ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

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Annales de la faculté des sciences de Toulouse 6^e série, tome 11, n° 3 (2002), p. 377-401

<http://www.numdam.org/item?id=AFST_2002_6_11_3_377_0>

Annales de la Faculté des Sciences de Toulouse Vol. XI, n° 3, 2002
pp. 377–401

Geometric mean curvature lines on surfaces immersed in $\mathbb{R}^3$ (*)

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RÉSUMÉ. — Dans ce travail on étudie les paires de feuilletages transverses avec singularités, définis dans la région elliptique d'une surface orientée plongée dans l'espace $\mathbb{R}^3$. Les feuilles sont les lignes de *courbure géométrique moyenne,* selon lesquelles la courbure normale est donnée par la moyenne géométrique $\sqrt{k_1 k_2}$ des courbures principales k_1, k_2. Les singularités sont les points ombilics (oú $k_1 = k_2$) et les courbes paraboliques (oú $k_1 k_2 = 0$).
On détermine les conditions pour la stabilité structurelle des feuilletages autour des points ombilics, des courbes paraboliques et des cycles de courbure géométrique moyenne (qui sont les feuilles compactes). La généricité de ces conditions est établie.
Munis de ces conditions on établit les conditions suffisantes, qui sont aussi vraisemblablement nécessaires, pour la stabilité structurelle des feuilletages. Ce travail est une continuation et une généralisation naturelles de ceux sur les lignes à *courbure arithmétique moyenne*, selon lesquelles la courbure normale est donnée par $(k_1 + k_2)/2$ et sur les lignes à *courbure nulle,* qui sont les *courbes asymptotiques*. Voir les articles [6], [7], [9].

ABSTRACT. — Here are studied pairs of transversal foliations with singularities, defined on the *Elliptic* region (where the Gaussian curvature $\mathcal{K}$ is positive) of an oriented surface immersed in $\mathbb{R}^3$. The leaves of the foliations are the *lines of geometric mean curvature,* along which the normal curvature is given by $\sqrt{\mathcal{K}}$, which is the *geometric mean curvature* of the principal curvatures k_1, k_2 of the immersion.
The singularities of the foliations are the *umbilic points* and *parabolic curves*, where $k_1 = k_2$ and $\mathcal{K} = 0$, respectively.

(*) Reçu le 30 mai 2002, accepté le 21 octobre 2002
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The first author was partially supported by FUNAPE/UFG. Both authors are fellows of CNPq. This work was done under the project PRONEX/FINEP/MCT – Conv. 76.97.1080.00 – Teoria Qualitativa das Equações Diferenciais Ordinárias and CNPq – Grant 476886/2001-5.

Here are determined the structurally stable patterns of geometric mean curvature lines near the *umbilic points*, *parabolic curves* and *geometric mean curvature cycles*, the periodic leaves of the foliations. The genericity of these patterns is established.
This provides the three essential local ingredients to establish sufficient conditions, likely to be also necessary, for *Geometric Mean Curvature Structural Stability*. This study, outlined at the end of the paper, is a natural analog and complement for the *Arithmetic Mean Curvature* and *Asymptotic Structural Stability* of immersed surfaces studied previously by the authors [6], [7], [9].

1. Introduction

In this paper are studied the *geometric mean curvature configurations* associated to immersions of oriented surfaces into $\mathbb{R}^3$. They consist on the *umbilic points* and *parabolic curves*, as singularities, and of the *lines of geometric mean curvature* of the immersions, as the leaves of the two transversal foliations in the configurations. Along these lines the normal curvature is given by the geometric mean curvature, which is the square root of the product of the principal curvatures (i.e of the Gaussian Curvature).

The two transversal foliations, called here *geometric mean curvature foliations*, are well defined and regular only on the non-umbilic part of the elliptic region of the immersion, where the Gaussian Curvature is positive. In fact, there they are the solution of smooth quadratic differential equations. The set where the Gaussian Curvature vanishes, the parabolic set, is generically a regular curve which is the border of the elliptic region. The umbilic points are those at which the principal curvatures coincide, generically are isolated and disjoint from the parabolic curve. See section 2 for precise definitions.

This study is a natural development and extension of previous results about the Arithmetic Mean Curvature and Asymptotic Configurations, dealing with the qualitative properties of the lines along which the normal curvature is the arithmetic mean of the principal curvatures (i.e. is the standard Mean Curvature) or is null. This has been considered previously by the authors; see [6], [9] and [7].

The point of departure of this line of research, however, can be found in the classical works of Euler, Monge, Dupin and Darboux, concerned with the lines of principal curvature and umbilic points of immersions. See [22], [24] for an initiation on the basic facts on this subject; see [12], [14] for a

discussion of the classical contributions and for their analysis from the point of view of structural stability of differential equations [16].

This paper establishes sufficient conditions, likely to be also necessary, for the structural stability of *geometric mean curvature configurations* under small perturbations of the immersion. See section 7 for precise statements.

This extends to the geometric mean curvature setting the main theorems on structural stability for the arithmetic mean curvature configuration and for the asymptotic configurations, proved in [6], [7], [9].

Three local ingredients are essential for this extension: the umbilic points, endowed with their geometric mean curvature separatrix structure, the geometric mean curvature cycles, with the calculation of the derivative of the Poincaré return map, through which is expressed the hyperbolicity condition and the parabolic curve, together with the parabolic tangential singularities and associated separatrix structure.

The conclusions of this paper, on the elliptic region, are complementary to results valid independently on the hyperbolic region (on which the Gaussian curvature is negative), where the separatrix structure near the parabolic curve and the asymptotic structural stability has been studied in [6], [9].

The parallel with the conditions for principal, arithmetic mean curvature and asymptotic structural stability is remarkable. This can be attributed to the unifying role played by the notion of Structural Stability of Differential Equations and Dynamical Systems, coming to Geometry through the seminal work of Andronov and Pontrjagin [1] and Peixoto [20].

The interest on lines of geometric mean curvature goes back to the paper of Occhipinti [17]. The work of Ogura [18] regards these lines in terms of his unifying notions *T-Systems* and *K-Systems* and makes a local analysis of the expressions of the fundamental quadratic forms in a chart whose coordinate curves are lines of geometric mean curvature. A comparative study of these expressions with those corresponding to other lines of geometric interest, such as the *principal, asymptotic, arithmetic mean curvature* and *characteristic lines*, was carried out by Ogura in the context of *T-Systems* and *K-Systems*. In [8] the authors have studied the foliations by *characteristic lines*, called harmonic mean curvature lines.

The authors are grateful to Prof. Erhard Heil for calling their attention to these papers, which seem to have remained unquoted along so many years.

No global examples, or even local ones around singularities, of geometric mean curvature configurations seem to have been considered in the literature on differential equations of classic differential geometry, in contrast with the situations for the principal and asymptotic cases mentioned above. See also the work of Anosov, for the global structure of the geodesic flow [2], and that of Banchoff, Gaffney and McCrory [3] for the parabolic and asymptotic lines.

