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Representations of posets: linear versus unitary

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Abstract. A number of recent papers treated the representation theory of partially ordered sets in unitary spaces with the so called orthoscalar relation. Such theory generalizes the classical theory which studies the representations of partially ordered sets in linear spaces. It happens that the results in the unitary case are well-correlated with those in the linear case. The purpose of this article is to shed light on this phenomena.

MSC 2010: 16G20, 53D20, 14L24.

1. Introduction

In the second part of XX century the new results concerning to representation theory in linear spaces were obtained, in particular, the results about relative position of several subspaces and partially ordered collections of subspaces in linear spaces (see, for example, [7, 24]). Representations of partially ordered sets (*posets* in sequel) were first introduced by L.Nazarova and A.Roiter (see [26]), using the language of "matrix problems", as a tool to prove the second Brauer-Thrall conjecture. Later P.Gabriel gave an equivalent (up to a finite number of indecomposables) definition of representations using "subspace" language (see, for example, [29, Chapter 3]). In the sequel we use the latter terminology.

One can try to develop a similar theory for unitary spaces. Namely, one can transfer straightforward the definitions from linear to unitary case keeping in mind that the morphisms between two representations should preserve their unitary structure; that is, the morphisms are unitary maps. With this restriction the classification problem becomes extremely difficult already in very simple situations. For example the problem of classifying unitary representations of the poset $\mathcal{P} = \{a_1, a_2, a_3 \mid a_2 \prec a_3\}$ is hopeless, since by [20, 21] it contains the problem of classifying any system of operators in unitary spaces (see [27, Chapter 3] for more details).

The idea, hence, is to consider those representations which satisfy some additional conditions. In such a way there were defined orthoscalar representations of quivers (see [18, 19]) and orthoscalar representations of posets (see, for example, [1, 28]). Certainly the additional conditions can be chosen differently. Orthoscalarity relation appears in

different areas of mathematics. For example, using [14, 16], one can show that every indecomposable orthoscalar representation of a poset generate a stable reflexible sheaf on certain toric variety. This is one of the motivation to study such representations.

Analyzing the results related to orthoscalar representations of quivers and posets (see [1, 18, 19, 28] and references therein) one can see the mysterious (on the first sight) connection with the classical results (more details in Section 1). The main goal of this paper is to explain this phenomena.

For a given finite poset \mathcal{P} we consider the variety $R_{\mathcal{P},d}$ of all representations of \mathcal{P} with dimension vector $d = (d_0; d_i)_{i \in \mathcal{P}}$. We show that the classes of unitary non-equivalent orthoscalar representations of \mathcal{P} with dimension vector d can be viewed as the symplectic quotient of $R_{\mathcal{P},d}$. On the other side we consider the GIT quotient of $R_{\mathcal{P},d}$ with respect to some polarization of this variety. This quotient can be viewed as the set of isomorphism classes of polystable linear representations of \mathcal{P} with dimension d . We show that the identification between these two quotients is a consequence of the fundamental Kirwan-Ness theorem (see, for example, [23]).

Let us mention that the idea of the symplectic reduction in similar contexts is not new. For example, it appeared in [10] (constructing moduli spaces for quivers) and in [15] (in connection with Horn's problem). In this text we apply it in the setting of partially ordered sets.

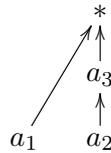
1.1. Structure of the paper.

In Section 2 we recall basic facts about the posets and their indecomposable representations in linear spaces and indecomposable orthoscalar representations in unitary spaces, in particular we compare some known results about the structure of representations in both cases. In Section 3 we give some basic facts about symplectic quotients. Section 4 is dedicated to the GIT quotients. We explain the construction and state the Kirwan-Ness theorem which connects symplectic and GIT quotients. In Section 5 we prove Theorem 3 which relates representations of posets in linear spaces with the ones in unitary spaces with orthoscalar condition. In Section 6 we estimate the dimension of the variety of uni-classes of representations of a given poset with a given dimension vector using the connection between posets and bound quivers.

2. Posets and their representations.

Let \mathcal{P} be a finite poset with partial order relation \prec . Denote by \mathcal{P}^* the extension of \mathcal{P} by a unique maximal element $*$. To \mathcal{P} we associate the Hasse quiver of \mathcal{P}^* which is denoted by $Q^{\mathcal{P}}$. The *order* of \mathcal{P} is the number of its elements. A poset is said to be *primitive* and denoted by (n_1, \dots, n_s) if it is a disjoint (cardinal) sum of linearly ordered sets with the orders n_i . A poset \mathcal{P} is primitive if and only if the corresponding quiver $Q_{\mathcal{P}}$ has a star-shaped form.

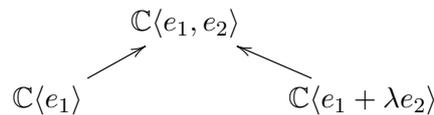
Example 1. Let $\mathcal{P} = (1, 2)$ be the poset consisting of three elements a_1, a_2, a_3 with a unique relation $a_2 \prec a_3$. Then the associated quiver $Q^{\mathcal{P}}$ has the following form:



2.1. Linear representations

Following P.Gabriel (see [29, Chapter 3]) we define the category $\mathcal{P} - sp$ of representations of \mathcal{P} in linear spaces. The objects of $\mathcal{P} - sp$ are systems $(V; V_i)_{i \in \mathcal{P}}$, in which V is a vector space, each V_i is a subspace of V and $V_i \subset V_j$ if $i \prec j$. The set of morphisms between two objects $(V; V_i)_{i \in \mathcal{P}}$ and $(\tilde{V}; \tilde{V}_i)_{i \in \mathcal{P}}$ consists of the linear maps $f : V \rightarrow \tilde{V}$ such that $f(V_i) \subset \tilde{V}_i$ for all $i \in \mathcal{P}$. Two objects are *isomorphic* if there exists an invertible $f : V \rightarrow \tilde{V}$ with $f(V_i) = \tilde{V}_i$ for all $i \in \mathcal{P}$. The category $\mathcal{P} - sp$ is additive with usual direct sums. An object $(V; V_i)_{i \in \mathcal{P}}$ is said to be *indecomposable* if it is not isomorphic to a direct sum of two non-zero objects.

Example 2. Let $\mathcal{P} = (1, 1)$ be a poset consisting of two incomparable elements. Fixing $\lambda \in \mathbb{C}$ we build the following linear two-dimensional representation of \mathcal{P}



It is easy to see that any such representation with $\lambda \neq 0$ is equivalent to the one with $\lambda = 1$ and hence it splits into one-dimensional representations:



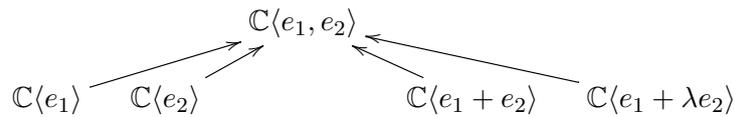
One can show also that this poset has only finite number of isomorphism classes of indecomposable objects in $\mathcal{P} - sp$.

