1}{p+q} \binom{p+n-1}{p}$
$\mathbb{H}$ $d=2$	$(\mathbb{S}^{4n+3}, \mathbf{h}(t))$	$(r,s) = (\frac{1}{t}, 1)$ $\lambda^{(p,q)}(t) = 4p(p+q+2n+1) + 4nq + q(q+2)\frac{1}{t^2}$ $m_{p,q} = \frac{(2p+q+2n+1)(q+1)^2}{(2n+1)(p+q+1)} \binom{p+q+2n}{p+q} \binom{p+2n-1}{p}$
$\mathbb{C}\mathrm{a}$ $d=4$	$(\mathbb{S}^{15}, \mathbf{k}(t))$	$(r,s) = (\frac{1}{t}, 2)$ $\lambda^{(p,q)}(t) = 4p(p+q+7) + 8q + q(q+6)\frac{1}{t^2}$ $m_{p,q} = \frac{2p+q+7}{7}(1+\frac{q}{3})\binom{p+q+6}{p+q}\binom{p+3}{p}\binom{q+5}{q}/\binom{p+q+3}{p+q}$

Proof of Theorem A. Replacing $d \in \{1,2,4\}$ in (1.4) and (1.5), one obtains $\lambda^{(p,q)}(t)$ and $m_{p,q}$ as listed in Table 1. By Propositions 3.1, 4.1, and 5.1, these are the eigenvalues and respective multiplicities of the Laplace–Beltrami operator on the corresponding sphere $(\mathbb{S}^{N-1}, \mathrm{g}(t))$, and $\lambda^{(p,q)}(t)$ is basic if $q = 0$. □

Recall that the distance sphere $S(r) \subset M$ is isometric to $(\mathbb{S}^{N-1}, \alpha^2\,\mathrm{g}(t))$, where (α, t) is $(\sin r, \cos r)$ or $(\sinh r, \cosh r)$ according to $M = \mathbb{K}P^{n+1}$ or $M = \mathbb{K}H^{n+1}$, cf. (1.2), and $N = \dim M = 2d(n+1)$. Rescaling all spaces in the Riemannian submersion (1.3) by α, since its fibers are totally geodesic, one obtains the inclusions of spectra

$$\tfrac{1}{\alpha^2}\mathrm{Spec}(\mathbb{K}P^n) \subset \mathrm{Spec}(S(r)) \subset \tfrac{1}{\alpha^2}\big(\mathrm{Spec}(\mathbb{K}P^n) + \mathrm{Spec}\big(\mathbb{S}^{2d-1}_t\big)\big), \tag{6.1}$$

where $+$ is the Minkowski sum of sets, $A + B = \{a + b \,:\, a \in A,\ b \in B\}$. These inclusions are also immediate from Theorem A, by analyzing the case $q = 0$ in (1.4).

However, there is another remarkable inclusion of spectra, given by the following:

Corollary 6.1. *The Laplace–Beltrami spectrum of $S(r) \subset M$ satisfies*

$$\mathrm{Spec}(S(r)) \subset \mathrm{Spec}\big(\mathbb{S}^{N-1}_\alpha\big) \pm \mathrm{Spec}\big(\mathbb{S}^{2d-1}_t\big), \tag{6.2}$$

where $+$ is used if M is projective, and $-$ if M is hyperbolic.

Let us first prove (6.2) with a geometric argument, assuming that $\mathbb{K} \in \{\mathbb{C}, \mathbb{H}\}$ and $M = \mathbb{K}P^{n+1}$ is a projective space, hence the base of the Riemannian submersion

$$\mathbb{S}^{2d-1}_1 \longrightarrow \mathbb{S}^{N+2d-1}_1 \longrightarrow \mathbb{K}P^{n+1} \tag{6.3}$$

whose totally geodesic fibers are precisely the orbits of the free action of the group $\mathbb{S}^{2d-1}_1 \subset \mathbb{K}^*$ of multiplicative units on the unit sphere

$$\mathbb{S}^{N+2d-1}_1 \subset \mathbb{R}^{N+2d} \cong \mathbb{K}^{N/2d+1}. \tag{6.4}$$

