RT-MAT 90-03

A critical study on the foundations of Geometry.

R.G. LINTZ

Junho 1990

A CRITICAL STUDY ON THE FOUNDATIONS OF GEOMETRY

by

R.G. Lintz - University of São Paulo, Brazil

Summary

A new way of approaching the study of the foundations of geometry is attempted here by locating the subject in its historical perspective. A critical and comparative discussion of the Greek and the Western concepts of geometry tries to emphasize their differences.

- §I -

Introduction

Transcendental Dialogue:

Euclid: "*Σημεῖόν εστιν, οὗ μέρος ουθέν.*"

Hilbert: "*Nein! Die Ebene ist ein System von Dingen, welche* Punkte *heißen.*"

1. In this work we intend to develop a critical study on the foundations of geometry under a broader philosophical perspective than that mostly often considered by the authors dealing with this subject. Usually geometry is understood as a science which started with the Greeks and had afterwards a "natural evolution" through many centuries under the tutelage of several other people like the Arabs, the Indus, etc. and finally reached its "higher perfection" in the hands of the Western mathematicians. We have always rejected this approach to the subject and instead we shall build a completely different picture of the situation, which will be the aim of this work. As the historical perspective will be of fundamental importance for us we start by recalling some basic facts of philosophy of history and also ideas developed in our previous works [1-a] and [1-c].

Any theory of history develops a certain scheme through which facts are collected and analysed leading to some conclusion explaining "what happened and what will happen". A theory is better than another if it can explain more facts with a minimum of general assumptions. It is our opinion that the line of thoughts developed in philosophy of history starting with G.B. Vico in the XVIII century and culminating with the monumental works of A. Toynbee [2] and O. Spengler [3] is according to the criterion above, the most general and profound theory of history. It allows us to locate and explain facts better than any other that we have noticed and therefore that is the philosophy we shall use in this work. The reason why these theories have been rejected by the majority of philosophers and scientists is easily explained by psychology: they shake the very foundations of beliefs we considered as solidly established and in particular hurt the pride of our Western Civilization by turning it down from its pedestal of the finest achievements of mankind to an equal ground, without any claim of superiority, with the other civilizations and historical cultures which preceeded us in time. But it is about time we put aside our pride and arrogance and try to analyse the facts in a real objective and unbiased perspective.

2. Any historical culture is an organism which exists and develops itself in time and space. The word **organism** is made precise in the following way. It is an object of our consideration (*μάθημα*) determined by:

a) a certain form which can be detected by us through some material media like figures in space, colours, sounds, written words or signs of any kind, which we shall denote by its **organogram**, expressed and conducted by a set of rules named its **syntax**;

b) a certain number of fundamental elements giving the organism its own identity and distinguishing it from other organisms which we call its **structure**;

c) a certain genetic code responsible for its evolution in time according to certain rules denoted by its **organogen**.

Let us illustrate this concept with a few examples and for more details we refer the reader to [1-a].

Take a flower: its appearance is given by its particular shape, colours, etc., which is its organogram and nature settles these elements according to rules depending on its species and genus which is its **syntax**. Its internal organization, namely, its physiology characterized by the proper functioning of its several organs provides its **structure** and finally its genetic code which conditions its development from birth to death is its **organogen**.

As a second example take a historical culture. Its **organogram** is given by all its expressive forms, namely an architecture, a sculpture, perhaps a music and also a mathematics. These are objective documents which show the existence of the historical culture in space and time. Its **syntax** is provided by the rules of succession of those forms in space and time. One of these rules is that the beginning of each historical culture is a mythological age and the end a technological age. The **structure** of a historical culture is given by its primitive symbols translating its feeling of the cosmos. For instance, in the Greek culture among these symbols we find a strong tendence for the finite, the visible and plastic space the **terror of the infinite and time** what explain why sculpture is its most characteristic expressive form rather than, say, music and geometry as the study of figures in space is their mathematics together with the integers as a collection of unities. There is here no room for the abstraction of set theory and real numbers. This consideration of mathematics, in particular geometry, as a **form of expression attached to a historical culture** will be of fundamental importance in this work. Finally its **organogen** is given by its law of evolution in time: its birth, youth, maturity and old age. Each of these stages emphasizes one or more particular expressive forms. In general for mathematics when it emerges as an expressive form this happens in the transition from maturity to old age, while on the contrary, architecture is usually the first great expressive form of a historical culture appearing in its youth.

Finally, the *leit-motiv* of this work will be the consideration of geometry as an expressive form or better an organism attached to a historical culture. In our case we shall be dealing with geometry in the Greek culture and in the Western culture. Here by Western culture we understand the historical culture born in the so called nordic mythology an dominating today all over the world and perhaps over a large part of the universe du to its deep passion for the infinite, the abstract space and time. Space is reduced to a abstract concept. Its strong opposition to the picture of the world created by the Greel civilization will be responsible for its opposite approach to geometry.

3. The study of the syntax of the organogram of a certain organism is an importan stage of the study of that organism and the discipline dealing with that we shall denominat **Inorganic Logic**. It corresponds in the case where the organogram is given by a languag to what is traditionally called Logic and more recently, **Mathematical Logic**. However there is another discipline concerned with the study of the structure and the organoge of a certain organism which we shall denominate **Organic Logic**. As far as we knov this discipline did not have the same attention as the Inorganic Logic eventhough i is already present in all our mental activities. One of its fundamental principles is wha we have called the **Principle of Analogy**, which allows, for instance, a naturalist t classify plants and animals and which is the base for all experimental sciences. By its ow nature, contrary to what happens to Inorganic Logic, it is not formalizable: it exists onl in intuitive and inductive reasonings and never in deductive ones related to the very act o understanding. In this form it seems to have been already noticed by Goethe through hi great and profound intuitive feelings of Nature as in [4]. More details can be seen at [1-a]

We summarize the situation in the diagram below:

ORGANISM
- { organogram, syntax } → Inorganic Logic
- { structure, organogen } → Organic Logic

After the introduction of the fundamental ideas above we can state with more precision our aim in this work. We intend to consider geometry as an organism attached to eithe

the Greek culture or the Western culture. Consequently, what we understand by the foundations of geometry is not the same what is usually known by that name. Being an organism to study geometry or its foundations it means to study its organogram with its syntax, its structure and its organogen, what requires the use not only of Inorganic Logic but as well of Organic Logic. What have been done by people studying the foundations of geometry has been really the study of its organogram and syntax and moreover only as conceived in the West, namely by the Western culture and its mathematicians and philosophers. Clearly this is a rather short perspective of the area and that will be not our philosophy in this work. We shall be dealing first with geometry as conceived by the Greek culture in its proper environment. Afterwards we shall consider its study in the Western culture and finally a parallel between both geometries. This attitude will certainly provide new insight into the subject with many surprising results shaking a little bit perhaps some of our most firm convictions!

- §II -

Foundations of Greek Geometry

1. The study of any manifestation of a historical culture depends on the documents available to us and one of the difficulties of the subject is the correct analysis and interpretation of those documents leading to a reconstruction of the situation which is always subjetive eventhough we try our best to be as impartial as possible. The historian faces here the same type of problem as does the archeologist who wants to reconstruct, for instance, a whole sculpture from scattered pieces of stone. In the case of geometry we rely on the texts available, namely, whatever remained, to reconstruct the Greek thought on the subject. Of course, it is outside the scope of this work and also of our competence to endeavour into the critical analysis of the texts. Rather we shall rely here on the work of

Sir T.L. Heath and his translations of so many texts of Greek mathematicians, which we believe are among the best available in English. Certainly, the celebrated works of Heiberg and Diels will be always in the background to help us in those more obscure and debatable points.

Every analysis of the foundations of geometry starts with the question: "what does exist at the base of geometry".

For the Greek mathematicians what existed in geometry was the geometrical figures as entities existing in a visible space and with their own identities, namely, they were realities given *a priori* as far as possible from pure abstraction. Indeed, it belongs to our intuition the feeling of straightness, flatness, the feeling of spaciousness given by our freedom of movements and felt with great intensity by a ballerina, etc. and in giving form to those sensations the Greeks explained the concept of straight line, point and plane with a strong plastic, visible and finite content.

Therefore what exists in mathematics is by one side the geometrical figure and by another side the numbers as discrete unities and completely set apart during the establishment of mathematics as a rigorous discipline in the hands of Eudoxus and Euclid. Indeed, the failure of the pythagorean approach to geometry, leading to the "crisis of the incommensurable" for one and a half century, originated in the idea of associating numbers to segments under the euphory that "number is the origin of everything". Their theory of measure of a segment with a certain unit produced the dilemma: either the result was a rational number or they had to assume the possibility of division of a segment *ad infinitum*, creating all the well known paradoxes of Zenon and others. It is attributed to the great Eudoxus the solution of the puzzle with the creation of his deep and profound theory of magnitudes which we know through Euclid in his book V of the "Elements", [5-a]. His fundamental idea was to eliminate the numbers from geometry, namely, the geometrical figures turned out to be the primordial elements of geometry, the initial data, and the measurement of length of segments, areas and volumes of figures in the plane and in the space was no more the business of the geometer; it belongs now to applied mathematics, or logistics ($\lambda o \gamma \iota \sigma \tau \iota \kappa \acute{\eta}$) the "art of calculation". That was useful for engineers, architects, physicists, etc., but it was not the concern of the mathematician or philosopher. As a matter of fact no only Plato but even Archimedes, the greatest of the "applied mathematicians" of Greece considered as "ignoble and vile the business of mechanics". Certainly nature provides challenges for the mathematician but these only attain a supreme level of dignity when properly treated by rigorous geometrical methods, namely, "*more geometrico*".

In this way, Eudoxus' theory of magnitudes is the foundation of the theory of similitude of figures and their equivalence in area and volume independently of measurements. For the particular case of "curved" figures, like the circle, the parabola, etc. the method of exhaustion furnished the means for their comparative study relative to size ($\pi\epsilon\lambda\iota\chi o \tau \eta s$).

This point alone would be already enough for anyone to repel the bastard idea of approaching the concept of real numbers to Greek geometry and not be surprise anymore for not finding in Euclid, Archimedes and Apollonius no trace of any "formulas" for "computing" the area of a triangle or any other figure. As we proceed this idea will become clearer.