This paper is organized as follows:

Section 2 is devoted to the general study of the differential equations and general properties of Geometric Mean Curvature Lines. Here are given the precise definitions of the Geometric Mean Curvature Configuration and of the two transversal Geometric Mean Curvature Foliations with singularities into which it splits. The definition of Geometric Mean Curvature Structural Stability focusing the preservation of the qualitative properties of the foliations and the configuration under small perturbations of the immersion, will be given at the end of this section.

In Section 3 the equation of lines of geometric mean curvature is written in a Monge chart. The condition for umbilic geometric mean curvature stability is explicitly stated in terms of the coefficients of the third order jet of the function which represents the immersion in a Monge chart. The local geometric mean curvature separatrix configurations at stable umbilics is established for C^4 immersions and resemble the three Darbouxian patterns of principal and arithmetic mean curvature configurations [5], [12].

In Section 4 the derivative of first return Poincaré map along a geometric mean curvature cycle is established. It consists of an integral expression involving the curvature functions along the cycle.

In Section 5 are studied the foliations by lines of geometric mean curvature near the parabolic set of an immersion, assumed to be a regular curve. Only two generic patterns of the three singular tangential patterns in common with the asymptotic configurations, the *folded node* and the *folded saddle*, exist generically in the case; the *folded focus* being absent. See [6].

Section 6 presents examples of Geometric Mean Curvature Configurations on the Torus of revolution and the quadratic Ellipsoid, presenting non-trivial recurrences. This situation, impossible in principal configurations, has been established for arithmetic mean curvature configurations in [7].

In Section 7 the results presented in Sections 3, 4 and 5 are put together to provide sufficient conditions for Geometric Mean Curvature Structural Stability. The genericity of these conditions is formulated at the end of this section, however its rather technical proof will be postponed to another paper.

2. Differential equation of geometric mean curvature lines

Let $\alpha : \mathbb{M}^2 \to \mathbb{R}^3$ be a C^r, $r \geqslant 4$, immersion of an oriented smooth surface $\mathbb{M}^2$ into $\mathbb{R}^3$. This means that $D\alpha$ is injective at every point in $\mathbb{M}^2$.

The space $\mathbb{R}^3$ is oriented by a once for all fixed orientation and endowed with the Euclidean inner product $<,>$.

Let N be a vector field orthonormal to α. Assume that (u, v) is a positive chart of $\mathbb{M}^2$ and that $\{\alpha_u, \alpha_v, N\}$ is a positive frame in $\mathbb{R}^3$.

In the chart (u, v), the *first fundamental form* of an immersion α is given by:

$I_\alpha =< D\alpha, D\alpha >= Edu^2 + 2Fdudv + Gdv^2$, with

$E =< \alpha_u, \alpha_u >$, $F =< \alpha_u, \alpha_v >$, $G =< \alpha_v, \alpha_v >$

The *second fundamental form* is given by:

$$II_\alpha =< N, D^2\alpha >= edu^2 + 2fdudv + gdv^2.$$

The normal curvature at a point p in a tangent direction $t = [du : dv]$ is given by:

$$k_n = k_n(p) = \frac{II_\alpha(t,t)}{I_\alpha(t,t)}.$$

The lines of geometric mean curvature of α are regular curves γ on $\mathbb{M}^2$ having normal curvature equal to the geometric mean curvature of the immersion, i.e., $k_n = \sqrt{\mathcal{K}}$, where $\mathcal{K} = \mathcal{K}_\alpha$ is the Gaussian curvature of α.

Therefore the pertinent differential equation for these lines is given by:

$$\frac{edu^2 + 2fdudv + gdv^2}{Edu^2 + 2Fdudv + Gdv^2} = \sqrt{\frac{eg - f^2}{EG - F^2}} = \sqrt{\mathcal{K}}$$

Or equivalently by

$$[g - \sqrt{\mathcal{K}}G]dv^2 + 2[f - \sqrt{\mathcal{K}}F]dudv + [e - \sqrt{\mathcal{K}}E]du^2 = 0. \qquad (1)$$

This equation is defined only on the closure of the *Elliptic region*, $\mathbb{EM}^2{}_\alpha$, of α, where $\mathcal{K} > 0$. It is bivalued and C^{r-2}, $r \geqslant 4$, smooth on the complement of the umbilic, $\mathcal{U}_\alpha$, and parabolic, $\mathcal{P}_\alpha$, sets of the immersion α. In fact, on $\mathcal{U}_\alpha$, where the principal curvatures coincide, the equation vanishes identically; on $\mathcal{P}_\alpha$, it is univalued.

Also, the above equation is equivalent to the quartic differential equation, obtained from the above one by eliminating the square root.

$$A_{40}du^4 + A_{31}du^3dv + A_{22}du^2dv^2 + A_{13}dudv^3 + A_{04}dv^4 = 0 \qquad (2)$$

where,

$$\begin{aligned}
A_{40} &= e^2(EG - F^2) - E^2(eg - f^2)\\
A_{31} &= 4ef(EG - F^2) - 4EF(eg - f^2)\\
A_{22} &= 6f^2EG - 6egF^2\\
A_{13} &= 4fg(EG - F^2) - 4FG(eg - f^2)\\
A_{04} &= g^2(EG - F^2) - G^2(eg - f^2)
\end{aligned}$$

The developments above allow us to organize the lines of geometric mean curvature of immersions into the *geometric mean curvature configuration,* as follows:

Through every point $p \in \mathbb{EM}^2{}_\alpha \backslash (\mathcal{U}_\alpha \cup \mathcal{P}_\alpha)$, pass two geometric mean curvature lines of α. Under the orientability hypothesis imposed on $\mathbb{M}$, the geometric mean curvature lines define two foliations: $\mathbb{G}_{\alpha,1}$, called the *minimal geometric mean curvature foliation*, along which the geodesic torsion is negative (i.e $\tau_g = -\sqrt[4]{\mathcal{K}}\sqrt{2\mathcal{H} - 2\sqrt{\mathcal{K}}}$), and $\mathbb{G}_{\alpha,2}$, called the *maximal geometric mean curvature foliations*, along which the geodesic torsion is positive (i.e $\tau_g = \sqrt[4]{\mathcal{K}}\sqrt{2\mathcal{H} - 2\sqrt{\mathcal{K}}}$).

By comparison with the arithmetic mean curvature directions, making angle $\pi/4$ with the minimal principal directions, the geometric ones are located between them and the principal ones, making an angle θ such that $tan\theta = \pm\sqrt[4]{\frac{k_1}{k_2}}$, as follows from Euler's Formula. The particular expression for the geodesic torsion given above results from the expression $\tau_g = (k_2 - k_1)sin\theta cos\theta$ [24]. It is found in the work of Occhipinti [17]. See also Lemma 1 in Section 4 below.

The quadruple $\mathbb{G}_\alpha = \{\mathcal{P}_\alpha, \mathcal{U}_\alpha, \mathbb{G}_{\alpha,1}, \mathbb{G}_{\alpha,2}\}$ is called the *geometric mean curvature configuration* of α. It splits into two foliations with singularities: $\mathbb{G}^i_\alpha = \{\mathcal{P}_\alpha, \mathcal{U}_\alpha, \mathbb{G}_{\alpha,i}\}$, $i = 1, 2$.