Being a distance sphere, the preimage of $S(r) \subset \mathbb{K}P^{n+1}$ under this submersion is the boundary of the tubular neighborhood of radius r of the fiber that corresponds to the central point of $S(r)$. Since this fiber is an orbit of the aforementioned action on (6.4), this boundary is a product of spheres, isometric to

$$\mathbb{S}^{N-1}_{\alpha} \times \mathbb{S}^{2d-1}_{t} = \mathbb{S}^{N+2d-1}_{1} \cap (\mathbb{R}^{N} \oplus \mathbb{R}^{2d}) \cong \mathbb{S}^{N+2d}_{1} \cap (\mathbb{K}^{N/2d} \oplus \mathbb{K}),$$

which proves (6.2) for $\mathbb{K} \in \{\mathbb{C}, \mathbb{H}\}$ and M projective. The same argument can be generalized to the case in which $M = \mathbb{K}H^{n+1}$ is hyperbolic, interpreting (6.4) as the unit pseudo-sphere in the pseudo-Riemannian vector space $\mathbb{K}^{N/2d} \oplus \mathbb{K}$ of signature $(N, 2d)$, analogous to the discussion in [4, Section 6].

Nevertheless, the above arguments *do not* apply to $\mathbb{K} = \mathbb{C}\mathrm{a}$ in either case because it is not associative; in particular, its unit sphere $\mathbb{S}^7_1$ is not a group. Moreover, it is well known that there are no fiber bundles $\mathbb{S}^{\ell} \to \mathbb{C}\mathrm{a}P^2$ such as (6.3) for topological reasons [13]. Thus, it is a somewhat surprising consequence of Theorem A that (6.2) still holds for $\mathbb{K} = \mathbb{C}\mathrm{a}$, in both projective and hyperbolic cases. In fact, (6.2) can be explicitly parametrized, for all $\mathbb{K} \in \{\mathbb{C}, \mathbb{H}, \mathbb{C}\mathrm{a}\}$ at once, using that, by (1.4),

$$\lambda^{(p,q)}(t) = (2p+q)(2p+q+N-2) + q(q+2d-2)\left(\frac{1}{t^2} - 1\right),$$

and, by (1.2), we have $\pm\frac{\alpha^2}{t^2} = (\frac{1}{t^2} - 1)$ according to $M = \mathbb{K}P^{n+1}$ or $M = \mathbb{K}H^{n+1}$.

7 | RESONANCE AND RIGIDITY OF DISTANCE SPHERES

In this section, we recall the variational and bifurcation framework for CMC hypersurfaces and prove Theorem B in the Introduction.

7.1 | CMC spheres

Given an N-dimensional Riemannian manifold (M, g), let $\mathrm{Emb}(\mathbb{S}^{N-1}, M)$ be the space of $C^{2,\alpha}$ unparametrized embeddings of $\mathbb{S}^{N-1}$ into M, that is, equivalence classes of embeddings $\mathbf{x} \colon \mathbb{S}^{N-1} \to M$ for the action of $\mathrm{Diff}(\mathbb{S}^{N-1})$ by right composition. Consider the family of functionals

$$\begin{aligned} f_H \colon\ & \mathrm{Emb}(\mathbb{S}^{N-1}, M) \longrightarrow \mathbb{R} \\ f_H(\mathbf{x}) &= \mathrm{Area}(\mathbf{x}) + H\ \mathrm{Vol}(\mathbf{x}), \end{aligned} \tag{7.1}$$

where $\mathrm{Area}(\mathbf{x})$ denotes the $(N-1)$-volume of $\mathbf{x}(\mathbb{S}^{N-1})$, and $\mathrm{Vol}(\mathbf{x})$ the N-volume of the region enclosed by $\mathbf{x}(\mathbb{S}^{N-1})$. It is well known that critical points of (7.1) are precisely the embedded spheres in M with CMC H. Moreover, the second variation of (7.1) at a critical point is represented by the Jacobi operator

$$J_{\mathbf{x}}(\phi) = \Delta_{\mathbf{x}}\phi - (\mathrm{Ric}(\vec{n}_{\mathbf{x}}) + \|A_{\mathbf{x}}\|^2)\phi, \tag{7.2}$$