Let us go now into the analysis of the basic elements of Greek geometry under the shadow of the ideas introduced in §I.

Looking to the organogram of Greek geometry we observe that it is constituted by a certain language, say Greek, plus geometrical figures as real entities existing in space, where space here is the inorganic space which for the Greek mind is a model retaining all the basic feelings of the organic space and not an abstract idea without any visible elements. True, it is hard and difficult for us, Western mathematicians, to grasp this idea *in totum*; we shall never be able to do that; the real and deep feelings of past civilizations is last forever! Only with great effort in trying to "think as a Greek" we shall be able to experience that feeling of the plastic space so magnificiently expressed not only in the Greek geometry but as well in the architecture of Callicrates and in the sculpture of Pheidias and Praxiteles. This plastic space will be the stage where all the drama of Greek geometry will be presented with its figures and demonstrations sometimes showing up like a *deus ex machina*!

The syntax of its organogram is made up of definitions, axioms, postulates and rules of inference which include not only Aristotelian logic but geometrical constructions as well. That is one of the most important aspects of the subject which has been completely overlooked and distorted by the late criticism from the Western point of view. Indeed, through the prism of our mathematical logic it is almost impossible to understand how a geometrical construction or figure can be part of the rules of inference of any logical system, because it cannot be formalized as such. The reason is that since Peano, Russell, Hilbert and all the other creators of "modern" logic and mathematics basically a logical system is a collection of signs and the rules of inference are nothing else but rules of manipulation of these signs, if we take, for instance, the point of view of Hilbert's formalist school. Therefore, there is no room for a geometrical figure to be part of the rules of inference! Of course, we are forgetting that signs or letters drawn in a paper are, afterall, geometrical figures!

For the rest of this paragraph we shall only be concerned almost all the time with the analysis of the organogram of Greek geometry and its syntax and therefore we wish, for completeness, to say only a few words about its structure and organogen which fall under the dominion of organic logic. In future works we intend to focus in this area with details. The structure of Greek geometry is given by the peculiar feeling of space by the Greek culture, namely, the organic space is here the essential element felt as a visible, finite and plastic object where the geometrical figure can move without deformation. The intuitive base for the concept of superposition of figures is rooted here in the structure of Greek geometry and consequently can only be understood under the laws of organic logic without any possibility of being formalized. This concept is used very often in proofs of theorems as a part of the rules of inference. Only through the postulates, to be discussed below, it is possible to render this concept a formal one. But a trace of the organic will always remain behind the scene. It is this closer relationship between organic and inorganic logic which is strange to us and it is so dear to the Greek mind. Finally the organogen of Greek geometry is responsible for its development in time in the sucession of expressive forms constituting the Greek culture. As discussed in details in [1-b and c] it evolves through three stages: **primitive ornamentation**, from the beginning up to Eudoxus, **art**, from Eudoxus through Euclid, Archimedes up to Apollonius and **posterior ornamentation**, from Apollonius up to the end of Greek culture around the V century A.D.

3. Let us go back to the discussion of the organogram and the syntax of Greek geometry. We start with the analysis of some of the definitions given in Book I of Euclid's Elements. As we know it today it begins abruptly with the definition of point:

"Point is that which has no parts."

Our first reaction to that definition is: "it is meaningless". But let us try to understand what really Euclid intended to say. The Greek word for point is *σημεῖόν*, which means a mark, or a visible sign and by saying that his mark positively has no parts, *οὖ μέρος ο'υθέν*, as emphasized by the adverb *ο'υθέν*, Euclid ties the concept of point with the definition of Eudoxus of magnitude. More precisely, a point is not a magnitude relative to size, because it cannot have multiples and submultiples. In this way when we talk about a point either it is given as an object by itself or as the intersection of two lines, or a line and a plane, etc. This excludes from the very beginning, what is very popular in the West, the possibility of defining a point P by a sequence of other points, namely

$$P = \lim_{n \to \infty} P_n.$$

This expression is meaningless in Greek geometry, first of all because the point P

is not given as an object by itself nor as the intersection of two appropriated figures and second because the concept of infinite was from the start eliminated from geometry as a "dangerous concept" leading to paradoxal situations.

This idea of a point as an object existing in space with its own individuality is typically Greek and cannot be translated in terms of an Western symbolic logic.

Let us consider now the concept of straight line:

"A straight line is a line which lies evenly with the points on itself."

Intuitively this definition, in the same way as for the point, is only intended to express the organic concept of straight line as something that proceeds always in the same "direction" without deviating either to the "right" or to the "left", namely it is trying to explain our feeling of straithness and as such cannot yet be taken as a formal definition of straight line. Therefore, the problem of the geometer is how to render those organic concepts into inorganic ones. This is achieved with the introduction of conveniently chosen postulates, and here is the place to make a few important comments.

When Greek geometry began to be analysed and studied by Western mathematicians its postulates were regarded only as part of the so called "logical structure of geometry". This misunderstanding had tragical consequences! Indeed, because of that nobody realized that the postulates besides setting the "rules of the game" where also intended to introduce the concepts of point, straight line, plane and other geometrical figures as **technical concepts**, adapted to inorganic logic and able to be used in the development of geometry as precise and rigorous tools. Otherwise, they could be only handled by organic logic which is not the right discipline to be used in the study of the organogram and the syntax of Greek geometry. As a matter of fact, traditionally when one talks about the foundations of geometry it is usually understood in the mathematical community, what we have called the study of the organogram and syntax of geometry what is completely different from the study of geometry as an organism. This confusion of an organism with its organogram had funny consequences, which we shall dicuss later. To render our ideas clearer we shall study now the postulates of Greek geometry as conceived by Euclid in his Elements.

4. Starting from the assumption that point, straight line, plane and other geometrical figures had been precisely introduced as objects existing in space with their own individualities, Euclid states the following postulates [5-a]:

Postulate 1. To draw a straight line from any point to any point.

Postulate 2. To produce a finite straight line continuously in a straight line.

Postulate 3. To describe a circle with any centre and distance.

Postulate 4. That all right angles are equal to one another.

Postulate 5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than the two right angles.

Concerning postulates 1 up to 4 we shall discuss only a few points interesting us here, directing the reader for more details to the magnificent translation of the Euclid's Elements by Sir T.L. Heath [5-a], which is our main reference in this work. However in the case of postulate 5 we shall deviate ourselves from its traditional interpretation and indeed we could say that the clarification of this point is one of the reasons why we decided to write the present work.

In postulate 1 Euclid translates in pure geometrical terms the organic feeling of going from one position in space to another in a "straight way", i.e., without going around in a crooked path. It is the strong space content when one looks to another place far from him. In this act of looking ahead is contained that "feeling of straightness", which is the deep feeling of spaceciousness expressed by that postulate. As a matter of fact, the use of the verb ἀγαγεῖν in the aorist tense infinitive mood emphasize the space characteristic of the line defined by two points independent of time. Also the uniqueness of the line is implicitly understood here as indicated by the reference of practically every mathematician after Euclid.

In particulate we call attention to the adverb $\sigma\upsilon\nu\epsilon\chi\grave{\epsilon}\zeta$ attached to the verb $\epsilon\kappa\beta\alpha\lambda\epsilon\hat{\iota}\nu$ that makes it clear that the line can be extended continuously, without gaps. We see that the notion of continuity of the line is an essential part of that concept as a space entity and not as an abstract one and consequently it has nothing to do with Dedekind's postulate as we shall discuss later in details. This postulate makes it also clear that the straight line is a finite object in space, not extending itself from $-\infty$ to $+\infty$ actually, but rather only potentially.

About postulate 3 we believe that what was really intended by Euclid was the following: given a point A and a straight line AB with extremities A and B, there is a circle with center A containing B in its circunference ($\pi\epsilon\rho\iota\varphi\acute{\epsilon}\rho\epsilon\iota\alpha$), because, as pointed out correctly by Heath in [1-a], Vol.I p.199, there was no Greek workd for radius. The word $\delta\iota\alpha\rho\tau\acute{\eta}\mu\alpha\tau\iota$ used in postulate 3 means distance but not in the numerical sense of the measure of a segment but rather as something existing in space. Due to that the definition of circle ($\kappa\acute{\upsilon}\kappa\lambda o s$) as a plane figure formed by all equal segments with one extremity in A, its center, has a space content and not a metric one. Here the crucial point is the word equal. Intuitively, two segments are equal when they coincide with each other by a rigid motion in space. Clearly, without some care this concept is tautological, because the definition of (rigid) translation in space requires the notion of equal segment if one is to avoid the introduction of metrical concepts. To handle that situation Euclid first postulates the possibility of rigid rotation of one segment over another having one extremity in common and second in proposition 2, Book I he proves the possibility of defining equality by superposition of segments in the plane in general, i.e., without having necessarily one extremity in common. In our present texts of Elements, due to its deterioration in time by innumerous interpolations and copist's errors the clarity of this concept of equality of segments which is fundamental in the logical structure of Greek geometry, is seriously jeopardized.

In the analysis of Euclid's Elements by Western mathematicians this assumption of rigid translation of figures in space has always been looked as something scandalous "proper of a primitive stage of the development of mathematics", what is clearly nonsense! The introduction of group theoretical concepts as proclaimed by F. Klein is completely strange to Greek geometry being meaningful only in the context of set theory. To summarize, Euclid proposes that by coupling the definition of a circle with the possibility of rigid rotations and the postulate 3 we can rigorously define in terms of Greek geometry the concept of equality of segments, by superposition, in the general case. It is amazing how these fundamental question have almost invariably escaped the attention of many critical analysis of Euclid's Elements.

Another important consequence of postulate 3 and the concept of superposition of segments is the equality of angles. The concept of angle by itself would deserve a careful discussion but we shall not do it here; we content ourselves in directing the reader to [5-a]. Assuming that an angle is a plane figure formed by two segments in the plane, its sides, with an extremity in common we define their equality as follows: let α be an angle with sides AB, AC and α', with sides $A'B'$, $A'C'$. Consider a segment AD contained in AB which is smaller than AB, AC, $A'B'$, $A'C'$, namely, AD is equal to segments contained in each side of α and α'. Draw a circle with center A and radius AD intercepting AC in a point E, whose existence is provided by the definition of circle. Indeed, its periphery contains the extremity of any segment with the other extremity at A and equal to AD and as AD is smaller than AC it is equal to a segment AE contained in AC and therefore E belongs to the circunference of the circle in question. In the same manner we can draw a circle with center A' and radius equal to a segment $A'D' = AD$ defining a point E' in $A'C'$. Now we put by definition: α is equal to α', writing $\alpha = \alpha'$ if $DE = D'E'$.

