RT-MAT 2005 - 25

Šilov boundary for holomorphic functions on some classical Banach spaces

María D. Acosta
and
Mary Lílian Lourenço

Dezembro 2005

ŠILOV BOUNDARY FOR HOLOMORPHIC FUNCTIONS ON SOME CLASSICAL BANACH SPACES

MARÍA D. ACOSTA AND MARY LILIAN LOURENÇO

ABSTRACT. Let $\mathcal{A}_\infty(B_X)$ be the Banach space of all bounded and continuous functions on the closed unit ball B_X of a complex Banach space X and holomorphic on the open unit ball, with sup norm, and let $\mathcal{A}_u(B_X)$ be the subspace of $\mathcal{A}_\infty(B_X)$ of those functions which are uniformly continuous on B_X. A subset $B \subset B_X$ is a boundary for $\mathcal{A}_\infty(B_X)$ if $\|f\| = \sup_{x\in B} |f(x)|$ for every $f \in \mathcal{A}_\infty(B_X)$. We prove that for the cases $X = d(w, 1)$ (the Lorentz sequence space) and for $X = C_1(H)$, the trace class operators, there is a minimal closed boundary for $\mathcal{A}_\infty(B_X)$. For a change, for the cases $X = \mathcal{S}$, the Schreier space and $X = K(\ell_p, \ell_q)$ $(1 \le p \le q < \infty)$, there is no minimal closed boundary for the corresponding spaces of holomorphic functions.

1. INTRODUCTION

A result of Šilov asserts that if $\mathfrak{A}$ is a unital separating algebra of $\mathcal{C}(K)$ (K is a compact Hausdorff topological space), there is a smallest closed subset $S \subset K$ such that every function of $\mathfrak{A}$ attains its norm at some point of S [6, Theorem I.4.2]. Bishop [4] proved that if K is metrizable, in fact, there is a minimal subset of K satisfying the above condition for every separating algebra of $\mathcal{C}(K)$. That subset is the set of peaks points for $\mathfrak{A}$ (see definition below).

Globevnik introduced the corresponding concepts for a subalgebra of $\mathcal{C}_b(\Omega)$, the space of bounded continuous functions on a topological space Ω not necessarily compact [9]. In fact, he considered the case $\Omega = B_X$, where X is a Banach space. If $\mathfrak{A}$ is a subspace of $\mathcal{C}_b(\Omega)$, we will say that a subset $B \subset \Omega$ is a boundary for $\mathfrak{A}$ if

$$\|f\| = \sup_{b\in B} |f(b)|, \quad \forall f \in \mathfrak{A} .$$

If there is a minimal closed boundary B for $\mathfrak{A}$, we will say that B is the Šilov boundary of $\mathfrak{A}$.

If X is a complex Banach space, we will denote by $\mathcal{A}_u(B_X)$ the space of uniformly continuous functions on the closed unit ball of X which are holomorphic on

1991 *Mathematics Subject Classification.* Primary 46G20, 46J20 Secondary 46B99.

Key words and phrases. holomorphic function, boundary, Šilov boundary.

The first author was supported in part by D.G.E.S., project BFM 2003-01681. The second author was partially supported by CNPq (project $n^{\underline{o}}$ 472416/2004-9).

the open unit ball. Globevnik [9] described the boundaries of $\mathcal{A}_u(B_{c_0})$. As a consequence of the description, he showed that this algebra has no Šhilov boundary. Aron, Choi, Lourenço and Paques [3] gave examples of boundaries for $\mathcal{A}_u(B_{\ell_\infty})$ and proved that there is no Šilov boundary for this algebra. They also showed that the unit sphere of ℓ_1 is the Šilov boundary for $\mathcal{A}_u(B_{\ell_1})$.

Moraes and Romero [14] gave a characterization of the boundaries of $\mathcal{A}_u(B_{d_*(w,1)})$, where $d_*(w,1)$ is the canonical predual of the Lorentz sequence space $d(w,1)$ when $w = \{\frac{1}{n}\}$. Later Acosta, Moraes and Romero [2] generalized that characterization proving it for any space $d_*(w,1)$ and obtained another one in terms of the strong peak sets of the unit ball. In this case, there is no Šilov boundary. Choi, García, Kim and Maestre [5] proved that there is no Šilov boundary for $\mathcal{A}_u(B_{C(K)})$, when K is infinite and scattered. Acosta showed the same result for every infinite K and also proved that for this space the subset of extreme points of the unit ball of $C(K)$ is a boundary for $\mathcal{A}_u(B_{C(K)})$ [1].

Before going on it is convenient to recall some definitions. Let $\mathcal{A}$ be a function space on a metric space Ω. An element $y \in \Omega$ is called a **peak point** for $\mathcal{A}$ if there is some $f \in \mathcal{A}$ such that $f(y) = 1$ and $|f(x)| < 1$, $\forall x \in \Omega \backslash \{y\}$. In this case we say that f **peaks** at y. An element $y \in \Omega$ is called a **strong peak point** for $\mathcal{A}$ if there is some $f \in \mathcal{A}$ satisfying $f(y) = 1$ and such that given any $\varepsilon > 0$ there is some $\delta > 0$ such that $\operatorname{dist}(x, y) > \varepsilon$ implies that $|f(x)| < 1 - \delta$. It is clear that every closed boundary for $\mathcal{A}$ contains all the strong peak points.

In this paper we prove that there is no Šilov boundary for $\mathcal{A}_u(B_X)$ when X is the Schreier space or the space $K(\ell_p, \ell_q)$ $(1 \leq p \leq q < \infty)$. For the spaces $X = C_1(H)$, the trace class operators on a complex Hilbert space H or $X = d(w,1)$, there is Šilov boundary for $\mathcal{A}_u(B_X)$. In fact, all the points in the unit sphere of $d(w,1)$ are strong peak points for $\mathcal{A}_u(B_{d(w,1)})$ and so, in this case the Šilov boundary is the unit sphere. For ℓ_1 the same result also holds. That fact was proved in [3] for the finite supported sequences in the unit sphere. If K is infinite, we also prove that there is no strong peak points for $\mathcal{A}_u(B_{C(K)})$. The subset of the peak points for $\mathcal{A}_u(B_{C(K)})$ is the set of extreme points of $B_{C(K)}$ if K is separable.