The *dimension vector* $d = (d_0; d_i)_{i \in \mathcal{P}} \in \mathbb{N}^{|\mathcal{P}|+1}$ of a given representation $(V; V_i)_{i \in \mathcal{P}}$ is defined as $d_0 = \dim V$, $d_i = \dim V_i$. Fixing a dimension vector d we define the variety $R_{\mathcal{P}, d}$ of representations of \mathcal{P} with dimension d

$$R_{\mathcal{P}, d} = \left\{ (V_i)_{i \in \mathcal{P}} \in \prod_{i \in \mathcal{P}} \text{Gr}(d_i, d_0) \mid V_i \subset V_j, i \prec j \right\}.$$

The group $GL(d_0)$ acts on $R_{\mathcal{P}, d}$ via simultaneous base change $g \cdot (V_i)_{i \in \mathcal{P}} = (g(V_i))_{i \in \mathcal{P}}$, so that the orbits of this action are in one-to-one correspondence with the isomorphism classes of representations of \mathcal{P} with the dimension vector d .

Example 3. Consider the poset $\mathcal{P} = (1, 1, 1, 1)$ consisting of four incomparable elements. L.Nazarova [24] and I.Gelfand with V.Ponomarev [7] completely classified indecomposable systems of four subspaces. Let us recall the description for the dimension vector $d = (2; 1, 1, 1, 1)$:



It was shown that non-isomorphic indecomposable representations are parametrized by the extended complex plane $\mathbb{C} \cup \{\infty\} \simeq S^2$ (when $\lambda = \infty$ the corresponding subspace $\mathbb{C}\langle e_1 + \lambda e_2 \rangle$ is defined to be $\mathbb{C}\langle e_2 \rangle$). One can see that when $\lambda = 0, 1$ or ∞ then the subspaces in corresponding representations are not in general position (these cases are exceptional). Therefore we have the sphere without three points of non-exceptional representations.

2.2. Unitary representations. Orthoscalar representations.

Define the category $\mathcal{P} - usp$ of unitary representations of a poset \mathcal{P} . The objects of $\mathcal{P} - usp$ are systems $(U; U_i)_{i \in \mathcal{P}}$, in which U is a unitary space, each U_i is a subspace of U and $U_i \subset U_j$ if $i \prec j$. The set of morphisms between two objects $(U; U_i)_{i \in \mathcal{P}}$ and $(\tilde{U}; \tilde{U}_i)_{i \in \mathcal{P}}$ consists of isometric maps $\varphi : U \rightarrow \tilde{U}$ such that $\varphi(U_i) \subset \tilde{U}_i$. Two systems are said to be *unitary equivalent* if the morphism between them can be chosen to be a unitary map. An object $(U; U_i)_{i \in \mathcal{P}}$ is said to be *indecomposable* if it is not unitary equivalent to an orthogonal sum of two non-zero objects in $\mathcal{P} - usp$.

Example 4. The poset $\mathcal{P} = (1, 1)$ contains a continuous family of unitary non-equivalent representations. Indeed, consider the representations of the poset \mathcal{P} as in Example 2. Taking usual scalar product in $\mathbb{C}\langle e_1, e_2 \rangle$ we view these representations as the objects in $\mathcal{P} - usp$. For different $\lambda \in (0, 1]$ they are indecomposable and unitary non-equivalent. Moreover, P.Halmos [8] proved that the representations as in Example 2 gives a complete list of all indecomposable objects in $\mathcal{P} - usp$.

Example 5. Consider the poset $\mathcal{P} = (1, 2)$. There exists only a finite number of indecomposable objects in $\mathcal{P} - usp$ up to the isomorphism. At the same time it is an extremely hard problem to classify the indecomposable objects in $\mathcal{P} - usp$ up to the unitary equivalence. By S.Kruglyak and Yu.Samoilenko [20, 21], such problem is **-wild*; that is, it contains the problem of classifying any system of operators in unitary spaces (see [27, Chapter 3] for more details).

Fix a weight $\chi = (\chi_0; \chi_i)_{i \in \mathcal{P}} \in \mathbb{Z}_+^{|\mathcal{P}|+1}$ and consider those objects $(U; U_i)_{i \in \mathcal{P}}$ in $\mathcal{P} - usp$ that satisfy

$$\sum_{i \in \mathcal{P}} \chi_i P_{U_i} = \chi_0 I,$$

where P_{U_i} is the orthogonal projection onto U_i . Such objects are called χ -*orthoscalar* ([1, 18, 19]) and form the subcategory denoted by $\mathcal{P}_\chi - usp$. One of the main motivations to study χ -orthoscalar objects is that any such object gives rise to a polystable reflexible sheaves on certain toric variety (see [14, 16] and references therein).

Example 6. Consider the poset $\mathcal{P} = (1, 1)$. The objects in $\mathcal{P}_\chi - usp$ are those systems $(U; U_1, U_2)$ that satisfy

$$\chi_1 P_{U_1} + \chi_2 P_{U_2} = \chi_0 I.$$

One can see that the projections P_{U_1} and P_{U_2} commute. Hence the indecomposable objects are at most one-dimensional. For the category \mathcal{P}_χ -usp to be non-empty χ_0 should be equal to $\chi_1 + \chi_2$ (trace identity). Moreover, if this condition is satisfied then the indecomposables in \mathcal{P}_χ -usp are in one-to-one correspondence with the indecomposables in $\mathcal{P} - sp$.

Example 7. Consider the poset $\mathcal{P} = (1, 1, 1, 1)$. As in Example 3 we consider the objects with the dimension vector $(2; 1, 1, 1, 1)$. Let us take the weight $\chi = (2; 1, 1, 1, 1)$. The description of unitary non-equivalent irreducible quadruples of projections that satisfy

$$P_{U_1} + P_{U_2} + P_{U_3} + P_{U_4} = 2I,$$

is the following (e.g. [27, Chapter 2.2]):

$$\begin{aligned} P_{U_1} &= \frac{1}{2} \begin{pmatrix} 1+a & -b-ic \\ -b+ic & 1-a \end{pmatrix}, & P_{U_2} &= \frac{1}{2} \begin{pmatrix} 1-a & b-ic \\ b+ic & 1+a \end{pmatrix}, \\ P_{U_3} &= \frac{1}{2} \begin{pmatrix} 1-a & -b+ic \\ -b-ic & 1+a \end{pmatrix}, & P_{U_4} &= \frac{1}{2} \begin{pmatrix} 1+a & b+ic \\ b-ic & 1-a \end{pmatrix}. \end{aligned}$$

Topologically the set of parameters a, b, c is a sphere S^2 without three points where representations split into one-dimensional. Moreover by [22], to each triple (a, b, c) one can associate $\lambda \neq 0, 1, \infty$ (and vice versa) such that the corresponding system $(U; U_i)_{i \in \mathcal{P}}$ is equivalent to a system $(V; V_i)_{i \in \mathcal{P}}$ which corresponds to λ as in Example 3. Hence, the uni-classes of indecomposable objects in $\mathcal{P}_{(2;1,1,1,1)} - usp$ with dimension vector $(2; 1, 1, 1, 1)$ are in one-to-one correspondence with the iso-classes of indecomposable non-exceptional objects in $\mathcal{P} - sp$ with dimension vector $(2; 1, 1, 1, 1)$.