In few words, repeating, with the help of postulates 1, 2 and 3 it is possible to lay down the foundations of a "rigorous" theory of rigid motions in space and equality by superposition. The meaning of the word rigorous will became clearer after the discussion in §IV ahead.

Postulate 4 provides a strong characteristic of that profound feeling of a visible and plastic space: the space of architecture! No one can ever conceive, say, a doric temple without the clear idea that the columns are perpendicular to the floor and the beams are perpendicular to the columns and "consequently" are parallell to the floor. The words vertical and horizontal, related to orthogonality and parallelism having a strong spatial content are felt organically as such. In the same way if one makes a door with edges which are not perpendicular to each other it will not close properly: here in the architectonic organic space of everyday life there is no room for non-euclidian geometry!

Euclid's problem, as in previous cases, was to translate that organic architectural idea of orthogonality in a working concept for mathematicians and we believe that his idea in expressing that into the postulate 4 was one of his many strokes of genius. Heath is again absolutely correct in pointing out that the equality of right angles is also equivalent to the concept of invariability of figures by translations and the homogeneity of space, another element to reinforce the conclusions from postulates 1, 2 and 3, as seen above.

Finally, we have the celebrated postulate 5, improperly known as the postulate of parallels, because it is equivalent to the statement that in a plane through a point not belonging to a straight line there is only one parallel to the given line. But, certainly the concept of straightness of the line is essential in the definition of parallel and the problem is: how to translate in a technical and precise language the concept of straightness, i.e., the concept of lying "evenly without deviating from its direction", etc. That is exactly the aim of postulate 5! It translates in a mathematical satisfactory form the intuitive ideas belonging to the organic concept of straight line in a way that it can be used formally in the proofs of theorems, etc., and in all situations concerning the organogram of Greek mathematics.

It is hard to understand how the original intention of Euclid, as we are firmly convinced, could be interpreted in later times only as the question of parallels, creating for more than 2.300 years the well known orgy of proofs of the V^{th} postulate! What lies behind all those proofs? Let us see. The postulate 5 talks about a characteristic property of two straight lines r and s in the plane intercepted by a third straight line and it says that r will intercept s if produced in a convenient way. This express exactly the fact that r, when produced remains "evenly to itself" and "does not deviate from its direction". Indeed, if r could "deviate from its directin" it could behave itself like an arc of hyperbole and never intercept its assymptote s. Now if we want to prove that postulate we have to express in one way or another that property of the straight line of "keeping its direction" in some formal statement apt to be used in the proof. That is exactly what invariably happened in all the proofs of that postulate. The proofs in themselves are in general correct, but they have to use somehow as said above, as **hypothesis**, some assumptions translating in technical terms that fundamental property of the straight line of "keeping its direction", which is not contained in the first four postulates! Otherwise, a proof in the organogram of Greek mathematics would be impossible because it is clear that one cannot prove a property of a certain object without using its definition either directly or indirectly. Certainly, all this sounds strange for a Western mathematicians used to deal with abstract entities deprived of any intuitive meaning. Later in §III and IV we shall come back to this question.

As we see it here the fundamental historical mistake is the confusion of the equivalent statements of the postulate 5 given by "its proofs" with the question of its independence. There is no question of independence at all in the sense understood by the symbolic logi created by Western mathematicians and philosophers. It is obvious from the very beginning that the V^{th} postulate cannot be a consequence of the remaining four postulates, in the organogram of Greek mathematics, because it states a property of the straight line which is not contained in them. Consequently it cannot be derived from them unless we use in one way or another a property of the straight line equivalent to that stated in postulate 5, translating the intuitive feeling of "proceeding evenly to itself", "keeping its direction", etc.

The root of the misunderstanding is the lack of historical perspective and the consequent confusion of an organism with its organogram. A straight line for a Greek mathematician is not an abstract object without any meaning, as it is the case for a Western mathematician, but on the contrary it is an object given initially with strong space content and characteristic properties expressed exactly by the postulates. The so called "proofs of independence" of the Vth postulate from the remaining ones by the construction of proper models and the creation of "non-euclidian" geometries will be one of the fundamental points to be treated in §III and IV.

To summarize our discussion the postulates translate among other things the concept of straight line in technical terms to render it possible its use in formal proofs. Consequently the attempt of proving the Vth postulate, for instance, is a clear nonsense from the very beginning because in such proof one has to use the concept of straight line and then some equivalent of that postualte has to be used therein, otherwise we are not using the whole concept of straightline. This fundamental mistake has plagued for over 2.300 years the understanding of euclidean or better Greek mathematics as such. Here the postulates are not merely conventional statements about abstract entities as conceived in the West, but they are rather statements about concrete objects given as initial data expressing, besides their proper definitions, some of their fundamental properties in technical terms apt to be used in formal proofs.

Now we can have a clear picture of the situation when we hear that Apollonius tried to "prove the postulates". One is really talking about substituting one set of postulates by another and not trying to prove the postualtes "from nothing", an attitude offensive to the genius of the Great Geometer!

- §III -

Geometry in the West

1. As pointed out before, in the West the concepts of space, number, magnitude, etc., are expressed through abstract forms strongly denying the visible and sensible space which is merged into the concept of pure number represented by sets, classes, etc.: geometry becomes a chapter of set theory. Going back to the basic ideas discussed in §I this implies that the organogram of Western mathematics is formed by a certain language (metalanguage) like English, French, German, plus a collection of symbols, axioms, rules of inference of a particular set theory selected to please the taste of the mathematician building the theory, namely, either in the line of the logistic or the formalist or the intuitionist school of thought. The final result is a collection of sheets of paper full of signs intelligible only by the initiated usually called a **book**, most often without figures! However, does not matter how abstract, how impartial one's point of view and how much one tries to erase any trace of subjectivity in such theories, in the background lies unavoidably the human mind with all its passions and historical roots. It is impossible to space from the profound archetypes of the human species which are the *leit-motifs* of all forms of expression related to man. Indeed, does not matter how formal or abstract or logical one intends to become, he cannot avoid himself and all his psychological tendencies, including repressions and all sort of neurosis conducting his actions, back-stage. And that is the reason why one is sympathetic to one school of thought rather than another. If one is materialist and affected perhaps in childhood by religious resentments or something else he eventually will be happy with, for instance, the philosophy of the Vienna Circle. Otherwise he would rather adhere to a school in tune perhaps with a platonic idealism allied with theology, etc. In few words, one invariably translates in this creative work the image of his soul and his state of mind.

One of the most characteristic and beautiful examples of an exposition of geometry in the Western style is the celebrated work of Hilbert [6] which we shall take as representative of Western geometry as we have done with Euclid's Elements relatively to Greek geometry.

Hilbert starts with an abstract set X containing some particular subsets of objects called point, lines, planes subjected to a certain number of postulates or axioms divided into five groups:

I, 1-8: *Axioms of relationships.*

II, 1-4: *Axioms of ordering.*

III, 1-5: *Axioms of congruence.*

IV, : *Axioms of of parallels.*

V, 1,2: *Axioms of continuity.*

The rules of inference are essentially those of classical logic which later were formalized in some abstract models in terms of symbolic logic as can be seen, for instance, at [8]. All that establishes the organogram and the syntax of Western geometry which are quite different from that of Greek geometry. Afterwards Hilbert shows how one can prove theorems from the axioms and basically if one has nothing better to do he can develops a chain of consequences from the axioms above building a system called, by mistake, euclidean geometry! It is a nice game and a good mental exercise, but later one start getting concerned about the question of a model for such a theory. By that it is meant the following: starting from objects whose existence we accept to define with them a set satisfying the axioms stated above. But what objects shall we assume to exist?

As seen before, for Euclid those objects are the geometrical figures existing in space with their own individualities, but for Hilbert the initial data or the objects which do exist are the numbers or more generally arithmetic supposed to be a consistent system. From there one can build the real numbers and finally a model for geometry is built by starting from the set E or ordered pairs (x, y) of real numbers, called plane. Each pair is called a point and linear combinations of them like

$$ax + by + c = 0$$

with a, b, c real numbers are called **straight lines**. Of course we can generalize that by considering triples (x, y, z) forming space and even n–plas $(x_1, \ldots, x_n)$ forming the n–dimensional euclidean space. However, we restrict ourselves here with the case of pairs.

Now, by assuming all properties of real numbers one can prove that all axioms I-V are satisfied in the set E and therefore the original theory is not "empty" namely it has at least a "model". Of course, we can also argue about the existence of numbers and this would take us into the business inaugurated by Frege, Peano and Russel whose success was later jeopardized a little bit by the celebrated results of K. Gödel. But this is a dangerous battlefield and prudence dictates that we should stay, at least for the time being, at a safe distance from it.

Another important consequence shown by Hilbert is that from the axioms alone we can associate to each segment in a straight line, a real number called its **length** and afterwards establish a **system of coordinates** in the line as well as in the plane. Do not forget that the words point, straight line and plane continue to be understood in the abstract sense; if we draw figures in a paper that is purely from the **psychological** point of view and not from the **logical** point of view. For instance, by a segment AB in a straight line r we understand the abstract set of points $P \in r$ such that $A \leq P \leq B$ where the relation $\leq$ is given by the ordering axioms II, 1-4 above and its pictorial representation in the paper is only a **psychological attitude** to "help" and (perhaps!) "guide" our thoughts! In this way we establish a one-to-one correspondence between our original set X and the set E of cartesian geometry defined by the pairs of real numbers allowing us to talk about the coordinates of a point, equation of a straight line, etc. In particular we can introduce the idea of distance in X which becomes a metric space an in no time we are dealing with more general sets called topological spaces. But let us leave those general considerations and return to the analysis of the concepts introduced above.

2. The fundamental question is: by considering the concepts of point, line and plane as understood by Euclid, is it possible to show that they satisfy the axioms I-V stated above? We intend to show that the answer is no and the main obstacle to a positive answer, namely, yes is the postulate of continuity.