Throughout this paper, all the Banach spaces considered are complex. For a Banach space X, B_X and S_X will be the closed unit ball and the unit sphere of X, respectively. We will denote by $\mathcal{A}_\infty(B_X)$, the Banach space of all bounded and continuous functions on B_X which are holomorphic on the open unit ball and $\mathcal{A}_u(B_X)$ the space of all functions in $\mathcal{A}_\infty(B_X)$ which are uniformly continuous.

2. Existence of Šilov boundary on the Lorentz sequence space

Given a decreasing sequence w of positive real numbers satisfying $w \in c_0 \backslash \ell_1$, the complex Lorentz sequence space $d(w,1)$ is given by

$$d(w,1) = \left\{x : \mathbb{N} \longrightarrow \mathbb{C} : \sup\left\{\sum_{n=1}^{\infty} |x(\sigma(n))| w_n : \sigma : \mathbb{N} \longrightarrow \mathbb{N} \text{ injective}\right\} < +\infty\right\}.$$

The norm is given by

$$\|x\| = \sup\left\{\sum_{n=1}^{\infty} w_n |x(\sigma(n))| : \sigma : \mathbb{N} \longrightarrow \mathbb{N} \text{ injective}\right\} \qquad (x \in d(w,1)).$$

It is well-known and easy to verify that the above supremum is attained for the decreasing rearrangement of x. The usual vector basis $\{e_n\}$ is a monotone Schauder basis (see [12]).

A canonical predual $d_*(w,1)$ of $d(w,1)$ is given by

$$d_*(w,1) = \left\{x \in c_0 : \lim_n \frac{\sum_1^n x^*(k)}{W_n} = 0\right\}$$

where $W_n = \sum_1^n w_k$ and x^* is the decreasing rearrangement of x. This space is a Banach space endowed with the norm

$$\|x\| = \sup_n \left\{\frac{\sum_1^n x^*(k)}{W_n}\right\}.$$

(see [16] and [7]). $d_*(w,1)$ has a Schauder basis whose sequence of biorthogonal functionals is, in fact, the canonical basis of $d(w,1)$.

We begin presenting some useful lemmas.

Lemma 2.1. *If $\{z_n\}$ is a bounded sequence of complex numbers such that the sequence $\{1+|z_n|-|1+z_n|\}$ converges to zero, then the sequence $\{|z_n|-z_n\}$ also converges to zero.*

Proof. We consider the following identity for a complex number z

$$\begin{gathered}(1+|z|-|1+z|)^2 = \\ 1+|z|^2+2|z|+|1+z|^2-2(1+|z|)\,|1+z| = \\ 2(\operatorname{Re} z-|z|)+2(1+|z|)(1+|z|-|1+z|)\ .\end{gathered}$$

If we apply the above identity to the sequence $\{z_n\}$ and use the assumption, we obtain that the sequence $\{|z_n| - \operatorname{Re} z_n\}$ converges to zero.

Now if we consider the following expression

$$\begin{gathered}(|z| - \operatorname{Re} z)^2 = \\ 2\operatorname{Re}^2 z + \operatorname{Im}^2 z - 2|z|\operatorname{Re} z = \\ = \operatorname{Im}^2 z + 2(\operatorname{Re} z - |z|)\operatorname{Re} z\ ,\end{gathered}$$

and we apply the identity to the sequence $\{z_n\}$, we deduce that $\{\operatorname{Im} z_n\} \to 0$. Hence

$$\{|z_n| - z_n\} = \{|z_n| - \operatorname{Re} z_n - i \operatorname{Im} z_n\} \to 0 .$$

■

Lemma 2.2. ([3, Lemma 9]) *Let $0 < a < 1$. The real valued function given by*

$$g_a(x) = \left(1 + \frac{x}{1-a}\right)\left(1 + \frac{1-x}{a}\right) \qquad (x \in \mathbb{R})$$

attains its maximum at $x = a$ and

$$g_a(x) < g_a(a) = \frac{1}{a(1-a)}, \quad \forall x \in \mathbb{R}\backslash\{a\}.$$

Lemma 2.3. *The subset of peak points in S_X for $\mathcal{A}_\infty(B_X)$ is invariant under surjective linear isometries on X. The same result also holds for the subset of strong peak points in S_X.*

Theorem 2.4. *The set of strong peak points for the space of the polynomials of degree less or equal than 2 on $d(w,1)$ contains the unit sphere of $d(w,1)$.*

Proof. Let $y_0 \in S_{d(w,1)}$. By Lemma 2.3 we can assume that supp y_0 is an interval of positive integers containing $\{1\}$ and it is also satisfied that

(1) $\qquad y_0(j) \in \mathbb{R}^+, \ \forall j \in \text{supp } y_0, \quad \text{and} \quad y_0(n) \geq y_0(n+1) \ \ \forall n \in \mathbb{N}.$

We will prove that y_0 is a strong peak for $\mathcal{A}_u(d(w,1))$.

If the support of y_0 contains just one element, then $y_0 = e_1$ and it is sufficient to consider the 1-degree polynomial given by

$$f(x) = 1 + x(1) \qquad (x \in d(w,1)).$$

Clearly $\|f\| = 2 = f(y_0)$. By using that in $S_{d(w,1)}$ the weak and the $\sigma(d(w,1), d_*(w,1))$ convergence coincide ([16, Proposition 2.2] and [10, Corollary III.2.15]) and that every point of the unit sphere is a point of continuity of the unit ball [13, Proposition 4], then it is easily checked that f strongly peaks the unit ball at y_0.

Otherwise we will assume that $J :=$ supp y_0 has at least two elements. Since $\|y_0\| = 1$, by (1), we know that $\sum_{i\in J} w_i y_0(i) = 1$ and so $0 < w_i y_0(i) < 1$ for every $i \in J$.

For every $k \in J$ we define the function given by

$$f_k(x) = \frac{1}{M_k}\left(1 + \frac{w_k x(k)}{1 - w_k y_0(k)}\right)\left(1 + \frac{1}{w_k y_0(k)} \sum_{j \in J\backslash\{k\}} w_j x(j)\right) \qquad (x \in d(w,1)) ,$$

where $M_k = \dfrac{1}{w_k y_0(k)(1 - w_k y_0(k))}$. Then f_k is clearly a non-homogeneous polynomial on $d(w,1)$ with degree 2 and satisfies that $f_k(y_0) = 1$. We will check that $\|f_k\| = 1$.