acting on the space of functions $\phi : \mathbb{S}^{n-1} \to \mathbb{R}$ with $\int_{\mathbb{S}^{N-1}} \phi = 0$, where $\Delta_{\mathbf{x}}$ is the Laplace–Beltrami operator on $\mathbb{S}^{N-1}$ with respect to the metric induced by the embedding $\mathbf{x} : \mathbb{S}^{N-1} \to M$, $\vec{n}_{\mathbf{x}}$ is a unit normal vector field to $\mathbf{x}(\mathbb{S}^{N-1}) \subset M$, and $\|A_{\mathbf{x}}\|$ is the Hilbert–Schmidt norm of its second fundamental form; for details, see, for example, [4, Section 2]. Functions $\phi \in \ker J_{\mathbf{x}}$ are called *Jacobi fields*, and the number $i_{\mathrm{Morse}}(\mathbf{x})$ of negative eigenvalues of $J_{\mathbf{x}}$, counted with multiplicity, is called the *Morse index* of $\mathbf{x}$. Moreover, $\mathbf{x}$ is *stable* if and only if $J_{\mathbf{x}}$ is positive-semidefinite, that is, $i_{\mathrm{Morse}}(\mathbf{x}) = 0$, and *non-degenerate* if and only if $\ker J_{\mathbf{x}} = \{0\}$.

7.2 | Equivariant rigidity and resonance

If a Lie group G acts isometrically on M, then (7.1) is clearly invariant under left composition with this action, so the entire G-orbit of a critical point is critical. Moreover, since (7.2) is G-equivariant, each Killing field $X \in \mathfrak{g}$ determines a Jacobi field $\phi_X = \langle X, \vec{n}_{\mathbf{x}} \rangle \in \ker J_{\mathbf{x}}$. In this context, we say that $\mathbf{x}$ is G-*equivariantly non-degenerate* if $\ker J_{\mathbf{x}}$ consists solely of such Jacobi fields induced by the G-action.

Let K be the G-isotropy of $x_0 \in M$, and assume that the K-action is transitive on all geodesic distance spheres $S(r) \subset M$ centered at x_0. In particular, the (unparametrized) embeddings

$$\mathbf{x}_r : \mathbb{S}^{N-1} \to M, \quad \mathbf{x}_r(\mathbb{S}^{N-1}) = S(r), \tag{7.3}$$

have CMC $H(S(r))$ for each r. Furthermore, assume that the map $r \mapsto H(S(r))$ is a diffeomorphism, so that $\mathbf{x}_r$ may also be parametrized by its mean curvature. In this context, an appropriate G-equivariant version of the implicit function theorem [11, Theorem 1.4] implies the following theorem.

Theorem 7.1. *Suppose that (7.3) is G-equivariantly non-degenerate if $r = r_*$. There exists $\varepsilon > 0$ such that, if an embedded sphere $\Sigma \subset M$ has CMC $H(\Sigma) = H(S(r))$, $r \in (r_* - \varepsilon, r_* + \varepsilon)$ and, up to isometries in G, is sufficiently close to $S(r)$ in $C^{2,\alpha}$-topology, then Σ is congruent to $S(r)$ via an isometry in G.*

The radii r_* for which the conclusion of Theorem 7.1 fails are called *resonant*.

Definition 7.2. We say that r_* is a *resonant radius* if there exist sequences r_j of radii converging to r_*, and $\Sigma_j \subset M$ of embedded CMC spheres converging to $S(r_*)$ in $C^{2,\alpha}$-topology, such that $H(\Sigma_j) = H(S(r_j))$ for all j, and Σ_j is not congruent to $S(r_j)$ via any isometry in G.

Clearly, by Theorem 7.1, a necessary condition for r_* to be resonant is that $\mathbf{x}_{r_*}$ is *not* G-equivariantly non-degenerate. The following sufficient condition for resonancy is a direct consequence of the equivariant bifurcation criterion [10, Theorem 5.4].

Theorem 7.3. *If for all $\varepsilon > 0$ sufficiently small, $\mathbf{x}_{r_*-\varepsilon}$ and $\mathbf{x}_{r_*+\varepsilon}$ are G-equivariantly non-degenerate and $i_{\mathrm{Morse}}(\mathbf{x}_{r_*-\varepsilon}) \neq i_{\mathrm{Morse}}(\mathbf{x}_{r_*+\varepsilon})$, then r_* is resonant.*

TABLE 2 Symmetric pairs (G,K) of rank one corresponding to the projective spaces $\mathbb{K}P^{n+1}$, and their non-compact duals, the hyperbolic spaces $\mathbb{K}H^{n+1}$