Recall that \mathcal{P} is said to be *representation-finite* (resp. **-representation-finite*) if it has only a finite number of isomorphism classes of indecomposables in $\mathcal{P} - sp$ (resp. in $\mathcal{P}_\chi - usp$ for any weight χ). P.Gabriel classified all representation-finite quivers together with their indecomposable representations. Correspondingly, M.Kleiner [13] classified representation-finite posets, and also their indecomposable representations. There is a direct analogue of the Gabriel's classification obtained by S.Kruglyak and A.Roiter [19] for unitary orthoscalar representations of quivers (see [19] for the definition of orthoscalarity for quivers). An analogue of the Kleiner's classification was obtained in [28]. In particular it turned out that

$$\mathcal{P} \text{ is representation-finite} \Leftrightarrow \mathcal{P} \text{ is } *-representation-finite.$$

2.3. Non-commutative Hopf fibration

Consider the subset of $R_{\mathcal{P},d}$ consisting of the representations which satisfy χ -orthoscalar condition:

$$R_{\mathcal{P},d}^\chi = \left\{ (V_i)_{i \in \mathcal{P}} \in R_{\mathcal{P},d} \mid \sum_{i \in \mathcal{P}} \chi_i P_{V_i} = \chi_0 I \right\},$$

where the projections P_{V_i} are taking with respect to the standard Hermitian metric on \mathbb{C}^{d_0} . The group $U(d_0)$ acts on $R_{\mathcal{P},d}$ as a subgroup of $GL(d_0)$. The orbits of this

action on $R_{\mathcal{P},d}^{\chi}$ are in one-to-one correspondence with the uni-classes of χ -orthoscalar representations of \mathcal{P} with dimension vector d .

We will see that the connection between orbit spaces $R_{\mathcal{P},d}/GL(d_0)$ and $R_{\mathcal{P},d}^{\chi}/U(d_0)$ is a generalization of the following commutative identification.

The group \mathbb{C}^* (identified with $GL(1)$) acts on \mathbb{C}^n by multiplication $c * z = (cz_1, \dots, cz_n)$. The corresponding orbit space is not Hausdorff (the orbit of 0 lies in any neighbourhood of any other orbit), but $\mathbb{C}^n - \{0\}/\mathbb{C}^*$ is Hausdorff and homeomorphic to a projective space $\mathbb{C}\mathbb{P}^{n-1}$. Consider the subset of \mathbb{C}^n consisting of those points (z_1, \dots, z_n) that satisfy $z_1\bar{z}_1 + \dots + z_n\bar{z}_n = 1$. This subset determines the sphere S^{2n-1} . The group S^1 (identified with the unitary group $U(1)$) acts on S^{2n-1} by rotations. The corresponding orbit space S^{2n-1}/S^1 (so called *Hopf-fibration*) is again a projective space $\mathbb{C}\mathbb{P}^{n-1}$. So we trivially have

$$\{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_1\bar{z}_1 + \dots + z_n\bar{z}_n = 1\}/U(1) \cong \mathbb{C}^n - \{0\}/GL(1).$$

Let \mathcal{P} be a poset with n non-comparable elements, d be dimension vector and χ be the weight. Formally substitute each z_i by the matrix $A_i \in M_{d_0 \times d_i}(\mathbb{C})$. The corresponding equation between z_i has the following form

$$A_1 A_1^* + \dots + A_n A_n^* = I.$$

Take $(\chi'_1, \dots, \chi'_n) = (\chi_1 \chi_0^{-1}, \dots, \chi_n \chi_0^{-1})$. Viewing each χ'_i as $\|A_i\|^2$ we write the non-commutative version of $\bar{z}_i z_i = |z_i|^2$ as $A_i^* A_i = \chi'_i I_{d_i}$. From last equation we get $A_i A_i^* = \chi'_i P_{\text{Im}(A_i)}$. We say that (A_1, \dots, A_n) and $(\tilde{A}_1, \dots, \tilde{A}_n)$ lie in the same equivalence class under the action of $U(d_0)$ if there exists $\varphi \in U(d_0)$ such that $\varphi(\text{Im}(A_i)) = \text{Im}(\tilde{A}_i)$ for all $i \in \mathcal{P}$. Then the quotient

$$\left\{ (A_i)_{i \in \mathcal{P}} \in (M_{d_0 \times d_i}(\mathbb{C}))_{i \in \mathcal{P}} \mid A_1 A_1^* + \dots + A_n A_n^* = I, \quad A_i^* A_i = \chi'_i I_{d_i} \right\} / U(d_0)$$

parametrizes the equivalence classes of χ -orthoscalar representations with dimension vector d of the poset \mathcal{P} and can be seen as a *non-commutative Hopf fibration*. On the other hand the quotient

$$\left\{ (A_i)_{i \in \mathcal{P}} \in (M_{d_0 \times d_i}(\mathbb{C}))_{i \in \mathcal{P}} \mid A_i^* A_i = \chi'_i I_{d_i} \right\} / GL(d_0)$$

can be identified with $R_{\mathcal{P},d}/GL(d_0)$. In what follows we show (see Section 4) that for any poset \mathcal{P} these two quotients are connected similarly to commutative example.

3. Symplectic quotient.

3.1. Lie groups and algebras. Coadjoint representations

We briefly recall necessary information about Lie groups, Lie algebras and coadjoint representations of Lie groups (more information see, for example, in [12]).

By G we denote a Lie group (which is assumed to be finite-dimensional), $\mathfrak{g} = \mathfrak{Lie}(G)$ its Lie algebra (the tangent space to the identity element of G) and \mathfrak{g}^* denotes the dual Lie algebra. A group G is called *complex matrix group* if G is a subgroup and at the

same time a smooth submanifold of $GL(n)$. In this paper we mainly consider the case when $G = U(n)$; that is, the group of unitary matrices in n -dimensional complex space \mathbb{C}^n . The corresponding Lie algebra $\mathfrak{u}(n)$ consists of skew-Hermitian matrices.

A Lie group G acts on itself by inner automorphisms

$$A : G \rightarrow \text{Aut}(G), \quad A(g) : h \mapsto ghg^{-1}.$$

Differentiating we get the *adjoint representation* $d(A(g))_e = \text{Ad}_g : \mathfrak{g} \rightarrow \mathfrak{g}$. In the case when G is a matrix group this representations is given by

$$\text{Ad}_g(x) = gxg^{-1}, \quad g \in G, \quad x \in \mathfrak{g}.$$

Let $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{K}$, $(\xi, x) \rightarrow \langle \xi, x \rangle = \xi(x)$ be natural pairing between \mathfrak{g} and \mathfrak{g}^* . The *coadjoint representation* $\text{Ad}^* : G \rightarrow \text{Aut}(\mathfrak{g}^*)$ is defined by

$$\langle \text{Ad}_g^* \xi, x \rangle = \langle \xi, \text{Ad}_{g^{-1}} x \rangle, \quad x \in \mathfrak{g}, \quad \xi \in \mathfrak{g}^*.$$

In the case when G is a matrix group the coadjoint representations is given by

$$\text{Ad}_g^*(\xi) = g\xi g^{-1}, \quad g \in G, \quad \xi \in \mathfrak{g}^*.$$

Recall that if \mathfrak{g}^* is semisimple then adjoint and coadjoint representations are equivalent. By the *coadjoint orbits* we understand the orbits of G on \mathfrak{g}^* .

Example 8. Let $G = U(n)$. Its dual Lie algebra $\mathfrak{u}(n)^*$ consists of the Hermitian matrices via the identification $\mathfrak{u}(n)^* \cong i\mathfrak{u}(n)$. Then each coadjoint orbit \mathcal{O}_λ is the set Hermitian matrices that have the spectrum $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$.