Let $\mathbb{M}^2$ be also compact. Denote by $\mathcal{M}^{r,s}(\mathbb{M}^2)$ be the space of C^r immersions of $\mathbb{M}^2$ into the Euclidean space $\mathbb{R}^3$, endowed with the C^s topology.

An immersion α is said *C^s-local geometric mean curvature structurally stable* at a compact set $K \subset \mathbb{M}^2$ if for any sequence of immersions α_n converging to α in $\mathcal{M}^{r,s}(\mathbb{M}^2)$, in a compact neighborhood V_K of K, there is a sequence of compact subsets K_n and a sequence of homeomorphisms mapping K to K_n, converging to the identity of $\mathbb{M}^2$, such that on V_K it maps umbilic, parabolic curves and arcs of the geometric mean curvature foliations $\mathbb{G}_{\alpha,i}$ to those of $\mathbb{G}_{\alpha_n,i}$ for $i = 1,\ 2$.

An immersion α is said *C^s-geometric mean curvature structurally stable* if the compact K above is the closure of $\mathbb{EM}^2{}_\alpha$.

Analogously, α is said *i-C^s-geometric mean curvature structurally stable* if only the preservation of elements of *i-th, i=1,2* foliation with singularities is required.

A general study of the structural stability of quadratic differential equations (not necessarily derived from normal curvature properties) has been carried out by Guíñez [11]. See also the work of Bruce and Fidal [4] for the analysis of umbilics for general quadratic differential equations.

3. Geometric mean curvature lines near umbilic points

Let 0 be an umbilic point of a C^r, $r \geqslant 4$, immersion α parametrized in a Monge chart (x, y) by $\alpha(x, y) = (x, y, h(x, y))$, where

$$h(x,y) = \frac{k}{2}(x^2+y^2) + \frac{a}{6}x^3 + \frac{b}{2}xy^2 + \frac{c}{6}y^3 + O(4) \tag{3}$$

This reduced form is obtained by means of a rotation of the x, y-axes. See [12], [14].

According to Darboux [5], [12], the differential equation of principal curvature lines is given by:

$$-[by + P_1]dy^2 + [(b-a)x + cy + P_2]dxdy + [by + P_3]dx^2 = 0, \tag{4}$$

Here the $P_i = P_i(x, y)$ are functions of the form $P_i(x, y) = O(x^2 + y^2)$.

As an starting point, recall the behavior of principal lines near Darbouxian umbilics in the following proposition.

PROPOSITION 1 [12], [14]. — *Assume the notation established in 3. Suppose that the transversality condition $T : b(b-a) \neq 0$ holds and consider the following situations:*

$D_1)$ $\Delta_P > 0$

$D_2)$ $\Delta_P < 0$ *and* $\dfrac{a}{b} > 1$

$D_3)$ $\dfrac{a}{b} < 1$

Here $\Delta_P = 4b(a-2b)^3 - c^2(a-2b)^2$

Then each principal foliation has in a neighborhood of 0, *one hyperbolic sector in the* D_1 *case, one parabolic and one hyperbolic sector in* D_2 *case and three hyperbolic sectors in the case* D_3. *These points are called principal curvature Darbouxian umbilics.*

PROPOSITION 2.— *Assume the notation established in 3. Suppose that the transversality condition* $T_g : kb(b-a) \neq 0$ *holds and consider the following situations:*

$G_1)$ $\Delta_G > 0$

$G_2)$ $\Delta_G < 0$ *and* $\dfrac{a}{b} > 1$

$G_3)$ $\dfrac{a}{b} < 1.$

Here $\Delta_G = 4c^2(2a-b)^2 - [3c^2 + (a-5b)^2][3(a-5b)(a-b) + c^2]$.

Then each geometric mean curvature foliation has in a neighborhood of 0, *one hyperbolic sector in the* G_1 *case, one parabolic and one hyperbolic sector in* G_2 *case and three hyperbolic sectors in the case* G_3. *These umbilic points are called geometric mean curvature Darbouxian umbilics.*

The geometric mean curvature foliations $\mathbb{G}_{\alpha,i}$ *near an umbilic point of type* G_k *has a local behavior as shown in Figure 1. The separatrices of these singularities are called umbilic separatrices.*

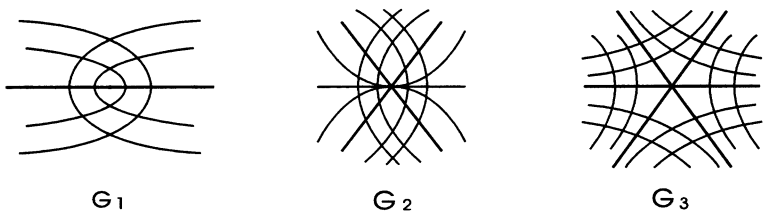

Figure 1. — Geometric mean curvature lines near the umbilic points G_i and their separatrices.

Proof. — Near 0 the functions $\sqrt{\mathcal{K}}$ and $\mathcal{H}$ have the following Taylor expansions. Assume here that $k > 0$, which can be achieved by means of an exchange in orientation. So it follows that,

$$\sqrt{\mathcal{K}} = k + \frac{1}{2}(a+b)x + \frac{1}{2}cy + O_1(2), \qquad \mathcal{H} = k + \frac{1}{2}(a+b)x + \frac{1}{2}cy + O_2(2).$$

Therefore the differential equation of the geometric mean curvature lines

$$[g - \sqrt{\mathcal{K}}G]dv^2 + 2[f - \sqrt{\mathcal{K}}F]dudv + [e - \sqrt{\mathcal{K}}E]du^2 = 0$$

is given by:

$$[(b-a)x + cy + M_1]dy^2 + [4by + M_2]dxdy - [(b-a)x + cy + M_3]dx^2 = 0 \quad (5)$$

where M_i, $i = 1, 2, 3$, represent functions of order $O((x^2 + y^2))$.

At the level of first jet the differential equation 5 is the same as that of the arithmetic mean curvature lines given by

$$[g - \mathcal{H}G]dv^2 + 2[f - \mathcal{H}F]dudv + [e - \mathcal{H}E]du^2 = 0.$$

The condition Δ_G coincides with the Δ_H condition established to characterizes the arithmetic mean curvature Darbouxian umbilics studied in detail in [7] reducing the analysis of that of hyperbolic saddles and nodes whose phase portrait is determined only by the first jet. □

Remark 1. — In the plane $b = 1$ the bifurcation diagram of the umbilic points of types G_i for the geometric mean curvature configuration and of types D_i for the principal configuration is as shown in Figure 2 below.

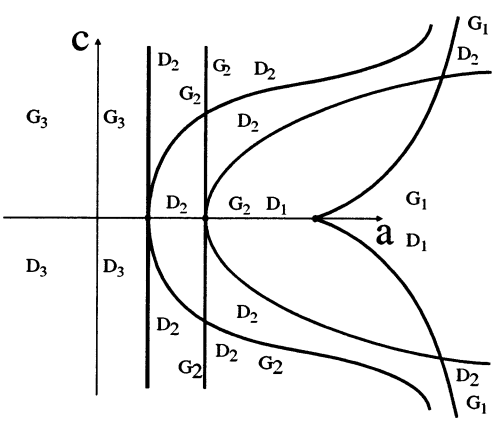

Figure 2. — Bifurcation diagram of points D_i and G_i.