First of all, to show that a certain object satisfies certain conditions we have to know the intrinsec properties of that object, namely, its definition. But whatt is the meaning of the verb satisfy?

Secondly, in what consists the act of verifying that an object satisfies certain properties? It is logical or psychological? Here we have one of the typical situations where we cannot separate the organic from the inorganic. The first stage of this process and as a matter of fact, of almost every process of knowledge is organic. Indeed, to verify if an object A satisfies a property p we have to use the Principle of Analogy. For example, suppose we say that a certain ball is red. Our conclusion is based on the fact tht we look to the ball and compare it in our mind, by analogy, with our concept of red, a pure organic attitude, impossible to formalize in a system of inorganic logic. In a second stage we express our organic knowledge in a language or a formal system what achieves its reduction to an inorganic knowledge. For instance, after concluding that the ball A is red we express that in a usual language or even a formal language: let E be the set of balls in the world and let $E(p)$ the set of red balls, then "$A \in E \Rightarrow A \in E(p) \Leftrightarrow A$ is red". In this way our primitive observation that A is red is reduced to a statement in propositional logic. In a future publication we shall discuss in detail the theory of knowledge under the pont of view of the organic and inorganic logic.

Having in mind all that let us consider a segment of straight line AB with extremities A and B. For Euclid the points A and B as well the segment AB are realities existing in space; for Hilbert they are abstract concepts, perhaps having as model the set of real numbers between two other real numbers A and B. Then our original question reduces to: is it possible to establish a one-to-one correspondence between the points of the Euclidean straight line and the set of real numbers? Let us start with an analysis of the concept of length of a segment.

3. The pythagoreans looked to this question in the following way: let us consider a segment AB and let us select a segment u as unity. As the "number is the origin of everything" it must be possible to associate a number to every segment. One first possibility is that by applying the segment u over AB it will fit in there an integer number m of times, in the sense this operation had in that period of history, namely, around VI B.C. In this case we say that the length of AB is m, measured in the unity u, writing

$$\ell(AB) = m.$$

Another possibility is that u does not fit exactly an integer number of times in AB and in this case we subdivide u in smaller unities u_1, say, $u_1 = 10^{-1}u$, and so on, up to $u_n = 10^{-n}u$ until we eventually have an integer number of segments u_n covering AB and we write:

$$\ell(AB) = u + 10^{-1}u + \ldots + 10^{-n}u = mu_n.$$

It was assumed that these are the only possible results of the operation of measuring a segment, otherwise we would proceed indefinitely and we could never associate a number to AB, contrary to the basic philosophical position of the pythagoreans. Notice that we repeat: number for a Greek mathematician was an integer (non negative!), namely, a collection of units, definition attributed to Thales.

Soon after, as we know, this point of view was seriously shaken by the discovery that no number could be associated to the diagonal of the square if we take one of its side as unity, or what comes to the same: there are segments which cannot be measured, in the sense described above, with a common unity, namely, they are said to be incommensurable.

Then a terrible crisis descended upon Greek geometry and only after almost two centuries later a way out from the impasse was found by the great genius of Eudoxus of

Cnidos. His solution for the riddle was to abandon the idea of associating numbers fo geometrical figures and to develop a theory of magnitudes independent of any process of measurement. It was the definite separation of number in one side as an independent concept whose study was arithmetic and geometrical figure in another side whose study was geometry. His theory of magnitudes, exposed in book V of Euclid's Elements and his principle of exhaustion remained as the back-bone of the whole of geometry, excluding any relationship with measurement of lenghts, areas and volumes, which were relegated to logistics, the analogue of our applied mathematics. Indeed, that question of measurements of lenghts, areas and volumes became the business of the engineer, the architect and the practical man, having nothing to do with the considerations of the geometer. This point of view has been in several occasions emphasized by Plato, Archimedes and most of the great mathematicians of Greece. That was the origin of the establishment of the concept of geometrical figure as the basic element given "a priori", leading in the hands of Euclid to that majestic monument which is one of the great expressive forms translating the concept of space as conceived by the Greek culture.

What happenned in the West? As said before whenever we embark ourselvesin the study of the foundatins of some branch of knowledge the first thing to decide is what concepts shall we take as primitive, i.e., given "a priori".

Following Hilbert we should take the concept of number to build a model for geometry initially given as an abstract set X with a certain structure defined by certain particular objects called point, straight line and plane subjected to a system of axioms or postulates. By using, in particular, the axioms of congruence and continuity we can attach to an interval AB, understood here as an abstract object, a real number called its length, written as $m(AB)$.

Following Dedekind to define real numbers we assume the existence of the rational numbers and consider the set R of all pairs of classes of rational numbers (A, B) such that:

$P_1)$ $A \neq \emptyset$, $B \neq \emptyset$ and $A \cap B = \emptyset$.

$P_2)$ Every rational number belongs either to A or to B.

$P_3)$ If $r \in A$ and $s \in B \Rightarrow r < s$.

A beautiful discussion of these ideas can be seen at [7]. Therefore the theory of length, or in technical terms, of measure of "segments" is reduced to a correspondence between abstract sets without any contact with "reality". But what happens in "practice"? Well, the engineer, the architect, the merchant, the physicist, etc., have the same attitude as in Greece, Egypt or anywhere else: they deal with concrete objects and there is no room for abstractions. It is the realm of the organic and consequently organic logic mainly through the Principle of Analogy reigns supreme! In the design of mechanisms, electrict circuits, public buildings, railroads, etc. **all numbers are rational!** There is no such a thing as real number! For instance, $\sqrt{2} = 1.414\ldots$ and the dots in the end just mean that if we need more accuracy in the results we can get extra decimals. In few words, our applied mathematics is what the Greeks called logistics, an amorph collection of rules and experimental data accumulated through ages and a such transmitted from generation to generation and from civilization to civilization. It is this accumulation of empirical knowledge which gives the impression of constant progress if we do not distinguish carefully the empirical from the symbolical. For instance, from the empirical point of view the engineers who built the pyramids, or a doric temple or a gothic cathedral faced similar technical problems like the stability of the structure, resistance of materials, etc. but as expressive forms, a pyramid, a doric temple and a gothic cathedral are three distinct symbolic representations of space of three distinct historical cultures.

Our discussion leads us naturally to a famous statement which we all heard from our teachers by the time we were students: "today we intend to show you how to establish an one-to-one corresponce between the points of a straight line and the set of real numbers". The embarassing question here is: what shall we understand by **straight line**? If it is

Hilbert's abstract concept, disregarding any model, we shall have exactly the situation decribed above of the correspondence between two abstract sets and nothing is gained from any "intuitive representation" in space. But if by straightline we understand an object represented by a model taken from analytic geometry than there is nothing to prove because this model is already the set of real numbers itself. The only alternative left to our teacher and that is what be probably had in view from the very beginning, is that by straight line we should understand the Greek of euclidian line as an object in space given "a priori". It is here that our troubles begin!

Indeed, the only way left to us was that of the pythagoreans and a Western mathemtician would proceed as follows: first of all we have to analyse the question of applying a unit segment u over a straight line r a certain number of times. That depends on the homogeneity of space and rigid motion of bodies and to define that we need to assume the rigidity of the unit u and we clearly get involved in a vicious circle. The only conclusion from an honest and diligent Western mathematician is that the procedure used by the phythagoreans is logically unattainable and could only be accepted from a n intuitive and practical point of view. But then we had to stop right here and declare as impossible any attempt of establish an one-to-one correspondence between the Greek straight line and the set of real numbers. But, suppose someone like a peace-maker come along and says: "Oh! Let us close our eyes for the time being to that vicious circle and let us assume that in some way or another we have the possibility of associating to any rational number a uniquely defined point in the Greek straight line." Now, let us consider all pair of classes of rational numbers (A, B) satisfying conditions D_1, D_2, D_3 above defining a Dedekind cut. To each pair (A, B) there corresponds a pair of classes (A_r, B_r), of points in r and then, Dedekind's continuity postulate says: there is a "point" $\alpha \in r$ corresponding to (A, B). Let us analyse this question more carefully.

First of all, even from the point of view of Western mathematicians the attempt of defining α as the supremum of A_r or the infimum of B_r requires a great logical sophistication, much more than originally thought, to avoid again the "circulus vitiosus" of H. Weyl, [9].

Secondly and that is the main question, from the point of view of Greek geometry the point α ($\sigma\eta\mu\epsilon\iota o\nu$!) is an object with proper identity existing in space and is either given from the beginning, like in the phrase, "let α be a point in the line r" or it is given by the intersection of two geometrical figures, like the phrase, "let α be the intersection of r with the straight line s". Consequently, the definition of the "point" α, as indicated above, by a pair of classes (A_r, B_r) is completely meaningless and could never be accepted by a Greek mathematician "as rigorously defined". As a matter of fact, the whole discussion around the celebrated method of Hippias for the squaring of the circle by using his quadratix is exactly motivated by the fact that a certain point is defined by a "limit process", as can be seen at [5-b].

Therefore, from the Greek point of view, there is in general no point corresponding to a Dedekind cut, what leads us to the final conclusion that it is logically impossible **from the point of view of Western mathematics to establish a one-to-one correspondence between the set of real numbers and the euclidean straight line.** That can only be acceptable in the so called "applciations of geometry" in everyday life, so useful indeed for the engineer, the architect and the physicist among others.

4. To finish this section we discuss briefly the other axioms given by Hilbert concerning their fullfiment in terms of Greek geometry.

About axioms of group I, i.e., axioms of relationship we see that 1. corresponds to postulate 1 of Euclid and 2. is understood as valid without it being explicitly said as we pointed out before. About # 3, that a straight line has at least two points, it is included in definition 3, Book I which says that "the extremities of a line are points". As we emphasized before, the concept of straight line correspond to an concept of segment. The existence of three points not in a straight line is provided by the definition 7 of plane surface. As a matter of fact some of the definitions of Euclid are only intuitive explanations but others are much more technical and precise as we shall see in the following. Concerning I, 4, 5, 6, 7, 8 which deal with the relationship between plane and straight line we come to the classical discussion of the insufficience of Euclid's postulates 1 up to 5 to cover geometry in space. This is absolutely correct and we would expect from Euclid some postulate turning the intuitive and vague definition 7 of plane surface into a operational and technical one. However this never happened in the Elements and indeed Book XI which starts the geometry in space has by no means the logical solidity of the Books I, II, etc. A vivid discussion of the subject can be seen at [5-a], vol.3, p.272.