G/K	G	K
$\mathbb{C}P^{n+1}$	$\mathsf{SU}(n+2)$	$\mathsf{S}(\mathsf{U}(n+1)\mathsf{U}(1))$
$\mathbb{H}P^{n+1}$	$\mathsf{Sp}(n+2)$	$\mathsf{Sp}(n+1)\mathsf{Sp}(1)$
$\mathbb{C}\mathrm{a}P^2$	F_4	$\mathsf{Spin}(9)$
$\mathbb{C}H^{n+1}$	$\mathsf{SU}(n+1,1)$	$\mathsf{S}(\mathsf{U}(n+1)\mathsf{U}(1))$
$\mathbb{H}H^{n+1}$	$\mathsf{Sp}(n+1,1)$	$\mathsf{Sp}(n+1)\mathsf{Sp}(1)$
$\mathbb{C}\mathrm{a}H^2$	F_4^{-20}	$\mathsf{Spin}(9)$

7.3 | Rank one symmetric spaces

We now briefly revisit some well-known aspects of the geometry of rank one symmetric spaces that are used in the proof of Theorem B. First, recall that the symmetric pairs (G,K) that give rise to such spaces $M=\mathsf{G}/\mathsf{K}$ are as listed in Table 2. These semisimple Lie groups G act transitively on M, and $\mathsf{K}\subset\mathsf{G}$ is identified with the isotropy of a point $x_0\in M$, so its Lie algebra is $\mathfrak{k}=\{X\in\mathfrak{g} : X_{x_0}=0\}$. We fix a Cartan decomposition

$$\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m}, \tag{7.4}$$

and recall that the space $\mathfrak{m}=\{X\in\mathfrak{g} : (\nabla X)_{x_0}=0\}$ of infinitesimal transvections at x_0 is naturally identified with $T_{x_0}M$; in particular, $\dim\mathfrak{m}=\dim M=N$. The codimension of K-orbits on distance spheres $S(r)\subset M$ is equal to $\mathrm{rank}(M)-1$, so all these K-actions are transitive in our rank one setting. Thus, the eigenvalues of the second fundamental form A_r of $S(r)$ with respect to the unit outward-pointing normal $\vec{n}_r$ are constant, and can be computed as follows, see, for example, [12, §6]:

$$\begin{cases} 2\cot(2r), & \text{with multiplicity } 2d-1 \\ \cot(r), & \text{with multiplicity } 2dn \end{cases}, \qquad \text{if } M=\mathbb{K}P^{n+1},$$

$$\begin{cases} 2\coth(2r), & \text{with multiplicity } 2d-1 \\ \coth(r), & \text{with multiplicity } 2dn \end{cases}, \qquad \text{if } M=\mathbb{K}H^{n+1},$$

where $d=\dim_{\mathbb{C}}\mathbb{K}$, as before. Thus, the mean curvature of $S(r)\subset M$ is:

$$H(S(r))=\begin{cases} (N-1)\cot r-(2d-1)\tan r, & \text{if } M=\mathbb{K}P^{n+1}, \\ (N-1)\coth r+(2d-1)\tanh r, & \text{if } M=\mathbb{K}H^{n+1}. \end{cases} \tag{7.5}$$

Note that $H(S(r))$ is always decreasing, since $N=\dim M=2d(n+1)$; in particular, the map $r\mapsto H(S(r))$ is a diffeomorphism. Moreover, we have:

$$\|A_r\|^2=\begin{cases} 2dn\cot^2(r)+4(2d-1)\cot^2(2r), & \text{if } M=\mathbb{K}P^{n+1}, \\ 2dn\coth^2(r)+4(2d-1)\coth^2(2r), & \text{if } M=\mathbb{K}H^{n+1}. \end{cases} \tag{7.6}$$

The following is essential to determine if $S(r)$ is G-equivariantly non-degenerate:

Lemma 7.4. *A Killing field $X \in \mathfrak{g}$ induces a non-zero Jacobi field $\phi_X = \langle X, \vec{n}_r\rangle$ on $S(r)$ if and only if $X \in \mathfrak{m}$. Thus, the space of Jacobi fields on $S(r)$ has dimension $\geqslant N$, and equality holds if and only if $S(r)$ is G-equivariantly non-degenerate.*

Proof. Clearly, $X \in \mathfrak{g}$ induces the trivial Jacobi field $\phi_X \equiv 0$ if and only if X is everywhere tangent to $S(r) \subset M$. This implies that the 1-parameter subgroup of isometries in G associated to such a Killing field X leaves invariant $S(r) = \{x \in M : \operatorname{dist}(x_0, x) = r\}$, and hence fixes $x_0 \in M$, so $X \in \mathfrak{k}$. Conversely, $\phi_X \equiv 0$ for all $X \in \mathfrak{k}$. The result now follows from (7.4) and the fact that $\dim \mathfrak{m} = N$. $\square$

Lastly, routine computations of the Einstein constants for these spaces give:

$$\mathrm{Ric}_{\mathbb{K}P^{n+1}} = 2dn + 4(2d-1), \quad \text{and} \quad \mathrm{Ric}_{\mathbb{K}H^{n+1}} = -2dn - 4(2d-1). \tag{7.7}$$

We now combine Theorem A with Theorems 7.1 and 7.3 to prove Theorem B.