THEOREM 1.— *An immersion $\alpha \in \mathcal{M}^{r,s}(\mathbb{M}^2)$, $r \geqslant 4$, is C^3-local geometric mean curvature structurally stable at $\mathcal{U}_\alpha$ if and only if every $p \in \mathcal{U}_\alpha$ is one of the types G_k, $k = 1, 2, 3$ of proposition 2.*

Proof.— Clearly proposition 2 shows that the conditions G_i, $i = 1, 2, 3$ and $T_g : kb(b-a) \neq 0$ included imply the C^3-local geometric mean curvature structural stability. This involves the construction of the homeomorphism (by means of canonical regions) mapping simultaneously minimal and maximal geometric mean curvature lines around the umbilic points of α onto those of a C^4 slightly perturbed immersion.

We will discuss the necessity of the condition $T_g : k(b-a)b \neq 0$ and of the conditions G_i, $i = 1, 2, 3$. The first one follows from its identification with a transversality condition that guarantees the persistent isolatedness of the umbilic points of α and its separation from the parabolic set, as well as the persistent regularity of the Lie-Cartan surface $\mathcal{G}$. Failure of T_g condition has the following implications:

a) $b(b-a) = 0$; in this case the elimination or splitting of the umbilic point can be achieved by small perturbations.

b) $k = 0$ and $b(b-a) \neq 0$; in this case a small perturbation separates the umbilic point from the parabolic set.

The necessity of G_i follows from its dynamic identification with the hyperbolicity of the equilibria along the projective line of the vector field $\mathcal{G}$. Failure of this condition would make possible to change the number of geometric mean curvature umbilic separatrices at the umbilic point by means a small perturbation of the immersion. □

4. Periodic geometric mean curvature lines

Let $\alpha : \mathbb{M}^2 \to \mathbb{R}^3$ be an immersion of a compact and oriented surface and consider the foliations $\mathbb{G}_{\alpha,i}$, $i = 1, 2$, given by the *geometric mean curvature lines.*

In terms of geometric invariants, here is established an integral expression for the first derivative of the return map of a periodic geometric mean curvature line, called *geometric mean curvature cycle.* Recall that the return map associated to a cycle is a local diffeomorphism with a fixed point, defined on a cross section normal to the cycle by following the integral curves through this section until they meet again the section. This map is called holonomy in Foliation Theory and Poincaré Map in Dynamical Systems, [16].

A geometric mean curvature cycle is called *hyperbolic* if the first derivative of the return map at the fixed point is different from one.

The geometric mean curvature foliations $\mathbb{G}_{\alpha,i}$ has no geometric mean curvature cycles such that the return map reverses the orientation. Initially, the integral expression for the derivative of the return map is obtained in class C^6; see Lemma 2 and Proposition 3. Later on, in Remark 3 it is shown how to extend it to class C^3.

The characterization of hyperbolicity of geometric mean curvature cycles in terms of local structural stability is given in Theorem 2 of this section.

LEMMA 1.— *Let $c : I \to \mathbb{M}^2$ be a geometric mean curvature line parametrized by arc length. Then the Darboux frame is given by:*

$$\begin{aligned} T' &= k_g N \wedge T + \sqrt{\mathcal{K}} N \\ (N \wedge T)' &= -k_g T + \tau_g N \\ N' &= -\sqrt{\mathcal{K}} T - \tau_g N \wedge T \end{aligned}$$

where $\tau_g = \pm\sqrt{2\mathcal{H} - 2\sqrt{\mathcal{K}}\sqrt[4]{\mathcal{K}}}$. The sign of τ_g is positive (resp. negative) if c is maximal (resp. minimal) geometric mean curvature line.

Proof.— The normal curvature k_n of the curve c is by the definition the geometric mean curvature $\sqrt{\mathcal{K}}$. From the Euler equation $k_n = k_1 \cos^2\theta + k_2 \sin^2\theta = \sqrt{\mathcal{K}}$, get $\tan\theta = \pm\sqrt{\frac{\sqrt{\mathcal{K}}-k_1}{k_2-\sqrt{\mathcal{K}}}}$. Therefore, by direct calculation, the geodesic torsion is given by $\tau_g = (k_2 - k_1)\sin\theta\cos\theta = \pm\sqrt{2\mathcal{H} - 2\sqrt{\mathcal{K}}\sqrt[4]{\mathcal{K}}}$. □

Remark 2.— The expression for the geodesic curvature k_g will not be needed explicitly in this work. However, it can be given in terms of the principal curvatures and their derivatives using a formula due to Liouville [24].

LEMMA 2.— *Let $\alpha : \mathbb{M} \to \mathbb{R}^3$ be an immersion of class C^r, $r \geqslant 6$, and c be a mean curvature cycle of α, parametrized by arc length s and of length L. Then the expression,*

$$\alpha(s,v) = c(s) + v(N \wedge T)(s) + [\frac{2\mathcal{H}(s) - \sqrt{\mathcal{K}(s)}}{2} v^2 + \frac{A(s)}{6} v^3 + v^3 B(s,v)]N(s)$$

where $B(s,0) = 0$, defines a local chart (s,v) of class C^{r-5} in a neighborhood of c.

Proof. — The curve c is of class C^{r-1} and the map $\alpha(s,v,w) = c(s) + v(N \wedge T)(s) + wN(s)$ is of class C^{r-2} and is a local diffeomorphism in a neighborhood of the axis s. In fact $[\alpha_s, \alpha_v, \alpha_w](s,0,0) = 1$. Therefore there is a function $W(s,v)$ of class C^{r-2} such that $\alpha(s,v,W(s,v))$ is a parametrization of a tubular neighborhood of $\alpha \circ c$. Now for each s, $W(s,v)$ is just a parametrization of the curve of intersection between $\alpha(\mathbb{M})$ and the normal plane generated by $\{(N \wedge T)(s), N(s)\}$. This curve of intersection is tangent to $(N \wedge T)(s)$ at $v = 0$ and notice that $k_n(N \wedge T)(s) = 2\mathcal{H}(s) - \sqrt{\mathcal{K}(s)}$. Therefore,

$$\begin{aligned} \alpha(s,v,W(s,v)) &= c(s) + v(N \wedge T)(s) \\ &\quad + [\tfrac{2\mathcal{H}(s) - \sqrt{\mathcal{K}(s)}}{2} v^2 + \tfrac{A(s)}{6} v^3 + v^3 B(s,v)] N(s), \end{aligned} \tag{6}$$

where A is of class C^{r-5} and $B(s,0) = 0$. □

We now compute the coefficients of the first and second fundamental forms in the chart (s,v) constructed above, to be used in proposition 3.

$$\begin{aligned} N(s,v) = \tfrac{\alpha_s \wedge \alpha_v}{|\alpha_s \wedge \alpha_v|} &= [-\tau_g(s)v + O(2)]T(s) \\ &\quad -[(2\mathcal{H}(s) - 2\sqrt{\mathcal{K}(s)})v + O(2)](N \wedge T)(s) + [1 + O(2)]N(s). \end{aligned}$$