If $x \in B_{d(w,1)}$, then it is satisfied that

(2)

$$|f_k(x)| = \frac{1}{M_k}\left|1 + \frac{w_k x(k)}{1 - w_k y_0(k)}\right| \left|1 + \frac{1}{w_k y_0(k)} \sum_{j \in J\setminus\{k\}} w_j x(j)\right| \leq$$

$$\frac{1}{M_k}\left(1 + \frac{w_k |x(k)|}{1 - w_k y_0(k)}\right)\left(1 + \frac{1}{w_k y_0(k)} \sum_{j\in J\setminus\{k\}} |w_j x(j)|\right) \leq \quad (\text{since } x \in B_X)$$

$$\leq \frac{1}{M_k}\left(1 + \frac{w_k|x(k)|}{1 - w_k y_0(k)}\right)\left(1 + \frac{1 - w_k|x(k)|}{w_k y_0(k)}\right) \leq \quad (\text{by Lemma 2.2})$$

$$\leq \frac{1}{M_k}\left(1 + \frac{w_k y_0(k)}{1 - w_k y_0(k)}\right)\left(1 + \frac{1 - w_k y_0(k)}{w_k y_0(k)}\right) = 1$$

Hence $\|f_k\| = 1$.

Our intention is to show that y_0 is a strong peak for the space of 2-degree polynomials. For this reason, we will prove the following

$$x_n \in B_{d(w,1)},\ \forall n, \quad \{|f_k(x_n)|\}_n \to 1 \Rightarrow \{x_n(k)\} \to y_0(k)\ . \tag{3}$$

For every k fixed, we will write $u_n = \frac{w_k x_n(k)}{1 - w_k y_0(k)}$ and $v_n = \sum_{\substack{j\in J\\ j\neq k}} \frac{w_j x_n(j)}{w_k y_0(k)}$.

We rewrite the inequality (2) in terms of the above sequences

$$|f_k(x_n)| = \frac{1}{M_k}|1 + u_n|\ |1 + v_n| \leq$$

$$\frac{1}{M_k}(1 + |u_n|)\ (1 + |v_n|) \leq 1\ .$$

If we assume that $\{f_k(x_n)\}_n \to 1$, then, let us note that the sequence $\{1 + v_n\}$ has no subsequence converging to zero. Because of the above inequality we deduce that

$$\{|1 + u_n| - 1 - |u_n|\} \to 0\ .$$

Since k is fixed, in view of Lemma 2.1 that condition implies that $\{|u_n| - u_n\}$ converges to zero, that is, $\{|x_n(k)| - x_n(k)\}_n \to 0$. Also by Lemma 2.2, we know that

$$\{w_k|x_n(k)|\}_n \to w_k y_0(k)\ .$$

Hence we deduce that $\{x_n(k)\}_n \to y_0(k)$.

Now we choose a sequence $\{\alpha_n\}$ in ℓ_1 such that supp $\alpha = J$, $\alpha_n > 0, \forall n \in J$ and $\sum_{n\in J} \alpha_n = 1$. Let us define the function given by

$$f(x) = \sum_{k\in J} \alpha_k f_k(x) \quad (x \in B_{d(w,1)})\ .$$

Then f is a polynomial of degree at most 2 in $d(w,1)$ and satisfies $\|f\| \leq 1 = f(y_0)$.

We will prove now that this function strongly peaks the unit ball of $d(w,1)$ at y_0. So assume that for some sequence $\{x_n\}$ in the unit ball it happens that $\{|f(x_n)\}| \to 1$. That clearly implies that $\{f_k(x_n)\}_n \to 1$ for every $k \in J$.

Since $y_0 \in S_{d(w,1)}$, by condition (3), we know that $\{x_n\}$ converges pointwise to y_0. All the elements involved in the argument are in the unit ball of $d(w,1)$ and so the sequence $\{x_n\}$ converges to y_0 in the $\sigma(d(w,1), d_*(w,1))$-topology. Since $d_*(w,1)$ is M-ideal in its dual (see [16, Proposition 2.2] or [10, Examples III.1.4c]), in the unit ball of $d(w,1)$, the weak and weak*-topologies coincides on points of the unit sphere, in view of [10, Corollary III.2.15]. By applying that fact to the element y_0, which is the w^*-limit of the sequence $\{x_n\}$, then we obtain that in fact $\{x_n\}$ converges weakly to y_0. Since all the points of the unit sphere of $d(w,1)$ are points of continuity [13, Proposition 4], then we obtain that $\{x_n\}$ converges in norm to y_0 and y_0 is a strong peak point, as we wanted to show. ∎

Corollary 2.5. *The Šilov boundary for the space of 2-degree polynomials on $d(w,1)$ is $S_{d(w,1)}$. Hence $S_{d(w,1)}$ is also the Šilov boundary for $\mathcal{A}_u(B_{d(w,1)})$ and $\mathcal{A}_\infty(B_{d(w,1)})$.*

It is known that all the finite supported elements in S_{ℓ_1} are strong peak points for the space of 2-degree polynomials on ℓ_1 [3, Theorem 10]. We will extend such result.

Theorem 2.6. *S_{ℓ_1} is the subset of strong peak points for the space of 2-degree polynomials on ℓ_1.*

Proof. If $y_0 \in S_{\ell_1}$, by Lemma 2.3, we can assume that $y_0(n) \geq 0$ for every n. If $|\text{supp } y_0| = 1$ and $\{n\} = \text{supp } y_0$, the function $x \mapsto 1 + x(n)$ strongly peaks the unit ball of ℓ_1 at y_0. Otherwise, if we assume that $J := \text{supp } y_0$ satisfies that $\|J\| \geq 2$, then the 2-degree polynomial given by

$$f_k(x) := \frac{1}{y_0(k)(1-y_0(k))}\left(1 + \frac{x(k)}{1-y_0(k)}\right)\left(1 + \frac{\sum_{i\neq k} x(i)}{y_0(k)}\right) \quad (x \in \ell_1)$$

satisfies that $f(y_0) = 1$. In view of Lemma 2.1, it also satisfies that $\|f_k\| = 1$ and now we can follow the same argument in the proof of Theorem 2.4. ∎

3. Boundaries on the Schreier space and $C(K)$.

A subset $E = \{n_1 < n_2 < \cdots < n_k\}$ of the natural numbers $\mathbb{N}$ is said to be *admissible* if $k \leq n_1$. The Schreier space S is the completion of the space c_{00} of all scalar sequence of finite support with respect the norm $\|x\| = \sup \sum_{j \in E} |x_j|$, where the supremum is taken over all admissible sets of natural numbers E.