For the axioms of group II dealing with order relation of points in a straight line we have to introduce an ordering in the euclidean line. This can be achieved by saying that the point D precedes E as points of a straight line with extremities A, B if the segment AD is contained in the segment AE. Afterwards, we can see without difficulty that II, 1, 2, 3 are valid, always having in mind that we are talking about the euclidean line and not about an abstract concept. Axiom II, 4 deserve some special comments. In many places the Greek geometers use without any particular remark that a triangle, a polygon, etc. separates the plane, namely, they define regions in the plane such that if we connected one point of one region with another point of another region with a straight line r, then r intersects the boundary of the given figure. This is not made explicitly through a postulate but it is a consequence of the definition 19 of rectilineal figure. Indeed, it is given by the word περιεχόμενα derived from the verb περιέχω which means to surround, to encircle in the sense of separating something from something else. Perhaps, by that time, the meaning of this word was so clear, even from the mathematical technical point of view, that Euclid found it unnecessary turning the definition 19 into a postulate, or maybe, he did and it was lost. Let us not forget that, as far as we know, one of the oldest Greek copies of Euclid's Elements is provided by Theor of Alexandria, more than 700 years after Euclid! In this way, with some indulgence to what was lost in time, the definition of rectilinear figure takes care of axiom II, 4.

The axiom group III of congruence is taken care, if with use in the Greek sense the concept of equality of segments by rigid translations in space, whose discussion we have already done before.

The axiom IV of parallels is a consequence of postulate 5 as well known.

To summarize, we have shown that under the proper perspective of Greek mathematics and trying to reason as a Greek geometer we can, in a satisfactory way, show that, with the expection of axioms V of continuity, the remaining axioms are valid for Greek geometry. Indeed, even axiom V, 1, namely, "Archimede's axiom" is true for Greek geometry being introduced in Book V as proposition 1, which really depends on the definition of comparable magnitudes of Book V. As discussed before, the method of exhaustion is the real substitute for the axioms of continuity whenever they are needed in Greek geometry.

- §IV -

Geometry and Logic

1. Traditionally when we talk about mathematics people immediatly get the idea of something precise, where we can "prove what we say" under the formalism: "if we know this then that'. In particular geometry since Euclid has always been a model of logical reasoning where we can establish facts by deduction from initial premisses in such a way that it is outside debates or points of view.

The investigations about the act of thinking in general with its principles and laws has in Aristotle one of the most celebrated pioneers eventhough in many other civilizations like China and India several philosophers and scholars flourished in comparable level to Greece. In the West the studies of logic began under the shadow of Aristotle in scholastic philosophy and this gave the impression that what we call mathematical logic today resulted from an evolution of the ideas of the Stagirite. That is wrong, in the same way that our geometry is not an evolution of Greek geometry. Of course, analogously as to what happened in mathematics our act of thinking has from the **organic point of view** the same background for any man from any historical culture. However, here again the **form** one expresses his ideas about logic is different in different historical cultures and this has an outmost influence in the foundations of geometry as we intend to show in this last part of our work.

Mathematical logic or inorganic logic in our nomenclature became established in the West on solid principles in the beginning of this century as a result of the works of Frege, Peano, Russel, founders of the logistic school, Hilbert, founder of the formalistic school and Brouwer, founder of the intuitionistic school, to name the most important tendencies in the area. Whatever their differences might be they are all concerned with the **organogram** of mathematics in particular its **syntax** in our nomenclature. Mathematics and in particular geometry as an **organism** with historical and cultural dimensions cannot be studied with mathematical logic alone but it requires also the help of organic logic. Consequently as the organogram of Greek geometry is different from the organogram of Western geometry the "logical structure" of one is essentially different from the other and it cannot be "deducted" one from another, or in other words, we cannot study critically Greek geometry by using the "logical structure" of Western geometry and vice-versa, eventhough this has been traditionally a deplorable practice with tragial consequences!

With those remarks in mind let us analyse comparatively the logical structures of geometry in Greece and in the West.

2. Recalling some ideas discussed before we have seen that the organogram of Greek geometry is formed by words of the Greek language and geometrical figures takens as initial data existing in space with their own individuality. The syntax of its organogram, namely, its rules of inference is formed by aristotelian logic as understood in Euclid's time and geometrical constructions, as we shall clarify in a moment with examples. For the Western mathematician this look like a very strange and incomprehensible attitude because he is used to the organogram of Western geometry, which as a chapter of set theory, is formed by a certain language, say English plus a collection of symbols and rules of inference given by mathematical logic adjusted to some particular school as indicated above. Definitely, geometrical figure as such is not part of the business, on the contrary, they are not needed at all as independent entities lying in space and rather they are taken as names, like point, straight line, plane, of particular objects attached to some abstract set and subjected to certain axioms.

Let us consider an example. It is a very popular attitude to refer ourselves to Proposition 1 in Book I of Euclid's Elements as defective because in a certain point it fails to use the postulate of continuity! this proposition says that given a segment AB there exists an equilateral triangle with AB as one of its sides. The proof depends on the fact that two circles with same radius AB and centers respectively at A and at B do intersect each other at a certain point C. The reaction of a Western mathematician is the following: to prove the existence of the point C we have to use the postulate of continuity. But, as seen before, that is impossible because segments, circles, etc. are understood in the Greek sense with their intrinsic spatial content. The only way to handle the situation should be translating everything in terms of Western geometry and "to solve" the question in terms of Hilbert's "grundlagen". This attitude would be similar to the following one: a Western architect visiting Athens concludes that the structure of the Parthenon is not strong enough and decides to rebuild the whole thing with reinforced concrete, materials of "better quality" and the final result would be in its appearance exactly like the original, but with a little difference: it would be not a doric architecture anymore! That is exactly what we do when we try do "adjust" Euclid's proof to an Western standard of "rigour". It is amazing how such crystal clear facts are completely overlooked in the usual criticism of the foundations of geometry! How come so many brilliant minds involved in these studies never realized those fundamental questions, that we do not understand – it is a mistery!

Now let us look to that same proposition from the point of view of Greek geometry. To be strictly logical and precise it is only allowed to use in the proof the syntax of the organogram of that geometry. Therefore, the rules of inference include aristotelian logic and geometrical constructions. Hence in the present situation it is logically admissible to accept the existence of the point C as a consequence of the properties of the circle as a figure in the plane. The geometrical figure itself of two circles as above includes the point C. However, this line of reasoning is inadmissible in a statement like: "in a triangle one side is smaller than the sum of the other two". In this case we are not talking about a particular triangle but rather about the "genus" triangle and then a proof without using data from a particular figure becomes necessary. As a matter of fact, Euclid had always been criticized by his "excess of rigour" in trying to prove something that is "evident even for an ass". Euclid is supposed to have replied to that criticism as follows: "indeed the ass knows that one side of a triangle is smaller than the sum of the other two, but he does not know that this statement is a true proposition!". Namely, the animals know that something is but not that something is true.

For us Western mathematicians it is very hard to accept that geometrical construction of a figure can be part of the rules of inference of a theory in particular in geometry when we reduce everything to set theory. Indeed, any set theory is expressed through a formalism in whose syntax there is no room for geometrical constructions as objects existing in space. In more technical terms, the organogram and the syntax of Western geometry does not contain any object with independent existence in a visual and plastic space.

We summarize now for better understanding the logical characteristics of both geometries in Greece and in the West and based on that we shall discuss, to finish this work, a series of questions connected with them.

In Greece geometry is an organism such that:

a) its organogram is formed by an usual language, most often ionic, with a syntax given by aristotelian logic plus geometrical constructions as objects existing in space;

b) its structure is formed by the concept of geometrical figure as objects given "a priori" with proper individuality and space is here considered as a real entity, finite and with no relationship with time. Number is understood as natural number, i.e., a collection of units;

c) its organogen is given by its "genetic code" providing its evolution inside the Greek culture through the stages of primitive ornamentation, art and posterior ornamentation as discussed in [1-a].

In the West geometry is an organism such that:

a) its organogram is formed by an usual language, for instance, English and a syntax given by some logical theory, say formal logic as conceived by Russel and the symbols and axioms of a theory of sets;

b) its structure is formed by abstract concepts named point, straight line, plane, etc. attached to some theory of sets and space is infinite, not visual and strongly connected with time, disguised in the form of sequence of points, limit processes, etc. Number is the abstract concept of real number.

c) its organogen is given by its law of evolution inside the Western culture in the same way as any other expressive form, through the three stages considered in [1-a].

In this way we see very clearly that Greek and Western geometries are distinct organism and its is impossible to reduce one to another and their foundations have to be studied in their proper environment inside the historical culture to which they belong. Let us consider now some questions related to both geometries.

3. We start by returning to the question of continuity. As we saw before in the West the idea of continuity of the straight line is expressed through Dedekind's cuts and it is meaningful only if straightline is understood in the abstract sense of Hilbert. It is

impossible to apply that concept to the euclidean straight line as seen before and in this case the natural question is: how did the Greek geometers look to the idea of continuity of the straight line? This question has a great importance for the foundations of geometry and also for the history of mathematics mainly because it is in a certain sense still an open question! Indeed, what we know about Greek mathematics, through documents which survived the destructive action of both time and man, is only a small part of what was done by the Greek geometers. Hence we find here and there some facts giving a faint idea of the use of continuity in geometry. One of those points is related to the question of existence of the fourth proportional.

Following Eudoxus and Euclid, suppose that we give three magnitudes a, b, c, where a and b are comparable and look to the problem of finding a magnitude d comparable with c called the fourth proportional of a, b, c such that

$$\frac{a}{b} = \frac{c}{d}.$$

Clearly that is not true for all classes of magnitudes. For instance, it is not true in general for numbers (natural numbers!) like 2, 3 and 5. However, the existence of the fourth proportional is always true for classes of magnitudes which are capable of "changing in size continuously", which is actually the fundamental hypothesis in the classical proof by De Morgan as reproduced by Heath in [2-a] following Prop. 18 in Book V.