Therefore it follows that $E =< \alpha_s, \alpha_s >$, $F =< \alpha_s, \alpha_v >$, $G =< \alpha_v, \alpha_v >$, $e =< N, \alpha_{ss} >$, $f =< N, \alpha_{sv} >$ and $g =< N, \alpha_{vv} >$ are given by

$$\begin{aligned} E(s,v) &= 1 - 2k_g(s)v + h.o.t \\ F(s,v) &= 0 + 0.v + h.o.t \\ G(s,v) &= 1 + 0.v + h.o.t \\ e(s,v) &= \sqrt{\mathcal{K}(s)} + v[\tau_g'(s) - 2k_g(s)\mathcal{H}(s)] + h.o.t \\ f(s,v) &= \tau_g(s) + \{[2\mathcal{H}(s) - \sqrt{\mathcal{K}(s)}]' + k_g(s)\tau_g(s)\}v + h.o.t \\ g(s,v) &= 2\mathcal{H}(s) - \sqrt{\mathcal{K}(s)} + A(s)v + h.o.t \end{aligned} \tag{7}$$

PROPOSITION 3. — *Let $\alpha : \mathbb{M} \to \mathbb{R}^3$ be an immersion of class C^r, $r \geqslant 6$ and c be a maximal (resp. minimal) geometric mean curvature cycle of α, parametrized by arc length s and of total length L. Then the derivative of the Poincaré map π_α associated to c is given by:*

$$ln\pi_\alpha'(0) = \int_0^L \left[\frac{[\sqrt{\mathcal{K}}]_v}{2\tau_g} + \frac{k_g}{\tau_g}(\mathcal{H} - \sqrt{\mathcal{K}}) \right] ds.$$

Here $\tau_g = \sqrt[4]{\mathcal{K}}\sqrt{2\mathcal{H} - 2\sqrt{\mathcal{K}}}$ (resp. $\tau_g = -\sqrt[4]{\mathcal{K}}\sqrt{2\mathcal{H} - 2\sqrt{\mathcal{K}}}$).

Proof. — The Poincaré map associated to c is the map $\pi_\alpha : \Sigma \to \Sigma$ defined in a transversal section to c such that $\pi_\alpha(p) = p$ for $p \in c \cap \Sigma$ and $\pi_\alpha(q)$ is the first return of the geometric mean curvature line through q to the section Σ, choosing a positive orientation for c. It is a local diffeomorphism and is defined, in the local chart (s, v) introduced in Lemma 2, by $\pi_\alpha : \{s = 0\} \to \{s = L\}$, $\pi_\alpha(v_0) = v(L, v_0)$, where $v(s, v_0)$ is the solution of the Cauchy problem

$$(g - \sqrt{\mathcal{K}}G)dv^2 + 2(f - \sqrt{\mathcal{K}}F)dsdv + (e - \sqrt{\mathcal{K}}E)ds^2 = 0, \quad v(0, v_0) = v_0.$$

Direct calculation gives that the derivative of the Poincaré map satisfies the following linear differential equation:

$$\frac{d}{ds}(\frac{dv}{dv_0}) = -\frac{N_v}{M}(\frac{dv}{dv_0}) = -\frac{[e - \sqrt{\mathcal{K}}E]_v}{2[f - \sqrt{\mathcal{K}}F]}(\frac{dv}{dv_0})$$

Therefore, using equation 7 it results that

$$\frac{[e - \sqrt{\mathcal{K}}E]_v}{2[f - \sqrt{\mathcal{K}}F]} = -\frac{\tau_g'}{2\tau_g} - \frac{[\sqrt{\mathcal{K}}]_v}{2\tau_g} - \frac{k_g}{\tau_g}(\mathcal{H} - \sqrt{\mathcal{K}}).$$

Integrating the equation above along an arc $[s_0, s_1]$ of geometric mean curvature line, it follows that:

$$\frac{dv}{dv_0}|_{v_0=0} = \frac{(\tau_g(s_1))^{\frac{-1}{2}}}{(\tau_g(s_0))^{\frac{-1}{2}}} exp[\int_{s_0}^{s_1} \left[\frac{[\sqrt{\mathcal{K}}]_v}{2\tau_g} + \frac{k_g}{\tau_g}(\mathcal{H} - \sqrt{\mathcal{K}})\right] ds. \qquad (8)$$

Applying 8 along the geometric mean curvature cycle of length L, obtain

$$\frac{dv}{dv_0}|_{v_0=0} = exp[\int_0^L \left[\frac{[\sqrt{\mathcal{K}}]_v}{2\tau_g} + \frac{k_g}{\tau_g}(\mathcal{H} - \sqrt{\mathcal{K}})\right] ds.$$

From the equation $\mathcal{K} = (eg - f^2)/(EG - F^2)$ evaluated at $v = 0$ it follows that $\mathcal{K} = \sqrt{\mathcal{K}}[2\mathcal{H} - \sqrt{\mathcal{K}}] - \tau_g^2$. Solving this equation it follows that $\tau_g = \sqrt[4]{\mathcal{K}}\sqrt{2\mathcal{H} - 2\sqrt{\mathcal{K}}}$. This ends the proof. □

Remark 3. — At this point we show how to extend the expression for the derivative of the hyperbolicity of geometric mean curvature cycles established for class C^6 to class C^3 (in fact we need only class C^4).

The expression 8 is the derivative of the transition map for a geometric mean curvature foliation (which at this point is only of class C^1), along an arc of geometric mean curvature line. In fact, this follows by approximating the C^3 immersion by one of class C^6. The corresponding transition map

(now of class C^4) whose derivative is given by expression 8 converges to the original one (in class C^1) whose expression must given by the same integral, since the functions involved there are the uniform limits of the corresponding ones for the approximating immersion.

Remark 4. — The expression for the derivative of the Poincaré map is obtained by the integration of a one form along the geometric mean curvature line γ. In the case of the arithmetic mean curvature cycles the correspondent expression for the derivative is given by: $\ln \pi'(0) = \frac{1}{2}\int_0^L \frac{\mathcal{H}_v}{\sqrt{\mathcal{H}^2-\mathcal{K}}} ds$. This was proved in [7].

PROPOSITION 4. — *Let $\alpha : \mathbb{M} \to \mathbb{R}^3$ be an immersion of class C^r, $r \geqslant 6$, and c be a maximal geometric mean curvature cycle of α, parametrized by arc length and of length L. Consider a chart (s, v) as in lemma 2 and consider the deformation*

$$\beta_\epsilon(s,v) = \beta(\epsilon, s, v) = \alpha(s,v) + \epsilon[\frac{A_1(s)}{6}v^3]\delta(v)N(s)$$

where $\delta = 1$ in neighborhood of $v = 0$, with small support and $A_1(s) = \tau_g(s) > 0$.

Then c is a geometric mean curvature cycle of β_ϵ for all ϵ small and c is a hyperbolic geometric mean curvature cycle for β_ϵ, $\epsilon \neq 0$.