The following theorem shows in particular that the intersection of all boundaries for $\mathcal{A}_\infty(B_S)$ is empty.

Theorem 3.1. *Let S be the Schreier space and B a boundary for $\mathcal{A}_\infty(B_S)$. If $x_0 \in B$ and $0 < r < 1$, then $B \backslash (x_0 + rB_S)$ is a boundary for $\mathcal{A}_\infty(B_S)$. As a consequence, there is no Šilov boundary for $\mathcal{A}_\infty(B_S)$.*

Proof. Assume that $h \in \mathcal{A}_\infty(B_S)$. For every $0 < \varepsilon < \frac{1-r}{2}$, there is $y_0 \in c_{00}$ such that $\|y_0\| < 1$ and

$$|h(y_0)| > \|h\| - \varepsilon \, .$$

We write $k = \max \operatorname{supp} y_0$. If we choose a positive integer n such that $n > \frac{k}{1-\|y_0\|}$ and $\|(I - P_n)(x_0)\| < \varepsilon$, we will check that the element $y_0 + zy \in B_S$, for every $z \in \mathbb{C}, |z| = 1$ and $y = \sum_{j=n+1}^{2n} \frac{1}{n} e_j$.

Let $A = E \cup F$ be an admissible set such that $E \subset \{1, \ldots, k\}$ and $\min F > k$. If $E \neq \emptyset$, then $|E| + |F| \leq k$ and

$$\sum_{i \in E \cup F} |y_0 + zy(i)| \leq$$

$$\sum_{i \in E} |y_0(i)| + \sum_{i \in F} |y(i)| \leq \|y_0\| + \frac{k}{n} \leq 1 \, .$$

If $E = \emptyset$, then $\sum_{i \in F} |y_0 + zy(i)| = \sum_{i \in F} |y(i)| \leq 1$. So $\|y_0 + zy\| \leq 1$.

Since $\|y\| = 1$, there is $y^* \in S_S$ such that $y^*(y) = 1, \; y^*(e_j) = 0, \forall j \leq n$.

Let $z \in \mathbb{C}$, such that $|z| = 1$ and $|h(y_0) + zy^*(y)| = |h(y_0)| + 1$. Now, we define the holomorphic function g given by

$$g(x) := h(x) + zy^*(x) \quad (x \in B_S) \, .$$

It easy to see that the function g belongs to $\mathcal{A}_\infty(B_S)$ and satisfies that

$$1 + \|h\| - \varepsilon < \|g\| \leq \|h\| + 1.$$

Since B is an boundary there is $z_0 \in B$, such that

$$|g(z_0)| > \|h\| - \varepsilon + 1.$$

On the other hand

$$|g(z_0)| \leq |h(z_0)| + |y^*(z_0)| \leq \|h\| + |y^*(z_0)| \leq \|h\| + 1 \, .$$

So, it implies $|y^*(z_0)| > 1-\varepsilon$. Hence

$$\|(I-P_n)(z_0)\| \geq |y^*(z_0)| > 1-\varepsilon.$$

Consequently,

$$\|z_0 - x_0\| \geq \|(I-P_n)(z_0-x_0)\| \geq$$
$$\|(I-P_n)(z_0)\| - \|(I-P_n)x_0\| \geq 1-2\varepsilon > r.$$

Also z_0 satisfies that $|h(z_0)|+1 \geq \|h\|+1-\varepsilon$ and so $|h(z_0)| > \|h\|-\varepsilon$. Therefore $z_0 \in B\backslash(x_0+rB_S)$ and this set is a boundary for $\mathcal{A}_\infty(B_S)$. As a consequence, there is no Šilov boundary for this space. This completes the proof. ■

We will recall that a point $x \in B_X$ is a $\mathbb{C}$-extreme point of the unit ball if it satisfies that

$$(y \in X, \|x+\lambda y\| \leq 1,\ \forall\lambda \in \mathbb{C}, |\lambda|=1) \ \Rightarrow\ y=0\ .$$

Theorem 3.2. *If K is any infinite compact Hausdorff topological space, then there is no strong peak points for $\mathcal{A}_\infty(B_{C(K)})$.*

If K is separable, then all the extreme points in $B_{C(K)}$ are peak points for the space of 1-degree polynomials on $C(K)$.

Proof. It is known that every peak point is a $\mathbb{C}$-extreme point [8, Theorem 4]. So we will prove that the $\mathbb{C}$-extreme points of $B_{C(K)}$ are no strong peak points. Assume that $x_0 \in S_{C(K)}$ is an extreme point of the unit ball. Since K is infinite, then there is a sequence $\{x_n\}$ of functions on $C(K)$ satisfying that

$$0 \leq x_n \leq 1,\ \|x_n\| = 1, \forall n,\ \ \operatorname{supp} x_n \cap \operatorname{supp} x_m = \emptyset, \forall n \neq m\ .$$

Assume that $h \in B_{\mathcal{A}_\infty(B_{C(K)})}$ is such that $h(x_0)=1$. Since $\{x_n\}$ is equivalent to the c_0-basis, then it converges weakly to zero. By Rainwater Theorem, the sequence $\{x_0(1-x_n)\}$ is in the unit ball of $C(K)$ and converges weakly to x_0. Since $C(K)$ has the Dunford-Pettis property, then it has also the polynomial Dunford-Pettis property [15], and so, if we follow the argument in the proof of [1, Proposition 4.1], then

$$\{h(x_0(1-x_n))\} \longrightarrow 1\}\ .$$

Since x_n are nonnegative elements in the unit sphere, then for every n there is $t_n \in K$ such that $x_n(t_n)=1$ and so

$$\|x_0(1-x_n)-x_0\| \geq \|x_0x_n\| \geq |x_0(t_n)x_n(t_n)| = 1\ .$$

Hence x_0 is not a strong peak point for $\mathcal{A}_\infty(B_{C(K)})$.