The question is to clarify the meaning of "changing in size continuously". For the case of the straight line we find in Postulate 2 the word σuνεχὲs in the neuter which is equivalent to the adverb σuνεχῶs meaning continuously, without gap, namely, it is assumed that a straight line can be increased in size continuously, as the result of the combination of σuνεχὲs with the verb ἐ κβαλεῖν which is the aorist tense mood of the infinitive εκβάλλω meaning "to throw" or "cut out". Therefore, from the Postulate 2 taking in consideration its writing in Greek, we have to assume that the continuity of the straight line was a datum "a priori" with spatial significance. Of course, a more technical approach to the idea of continuity had to be provided and this is exactly achieved by the existence of the fourth proportional coupled with the principle of exhaustion. We are preparing a more elaborate paper on this subject to be pusblished elsewhere and hence, right now, we content ourselves with some general remarks and examples.

According to Book VI, Prop.12 of Euclid's Elements, the fourth proportional exists for the case of straight lines and by Prop.1 of the same book the result can also be extended to polygons, namely, if a, b are polygons and c is a straight line, then there is a straight line d which is the fourth proportional of a, b, c. Afterwards, by using the method of exhaustion it is possible to extend the result to figures A having the property: there are two polygons P and Q with Q contained in A and P containing A such that the figure, difference $P-R$, can be made smaller than a given square B. That is, of course, quite similar to the definition of Peano-Jordan measure but only formally. Indeed, to say that $P-Q$ is smaller than B it is understood that $P-Q$ is equivalent to a square B' which fits inside B with complete absence of any numerical content. On the contrary, the Peano-Jordan measure relies heavily upon the concept of real number, an abstraction incompatible with the Greek thought.

Assuming as well defined the concept of magnitude "changing continously" the Greek geometers used very often the principle of the existence of the fourth proportional in the following context: suppose that we have to prove that, for magnitudes a, b, c, d it is true that

$$\frac{a}{b} = \frac{c}{d}. \tag{1}$$

As for relations we only have either $=$ or $\gtrless$ then by assuming that (1) is false we have either $>$ or $<$. If it is $>$ then there is a magnitude $b' > b$ such that

$$\frac{a}{b'} = \frac{c}{d} \tag{2}$$

and afterwards we try to find a contradiction as a consequence of (2) and similarly for $<$. It is clear that this type of reasoning, namely the existence of b', depends on the "changing with continuity" of the magnitude b and the puzzling question is that, as said before, this technique is used very often by the Greek geometers, as for instance in Book XII, Prop.2 of Euclid's Elements, without any comments, what leads us to think that it must have been a well known result in Greece, in the same manner that in the West everytime we use the idea of continuity in one way or another we do not bother ourselves to recall its definition all the time. By another side, so far we have been unable to find some work by a Greek geometer dealing specifically with that question. We are convinced that they did it but it has been lost forever as so many great achievements of the great geniuses from Greece. It is an open question anyway!

4. Let us study now the question of the non-euclidean geometries. It is hard to find an area of mathematics where the misinterpreation of facts have gone so far and so deep. For about 23 centuries the "problem of parallels" got the attention of mathematicians both in Greece and in the West. The great comedy (or tragedy!) has been: **it was never really a problem and its "solution" found in the XIX century has never been a solution!** We dedicate this last part of our work to the clarification of this statement.

The word **parallel** is itself Greek, of course, and it means "side by side" from *παράλληλος*. For instance, in architecture we say that two beams run parallel to each other, i.e., side by side similarly. This concept is deeply connected with the principle of analogy of organic logic. Indeed when we draw parallels in a sheet of paper we start with one line and afterwards we imitate that line by analogy, drawing other lines parallel to it. Without this organic notion of parallels architecture cannot exist. That is the reason why from the very beginning the notion of parallels was associated with three other notions:

1) two straight lines in the plane without a common point;

2) two straight lines in the plane with the same direction;

3) two straight lines in the plane with constant distance from each other.

Due to the fact that an excellent historical-critical study of the relationship among the three notions above with the concept of parallels is done by Heath in [5-a] when discussing def. 23 of Book I of Euclid's Elements, we only focus in here into those aspects of the question which are relevant to our point of view in this work.

Euclid takes 1) as the definition of parallels in the more precise statement of def. 23, Book I, where he ues the idea of "producing a straight line indefinitely". The word ἄπειρον, which means generally without limits and already used by the philosopher Anaximander in relation to the Universe, in geometry is connected with the idea of some magnitude which can be increased as much as needed but never infinitely in the sense of atuality, but rather only in the sense of potentiality. For a Greek geometer it is meaningless to say that the straight line is infinite. From the very beginning they realized that to avoid troubles and paradoxes it was better to get rid of the idea of actual infinite from the start! That is why this notion was banned forever from geometry and as a matter of fact from the whole Universe as felt by the Greek soul.

The idea of using 2) or 3) as a definition of parallels was rejected, among others, by Aristotle as leading very easily to a "petitio principii" and consequently discarted by Euclid. However, from the organic or intuitive point of view they are very appealing and indeed are the "everyday used" definition of parallels. Suppose, for instance, that a company decides to build a railroad in South America from South to North in a "straight line". One of the rails could follow exactly a meridian, the other running "parallel" to it. Of course, parallel here means to keep the same distance from each other along the way. The notion of distance here is very clear for the engineers and working people: it is given by the modulus which is an iron bar used to fix the rails at equal distance before nailing them to the beams. Therefore, irrespective of the curvature of the earth for the railroad builders the geometry to be used is the Euclidean and not the Riemannian one, namely, if the rail is taken as a straight line the other rail is also a straight line parallel to it!

Another example: two airplanes are to fly from south to north keeping the "same direction". Clearly this means that they are supposed to fly "parallel to each other", namely, they should not follow two meridians; otherwise they would collide in the north pole! To savely circumvent the earth they have to keep parallel to each other in the sense of never deviating themselves from a "fixed direction". Here we realize that the idea of direction is an organic concept and cannot be reduced to the formalism of the inorganic logic. This does not mean of course that with suitable convention we could not formalize this concept in an abstract formalism. Indeed, with the help of algebra in topology, differentiable geometry, etc. the concept of orientation of a manifold, for example, can be introduced in abstract terms, but never in terms of Greek geometry. The organic intuition of space is present in our "visualization of the orientation", like the rule of three fingers in the "geometrical" definition of the cross product of two vectors on the momentum of a force, etc.

Considering all that Euclid had to introduce the notion of parallel in geometry in such way as to keep in the background the concept of equal distance, same direction and do not having a point in common. On top of that, what is a fundamental assumption, to preserve the definition of straight line. All that is achieved with the genial introduction of the V^{th} postulate. As discussed before, this postulate is only meaningful when understood as a tehcnical procedure for connecting the definition of straight line as something which "proceeds evenly without deviating from its direction" with the notion of parallel. Therefore, any attempt to "prove" this postulate has to use some alternative definition of straight line and parallel which is the same thing as to substitute that postulate by something equivalent to it, preserving the concepts of straight line and parallels as entities and properties existing in space and given "a priori" with their own individuality and not as abstract entities belonging to some set theory!

Consequently this postulate is clearly improvable from the very beginning using only the other four postulates, because we cannot prove something about some object without using all its essential properties known "a priori". Now, "to proceed evenly without deviating from its direction" is an essential property of the straight line and indeed if we analyse all "proofs" of the V^{th} postulate they invariably use along the way some assumption translating in one way or another that essential property of the straight line of "proceed evenly..."

To summarize, regarding the V^{th} postulate either we provide a proof, by using some equivalent hypothesis to include the concept of straight line and parallel and in this case the proof itself is correct as the classical case of Ptolomy's proof, for example, or we do not use any equivalent assumption and in this case the V^{th} postulate is not a consequence of the other four postulates by the reasons explained above. That is what we meant before by the statement that there was no "problem" of the independence of the V^{th} postulate from the remaining four! Now about the solution!

In the XIX^{th} century geometry has to be understood in the Western sense and consequently the words point, straight line and plane do not have any spatial content but they are rather abstract concepts. A straight line, for instance can be "curved", like meridians on the surface of a sphere. For the Greek geometers a spherical triangle was not a triangle in the euclidean sense but just a figure drawn on the sphere whose sides were arc of meridians. As a matter of fact, there existed a whole branch of geometry called **Sphaerica**, dealing with figures drawn on a sphere, of great importance to astronomy, connected with the names of Menelaus, Ptolomy and others. But they never looked to that as a "new geometry"!

When Gauss, Bolyai and Lobachewsky started their celebrate research in the theory of parallels set theory did not exist and the concepts of point, straight line and plane were not yet clearly conceived as abstract entities in the sense of Hilbert; on the contrary, euclidean tradition was still very strongly rooted in their minds eventhough the Greek concept of space had already been erased by the action of time. The real problem of Gauss, for instance, when he got disturbed by imagining a "non euclidean geometry", was his fear of contradiction with the usual (euclidean?) concepts of points, straight and plane and the reaction of the public, what lead that great genius to hide his discovery for the time being.

Bolyai and Lobachewsky were less concerned with "plebis opinio" and decided to prove theorems until the eventual finding of a contradiction resulting from the substitution of the fifth postulate by something else, as for example: through a point P outside a straight line r in the plane we have two families of straight lines formed by those lines which intersect r from one side and by those which do not intersect r by another side; the boundary lines of both families were named parallels to r. Of course, no contradiction was found with "euclidean geometry" because the concept of straight line used by them was not the euclidean one in the Greek sense, namely "proceeding evenly..." If they had decided to use the Greek concept of straight line then they would have to introduce this concept through some hypothesis equivalent to the V^{th} postulate, as discussed before and in this case they would have, for sure, reached a contradiction!