Proof. — In the chart (s, v), for the immersion β, it is obtained that:

$$\begin{aligned}
E(s,v) &= 1 - 2k_g(s)v + h.o.t \\
F(s,v) &= 0 + 0.v + h.o.t \\
G(s,v) &= 1 + 0.v + h.o.t \\
e(s,v) &= \sqrt{\mathcal{K}(s)} + v[\tau_g'(s) - 2k_g(s)\mathcal{H}(s)\ )] + h.o.t \\
f(s,v) &= \tau_g(s) + [2\mathcal{H}(s) - \sqrt{\mathcal{K}(s)}]'v + h.o.t \\
g(s,v) &= 2\mathcal{H}(s) - \sqrt{\mathcal{K}(s)} + v[A(s) + \epsilon A_1(s)] + h.o.t
\end{aligned}$$

In the expressions above $E =< \beta_s, \beta_s >$, $F =< \beta_s, \beta_v >$, $G =< \beta_v, \beta_v >$, $e =< \beta_{ss}, N >$, $f =< N, \beta_{sv} >$, $g =< N, \beta_{vv} >$ and $N = \beta_s \wedge \beta_v / \mid \beta_s \wedge \beta_v \mid$.

Therefore c is a maximal geometric mean curvature cycle for all β_ϵ and at $v = 0$ it follows from equation $\mathcal{K} = (eg - f^2)/(EG - F^2)$ that

$$\mathcal{K}_v = \epsilon\sqrt{\mathcal{K}}A_1(s) + f(k_g, \tau_g, \sqrt{\mathcal{K}}, \mathcal{H})(s).$$

Therefore, assuming $A_1(s) = \tau_g(s) > 0$, it results that,

$$\frac{d}{d\epsilon}(ln\pi'(0))|_{\epsilon=0} = -\int_0^L \frac{d}{d\epsilon}\left(\frac{(\sqrt{\mathcal{K}})_v}{2\tau_g}\right) ds = -\frac{1}{4}L < 0. \qquad \square$$

As a synthesis of propositions 3 and 4, the following theorem is obtained.

THEOREM 2.— *An immersion $\alpha \in \mathcal{M}^{r,s}(\mathbb{M}^2)$, $r \geqslant 6$, is C^6-local geometric mean curvature structurally stable at a geometric mean curvature cycle c if only if,*

$$\int_0^L \left[\frac{[\sqrt{\mathcal{K}}]_v}{2\tau_g} + \frac{k_g}{\tau_g}(\mathcal{H} - \sqrt{\mathcal{K}})\right] ds \neq 0.$$

Proof.— Using propositions 3 and 4, the local topological character of the foliation can be changed by small perturbation of the immersion, when the cycle is not hyperbolic. $\square$

5. Geometric mean curvature lines near the parabolic line

Let 0 be a parabolic point of a C^r, $r \geqslant 6$, immersion α parametrized in a Monge chart (x, y) by $\alpha(x, y) = (x, y, h(x, y))$, where

$$\begin{aligned} h(x,y) = {} & \tfrac{k}{2}y^2 + \tfrac{a}{6}x^3 + \tfrac{b}{2}xy^2 + \tfrac{d}{2}x^2y + \tfrac{c}{6}y^3 \\ & + \tfrac{A}{24}x^4 + \tfrac{B}{6}x^3y + \tfrac{C}{4}x^2y^2 + \tfrac{D}{6}xy^3 + \tfrac{E}{24}y^4 + O(5) \end{aligned} \tag{9}$$

The coefficients of the first and second fundamental forms are given by:

$$\begin{aligned} E(x,y) &= 1 + O(4) \\ F(x,y) &= dkxy^2 + \tfrac{bk}{2}y^3 + O(4) \\ G(x,y) &= 1 + k^2y^2 + 2kbxy^2 + kcy^3 + O(4) \\ e(x,y) &= ax + dy + \tfrac{A}{2}x^2 + Bxy + \tfrac{C}{2}y^2 - \tfrac{1}{2}dk^2y^3 + O(4) \\ f(x,y) &= dx + by + \tfrac{B}{2}x^2 + Cxy + \tfrac{D}{2}y^2 - \tfrac{1}{2}dk^2xy^2 - \tfrac{1}{2}bk^2y^3 + O(4) \\ g(x,y) &= k + bx + cy + \tfrac{C}{2}x^2 + Dxy + \tfrac{1}{2}(E - k^3)y^2 \\ & \quad - \tfrac{1}{2}k^2dx^2y - \tfrac{3}{2}bk^2xy^2 + dk^2xy^2 + (\tfrac{b}{2} - c)k^2y^3 + O(4) \end{aligned} \tag{10}$$

The Gaussian curvature is given by

$$\begin{aligned} \mathcal{K}(x,y) = {} & k(ax + dy) + \tfrac{1}{2}(Ak + 2ab - 2d^2)x^2 + (Bk + ac - bd)xy \\ & + \tfrac{1}{2}(Ck + 2cd - 2b^2)y^2 + O(3). \end{aligned} \tag{11}$$

The coefficients of the quartic differential equation 2 are given by

$$\begin{aligned}
A_{40} &= -k(ax+dy)+\tfrac{1}{2}(2a^2+2d^2-2ab-Ak)x^2\\
&\quad +(2ad-Bk+bd-ac)xy+\tfrac{1}{2}(2b^2+2d^2-2cd-Ck)y^2\\
&\quad +\tfrac{1}{6}(6dB-3Ab)x^3+\tfrac{1}{2}(2dA+3dC-Ac)x^2y\\
&\quad +\tfrac{1}{2}(4dB+3Cb-2Bc)xy^2\\
&\quad +\tfrac{1}{6}(12dk^3+6dC+6bD-3cC-3dE)y^3+O(4)\\
A_{31} &= 4adx^2+4(ab+d^2)xy+4bdy^2+2Adx^3\\
&\quad +(6dB+2Ab)x^2y+(4bB+6dC)xy^2+2(dD+bC)y^3+O(4)\\
A_{22} &= 6d^2x^2+12bdxy+6b^2y^2+(3AD+6dB)x^3\\
&\quad +6(BD+bB+4dC)x^2y+(4dD+3CD+12bC)xy^2+6bDy^3+O(4)\\
A_{13} &= 4k(dx+by)+(2Bk+4bd)x^2+4(Ck+cd+b^2)xy+(2kD+4bc)y^2\\
&\quad +(6BD+2bB+2dC)x^3+(12CD+2Bc+6Cb)x^2y\\
&\quad +(6D^2+4cC+2dE+2bD-4dk^3)xy^2+(2bE+2cD-4bk^3)y^3+O(4)\\
A_{04} &= k^2+k(2b-a)x+k(2c-d)y+\tfrac{1}{2}[-2ab+2b^2+(2C-A)k+2d^2]x^2\\
&\quad +[(2D-B)k+c(2b-a)+bd]xy+[c^2+\tfrac{1}{2}(2E-C)k-k^4-cd+b^2]y^2\\
&\quad +\tfrac{1}{6}(18CD+6dB-3Ab+6Cb)x^3+\tfrac{1}{2}(3dC-Ac+2cC)x^2y\\
&\quad +\tfrac{1}{2}(2bE+6DE-6Dk^3-2bk^3-2Bc+3Cb)xy^2\\
&\quad +\tfrac{1}{6}(6cE-6ck^3-3cC-3dE+6bD)y^3+O(4)
\end{aligned} \tag{12}$$

LEMMA 3. — *Let* 0 *be a parabolic point and consider the parametrization* $(x,y,h(x,y))$ *as above. If* $k>0$ *and* $a^2+d^2\neq 0$ *then the set of parabolic points is locally a regular curve normal to the vector* (a,d) *at* 0.