We say that a Lie group $G^{\mathbb{C}}$ is a *complexification* of G , if G is a closed sub-Lie group of $G^{\mathbb{C}}$ and $\mathfrak{g}^{\mathbb{C}} = \mathfrak{Lie}(G^{\mathbb{C}})$ is a vector-space complexification of $\mathfrak{g} = \mathfrak{Lie}(G)$; that is, $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$.

Example 9. The complexification of $G = SL(n, \mathbb{R})$ is the group $SL(n, \mathbb{R})^{\mathbb{C}} = SL(n, \mathbb{C})$, due to obvious identification $\mathfrak{sl}(n, \mathbb{R})^{\mathbb{C}} = \mathfrak{sl}(n, \mathbb{C})$. The complexification of $G = U(n)$ is $GL(n)$, because $\mathfrak{u}(n)^{\mathbb{C}}$ is the full matrix algebra.

3.2. Symplectic manifolds.

Let us recall some basic facts about symplectic manifolds (more details see, for example, [5]). Let M be a manifold, and ω be a closed 2-form on the tangent space which assumed to be non-degenerate; that is, for any tangent vector $x_1 \in T_p M$, there exists another vector $x_2 \in T_p M$ with $\omega(x_1, x_2)$ nonzero. Due to non-degeneracy of ω the manifold M has to be even dimensional.

Example 10. Consider the space \mathbb{R}^{2n} with basis $\{x_1, \dots, x_n, y_1, \dots, y_n\}$ and the form ω acting by $\omega(x_i, x_j) = \omega(y_i, y_j) = 0$, and $\omega(x_i, y_j) = \omega(y_j, x_i) = \delta_{ij}$ (Kronecker delta). Such form ω is called *standard*.

If (M, ω) is locally isomorphic to \mathbb{R}^{2n} with standard ω then M is called a *symplectic manifold*, and ω is its *symplectic form*.

Example 11. Complex projective space $\mathbb{C}\mathbb{P}^n$ can be equipped with the Fubini-Study form which makes it into a symplectic manifold (see, for example, [5, Section 16]). We do not define this form, just note that one can think of it as the one realized on the quotient space after the identification $\mathbb{C}\mathbb{P}^n = S^{2n+1}/S^1$.

The coadjoint orbits have a structure of symplectic manifold, moreover each coadjoint orbit possesses a G -invariant symplectic structure (see [11, Chapter 1]). In some cases the opposite is also true: a symplectic Hausdorff manifold (M, ω) , with the action of a Lie group G which preserves ω , is isomorphic to a coadjoint orbit of G (see, for example, [17]).

Example 12. Describe the symplectic form on a coadjoint orbit of $U(n)$. It is enough to calculate it in a generic point $p \in \mathfrak{u}(n)^*$. Any tangent vector to p looks like $ad_p(x) = [p, x]$ for some $x \in \mathfrak{u}(n)$. Then the symplectic form is given by $\omega_p(ad_p(x_1), ad_p(x_2)) = tr(x_1px_2 - x_2px_1)$.

Example 13. Consider a coadjoint orbit \mathcal{O}_λ of $U(n)$ with $\lambda_1 \neq \lambda_2 = \lambda_3 = \dots = \lambda_n$. The eigenspaces corresponding to elements of \mathcal{O}_λ are the line and the orthogonal hyperplane. Hence, \mathcal{O}_λ is homeomorphic to $\mathbb{C}\mathbb{P}^{n-1}$ and by the Kirillov's construction we have many symplectic forms on $\mathbb{C}\mathbb{P}^{n-1}$: one for each distinct pair of real numbers. In particular, the Fubini-Study form (Example 11) corresponds to the choice $\lambda_1 = 1, \lambda_2 = 0$. The corresponding coadjoint orbit is just the set of one-dimensional orthogonal projections.

3.3. Moment map

For a given function $f : M \rightarrow \mathbb{C}$ its *symplectic gradient* $X_f : M \rightarrow T_*M$ (called the *Hamiltonian vector field*) is defined by the following equation

$$d_x f = \omega(x, X_f(p)),$$

where x and $X_f(p)$ are tangent vectors to the point $p \in M$ and $d_x f$ is the derivation of the function f in the direction $x \in T_p M$. Since ω is non-degenerate this defines X_f uniquely.

Let G be a connected Lie group acting on some symplectic manifold M smoothly preserving the symplectic form. Then it generates a Lie algebra homomorphism from \mathfrak{g} to the Lie algebra of smooth vector fields on M . Indeed, an element of G near the identity gives a diffeomorphism of M close to the identity; differentiating we get that each tangent vector to the identity generates a vector field on M . By $p \mapsto a_p$ we denote the vector field on M associated to $a \in \mathfrak{g}$.

We say that $\mu : M \rightarrow \mathfrak{g}^*$ is a *moment map* for the action of G on M if the following two conditions are satisfied:

- (i) μ is G -equivariant with respect to the action of G on M and to the coaction of G on \mathfrak{g}^* ; that is, the following holds

$$\mu(g \cdot p) = g\mu(p)g^{-1}, \quad p \in M, \quad g \in G;$$

- (ii) $d_x \langle \mu(p), a \rangle = \omega(x, a_p)$, $\xi \in T_x M$ and $a \in \mathfrak{g}$. This property means that the vector field associated to any $a \in \mathfrak{g}$ equals to the symplectic gradient of the function $f_a(p) = \langle \mu(p), a \rangle : M \rightarrow \mathbb{C}$.

If an action has moment map it is said to be *Hamiltonian*, and the corresponding moment map is uniquely determined up to adding a constant in \mathfrak{g}^* . On the other hand given a moment map, one can recover the action of the Lie algebra and the Lie group on the manifold.

Example 14. Let $M = \mathbb{R}^2$ and the Lie group S^1 acts by rotations on M . It obviously preserves ω , and the corresponding moment map $\mu : \mathbb{R}^2 \rightarrow \mathbb{R}$ is defined as $\mu(x) = x_1^2 + x_2^2 + \text{const}$.

Example 15. Assume that M is a coadjoint orbit of G . There exists a unique symplectic structure on M (called Kirillov-Konstant-Souriau) such that the moment map is the embedding

$$\mu : M \rightarrow \mathfrak{u}(n)^*, \quad \mu : x \mapsto x.$$

Example 16. Assume the $M = M_{n \times k}(\mathbb{C})$ with the symplectic form $\omega(A, B) = \text{tr}(A^*B) - \text{tr}(B^*A)$, and with the natural action of $U(k)$ which obviously preserves the symplectic structure. Then $\mu : M \rightarrow \mathfrak{u}(k)^*$ is given by $\mu : A \mapsto \frac{i}{2}AA^* + \text{const} \cdot I$.

An important property of symplectic manifolds is that having two manifolds (M_1, ω_1) and (M_2, ω_2) one can form the product manifold $M_1 \times M_2$ with symplectic structure $\pi_1^*\omega_1 + \pi_2^*\omega_2$, where $\pi_i : M_1 \times M_2 \rightarrow M_i$ is the projection onto i -th factor, and π_i^* is the pull-back of π_i .