After Beltrani, Poincaré and others a series of models of non-euclidean geometries have been built, but in all of them the concept of straight line is not the euclidean one and consequently they do not solve the "problem" of the V^{th} postulate. All they do is the presentation of **sets with certain structures** which obey similar statements to the postulates I up to IV of Euclid's Elements but not the V^{th} one! That is what we meant before by "the solutions were no solutions", because there was nothing to solve! To render this point clearer let us analyse briefly one of those models. Let us call "plane" the interior of a disc D and straight lines all segments inside D with extremities in its boundary; "points" will be the "usual ones". The notion of distance of two points P, Q is defined by the log of the absolute value of the double ratio of P, Q and the points A, B defined by the intersection of the line r through P, Q with the boundary of D. Details can be seen either in [6] or in [10]. Now taking a "straight line" r in this model and a point P outside r we have two "straight lines" intersecting r in the boundary of D (at infinite!) at points A and B and hence they are "parallel to r". But, of course, this models does not show that the V^{th} postulate is independent of the other because the concept of straight line used there is not the same as Euclid's. All it is proved is: if a set

E with abstract concepts named point, straight line and plane satisfy formally the postulates I up to V of Euclid's Elements then there is another model E' which satisfies I up to IV but not V! Therefore postulate V is independent of postulates I up to IV of model E: "Elementary my dear Watson!" And this, of course, has nothing to do with geometry in the Greek sense! As a matter of fact, a "small being" living in our previous model in the disc D perhaps would have a feeling that the segments AP and PB above together would form a "straight line" and therefore in his intuitive perception of facts in his Universe maybe the V^{th} postulate would be true afterall!

Finally we come to the question: in the real world of our perceptions is the euclidean geometry experimentally verified or not? Here we are dealing with organic logic and cosequently everything depends on our intuitive feeling of point, straight line, plane and space. As seen before, for the architect and the engineer definitely the geometry to be used is the euclidean one. Indeed, we doubt if any one of us would ever buy an apartment in a building knowing that the architect used "non-euclidean geometry" in its design! Or, what tragical consequences would result if an engineer would build a railroad assuming as "straight lines" the meridian on earth. By another side, how about the physicist who assumes as a straight line the distance between two points P and Q defined by a beam of light from P or Q? It is well known, how starting from here, Einstein and others built a model for the Universe which is a four dimensional riemannian manifold. Of course, that is only an abstract object such that conclusions deducted from it, so far, in a large range of phenomena are confirmed by experience. It has no claim to be the real world, which is organic and whose existence is felt by our intuition through our senses. Here more than ever the distinction between the organic and the inorganic is fundamental. In [1-b] this question is discussed in detail and right now we shall only recall a few important points of this subject.

In [11] Kant discusses the question: is the concept of space synthetic a *priori* or analytic? The crucial point here is to clarify the meaning of the word space. In his criticism to Kant, Gauss makes it clear that for him it was not correct to suppose that euclidean geometry was necessarily the geometry of the real world. The whole point of disagreement between the two giants was simply that they were talking about different concepts! Gauss intended to show that other **geometrical models** for the world as a reality could be proposed besides the euclidean one. We see clearly that he was talking about **inorganic space** as a model of formal representations of our intuition of the space of the real world. By another side, Kant was talking about **organic space**. Indeed, if we analyse deeper, the kantian concept of synthetic judgement a *priori* we realize that it fits only in the realm of organic logic while analytic judgement belongs to inorganic logic. Consequently, when the great philosopher of Königsberg says that space is a concept **synthetic a priori** he is absolutely correct because he is talking, perhaps without a clear idea of it, about **organic space**. This misunderstanding of Kant's ideas about space has been rooted like a weed in the Western thought until our days. Indeed, in the work of most philosophers and mathematicians in our Western culture we find vestiges of this confusion between organic and inorganic space.

Here we feel with great intensity the difference between Greek and Western geometry: the **inorganic representation** of the **organic space** built by the Greek geometers were close to each other while in the West they are as far apart as possible: two distinct historical cultures, two distinct concepts of space, two distinct geometries!

REFERENCES

[1] Lintz, R.G. a) *Organic and Inorganic Logic and the Foundations of Mathematics.* Philosophia Naturalis, Band 16, 4, (1977), pp.401-420. b) *Non-deterministic foundations of Mechanics.* Technical Report 01/89, Univ. Estadual de Londrina, Brazil (1989), 227 pages. c) *História da Matemática*, vol.I, Technical Report

07/88, Univ. Estadual de Londrina, Brazil (1988), 386 pages.

[2] Toynbee, A. *Study of History*. 12 Vols., Oxford, 1934-1961.

[3] Spengler, O. *Der Untergang des Abendlandes*. C.H. Beck'sche Verlag, München, (1923).

[4] Goethe, J.W. *Metamorphose der Pflanzen*. Jubiläums Ausgabe, ed. E. von Hellen (1902-12).

[5] Heath, T.L. a) *The thirteen books of Euclid's Elements*. 3 vols., Dover Publ., (1956).
b) *A History of Greek Mathematics*; 2 vols., Dover Publ. (1981).

[6] Hilbert, D. *Grundlagen der Geometrie*. Teubner Verlag (1930).

[7] Dedekind, R. *Was sind und was sollen die Zahlen?* Braunschweig (1911).

[8] Artin, E. *Geometric Algebra*. Interscience Publ., (1966).

[9] Weyl, H. *Das Kontinuum*. Chelsea Publ., New York.

[10] Klein, F. *Vorlesungen über Nicht-Euklidische Geometrie*. Springer–Verlag (1928).

[11] Kant, I. *Kritik der reinen Vernunft*. R. Schmidt ed., Leipzig (1926).

TRABALHOS DO DEPARTAMENTO DE MATEMATICA

TITULOS PUBLICADOS

80-01 PLETCH, A. Local freeness of profinite groups. 10 p.
80-02 PLETCH, A. Strong completeness in profinite groups. 8 p.
80-03 CARNIELLI, W.A. & ALCANTARA, L.P. de Transfinite induction on ordinal configurations. 22 p.
80-04 JONES RODRIGUES, A.R. Integral representations of cyclic p-groups. 13 p.
80-05 CORRADA, M. & ALCANTARA, L.P. de Notes on many-sorted systems. 25 p.
80-06 POLCINO MILIES, F.C. & SEHGAL, S.K. FC-elements in a group ring. 10 p.
80-07 CHEN, C.C. On the Ricci condition and minimal surfaces with constantly curved Gauss map. 10 p.
80-08 CHEN, C.C. Total curvature and topological structure of complete minimal surfaces. 21 p.
80-09 CHEN, C.C. On the image of the generalized Gauss map of a complete minimal surface in R4. 8 p.
81-10 JONES RODRIGUEZ, A.R. Units of ZCpn. 7 p.
81-11 KOTAS, J. & COSTA, N.C.A. da Problems of model and discussive logics. 35 p.
81-12 BRITO, F.B. & GONÇALVES, D.L. Algebras não associativas, sistemas diferenciais polinomiais homogêneos e classes características. 7 p.
81-13 POLCINO MILIES, F.C. Group rings whose torsion units from a subgroup II. Iv. (não paginado).
81-14 CHEN, C.C. An elementary proof of Calabi's theorems on holomorphic curves. 5 p.
81-15 COSTA, N.C.A. da & ALVES, E.H. Relations between paraconsistent logic and many-valued logic. 8 p.
81-16 CASTILLA, M.S.A.C. On Przymusinski's theorem. 6 p.
81-17 CHEN, C.C. & GOES, C.C. Degenerate minimal surfaces in R4. 21 p.
81-18 CASTILLA, M.S.A.C. Imagens inversas de algumas aplicações fechadas. 11 p.
81-19 ARAGONA VALLEJO, A.J. & EXEL FILHO, R. An infinite dimensional version of Hartogs' extension theorem. 9 p.
81-20 GONÇALVES, J.Z. Groups rings with solvable unit groups. 15p.
81-21 CARNIELLI, W.A. & ALCANTARA, L.P. de Paraconsistent algebras. 16 p.
81-22 GONÇALVES, D.L. Nilpotent actions. 10 p.
81-23 COELHO, S.P. Group rings with units of bounded exponent over the center. 25 p.
81-24 PARMENTER, M.M. & POLCINO MILIES, F.C. A note on isomorphic group rings. 4 p.
81-25 MERKLEN GOLDSCHMIDT, H.A. Hereditary algebras with maximum spectra are of finite type. 10 p.
81-26 POLCINO MILIES, F.C. Units of group rings: a short survey. 32 p.

81-27 CHEN, C.C. & GACKSTATTER, F. Elliptic and hyper-elliptic functions and complete minimal surfaces with handles. 14 p.

81-28 POLCINO MILIES, F.C. A glance at the early history of group rings. 22 p.

81-29 FERRER SANTOS, W.R. Reductive actions of algebraic groups on affine varieties. 52 p.

81-30 COSTA, N.C.A. da The philosophical import of paraconsistent logic. 26 p.

81-31 GONÇALVES, D.L. Generalized classes of groups, spaces c-nilpotent and "the Hurewicz theorem". 30 p.

81-32 COSTA, N.C.A. da & MORTENSEN, Chris. Notes on the theory of variable binding term operators. 18 p.

81-33 MERKLEN GOLDSCHMIDT, H.A. Homogenes l-hereditary algebras with maximum spectra. 32 p.

81-34 PERESI, L.A. A note on semiprime generalized alternative algebras. 10 p.

81-35 MIRAGLIA NETO, F. On the preservation of elementary equivalence and embedding by filtred powers and structures of stable continuous functions. 9 p.

81-36 FIGUEIREDO, G.V.R. Catastrophe theory: some global theory a full proof. 91 p.

82-37 COSTA, R.C.F. On the derivations of gametic algebras. 17 p.

82-38 FIGUEIREDO, G.V.R. A shorter proof of the Thom-Zeeman global theorem for catastrophes of cod <= 5. 7 p.

82-39 VELOSO, J.M.M. Lie equations and Lie algebras: the intrasitive case. 97 p.

82-40 GOES, C.C. Some results about minimal immersions having flat normal bundle. 37 p.

82-41 FERRER SANTOS, W.R. Cohomology of comodules II. 15 p.

82-42 SOUZA, V.H.G. Classification of closed sets and diffeos of one-dimensional manifolds. 15 p.

82-43 GOES, C.C. The stability of minimal cones of codimension greater than one in Rn. 27 p.

82-44 PERESI, L.A. On automorphisms of gametic algebras. 27 p.

82-45 POLCINO MILIES, F.C. & SEHGAL, S.K. Torsion units in integral group rings of metacyclic groups. 18 p.

82-46 GONÇALVES, J.Z. Free subgroups of units in group rings. 8 p.

82-47 VELOSO, J.M.M. New classes of intransitive simple Lie pseudogroups. 8 p.