Proof of Theorem B. The Jacobi operator J_r of the distance sphere $S(r) \subset M$ can be computed using (7.2), (7.6), and (7.7), and simplifies to

$$J_r(\phi) = \Delta_r \phi - V(r)\phi, \tag{7.8}$$

where $\Delta_r = \frac{1}{\alpha^2}\Delta_{\mathrm{g}(t)}$, with α and t as in (1.2), and

$$V(r) = \begin{cases} (N-1)\csc^2 r + (2d-1)\sec^2 r, & \text{if } M = \mathbb{K}P^{n+1}, \\ (N-1)\operatorname{csch}^2 r - (2d-1)\operatorname{sech}^2 r, & \text{if } M = \mathbb{K}H^{n+1}. \end{cases}$$

First, let us analyze the projective case $M = \mathbb{K}P^{n+1}$, where $\alpha = \sin r$ and $t = \cos r$. By Theorem A, the eigenvalues of $\alpha^2 J_r$ are:

$$\begin{aligned} \lambda^{(p,q)}(t) - \alpha^2 V(r) &= 4p\left(p + q + \tfrac{N}{2} - 1\right) + 2dnq + q(q+2d-2)\sec^2 r - V(r)\sin^2 r \\ &= 4p\left(p + q + \tfrac{N}{2} - 1\right) + 2dnq - (N-1) \\ &\quad + (q(q-1) + (2d-1)(q - \sin^2 r))\sec^2 r, \end{aligned}$$

for all $(p,q) \in \mathbb{N}_0^2 \setminus \{(0,0)\}$. In particular, for all $p \in \mathbb{N}$,

$$\begin{aligned} \lambda^{(p,0)}(\cos r) - \sin^2 r\, V(r) &= 4p\left(p + \tfrac{N}{2} - 1\right) - (N-1) - (2d-1)\tan^2 r \\ &= 4p(p-1) + N(2p-1) + 1 - (2d-1)\tan^2 r \end{aligned}$$

is a decreasing function of $0 < r < \frac{\pi}{2}$, with a unique zero at:

$$r_p := \arctan \sqrt{\frac{4p(p-1)+N(2p-1)+1}{2d-1}}. \tag{7.9}$$

Note that $r_1 = \arctan \sqrt{\frac{N+1}{2d-1}}$, and $r_p \nearrow \frac{\pi}{2}$ as $p \nearrow +\infty$. Moreover, for all r,

$$\lambda^{(0,1)}(\cos r) - \sin^2 r\, V(r) = 0, \tag{7.10}$$

while, if $q \geqslant 2$, then

$$\begin{aligned}\lambda^{(0,q)}(\cos r) - \sin^2 r\, V(r) &\geqslant \lambda^{(0,2)}(\cos r) - \sin^2 r\, V(r)\\ &= 2dn + (2d+1)\sec^2 r\\ &\geqslant N + 1 > 0,\end{aligned}$$

and, if both $p \geqslant 1$ and $q \geqslant 1$, then

$$\begin{aligned}\lambda^{(p,q)}(\cos r) - \sin^2 r\, V(r) &\geqslant 4\left(\frac{N}{2}+1\right) + 2dn - (N-1) + (2d-1)\\ &= 2N + 4 > 0.\end{aligned}$$

Thus, if $r \notin \{r_p \,:\, p \in \mathbb{N}\}$, the only zero eigenvalues of J_r are (7.10), and hence $\dim \ker J_r$ coincides with the dimension of the eigenspace of $\Delta_{\mathrm{g}(t)}$ associated to $\lambda^{(0,1)}(t)$, which is $m_{0,1} = N$, by Theorem A. Therefore, it follows from Lemma 7.4 that $S(r)$ is G-equivariantly non-degenerate for all $r \notin \{r_p \,:\, p \in \mathbb{N}\}$.