Assume that M_1 and M_2 are symplectic manifolds with the action of the same Lie group G and corresponding moment maps μ_1 and μ_2 . Then the symplectic manifold $M_1 \times M_2$ possesses the diagonal action of G with the moment map

$$\mu : M_1 \times M_2 \rightarrow \mathfrak{g}^*, \quad \mu : (x, y) \mapsto \mu_1(x) + \mu_2(y).$$

Example 17. Let $M = \mathbb{R}^{2n}$ with the action of S^1 . Then $\mu : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ is defined as $\mu(x) = x_1^2 + \dots + x_{2n}^2 + \text{const}$.

Example 18. Consider the product $\mathcal{O}_{\lambda^{(1)}} \times \dots \times \mathcal{O}_{\lambda^{(n)}}$ of coadjoint orbits of $U(n)$. Moment map takes the set of Hermitian matrices (A_1, \dots, A_n) with $\sigma(A_i) = \lambda^{(i)}$ to their sum:

$$\mu : (A_1, \dots, A_n) \mapsto A_1 + \dots + A_n + \text{const} \cdot I.$$

3.4. Symplectic quotient

Assume that M is a manifold with Hamiltonian action of a Lie group G and with the corresponding moment map $\mu : M \rightarrow \mathfrak{g}^*$. Then G acts on the fiber $\mu^{-1}(0) \subset M$. The following theorem says that the corresponding orbit space has the structure of a symplectic manifold.

Theorem 1. (Marsden-Weinstein) *The quotient space $\mu^{-1}(0)/G$ is a symplectic manifold. If the action of G is free on $\mu^{-1}(0)$ then $\mu^{-1}(0)/G$ has dimension $\dim M - 2 \dim G$.*

The manifold $\mu^{-1}(0)/G$ is called the *symplectic quotient*.

Example 19. Consider the manifold and moment map as in Example 17. The constant in the moment map can be chosen to be -1 , hence $\mu^{-1}(0)$ is the sphere S^{2n-1} and the symplectic quotient is homeomorphic to $\mathbb{C}\mathbb{P}^{n-1}$ (Hopf fibration).

Example 20. Let M be the manifold as in Example 16. We choose the constant in the moment map to be equal to $-\frac{i}{2}$. Then $\mu^{-1}(0) = \{A \in M_{n \times k}(\mathbb{C}) \mid AA^* = I\}$ and the quotient $\mu^{-1}(0)/U(k)$ is the set of k -dimensional subspaces in n -dimensional space; that is, $\text{Gr}(k, n)$.

4. Quick GIT. Stability conditions and Kirwan-Ness theorem

Assume that M is a variety and G is a linear algebraic group acting on M . Group G is assumed to be *reductive*; that is, it is a complexification of some compact Lie group. Geometric Invariant Theory tries to build the quotient of M by G . The main problem is that the quotient M/G may not exist in the category of algebraic or projective varieties (especially when the group G is not finite). One of the possible solution is to remove some points from M , by taking an open subset M' of M as large as possible such that M'/G is a variety. The following elementary example explains the idea. The space $\mathbb{C}\mathbb{P}^n$ can be described as the GIT quotient of the space \mathbb{C}^{n+1} by \mathbb{C}^* , namely $\mathbb{C}\mathbb{P}^n = (\mathbb{C}^{n+1} - 0)/\mathbb{C}^*$.

4.1. Projective spectrum and GIT quotients

Let $R = \bigoplus_{k \in \mathbb{N}} R_k$ be a *graded ring* (the product of an element from R_n with an element from R_m lies in R_{n+m}). The *projective spectrum* of the ring R (denoted by $\text{Proj}(R)$) is defined as the set of maximal graded ideals of R . There is alternative definition in terms of ordinary spectrum of the ring. We have the projection $R \rightarrow R_0$, hence $\text{Spec}(R_0) \hookrightarrow \text{Spec}(R)$ ($\text{Spec}(\cdot)$ is contravariant). The group \mathbb{C}^* acts on R by rotating R_k 's. Then

$$\text{Proj}(R) = (\text{Spec}(R) \setminus \text{Spec}(R_0))/\mathbb{C}^*.$$

Example 21. Let $R = \mathbb{C}[x_1, \dots, x_n]$. Then $R_0 = \mathbb{C}$. Hence, $\text{Proj}(\mathbb{C}[x_1, \dots, x_n]) = (\mathbb{C}^n \setminus \{0\})/\mathbb{C}^*$ which is identified with $\mathbb{C}\mathbb{P}^{n-1}$.

A *polarization* L of M is a line bundle $L \rightarrow M$, equipped with the action of G . Graded ring $R(M)$ associated to M is defined as

$$R(M) = \bigoplus_{n \geq 0} \Gamma(M, L^n),$$

where L^n is the n -th tensor power of L and $\Gamma(M, L^n)$ is the set of G -invariant sections of L^n . The *GIT quotient* of M by G with respect to the line bundle L is defined as $\text{Proj}(R(M))$ and denoted by $M//G$.

A point $p \in M$ is:

- *semistable* if $s(p) > 0$ for some section $s \in \Gamma(M, L^n)$, in which $n > 0$;
- *polystable* if p is semistable and the orbit $\{g \cdot p \mid g \in G\}$ is closed;
- *stable* if p is polystable with a finite stabilizer;

- *unstable* if m is not semistable.

Let M^{ss} , M^{ps} , M^s and M^{us} be semistable, polystable, stable and unstable locus respectively. D.Mumford gave a numerical criterion to decide the type of the point $m \in M$ and proved that the GIT quotient $M//G$ can be identified with M^{ps}/G (see [23, Chapter 2] for the details).

Example 22. ([23, Chapter 8]) Let $M = (\mathbb{CP}^1)^n$ with $G = SL(2)$ acting diagonally. Then we have:

$$\begin{aligned} M^s &= \{(m_1, \dots, m_n) \in (\mathbb{CP}^1)^n \mid \text{at most } n/2 \text{ points equal}\}, \\ M^{ss} &= \{(m_1, \dots, m_n) \in (\mathbb{CP}^1)^n \mid \text{less than } n/2 \text{ points equal}\}, \\ M^{ps} - M^s &= \{(m_1, \dots, m_n) \in (\mathbb{CP}^1)^n \mid \#\{m_1, \dots, m_n\} = 2\}. \end{aligned}$$

4.2. Kirwan-Ness Theorem

Assume that M is a symplectic manifold with an action of a compact group G , which preserves symplectic form and has the moment map μ . Then there exists a unique continuation of the action of G to the action of $G^{\mathbb{C}}$ (its complexification). By [23], there exists an inclusion of $\mu^{-1}(0)$ into M^{ps} . The following theorem is a fundamental fact (proved by Kirwan and Ness independently) connecting the symplectic and the GIT quotients.

Theorem 2. (Kirwan-Ness) *The inclusion $\mu^{-1}(0)$ into M^{ps} induces a homeomorphism*

$$\mu^{-1}(0)/G \cong M//G^{\mathbb{C}}.$$

Example 23. Let $M = \mathbb{R}^{2n}$, and let S^1 acts by rotations on M . The complexification of S^1 is \mathbb{C}^* which acts by multiplication on M . The Kirwan-Ness theorem is just an identification between S^{2n-1}/S^1 and \mathbb{CP}^{n-1} .