If K is separable and $\{t_n : n \in \mathbb{N}\}$ is a dense set in K, we will prove that the function u such that $u(K)=\{1\}$ is a peak point for the space of 1-degree polynomials. In view of Lemma 2.3, this proves the stated assertion.

Just define the function f given by

$$f(x) := \sum_{n=1}^{\infty} \alpha_n(1 + x(t_n)) \quad (x \in C(K)) ,$$

where $\{\alpha_n\}$ is a sequence in S_{ℓ_1} such that $\alpha_n > 0$ for every n. Then f is clearly a 1-degree polynomial on $C(K)$ and satisfies that $f(u) = \|f\| = 2$. If $x \in B_{C(K)}$ and $|f(x)| = 2$, then $|1 + x(t_n)| = 2$ for every n and so $x(t_n) = 1$ for all n, that is, $x = u$. ∎

Since ℓ_∞ has a countable subset of functionals that separates points and attains the norm at the same element of the unit ball, we can also obtain:

Corollary 3.3. *([3]) All the extreme points in B_{ℓ_∞} are peak points for the space of 1-degree polynomials on ℓ_∞.*

4. Šilov boundary on the trace class operators

Let H be a complex Hilbert space. An operator $T : H \longrightarrow H$ is called a trace class operator if there is an orthonormal basis B such that $\sum_{e \in B} < T^*(Te), e >< \infty$, where T^* is the adjoint operator of T. We denote by $C_1(H)$ the Banach space of all trace class operators on H with the trace norm $\|T\|_1 = \sum_{e \in B} < T^*(Te), e >$.

Theorem 4.1. *If H is a complex Hilbert space, there is a Šilov boundary for $\mathcal{A}_u(C_1(H))$ and $\mathcal{A}_\infty(C_1(H))$ and both coincide.*

Proof. Assume that $\{e_i : i \in I\}$ is an orthonormal basis of H and $F \subset I$ is a subset, then the operator Π_F given by

$$\Pi_F(T) := P_F T P_F, \quad (T \in C_1(H)),$$

where $P_F(x) = \sum_{i \in F} x(i)e_i$, $(x \in H)$ is a norm one projection on $C_1(H)$. Since $\mathrm{Lin}\{e_i \otimes e_j : i, j \in I\}$ is dense in $C_1(H)$, then for every $h \in \mathcal{A}_\infty(B_{C_1(H)})$, it holds that

$$\|h\| := \sup_{\substack{F \subset I \\ F \text{ finite}}} \|h \circ \Pi_F\| .$$

For every complex finite dimensional space Y, it holds that the subset of peak points of B_Y is a boundary for $\mathcal{A}_u(B_Y)$ [4, Theorem 1]. We will prove that for every $F \subset I$ finite subset, every peak point of the unit ball of $\Pi_F(C_1(H))$ for the space of bounded and continuous functions on the unit ball of $\Pi_F(C_1(H))$ which are holomorphic on the open unit ball, is a strong peak point for $\mathcal{A}_u(B_{C_1(H)})$.

Let $T_0 \in S_{C_1(H)} \cap \Pi_F(C_1(H))$ be a peak point. Hence there is a function g defined on the unit ball of $\Pi_F(C_1(H))$ such that

$$g(T_0) = \|g\| = 1 \text{ and } |g(T)| < 1, \forall T \in (B_{C_1(H)} \cap \Pi_F(C_1(H)))\backslash\{T_0\} .$$

Now we extend g to $B_{C_1(H)}$ by

$$\tilde{g}(T) = g(\Pi_F(T)) \quad (T \in B_{C_1(H)}) .$$

We clearly have that $\tilde{g} \in \mathcal{A}_u(B_{C_1(H)})$ and it is satisfied that $\|\tilde{g}\| \le \|g\| = 1$ and $\tilde{g}(T_0) = 1$. Assume that $\{T_n\}$ is a sequence in $B_{C_1(H)}$ such that $\{|\tilde{g}(T_n)|\} \to 1$, that is $\{|g(\Pi_F(T_n))|\} \to 1$. Since $\Pi_F(C_1(H))$ is a finite-dimensional space and T_0 is a peak point, then $\{\Pi_F(T_n)\} \to T_0$. Since $\|T_0\| = 1$, then $\{\|\Pi_F(T_n)\|_n\} \to 1$. By using [11, Proposition 2.2], it holds that

$$\|P_F T_n P_F\|^2 + \|P_F T_n (I - P_F)\|^2 + \|(I - P_F) T_n P_F\|^2 + \|(I - P_F) T_(I - P_F)\|^2 \le \|T_n\|^2 \le 1 ,$$

and so $\|\Pi_F T_n - T_n\| = \|P_F T_n P_F - T_n\| \to 0$. Since we knew that $\{\Pi_F(T_n)\}$ converges to T_0, then $\{T_n\}$ also converges to T_0 and T_0 is a strong peak point, as we wanted to show. Since the strong peak points are contained in any closed boundary and in this case the subset of strong peak points is a boundary for $\mathcal{A}_u(B_{C_1(H)})$, then the Šilov boundary for this space is the closure of the strong peak points of $\mathcal{A}_u(B_{C_1(H)})$. We can follow the same argument for the space $\mathcal{A}_\infty(C_1(H))$.

■

5. Boundaries on $K(\ell_p, \ell_q)$

We now restrict our attention to study properties of the boundaries for $\mathcal{A}_\infty(B_X)$, where the Banach space X is the space of all compact operators on ℓ_p, for $1 \le p < \infty$.

Theorem 5.1. *If $1 \le p \le q < \infty$, there is no Šilov boundary for $\mathcal{A}_\infty(B_{K(\ell_p,\ell_q)})$. In fact, if B is a boundary for $\mathcal{A}_\infty(B_{K(\ell_p,\ell_q)})$, $0 < r < 1$ and $S_0 \in B$, the set $B\backslash(S_0 + rB_{K(\ell_p,\ell_q)})$ is also a boundary for $\mathcal{A}_\infty(B_{K(\ell_p,\ell_q)})$.*

There are closed boundaries A, B for $\mathcal{A}_\infty(B_{K(\ell_p,\ell_q)})$ such that $\operatorname{dist}(A, B) \ge 1$.