Furthermore, it follows from the above spectral analysis that

$$i_{\mathrm{Morse}}(\mathbf{x}_r) = \sum_{\{p \in \mathbb{N} : r_p < r\}} m_{p,0}.$$

Thus, the claims in Theorem B regarding $M = \mathbb{K}P^{n+1}$ follow from applying Theorem 7.1 to each $r_* \notin \{r_p \,:\, p \in \mathbb{N}\}$, and Theorem 7.3 to each $r_* \in \{r_p \,:\, p \in \mathbb{N}\}$.

Second, let us analyze the hyperbolic case $M = \mathbb{K}H^{n+1}$, where $\alpha = \sinh r$ and $t = \cosh r$. Similarly to the above case, by Theorem A, the eigenvalues of $\alpha^2 J_r$ are:

$$\begin{aligned}\lambda^{(p,q)}(t) - \alpha^2 V(r) &= 4p\left(p+q+\frac{N}{2}-1\right) + 2dnq + q(q+2d-2)\operatorname{sech}^2 r\\ &\quad - V(r)\sinh^2 r\\ &= 4p\left(p+q+\frac{N}{2}-1\right) + 2dnq - (N-1)\\ &\quad + (q(q-1) + (2d-1)(q+\sinh^2 r))\operatorname{sech}^2 r,\end{aligned}$$

for all $(p, q) \in \mathbb{N}_0^2 \setminus \{(0,0)\}$. In particular, we have that, for all r,

$$\lambda^{(0,1)}(\cosh r) - \sinh^2 r\, V(r) = 0, \tag{7.11}$$

while, if $q \geqslant 2$, then

$$\begin{aligned} \lambda^{(0,q)}(\cosh r) - \sinh^2 r\, V(r) &\geqslant \lambda^{(0,2)}(\cosh r) - \sinh^2 r\, V(r) \\ &= 2dn + (2d+1)\operatorname{sech}^2 r \\ &\geqslant 2dn > 0, \end{aligned}$$

and, for all $p \geqslant 1$ and $q \in \mathbb{N}_0$,

$$\lambda^{(p,q)}(\cosh r) - \sinh^2 r\, V(r) \geqslant N + 1 > 0.$$

Thus, the only zero eigenvalues of J_r are (7.11), and all other eigenvalues are strictly positive, so $i_{\mathrm{Morse}}(\mathbf{x}_r) = 0$ for all $r > 0$, that is, $S(r)$ is stable for all $r > 0$. As before, $\dim\ker J_r$ coincides with the dimension of the eigenspace of $\Delta_{\mathrm{g}(t)}$ associated to $\lambda^{(0,1)}(t)$, which is $m_{0,1} = N$, by Theorem A; so Lemma 7.4 implies that $\mathbf{x}_r$ is G-equivariantly non-degenerate for all $r > 0$, hence non-resonant by Theorem 7.1. □

Remark 7.5. It was known that a sequence of resonant radii $r_p \nearrow \frac{\pi}{2}$ existed for distance spheres $S(r)$ in $\mathbb{C}P^{n+1}$ and $\mathbb{H}P^{n+1}$ centered at any point x_0 due to basic eigenvalues for the Riemannian submersion $\mathbb{S}^{2d-1} \to S(r) \to \mathrm{Cut}(x_0)$, see [9, Example 6.1]. However, neither their exact location (7.9) nor the fact that *only* basic eigenvalues give rise to such bifurcations was previously known. Moreover, the study of local rigidity and resonance for geodesic spheres in $\mathbb{C}\mathrm{a}P^2$ was also not possible in [9] since none of the group normality assumptions $\mathsf{H} \lhd \mathsf{K}$ or $\mathsf{K} \lhd \mathsf{G}$ are satisfied in this case. The fact that it was possible to overcome these difficulties in Theorem B might suggest that a different approach, for example, using Mean Curvature Flow, cf. [9, Rem. 2.13], may lead to even more general bifurcation results.