Example 24. Consider the space $M = (\mathbb{CP}^1)^n$ with diagonal action of $G = SL(2)$. By Example 22, $M^s = M^{ss}$ if and only if n is odd. Take the maximal compact subgroup $SU(2)$ of G . Its dual Lie algebra can be identified with the real vector space \mathbb{R}^3 . The corresponding moment map takes the n -tuple of points on unit sphere to their sum in \mathbb{R}^3 . Then the fiber $\mu^{-1}(0)$ consists of the points with the center of gravity in the origin. It is not hard to check directly that $\mu^{-1}(0) \subset M^{ss}$. In the case when $n = 4$ it can be straightforwardly proved that $(\mathbb{CP}^1)^n//SL(2)$ and $\mu^{-1}(0)/SU(2)$ are isomorphic to a sphere \mathbb{CP}^1 without three removed points. We refer the reader to [23, Chapter 8] for the details.

Note that the correspondence in previous example is the same as the correspondence between Example 3 and Example 7.

5. Correspondence between unitary and linear representations of the posets

For a given poset \mathcal{P} fix a dimension vector $d = (d_0; d_i)_{i \in \mathcal{P}}$ and a weight $\chi = (\chi_0; \chi_i)_{i \in \mathcal{P}} \in \mathbb{N}^{|\mathcal{P}|+1}$, such that $\sum_{i \in \mathcal{P}} \chi_i d_i = \chi_0 d_0$. Let $\lambda^{(i)} = (\underbrace{\chi_i, \dots, \chi_i}_{d_i}, \underbrace{0, \dots, 0}_{d_0 - d_i})$, $i \in \mathcal{P}$. The variety

$R_{\mathcal{P}, d}$ is a subset of the product of Grassmannians $\prod_{i \in \mathcal{P}} \text{Gr}(d_i, d_0)$. Given an Hermitian

metric on \mathbb{C}^{d_0} one can view each $\text{Gr}(d_i, d_0)$ as the set of $d_0 \times d_0$ matrices $\chi_i P_i$, where P_i is the d_i -dimensional orthoprojection. Therefore $\text{Gr}(d_i, d_0)$ is identified with the coadjoint orbit $\mathcal{O}_{\lambda^{(i)}}$. Our first aim is to show how to embed the variety $R_{\mathcal{P},d}$ into some projective space so that its symplectic structure coincides with symplectic structure on the product of corresponding coadjoint orbits. We use slightly modified standard construction (see, for example, [6, Chapter 11] and [15]).

A standard way to embed $\text{Gr}(d_i, d_0)$ into a projective space is via Plucker embedding; that is, for an element $V_i \in \text{Gr}(d_i, d_0)$ we take its basis vectors a_j and wedge them together $a_1 \wedge \dots \wedge a_{d_i}$ obtaining an element of $\mathbb{P}(\wedge^{d_i} \mathbb{C}^{d_0})$ (this is well-defined because if we change a basis then $a_1 \wedge \dots \wedge a_{d_i}$ changes by a scalar).

If V is some vector space then one can form a symmetric tensor d -power of V denoted $\text{Sym}^d(V)$ (for $d < 0$ one takes $\text{Sym}^{-d}(V^*)$, for $d = 0$ we have $\text{Sym}^0(V) = \mathbb{C}$). We take the symmetric tensor χ_i -power of the space $\wedge^{d_i} \mathbb{C}^{d_0}$. By Veronese map, we embed the projective space $\mathbb{P}(V)$ into the space $\mathbb{P}(\text{Sym}^{\chi_i}(V))$. Hence, we have the following sequences of inclusions:

$$\text{Gr}(d_i, d_0) \hookrightarrow \mathbb{P}(\wedge^{d_i} \mathbb{C}^{d_0}) \hookrightarrow \mathbb{P}(\text{Sym}^{\chi_i}(\wedge^{d_i} \mathbb{C}^{d_0})).$$

Now it is a routine to check that the symplectic form on $\text{Gr}(d_i, d_0)$ (taken as the restriction of the Fubiny-Study form on $\mathbb{P}(\text{Sym}^{\chi_i}(\wedge^{d_i} \mathbb{C}^{d_0}))$) coincides with the symplectic form on the corresponding coadjoint orbit $\mathcal{O}_{\lambda^{(i)}}$.

Correspondingly for the product of Grassmanians $\prod_{i \in \mathcal{P}} \text{Gr}(d_i, d_0)$ we have the embedding

$$\prod_{i \in \mathcal{P}} \text{Gr}(d_i, d_0) \hookrightarrow \prod_{i \in \mathcal{P}} \mathbb{P}(\text{Sym}^{\chi_i}(\wedge^{d_i} \mathbb{C}^{d_0})).$$

Using the Segre map $\mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{(n+1)(m+1)-1}$ we embed the last product into

$$\mathbb{P} \left(\prod_{i \in \mathcal{P}} \text{Sym}^{\chi_i}(\wedge^{d_i} \mathbb{C}^{d_0}) \right).$$

Consider the moment map of the action of $U(d_0)$ on $\prod_{i \in \mathcal{P}} \text{Gr}(d_i, d_0)$ after this embedding. As we mentioned above the Fubiny-Study form on $\mathbb{P}(\prod_{i \in \mathcal{P}} \text{Sym}^{\chi_i}(\wedge^{d_i} \mathbb{C}^{d_0}))$ coincides with the symplectic form on the product of coadjoint orbits $\prod_{i \in \mathcal{P}} \mathcal{O}_{\lambda^{(i)}}$. Knowing the form of the moment map $\mu : (P_i)_{i \in \mathcal{P}} \mapsto \mathfrak{u}(d_0)^*$ (see Example 18) and taking a constant in the moment map to be equal to $-\chi_0$ we have the following

$$\mu^{-1}(0) = \left\{ (P_i)_{i \in \mathcal{P}} \in (M_{d_0}(\mathbb{C}))_{i \in \mathcal{P}} \mid \begin{array}{l} P_i = P_i^* = P_i^2, \text{rank}(P_i) = d_i, \quad i \in \mathcal{P}, \\ P_i P_j = P_i P_j = P_i, \quad i \prec j, \quad \sum_{i \in \mathcal{P}} \chi_i P_i = \chi_0 I \end{array} \right\}.$$

The continuation of the action of $U(d_0)$ to its complexification $GL(d_0)$ coincides with the action of $GL(d_0)$ on $R_{\mathcal{P},d}$. Using Muformd's numerical criterion it was shown in [6, Chapter 11] and [9, Theorem 2.2] the set of stable representations with respect to induced action of $GL(d_0)$ consists of those representations $(V; V_i)_{i \in \mathcal{P}}$ that satisfy

$$\frac{1}{\dim K} \sum_{i \in \mathcal{P}} \chi_i \dim(V_i \cap K) < \frac{1}{\dim V} \sum_{i \in \mathcal{P}} \chi_i \dim V_i$$

for each proper subspace $K \subset V$.