The same assertions also hold for $\mathcal{A}_u(B_{K(\ell_p,\ell_q)})$.

Proof. We will denote by $\{P_n\}$ and $\{Q_n\}$ the sequences of canonical projections associated to the usual basis of ℓ_p and ℓ_q, respectively.

Assume that $B \subset B_{K(\ell_p,\ell_q)}$ is a boundary of $\mathcal{A}_\infty(B_{K(\ell_p,\ell_q)})$, $0 < r < 1$ and $S_0 \in B$. If $h \in \mathcal{A}_\infty(B_{K(\ell_p,\ell_q)})$ and $0 < \varepsilon < \frac{1-r}{2}$, there is $N \in \mathbb{N}$ and $F \in B_{K(\ell_p,\ell_q)}$ such that $Q_N F P_N = F$ and

$$|h(F)| > \|h\| - \varepsilon .$$

Since S_0 is a compact operator, then there is $n > N$ such that

$$\|(I - Q_n)S_0(I - P_n)\| < \varepsilon.$$

Now we choose an operator $R \in S_{K(\ell_p,\ell_q)}$ such that

$$(I - Q_n)R(I - P_n) = R$$

and an element $x \in S_{\ell_p}$ such that $P_n x = 0$ and $\|R(x)\| = 1$ and $y^* \in S_{\ell_q^*}$ such that $Q_n^*(y^*) = 0$ and $y^*(R(x)) = 1$. By using the Maximum Modulus Theorem, there is $z_0 \in \mathbb{C}$ such that $|z_0| = 1$ and

$$\sup_{|z|=1} |h(F + z_0 R)| = |h(F + z_0 R)|.$$

If $\lambda \in \mathbb{C}$ is a modulus one scalar such that

$$|h(F + z_0 R) + \lambda y^*(Rx)| = |h(F + z_0 R)| + 1,$$

we define he holomorphic function g given by

$$g(T) := h(T) + \lambda y^*(Tx) .$$

Clearly the function g belongs to $\mathcal{A}_\infty(B_{K(\ell_p,\ell_q)})$ and satisfies that

$$\|g\| \geq |g(F + z_0 R)| \geq |h(F)| + |y^*(Rx)| > \|h\| - \varepsilon + 1 .$$

Since B is a boundary for $\mathcal{A}_\infty(B_{K(\ell_p,\ell_q)})$, then there is $S \in B$ such that $|g(S)| > \|g\| - \varepsilon$. Hence

$$\|h\| - 2\varepsilon + 1 \leq \|g\| - \varepsilon < |g(S)| \leq |h(S)| + |y^*(Sx)| , \tag{4}$$

and so,

$$|y^*(Sx)| \geq 1 - \varepsilon .$$

By the choice of x and y^*, then

$$\|(I - Q_n)S(I - P_n)\| \geq |y^*(I - Q_n)S(I - P_n)x| = |y^*(Sx)| \geq 1 - \varepsilon .$$

Finally, we deduce that

$$\|S - S_0\| \geq \|(I - Q_n)(S - S_0)(I - P_n)\| \geq$$

$$\|(I - Q_n)S(I - P_n)\| - \|(I - Q_n)S_0(I - P_n)\| \geq 1 - 2\varepsilon > r .$$

From inequality (4), we also obtain that

$$|h(S)| \geq \|h\| - 2\varepsilon .$$

We just proved that $B \backslash (S_0 + rB_{K(\ell_p,\ell_q)})$ is a boundary for $\mathcal{A}_\infty(B_{K(\ell_p,\ell_q)})$. As a consequence, there is no Šilov boundary for this space.

Now we will show a procedure to construct boundaries for $\mathcal{A}_\infty(B_{K(\ell_p,\ell_q)})$. Since $\mathrm{Lin}\{x \otimes y : x \in (\ell_p)^*, y \in \ell_q, \mathrm{supp}\ x, \mathrm{supp}\ y \text{ is finite}\}$ is dense in $K(\ell_p, \ell_q)$, then for every $h \in \mathcal{A}_\infty(B_{K(\ell_p,\ell_q)}))$, it is satisfied that

$$\|h\| = \sup\{\|h_F\| : F \subset \mathbb{N} \text{ finite}\} ,$$

where $h_F(T) := h(Q_F T P_F)$ for $T \in K(\ell_p, \ell_q)$ and Q_F, P_F are the projections given by

$$P_F(x) = \sum_{n \in F} x(n) e_n, \quad (x \in \ell_p), \quad Q_F(x) = \sum_{n \in F} x(n) e_n, \quad (x \in \ell_q) .$$

Note also that for $F \subset G$, then $\|h_F\| \leq \|h_G\|$.

Assume that $\{F_n\}$ is an increasing sequence of finite sets of $\mathbb{N}$ such that $G_n := F_{n+1} \backslash F_n$ is non empty and $\cup_n F_n = \mathbb{N}$. We consider the subsets A_n whose elements are those operators $T \in B_{K(\ell_p, \ell_q)}$ such that T admits a decomposition as sum of two operators $T = R + S$ satisfying the following

$$\|R\| = \|S\| = 1, \; R = Q_{F_n} R P_{F_n}, \; Q_{F_n} S P_{F_n} = 0, \; Q_{G_n} S P_{G_n} = S .$$

Let us note that A_n is closed for every n.

We will check that $B = \cup_n A_n$ is a closed boundary for $\mathcal{A}_\infty(B_{K(\ell_p, \ell_q)})$. Given $h \in \mathcal{A}_\infty(B_{K(\ell_p, \ell_q)})$ and $\varepsilon > 0$, there is some finite set $F \subset N$ such that $\|h_F\| > \|h\| - \varepsilon$. If m satisfies that $F \subset F_m$, then also $\|h_{F_m}\| \geq \|h\| - \varepsilon$. Hence there is an operator $R \in S_{K(\ell_p, \ell_q)}$ such that $Q_{F_m} R P_{F_m} = R$ where h_{F_m} attains its norm and so

$$|h(R)| \geq \|h\| - \varepsilon .$$

If $S \in S_{K(\ell_p, \ell_q)}$ satisfies that $Q_{F_m} S P_{F_m} = 0$ and $Q_{G_m} S P_{G_m} = S$, then the operator $R + zS$ is in the unit ball of $K(\ell_p, \ell_q)$, for every complex number z in the unit disk. If we apply the Maximum Modulus Theorem to the function $z \mapsto h(R + zS)$ defined on the complex unit disk, then there is a complex number z_0 with $|z_0| = 1$ and such that

$$|h(R + z_0 S)| \geq |h(R)| \geq \|h\| - \varepsilon .$$

Since the element $R + z_0 S \in A_m$, then B is a boundary for $\mathcal{A}_\infty(B_{K(\ell_p, \ell_q)})$.