If $a\neq 0$ *the parabolic curve is transversal to the minimal principal direction* $(1,0)$.

If $a=0$ *then the parabolic curve is tangent to the principal direction given by* $(1,0)$ *and has quadratic contact with the corresponding minimal principal curvature line if* $dk(Ak-3d^2)\neq 0$.

Proof. — If $a\neq 0$, from the expression of K given by equation 11 it follows that the parabolic line is given by $x=-\frac{d}{a}y+O_1(2)$ and so is transversal to the principal direction $(1,0)$ at $(0,0)$.

If $a = 0$, from the expression of $\mathcal{K}$ given by equation 11 it follows that the parabolic line is given by $y = \frac{2d^2 - Ak}{2dk}x^2 + O_2(3)$ and that $y = -\frac{d}{2k}x^2 + O_3(3)$ is the principal line tangent to the principal direction $(1, 0)$. Now the condition of quadratic contact $\frac{2d^2 - Ak}{2dk} \neq -\frac{d}{2k}$ is equivalent to $dk(Ak - 3d^2) \neq 0$. □

PROPOSITION 5.— *Let 0 be a parabolic point and the Monge chart (x, y) as above.*

If $a \neq 0$ then the mean geometric curvature lines are transversal to the parabolic curve and the mean curvatures lines are shown in the picture below, the cuspidal case.

If $a = 0$ and $\sigma = dk(Ak - 3d^2) \neq 0$ then the mean geometric curvature lines are shown in the picture below. In fact, if $\sigma > 0$ then the mean geometric curvature lines are folded saddles. Otherwise, if $\sigma < 0$ then the mean geometric curvature lines are folded nodes. The two separatrices of these tangential singularities, folded saddle and folded node, as illustrate in the Figure 3 below, are called parabolic separatrices.

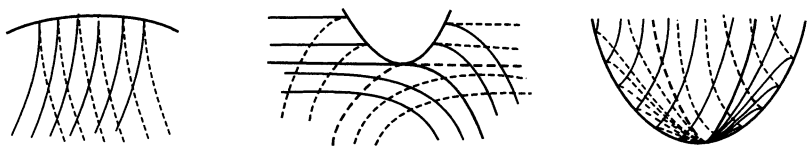

Figure 3. — Geometric mean curvature lines near a parabolic point (cuspidal, folded saddle and folded node) and their separatrices.

Proof. — Consider the quartic differential equation

$$H(x, y, p) = A_{04}p^4 + A_{13}p^3 + A_{22}p^2 + A_{31}p + A_{40} = 0$$

where $p = [dx : dy]$ and the Lie-Cartan line field of class C^{r-3} defined by

$$\begin{aligned} x' &= H_p \\ y' &= pH_p \\ p' &= -(H_x + pH_y), \qquad p = \tfrac{dy}{dx} \end{aligned}$$

where A_{ij} are given by equation 12.

If $a \neq 0$ the vector Y is regular and therefore the mean geometric curvature lines are transversal to the parabolic line and at parabolic points these lines are tangent to the principal direction $(1, 0)$.

If $a = 0$, direct calculation gives $H(0) = 0$, $H_x(0) = 0$, $H_y(0) = -kd$, $H_p(0) = 0$.

Therefore, solving the equation $H(x, y(x,p), p) = 0$ near 0 it follows, by the Implicit Function Theorem, that:

$$y = y(x,p) = \frac{2d^2 - Ak}{2kd}x^2 - \frac{2Abdk - BAk^2 - 2d^3b}{2k^2d^2}x^3 + O(4).$$

Therefore the vector field Y given by the differential equation below

$$x' = H_p(x, y(x,p), p)$$
$$p' = -(H_x + pH_y)(x, y(x,p), p)$$

is given by

$$x' = 4\frac{d^3}{k}x^3 + 12d^2x^2p + 12dkxp^2 + 4k^2p^3 + O(4)$$
$$p' = (Ak - 2d^2)x + dkp + O(2).$$

The singular point 0 is isolated and the eigenvalues of the linear part of Y are given by $\lambda_1 = 0$ and $\lambda_2 = dk$. The correspondent eigenvectors are given by $f_1 = (1, (2d^2 - Ak)/dk)$ and $f_2 = (0, 1)$.

Performing the calculations, restricting Y to the center manifold W^c of class C^{r-3}, $T_0W^c = f_1$, it follows that

$$Y_c = -4\frac{(Ak - 3d^2)^3}{kd^3}x^3 + 0(4)$$

It follows that 0 is a topological saddle or node of cubic type provided $\sigma(Ak - 3d^2)kd \neq 0$. If $\sigma > 0$ then the mean geometric curvature lines are folded saddles and if $\sigma < 0$ then the mean geometric curvature lines are folded nodes. In the case $\sigma > 0$, the center manifold W^c is unique, [23], cap. V, page 319, and so the saddle separatrices are well defined. See Figure 4 below.

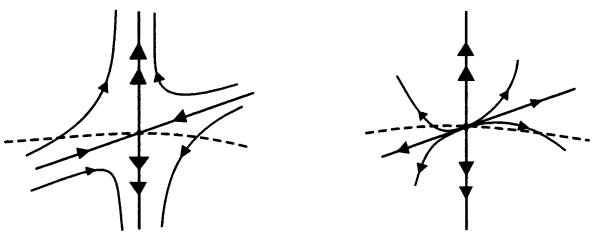

Figure 4. — Phase portrait of the vector field Y near singularities.

The reader may find a more complete study of this structure, which can be expressed in the context of normal hyperbolicity, in the paper of Palis and Takens [19].

Notice that due to the constrains of the problem treated here, the non hyperbolic saddles and nodes, which in the standard theory would bifurcate into three singularities, are actually structurally stable (do not bifurcate).

For a deep analysis of the lost of the hyperbolicity condition and the consequent bifurcations, the reader is addressed to the book of Roussarie [21]. □

THEOREM 3.— *An immersion $\alpha \in \mathcal{M}^{r,s}(\mathbb{M}^2)$, $r \geqslant 6$, is C^6-local geometric mean curvature structurally stable at a tangential parabolic point p if only if, the condition $\sigma \neq 0$ in proposition 5 holds.*

Proof.— Direct from Lemma 3 and proposition 5, the local topological character of the foliation can be changed by small perturbation of the immersion when $\sigma = 0$. □

6. Examples of geometric mean curvature configurations

As mentioned in the Introduction, no examples of geometric mean curvature foliations are given in the literature, in contrast with the principal and asymptotic foliation. In this section are studied the geometric mean curvature configurations in two classical surfaces: The Torus and the Ellipsoid. In contrast with the principal case [22], [24] (but in concordance with the arithmetic mean curvature one [7]) non-trivial recurrence can occur here.