Straightforward calculations show that if $(P_i)_{i \in \mathcal{P}} \in \mu^{-1}(0)$ then the system of subspaces $(\mathbb{C}^{d_0}; \text{Im} P_i)_{i \in \mathcal{P}}$ is χ -polystable; that is, it decomposes into a direct sum of stable representations $(\tilde{V}; \tilde{V}_i)_{i \in \mathcal{P}}$ which satisfy $\sum_{i \in \mathcal{P}} \chi_i \dim \tilde{V}_i = \chi_0 \dim \tilde{V}_0$. Denote by Φ the corresponding map from $Ob(\mathcal{P}_\chi - usp)$ to $Ob(\mathcal{P} - sp)$. As a consequence of Kirwan-Ness theorem we have the following:

Theorem 3. *Let \mathcal{P} be a poset, $(d_0; d_i)_{i \in \mathcal{P}}$ be a dimension vector and $\chi = (\chi_0; \chi_i)_{i \in \mathcal{P}}$ be a weight such that $\sum_{i \in \mathcal{P}} \chi_i d_i = \chi_0 d_0$. Then Φ induces a homeomorphism (with respect to the usual orbit-space topology) between*

$$\left\{ (P_i)_{i \in \mathcal{P}} \in (M_{d_0}(\mathbb{C}))_{i \in \mathcal{P}} \mid \begin{array}{l} P_i = P_i^* = P_i^2, \text{ rank}(P_i) = d_i, \quad i \in \mathcal{P}, \\ P_i P_j = P_i P_j = P_i, \quad i \prec j, \quad \sum_{i \in \mathcal{P}} \chi_i P_i = \chi_0 I \end{array} \right\} / U(d_0)$$

and

$$\left\{ (V; V_i)_{i \in \mathcal{P}} \in R_{\mathcal{P}, d} \mid (V; V_i)_{i \in \mathcal{P}} \text{ is } \chi\text{-polystable} \right\} / GL(d_0).$$

Let us show some consequences of this theorem.

- (i) In [18] it was shown that if two objects in $\mathcal{P}_\chi - usp$ are equivalent as the objects in $\mathcal{P} - sp$ then they are equivalent in $\mathcal{P}_\chi - usp$. This follows as a consequence of Theorem 3.
- (ii) In [18] it was shown that each indecomposable object in $\mathcal{P}_\chi - usp$ is Schurian; that is, it has trivial endomorphism ring. In fact this result can be deduced from the previous observations. Indeed, each indecomposable object in $\mathcal{P}_\chi - usp$ corresponds to some χ -stable object in $\mathcal{P} - sp$. But stable objects are Schurian.

6. Calculating the dimensions of the quotients

6.1. Bound quivers

For a finite quiver Q denote by Q_0 the set of its vertices and by Q_1 the set of its arrows denoted by $\rho : i \rightarrow j$ for $i, j \in Q_0$. In the following we only consider quivers without oriented cycles. A finite-dimensional *representation* of Q is given by a tuple

$$X = ((X_i)_{i \in Q_0}, (X_\rho)_{\rho \in Q_1} : X_i \rightarrow X_j)$$

of finite-dimensional vector spaces and linear maps between them. We say that X is *subspace representation* if all maps X_ρ are injective. The dimension vector $\underline{\dim} X \in \mathbb{N}^{|Q_0|}$ of X is defined by

$$\underline{\dim} X = (\dim X_i)_{i \in Q_0}.$$

The variety $R_{Q, d}$ of representations of Q with the dimension vector $d \in \mathbb{N}^{|Q_0|}$ is defined as the affine complex space

$$R_{Q, d} = \bigoplus_{\rho: i \rightarrow j} \text{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j}).$$

The algebraic group $GL_d = \prod_{i \in Q_0} GL(d_i)$ acts on $R_{Q,d}$ via simultaneous base change; that is,

$$(g_i)_{i \in Q_0} \cdot (X_\rho)_{\rho \in Q_1} = (g_j X_\rho g_i^{-1})_{\rho: i \rightarrow j}.$$

The orbits of this action are in a bijection with the isomorphism classes of representations of Q with the dimension vector d .

Let $\mathbb{C}Q$ be the path algebra of Q (see, for example, [2, Chapter 2] for the definition) and let RQ be the arrow ideal. A relation in Q is a \mathbb{C} -linear combination of paths of length at least two which have the same head and tail. For a set of relations $(r_j)_{j \in J}$ we can consider the admissible ideal I generated by these relations, that means that $RQ^m \subseteq I \subseteq RQ^2$ for some $m \geq 2$. We say that a representation X of Q is *bound* by I (or a representation of the *bound quiver* (Q, I)) if $X_{r_j} = 0$ for all $j \in J$. For every dimension vector this defines a closed subvariety of $R_{Q,d}$ denoted by $R_{(Q,I),d}$. Fixing a minimal set of relations generating I , we denote by $r(i, j, I)$ the number of relations with starting vertex i and terminating vertex j . Following [3], for the dimension of $R_{(Q,I),d}$ we have the following lower bound

$$\dim R_{(Q,I),d} \geq \dim R_{Q,d} - \sum_{(i,j) \in (Q_0)^2} r(i, j, I) d_i d_j.$$

Let $C_{(Q,I)}$ be the Cartan matrix of (Q, I) ; that is, $c_{j,i} = \dim e_i(\mathbb{C}Q/I)e_j$ where e_i denotes the primitive idempotent (resp. the trivial path) corresponding to the vertex i . On $\mathbb{Z}^{|Q_0|}$ a non-symmetric bilinear form is defined by

$$\langle d, \tilde{d} \rangle := d^t (C_{(Q,I)}^{-1})^t \tilde{d}.$$

For two representation X and Y we have the following homological interpretation of this form:

$$\langle X, Y \rangle := \langle \underline{\dim} X, \underline{\dim} Y \rangle = \sum_{i=0}^{\infty} (-1)^i \dim \text{Ext}^i(X, Y).$$

If Q is unbound, for two representations X, Y of Q with $\underline{\dim} X = d$ and $\underline{\dim} Y = \tilde{d}$ we have $\text{Ext}^i(X, Y) = 0$ for $i \geq 2$ and

$$\langle X, Y \rangle = \dim \text{Hom}(X, Y) - \dim \text{Ext}(X, Y) = \sum_{q \in Q_0} d_q \tilde{d}_q - \sum_{\rho: i \rightarrow j \in Q_1} d_i \tilde{d}_j.$$

6.2. Bound quiver and posets

Recall briefly the relation between representations of posets and representations of bound quivers.

Let \mathcal{P} be a poset and $Q^{\mathcal{P}}$ its Hasse quiver. All arrows of $Q^{\mathcal{P}}$ are oriented to one vertex q_0 which is called the root. Let $d \in \mathbb{N}^{|\mathcal{P}|+1}$ be a dimension vector. By $S_{Q^{\mathcal{P}},d} \subset R_{Q^{\mathcal{P}},d}$ we denote the open subvariety of subspace representations. For every (non-oriented) cycle $\rho_1 \dots \rho_n \tau_k^{-1} \dots \tau_1^{-1}$ with $\rho_i, \tau_j \in Q_1$ and $\rho_i \neq \tau_j$ we define a relation

$$r = \rho_1 \dots \rho_n - \tau_1 \dots \tau_k.$$

Let I be the ideal generated by all such relations.

Let $\pi = (V; V_i)_{i \in \mathcal{P}}$ be a representation of \mathcal{P} with the dimension vector d . This defines a representation $X(\pi) \in S_{(Q^{\mathcal{P}}, I), d}$ satisfying the stated relations. Indeed, every inclusion $V_i \subset V_j$ defines an injective map $X(\pi)_{\rho: i \rightarrow j} : V_i \rightarrow V_j$. Thus it defines an element of $S_{(Q^{\mathcal{P}}, I), d}$. For two arbitrary representations $\pi = (V; V_i)_{i \in \mathcal{P}}$ and $\pi' = (W; W_i)_{i \in \mathcal{P}}$ a morphism $f : \pi \rightarrow \pi'$ induces a morphism $X(f) : X(\pi) \rightarrow X(\pi')$, where $X(f)_i := f|_{V_i} : X(V)_i \rightarrow X(W)_i$.