82-48 CHEN, C.C. The generalized curvature ellipses and minimal surfaces. 10 p.

82-49 COSTA, R.C.F. On the derivation algebra of zygotic algebras for polyploidy with multiple alleles. 24 p.

83-50 GONÇALVES, J.Z. Free subgroups in the group of units of group rings over algebraic integers. 3 p.

83-51 MANDEL, A. & GONÇALVES, J.Z. Free k-triples in linear groups. 7 p.

83-52 BRITO, F.G.B. A remark on closed minimal hypersurfaces of S4 with second fundamental form of constant length. 12 p.

83-53 KIIHL, J.C.S. U-structures and sphere bundles. 8 p.

83-54 COSTA, R.C.F. On genetic algebras with prescribed derivations. 23 p.

83-55 SALVITTI, R. Integrabilidade das distribuições dadas por subálgebras de Lie de codimensão finita no gh(n,C). 4 p.

83-56 MANDEL, A. & GONÇALVES, J.Z. Construction of open sets of free k-Tuples of matrices. 18 p.

83-57 BRITO, F.G.B. A remark on minimal foliations of codimension two. 24 p.

83-58 GONÇALVES, J.Z. Free groups in subnormal subgroups and the residual nilpotence of the group of units of group rings. 9 p.

83-59 BELOQUI, J.A. Modulus of stability for vector fields on 3-manifolds. 40 p.

83-60 GONÇALVES, J.Z. Some groups not subnormal in the group of units of its integral group ring. 8 p.

84-61 GOES, C.C. & SIMOES, P.A.Q. Imersões mínimas nos espaços hiperbólicos. 15 p.

84-62 GIANBRUNO, A.; MISSO, P. & POLCINO MILIES, F.C. Derivations with invertible values in rings with involution. 12 p.

84-63 FERRER SANTOS, W.R. A note on affine quotients. 6 p.

84-64 GONÇALVES, J.Z. Free-subgroups and the residual nilpotence of the group of units of modular and p-adic group rings. 12 p.

84-65 GONÇALVES, D.L. Fixed points of S1-fibrations. 18 p.

84-66 RODRIGUES, A.A.M. Contact and equivalence of submanifolds of homogenous spaces. 15 p.

84-67 LOURENCO, M.L. A projective limit representation of (DFC)-spaces with the approximation property. 20 p.

84-68 FORNARI, S. Total absolute curvature of surfaces with boundary. 25 p.

84-69 BRITO, F.G.B. & WALCZAK, P.G. Totally geodesic foliations with integral normal bundles. 6 p.

84-70 LANGEVIN, R. & POSSANI, C. Quase-folheacões e integrais de curvatura no plano. 26 p.

84-71 OLIVEIRA, M.E.G.G. de Non-orientable minimal surfaces in RN. 41 p.

84-72 PERESI, L.A. On baric algebras with prescribed automorphisms. 42 p.

84-73 MIRAGLIA NETO, F. & ROCHA FILHO, G.C. The measurability of Riemann integrable-function with values in Banach spaces and applications. 27 p.

84-74 MERKLEN GOLDSCHMIDT, H.A. Artin algebras wich are equivalent to a hereditary algebra modulo preprojectives. 38 p.

84-75 GOES, C.C. & SIMOES, P.A.Q. The generalized Gauss map of minimal surfaces in H3 and H4. 16 p.

84-76 GONÇALVES, J.Z. Normal and subnormal subgroups in the group of units of a group rings. 13 p.

85-77 ARAGONA_VALLEJO, A.J. On existence theorems for the ∂-operator on generalized differential forms. 13 p.

85-78 POLCINO MILIES, C.; RITTER, J. & SEHGAL, S.K. On a conjecture of Zassenhaus on torsion units in integral group rings II. 14 p.

85-79 JONES RODRIGUEZ, A.R. & MICHLER, G.O. On the structure of the integral Green ring of a cyclic group of order p2. The Jacobson radical of the integral Green ring of a cyclic group of order p2. 26 p.

85-80 VELOSO, J.M.M. & VERDERESI, J.A. Three dimensional Cauchy-Riemann manifolds. 19 p.

85-81 PERESI, L.A. On baric algebras with prescribed automorphisms II. 18 p.

85-82 KNUDSEN, C.A. O impasse aritmo-geométrico e a evolução do conceito de número na Grécia antiga. 43p.

85-83 VELOSO, J.M.M. & VERDERESI, J.A. La géometrie, le probleme d'équivalence et le classification des CR-varietés homogenes en dimension 3. 30 p.

85-84 GONÇALVES, J.Z. Integral group rings whose group is solvable, an elementary proof. 11 p.

85-85 LUCIANO, O.O. Nebuleuses infinitesimelement fibrées. 5 p.

85-86 ASPERTI, A.C. & DAJCZER, M. Conformally flat Riemannian manifolds as hypersurfaces of the ligth cone. 8 p.

85-87 BELOQUI, J.A. A quasi-transversal Hopf bifurcations. 11 p.

85-88 POLCINO MILIES, F.C. & RAPHAEL, D.M. A note on derivations with power central values in prime rings. 7 p.

85-89 POLCINO MILIES, F.C. Torsion units in group rings and a conjecture of H.J.Zassenhaus. 14 p.

86-90 LOURENÇO, M.L. Riemann domains over (DFC) spaces. 32 p.

86-91 ARAGONA VALLEJO, A.J. & FERNANDES,J.C.D. The Hartogs extension theorem for holomorphic generalized functions. 9 p.

86-92 CARRARA ZANETIC, V.L. Extensions of immersions in dimension two. 27 p.

86-93 PERESI, L.A. The derivation algebra of gametic and zygotic algebras for linked loci. 25 p.

86-94 COELHO, S.P. A note on central idempotents in group ring. 5 p.

86-95 PERESI, L.A. On derivations of baric algebras with prescribed automorphisms. 21 p.

86-96 COELHO, F.U. A generalization of a theorem of Todorov on preprojectives partitions. 18 p.

86-97 ASPERTI, A.C. A note on the minimal immersions of the two-sphere. 11 p.

86-98 COELHO, S.P. & POLCINO MILIES, F.C. A note on central idempotents in group rings II. 8 p.

86-99 EXEL FILHO, R. Hankel matrices over right ordered amenable groups. 18 p.

NOVA SERIE

86-01 GOODAIRE, E.G. & POLCINO MILIES, F.C. Isomorphisms of integral alternative loop rings. 11 p.

86-02 WALCZAK, P.G. Foliations which admit the most mean curvature functions. 11 p.

86-03 OLIVEIRA, M.E.G.G. Minimal Klein bottles with one end in R3 and R4. 12 p.

86-04 MICALI, A. & VILLAMAYOR, O.E. Homologie de Hochschild de certaines algebres de groupes. 11 p.

86-05 OLIVEIRA, M.E.G.G. Minimal Klein bottles in R3 with finite total curvature. 9 p.

86-06 CARRARA ZANETIC, V.L. Classification of stable maps between 2-manifolds with given singular set image. 22 p.

87-01 BRITO, F.G.B. & WALCZAK, P.G. Total curvature of orthogonal vector fields on three-manifolds. 4 p.

87-02 BRITO, F.G.B. & LEITE, M.L. A remark on rotational hypersurfaces of Sn. 13 p.

87-03 GONÇALVES, J., RITTER, J. & SEHGAL, S. Subnormal sub groups in U(ZG). 13 p.

87-04 ARAGONA VALLEJO, A.J. & COLOMBEAU, J.F. The interpolation theorem for holomorphic generalized functions. 12 p.

87-05 ALMEIDA, S.C. de & BRITO, F.G.B. Immersed hypersurfaces of a space form with distinct principal curvatures. 9 p.

87-06 ASPERTI, A.C. Generic minimal surfaces. 21 p.

87-07 GOODAIRE, E.G. & POLCINO MILIES, F.C. Torsion units in alternative group rings. 17 p.

87-08 MERKLEN GOLDSCHMIDT, H.A. Algebras which are equivalent to a hereditary algebra modulo preprojectives II. 27 p.

87-09 REYNOL FILHO, A. P-localization of some classes of groups. 30 p.

88-01 CARLSON, J.F. & JONES, A. An exponential property of lattices over group rings. 22 p.

88-02 CARLSON, J.F. & JONES, A. Wild categories of periodic modules. 6 p.

88-03 SALLUM, E.M. The nonwandering set of flows on a Reeb foliation. 13 p.

88-04 ALMEIDA, R. Cohomologie des suites d'Atiyah. 14 p.

88-05 GUZZO JR., H. Alguns teoremas de caracterização para álgebras alternativas à direita. 16 p.

88-06 HARLE, C.E. Subvariedades isoparamétricas homogêneas. 5 p.

88-07 LANGEVIN, R. Vers une classification des diffeomorphismes Morse-Smale d'une surface. 18 p.

88-08 ARAGONA VALLEJO, A. J. & BIAGIONI, H. A. An intrinsic definition of the Colombeau algebra of generalized functions. 48 p.

88-09 BORSARI, H.D. A Cohomological characterization of redutive algebraic groups. 20 p.

88-10 COELHO, S.P. & POLCINO MILIES, F.C. Finite conjugacy in group rings. 20 p.

88-11 FERNANDEZ, R. Characterization of the dual of an Orlicz Space. 19 p.

89-01 GONÇALVES, J.Z. & MANDEL, A. Embedding the Free Group Ring into Formal Series. 4 p.

89-02 REYNOL Fº, A. Nilpotent spaces: some inequalities on nilpotency degrees. 17 p.

89-03 GOUVÊA, F.Q. Deforming Galois Representations: Controlling the Conductor. 22 p.

89-04 CORDARO, P. & TREVES, F. Homology & Cohomology in Hypo-Analytic Structures of the Hypersurfaces Type. 108p.

89-05 ARAGONA VALLEJO, A. J. & VILLARREAL ALVARADO, F. Colombeau's theory and shock Waves in a problem of Hydrodynamics. 48 p.

90-01 MANDEL, A. & GONÇALVES, J.Z. Semigroup laws and free semigroups in the group of units of group rings. 11 p.

90-02 LINTZ, R.G. Generation of Quantum-fields by non-deterministic fields. 32p.

90-03 LINTZ, R.G. A critical study on the foundations of Geometry. 41p.