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REFERENCES

1. T. Arias-Marco and D. Schueth, *Local symmetry of harmonic spaces as determined by the spectra of small geodesic spheres*, Geom. Funct. Anal. **22** (2012), 1–21.
2. J. C. Baez, *The octonions*, Bull. Amer. Math. Soc. (N.S.) **39** (2002), 145–205.
3. M. W. Baldoni-Silva, *Branching theorems for semisimple Lie groups of real rank one*, Rend. Sem. Mat. Univ. Padova **61** (1979), 229–250 (1980).
4. J. L. Barbosa, M. do Carmo, and J. Eschenburg, *Stability of hypersurfaces of constant mean curvature in Riemannian manifolds*, Math. Z. **197** (1988), 123–138.
5. G. Besson and M. Bordoni, *On the spectrum of Riemannian submersions with totally geodesic fibers*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **1** (1990), 335–340.
6. L. Bérard-Bergery and J.-P. Bourguignon, *Laplacians and Riemannian submersions with totally geodesic fibres*, Illinois J. Math. **26** (1982), 181–200.
7. R. G. Bettiol, E. A. Lauret, and P. Piccione, *The first eigenvalue of a homogeneous CROSS*, J. Geom. Anal. **32** (2022), no. 3, Paper No. 76, 63 pp.
8. R. G. Bettiol and P. Piccione, *Bifurcation and local rigidity of homogeneous solutions to the Yamabe problem on spheres*, Calc. Var. Partial Differential Equations **47** (2013), 789–807.
9. R. G. Bettiol and P. Piccione, *Delaunay-type hypersurfaces in cohomogeneity one manifolds*, Int. Math. Res. Not. IMRN **2016** (2016), 3124–3162.
10. R. G. Bettiol, P. Piccione, and G. Siciliano, *Equivariant bifurcation in geometric variational problems*, Analysis and topology in nonlinear differential equations, Progress in Nonlinear Differential Equations, vol. 85, Birkhäuser/Springer, Cham, 2014, pp. 103–133.
11. R. G. Bettiol, P. Piccione, and G. Siciliano, *On the equivariant implicit function theorem with low regularity and applications to geometric variational problems*, Proc. Edinb. Math. Soc. (2) **58** (2015), 53–80.
12. J.-P. Bourguignon and H. Karcher, *Curvature operators: pinching estimates and geometric examples*, Ann. Sci. École Norm. Sup. (4) **11** (1978), 71–92.
13. W. Browder, *Higher torsion in H-spaces*, Trans. Amer. Math. Soc. **108** (1963), 353–375.
14. H. Gluck, F. Warner, and W. Ziller, *The geometry of the Hopf fibrations*, Enseign. Math. (2) **32** (1986), 173–198.
15. R. Goodman and N. R. Wallach, *Symmetry, representations, and invariants*, Graduate Texts in Mathematics, vol. 255, Springer, Dordrecht, 2009.
16. L. Guijarro and G. Walschap, *When is a Riemannian submersion homogeneous?* Geom. Dedicata **125** (2007), 47–52.
17. B. Hall, *An elementary introduction, Lie groups, Lie algebras, and representations*, 2nd ed., Graduate Texts in Mathematics, vol. 222, Springer, Cham, 2015.
18. A. W. Knapp, *Lie groups beyond an introduction*, 2nd ed., Progress in Mathematics, vol. 140, Birkhäuser Boston, Inc., Boston, MA, 2002.
19. M. Lackmann, *The octonionic projective plane*, 2019–20 MATRIX annals, vol. 4 of MATRIX Book Series, Springer, Cham, 2021, pp. 135–145. Record of the ten programs in 2019 and two programs in January 2020 held at MATRIX, Creswick. Edited by David R. Wood, Jan de Gier, Cheryl E. Praeger and Terence Tao.
20. E. A. Lauret, *The smallest Laplace eigenvalue of homogeneous 3-spheres*, Bull. Lond. Math. Soc. **51** (2019), 49–69.
21. H. Mutô and H. Urakawa, *On the least positive eigenvalue of Laplacian for compact homogeneous spaces*, Osaka J. Math. **17** (1980), 471–484.
22. H. J. Rivertz and P. Tomter, *Stability of geodesic spheres*, Geometry and topology of submanifolds, VIII (Brussels, 1995/Nordfjordeid, 1995), World Sci. Publ., River Edge, NJ, 1996, pp. 320–324.
23. S. Tanno, *The first eigenvalue of the Laplacian on spheres*, Tôhoku Math. J. (2) **31** (1979), 179–185.
24. S. Tanno, *Some metrics on a* $(4r+3)$*-sphere and spectra*, Tsukuba J. Math. **4** (1980), 99–105.
25. N. R. Wallach, *Harmonic analysis on homogeneous spaces*, Pure and Applied Mathematics, vol. 19, Marcel Dekker, Inc., New York, 1973.