Let us note that for two positive integers $n < m$, if $T_n \in A_n, T_m \in A_m$, then

$$\|T_m - T_n\| \geq \|Q_{G_m}(T_m - T_n) P_{G_m}\| = \|Q_{G_m} T_m P_{G_m}\| = 1 . \tag{5}$$

Since every A_n is closed, from the above inequality, it follows that B is also closed.

By using the same argument, then the subsets $\cup_n A_{2n}$ and $\cup_n A_{2n-1}$ are also closed boundaries for $\mathcal{A}_\infty(B_{K(\ell_p, \ell_q)})$. In view of (5), then the distance between both sets is at least 1. ∎

Acknowledgements. This note was prepared while the first author visited the Universidade de São Paulo and the second visited the Universidad de Granada. The authors thank both institutions for the support received and the hospitality.

References

[1] M.D. Acosta, *Boundaries for spaces of holomorphic functions on $C(K)$*, Publ. Res. Inst. Math. Sci., to appear.

[2] M.D. Acosta, L.A. Moraes and L. Romero Grados, *On boundaries on the predual of the Lorentz sequence space*, preprint.

[3] R.M. Aron, Y.S. Choi, M.L. Lourenço and O.W. Paques, *Boundaries for algebras of analytic functions on infinite dimensional Banach spaces*, in *Banach Spaces* (B.L. Lin and W.B. Johnson, Eds.), *Contemporary Mathematics*, AMS, **144** (1993), 15–22.

[4] E. Bishop, *A minimal boundary for function algebras*, Pacific J. Math. **9** (1959), 629–642.

[5] Y. S. Choi, D. García, S. G. Kim and M. Maestre, *Norm or numerical radius attaining polynomials on $C(K)$*, J. Math. Anal. Appl. **295** (2004), 80–96.

[6] T.W. Gamelin, *Uniform Algebras*, Chelsea, New York, 1984.

[7] D.J.H. Garling, *On symmetric sequence spaces*, Proc. London Math. Soc. **16** (1966), 85–106.

[8] J. Globevnik, *On interpolation by analytic maps in infinite dimensions*, Math. Proc. Cambridge Philos. Soc. **83** (1978), 243–252.

[9] J. Globevnik, *Boundaries for polydisc algebras in infinite dimensions*, Math. Proc. Cambridge Philos. Soc. **85** (1979), 291–303.

[10] P. Harmand, D. Werner and W. Werner, *M-ideals in Banach spaces and Banach algebras*, Lecture Notes in Math. **1547**, Springer-Verlag, Berlin, 1993.

[11] C. Lennard, *C_1 is uniformly Kadec-Klee*, Proc. Amer. Math. Soc. **109** (1990), 71–77.

[12] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I and II*, Springer Monographs in Math., Springer-Verlag, London, 1991.

[13] A. Moltó, V. Montesinos and S. Troyanski, *On quasi-denting points, denting faces and the geometry of the unit ball of $d(w,1)$*, Arch. Math. **63** (1994), 45–55.

[14] L.A. Moraes and L. Romero Grados, *Boundaries for algebras of holomorphic functions*, J. Math. Anal. Appl. **281** (2003), 575–586.

[15] R.A. Ryan, Dunford-Pettis properties, *Bull. Acad. Polon. Sci. Math.* **27** (1979), 373–379.

[16] D. Werner, *New classes of Banach spaces which are M-ideals in their biduals*, Math. Proc. Cambridge Philos. Soc. **111** (1992), 337–354.

Universidad de Granada
Facultad de Ciencias
Departamento de Análisis Matemático
18071 Granada (SPAIN)
E-mail address: dacosta@goliat.ugr.es

Universidade de São Paulo
Departamento de Matemática e Estatística
05508-090 São Paulo (BRAZIL)
E-mail address: mllouren@ime.usp.br

TRABALHOS DO DEPARTAMENTO DE MATEMÁTICA

TÍTULOS PUBLICADOS

2004-01 ASSEM, I., COELHO. F.U., LANZILOTTA, M., SMITH, D.and TREPODE, SONIA Algebras determined by their left and right parts. 34p.

2004-02 FUTORNY, V., MOLEV, A. and OVSIENKO, S. Harish-Chandra Modules for Yangians. 29p.

2004-03 COX, B. L. and FUTORNY, V. Intermediate Wakimoto modules for affine *sl(n+1, C)*. 35p.

2004-04 GRISHKOV, A. N. and ZAVARNITSINE, A. V. Maximal subloops of simple Moufang loops. 43p.

2004-05 GREEN, E.L. and MARCOS, E. δ-Koszul Algebras. 15p.

2004-06 GORODSKI, C. Taut Representantions of compact simple lie groups. 16p.

2004-07 ASPERTI, A.C. and VALÉRIO, B.C. Ruled helicoidal surfaces in a 3-dimensional space form. 9p.

2004-08 ASSEM, I., CASTONGUAY, D., MARCOS, E.N. and TREPODE, S. Strongly simply connected schurian algebras and multiplicative bases. 26p.

2004-09 RODRIGUES, A.A.M., MIRANDA FILHO, R.C. and SOUZA, E.G. Definability and Invariance in First Order Structures. 15p.

2004-10 GIAMBRUNO, A. and MILIES, C. P. Free groups and involutions in the unit group of a group algebra. 5p.

2004-11 RAO, S. ESWARA and FUTORNY, V. Classification of integrable modules for affine lie superalgebras. 20p.

2004-12 GRISHKOV, A. N. and ZAVARNITSINE, A. V. Groups with triality. 22p.