Conversely, let $X \in S_{(Q^{\mathcal{P}}, I), d}$. This gives rise to a representation $\pi(X)$ of \mathcal{P} by defining $\pi(X)_q = X_{\rho_n^q} \circ \dots \circ X_{\rho_1^q}(X_q)$ for some path $p_q = \rho_1^q \dots \rho_n^q$ from q to q_0 . This definition is independent of the chosen path. Moreover, every morphism $\varphi = (\varphi_q)_{q \in Q_0} : X \rightarrow Y$ defines a morphism $\pi(\varphi)$ which is induced by $\varphi_{q_0} : X_0 \rightarrow Y_0$.

Therefore we get an equivalence between the categories of representations of \mathcal{P} and subspace representations of $Q^{\mathcal{P}}$ bound by I . This equivalence also preserves dimension vectors.

6.3. GIT quotient of quivers

A representation X of (Q, I) is *semistable* (resp. *stable*) with respect to a form $\Theta \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}^{|Q_0|}, \mathbb{Z})$ if $\Theta(\underline{\dim} X) = 0$ and for all subrepresentations $Y \subset X$ (resp. all proper subrepresentations $0 \neq Y \subsetneq X$) we have:

$$\Theta(\underline{\dim} Y) \geq 0 \quad (\text{resp. } \Theta(\underline{\dim} Y) > 0).$$

Denote the set of semistable (resp. stable) points by $R_{(Q, I), d}^{ss}$ (resp. $R_{(Q, I), d}^s$). A King in [10] built the GIT quotient $R_{(Q, I), d} // GL_d$ for quivers. This quotient parametrizes polystable representations. For a stable representation X we have that its orbit is of maximal possible dimension. The scalar matrices act trivially on $R_{(Q, I), d}$, hence the isotropy group is one-dimensional. Therefore, if the quotient $R_{(Q, I), d} // GL_d$ is not empty, for its dimension we have the lower bound

$$\begin{aligned} \dim R_{(Q, I), d} // GL_d &= \dim R_{(Q, I), d} - (\dim G_d - 1) \\ &\geq 1 - \sum_{i \in Q} d_i^2 + \sum_{\rho: i \rightarrow j \in Q_1} d_i d_j - \sum_{(i, j) \in (Q_0)^2} r(i, j, I) d_i d_j. \end{aligned}$$

In the case $I = 0$ we have $\dim R_{(Q, I), d} // GL_d = 1 - \langle d, d \rangle$.

6.4. Dimension of symplectic quotients of the posets

Assume that the representation $\pi = (V; V_i)_{i \in \mathcal{P}}$ is stable with the weight $(\chi_0; \chi_i)_{i \in \mathcal{P}} \in \mathbb{Z}^{|\mathcal{P}|+1}$; that is

$$\frac{1}{\dim K} \sum_{i \in \mathcal{P}} \chi_i \dim(V_i \cap K) < \frac{1}{\dim V} \sum_{i \in \mathcal{P}} \chi_i \dim V_i$$

for each proper subspace $K \subset V$ and $\sum_{i \in \mathcal{P}} \chi_i \dim V_i = \chi_0 \dim V$. Then the corresponding representation $X(\pi)$ of the bound quiver $Q^{\mathcal{P}}$ is stable with respect to the form $\Theta = (\chi_0; -\chi_i)_{i \in \mathcal{P}}$ and vice versa. Therefore, using Theorem 3 we have that if

there exists at least one object in $\mathcal{P}_\chi - usp$ with dimension vector $(d_0; d_i)_{i \in \mathcal{P}}$ then the unitary classes of all objects with this dimension vector depend on at least

$$1 - \sum_{i \in Q_0^{\mathcal{P}}} d_i^2 + \sum_{\rho: i \rightarrow j \in Q_1^{\mathcal{P}}} d_i d_j - \sum_{(i,j) \in (Q_0^{\mathcal{P}})^2} r(i, j, I) d_i d_j.$$

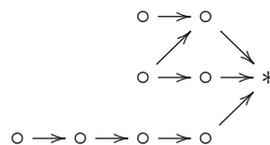
complex parameters. When the poset is primitive we have that the number of parameters is $1 - \sum_{i \in Q_0^{\mathcal{P}}} d_i^2 + \sum_{\rho: i \rightarrow j \in Q_1^{\mathcal{P}}} d_i d_j = 1 - \langle d, d \rangle$.

Example 25. Consider the case when \mathcal{P} is primitive and representation-tame. Hence the corresponding quiver $Q^{\mathcal{P}}$ is one of the affine Dynkin quivers; that is, $Q^{\mathcal{P}} = \tilde{D}_4, \tilde{E}_6, \tilde{E}_7$ or \tilde{E}_8 . Let us take the dimension vector $d_{\mathcal{P}}$ to be the minimal imaginary root of $Q^{\mathcal{P}}$:

$$\begin{aligned} d_{(1,1,1,1)} &= (2; 1, 1, 1, 1), \\ d_{(2,2,2)} &= (3; 1, 2; 1, 2; 1, 2), \\ d_{(1,3,3)} &= (4; 2; 1, 2, 3; 1, 2, 3), \\ d_{(1,2,5)} &= (6; 3; 2, 4; 1, 2, 3, 4, 5). \end{aligned}$$

Using Theorem 3 and the dimension formula we have at most $1 - \langle d_{\mathcal{P}}, d_{\mathcal{P}} \rangle = 1$ -parametric family of uni-classes of indecomposable objects in $\mathcal{P}_\chi - usp$ with dimension $d_{\mathcal{P}}$. One can check that it is always possible to choose the weight $\chi_{\mathcal{P}}$ such that there exists at least one stable representation, hence for these weights set of uni-classes of orthoscalar representations of \mathcal{P} in dimension $d_{\mathcal{P}}$ is isomorphic to $\mathbb{C}P^1$ (because the GIT quotient is one-dimensional and rational).

Example 26. Consider representation-tame non-primitive critical poset $\mathcal{P} = (N, 4)$, which has the following Hasse quiver



Fix the dimension vector $d_{(N,4)} = (5; 2, 4, 3, 2; 1, 2, 3, 4)$. One can check that with respect to the weight $\chi = (5; 2, 1, 1, 2; 1, 1, 1, 1)$ there exists at least one stable representation in dimension $d_{(N,4)}$, hence the corresponding GIT quotient is not empty. Due to the dimension formula we have at least $1 - \langle d_{\mathcal{P}}, d_{\mathcal{P}} \rangle + 2 \cdot 5 = 1$ -parametric family of uni-classes of representation of $\mathcal{P}_\chi - usp$. Moreover, it is possible to show that this is a maximal number of parameters. Therefore, we have that for the weight $\chi = (5; 2, 1, 1, 2; 1, 1, 1, 1)$ the set of uni-classes of orthoscalar representations of \mathcal{P} in dimension $d_{(N,4)}$ is isomorphic to $\mathbb{C}P^1$.

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