2004-13 BUSTAMANTE, J. C. and CASTONGUAY, D. Fundamental groups and presentations of algebras. 10p.

2004-14 FERNANDES, S.M. and MARCOS, E.N. On the Fundamental Derivation of a Finite Dimensional Algebra. 15p.

2004-15 CARRIÓN, H.; GALINDO, P. and LOURENÇO, M.L. Banach spaces whose bounded sets are bounding in the bidual. 10p.

2004-16 KOSZMIDER, P. Projections in weakly compactly generated Banach spaces and Chang's conjecture. 15p.

2004-17 KOSZMIDER, P. On a problem of Rolewicz about Banach spaces that admit support sets. 19p.

2004-18 GOODAIRE, E.G., MILIES, C.P. and PARMENTER, M.M. Central units in alternative loop rings. 8p.

2004-19 GIAMBRUNO, A. and MILIES, C. P. Free groups and involutions in the unit group of a group algebra. 5p.

2004-20 CARRIÓN, H. Entire Functions on l_1. 16p.

2004-21 SHESTAKOV, I. and ZHUKAVETS, N. Universal multiplicative envelope of free Malcev superalgebra on one odd generator. 28p.

2004-22 ASSEM, I., BUSTAMANTE, J.C., CASTONGUAY, D. and NOVOA, C. A note on the fundamental group of a one-point extension. 6p.

2004-23 BAHTURIN, Y. A., SHESTAKOV, I. P. and ZAICEV, M . V. Gradings on Simple Jordan and Lie Algebras. 25p.

2004-24 DOKUCHAEV. M., KIRICHENKO. V. and MILIES, C. P. Engel subgroups of triangular matrices over local rings. 16p.

2004-25 ADVÍNCULA, F. H. and MARCOS, E. N., Stratifications of algebras with radical square zero. 7p.

2004-26 ADVÍNCULA, F. H. and MARCOS, E. N., Algebras which are standardly stratified in all orders. 8p.

2004-27 DOKUCHAEV, M. A., KIRICHENKO, V. V., ZELENSKY, A. V. and ZHURAVLEV, V. N. Gorenstein matrices. 22p.

2004-28 PEREIRA, A. L. and PEREIRA, M. C. An extension of the method of rapidly oscillating solutions. 15p.

2004-29 DOKUCHAEV, M. and EXEL, R. Associativity of crossed products by partial actions, enveloping actions and partial representations. 24p.

2004-30 FUTORNY, V. and USTIMENKO, V. Small world semiplanes with generalised Schubert cells. 10p.

2004-31 FERRAZ, R. A. and MILIES, C. P. Idempotents in group algebras and minimal abelian codes. 11p.

2004-32 CRISTO, O. B. and MILIES, C. P. Symmetric elements under oriented involutions in group rings. 12p.

2005-01 PEREIRA, A. L. and PEREIRA, M. C. Generic simplicity for the solutions of a nonlinear plate equation. 22p.

2005-02 ARAGONA, J. Colombeau generalized functions on quasi-regular sets. 25p.

2005-03 PEREIRA. A. L. Global attractor and nonhomogeneous equilibria for a non local evolution equation in an unbounded domain. 26p.

2005-04 SHESTAKOV, I. and ZHUKAVETS, N. Malcev Poisson superalgebra of free Malcev superalgebra on one odd generator. 19p.

2005-05 SHESTAKOV, I. and ZHUKAVETS, N. Speciality of Malcev superalgebras on one odd generator. 15p.

2005-06 BILLIG, Y., FUTORNY, V. and MOLEV, A. Verma modules for Yangians. 18p.

2005-07 GORODSKI, C. Jacobi fields on normal homogeneous Riemannian manifolds. 6p.

2005-08 GRISHKOV, A. Notes on simple Lie algebras over a field of characteristic 2. 7p.

2005-09 MIRANDA, J. C. S. and MORETTIN, P. A. Estimation of the density of point processes on $\mathbb{R}^m$ via wavelets. 15p.

2005-10 KASHUBA, I., OVSIENKO, S. and SHESTAKOV, I. Jordan algebras versus associative algebras. 35p.

2005-11 IWAKI, E., JESPERS, E. and JURIAANS, S. O. The hypercentre of the unit group of an integral group ring. 11p.

2005-12 FUTORNY, V. and OVSIENKO, S. Theory of Galois algebras. 54p.

2005-13 MIRANDA, J. C. S. Characterization of the intensity for a class of point processes. 16p.

2005-14 MIRANDA, J. C. S. Neglecting higher order infinitesimals and the study of some measures. 16p.

2005-15 MIRANDA, J. C. S. Some wide classes of point processes interrelationships and three working properties. 15p.

2005-16 MIRANDA, J. C. S. Sure inference analysis. 27p.

2005-17 GRISHKOV, A. N. and ZAVARNITSINE, A. V. Sylow's theorem for Moufang loops I. 14p.

2005-18 LOURENÇO, M. L. and TOCHA, N. N. Zeros of complex homogeneous polynomials. 7p.

2005-19 ALEXANDRINO, M. M. Proofs of conjectures about singular riemannian foliations. 19p.

2005-20 MELO, S. T. and SILVA, C. C. K-Theory of pseudodifferential operators with semi-periodic symbols. 13p

2005-21 COX, B. L. and FUTORNY, V. Structure of intermediate Wakimoto modules. 20p.

2005-22 BAHTURIN, Y. A., SHESTAKOV, I. P. and ZAICEV, M. V. Group Gradings on Jordan Algebras $M_n^{(+)}$.30p.

2005-23 BILIOTTI, L. and GORODSKI, C. Polar actions on compact irreducible symmetric spaces are taut. 7p.

2005-24 LOURENÇO, M. L. and PELLEGRINI, L. Interpolation by analytic functions on preduals of Lorentz sequence spaces. 9p.

2005-25 LOURENÇO, M. L. and ACOSTA, M. D. Šilov boundary for holomorphic functions on some classical Banach spaces. 13p.

Nota: Os títulos publicados nos Relatórios Técnicos dos anos de 1980 a 2005 estão à disposição no Departamento de Matemática do IME-USP.
Cidade Universitária "Armando de Salles Oliveira"
Rua do Matão, 1010 - Cidade Universitária
Caixa Postal 66281 - CEP 05315-970