PROPOSITION 6.— *Consider a torus of revolution $T(r,R)$ obtained by rotating a circle of radius r around a line in the same plane and at a distance R, $R > r$, from its center. Define the function*

$$\rho = \rho(\frac{r}{R}) = 2(\frac{r}{R})^{\frac{3}{4}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{ds}{\sqrt[4]{\cos s}(1+\frac{r}{R}\cos s)^{\frac{3}{4}}}.$$

Then the geometric mean curvature lines on $T(r,R)$, defined in the elliptic region are all closed or all recurrent according to $\rho \in \mathbb{Q}$ or $\rho \in \mathbb{R} \setminus \mathbb{Q}$. Furthermore, both cases occur for appropriate (r,R).

Proof.— The torus of revolution $T(r,R)$ is parametrized by

$$\alpha(s,\theta) = ((R + r\cos s)\cos\theta, (R + r\cos s)\sin\theta,\ r\sin s).$$

Direct calculation shows that $E = r^2$, $F = 0$, $G = [R + r\cos s]^2$, $e = -r$, $f = 0$ and $g = -\cos s(R + r\cos s)$. Clearly (s,θ) is a principal chart.

The differential equation of the geometric mean curvature lines, in the principal chart (s,θ), is given by $\sqrt{eE}ds^2 - \sqrt{gG}d\theta^2 = 0$. This is equivalent to

$$(\frac{ds}{d\theta})^2 = \sqrt{\frac{\cos s(R + r\cos s)^3}{r^3}}$$

Solving the equation above it follows that,

$$\int_{\theta_0}^{\theta_1} d\theta = (\frac{r}{R})^{\frac{3}{4}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{ds}{\sqrt[4]{\cos s}(1 + \frac{r}{R}\cos s)^{\frac{3}{4}}}.$$

So the two Poincaré maps, $\pi_\pm : \{s = -\frac{\pi}{2}\} \to \{s = \frac{\pi}{2}\}$, defined by $\pi_\pm(\theta_0) = \theta_0 \pm 2\pi\rho(\frac{r}{R})$ have rotation number equal to $\pm\rho(\frac{r}{R})$. Direct calculations gives that $\rho(0) > 0$ and $\rho'(0) < 0$. Therefore, both the rational and irrational cases occur. This ends the proof. □

PROPOSITION 7.— *Consider an ellipsoid $\mathbb{E}_{a,b,c}$ with three axes $a > b > c > 0$. Then $\mathbb{E}_{a,b,c}$ have four umbilic points located in the plane of symmetry orthogonal to middle axis; they are of the type G_1 for geometric mean curvature lines and of type D_1 for the principal curvature lines.*

Proof. — This follows from proposition 2 and the fact that the arithmetic mean curvature lines have this configuration, as established in [7]. □

PROPOSITION 8.— *Consider an ellipsoid $\mathbb{E}_{a,b,c}$ with three axes $a > b > c > 0$. On the ellipse $\Sigma \subset \mathbb{E}_{a,b,c}$, containing the four umbilic points, p_i, $i = 1, \cdots, 4$, counterclockwise oriented, denote by S_1 (resp. S_2) the distance between the adjacent umbilic points p_1 and p_4 (resp. p_1 and p_2). Define $\rho = \frac{S_2}{S_1}$.*

Then if $\rho \in \mathbb{R} \setminus \mathbb{Q}$ (resp. $\rho \in \mathbb{Q}$) all the geometric mean curvature lines are recurrent (resp. all, with the exception of the geometric mean curvature umbilic separatrices, are closed). See Figure 5 below.

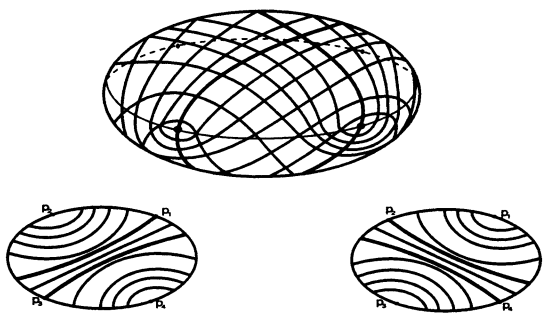

Figure 5. — Upper view of geometric mean curvature lines on the ellipsoid.

Proof. — The ellipsoid $\mathbb{E}_{a,b,c}$ belongs to the triple orthogonal system of surfaces defined by the one parameter family of quadrics, $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1$ with $a > b > c > 0$, see also [24] and [22].

The following parametrization of $\mathbb{E}_{a,b,c}$.

$$\alpha(u,v) = \left(\pm \sqrt{\frac{M(u,v,a)}{W(a,b,c)}}, \pm \sqrt{\frac{M(u,v,b)}{W(b,a,c)}}, \pm \sqrt{\frac{M(u,v,c)}{W(c,a,b)}} \right)$$

where,
$M(u,v,w) = w^2(u+w^2)(v+w^2)$ and $W(a,b,c) = (a^2-b^2)(a^2-c^2)$, define the ellipsoidal coordinates (u,v) on $\mathbb{E}_{a,b,c}$, where $u \in (-b^2,-c^2)$ and $v \in (-a^2,-b^2)$.

The first fundamental form of $\mathbb{E}_{a,b,c}$ is given by:

$$I = ds^2 = Edu^2 + Gdv^2 = \frac{1}{4}\frac{(u-v)u}{h(u)}du^2 + \frac{1}{4}\frac{(v-u)v}{h(v)}dv^2$$

The second fundamental form is given by

$$II = edu^2 + gdv^2 = \frac{abc(u-v)}{4\sqrt{uv}h(u)}du^2 + \frac{abc(v-u)}{4\sqrt{uv}h(v)}dv^2,$$

where $h(x) = (x+a^2)(x+b^2)(x+c^2)$. The four umbilic points are $(\pm x_0, 0, \pm z_0)$ $= (\pm a\sqrt{\frac{a^2-b^2}{a^2-c^2}}, 0, \pm c\sqrt{\frac{c^2-b^2}{c^2-a^2}})$.

The differential equation of the geometric mean curvature lines is given by:

$$\left(\frac{dv}{du}\right)^2 = \sqrt{\frac{eE}{gG}} = \sqrt{\frac{-\frac{u}{h(u)}}{-\frac{v}{h(v)}}}.$$

Define $d\sigma_1 = \sqrt[4]{-\frac{u}{h(u)}}du$ and $d\sigma_2 = \sqrt[4]{-\frac{v}{h(v)}}dv$. By integration, this leads to the chart (σ_1, σ_2), in which the differential equation of the geometric mean curvature lines is given by

$$d\sigma_1^2 - d\sigma_2^2 = 0.$$

On the ellipse $\Sigma = \{(x,0,z) | \frac{x^2}{a^2} + \frac{z^2}{c^2} = 1\}$ the distance between the umbilic points $p_1 = (x_0,0,z_0)$ and $p_4 = (x_0,0,-z_0)$ is given by $S_1 = \int_{-b^2}^{-c^2} \frac{\sqrt[4]{-v}}{\sqrt{h(v)}}dv$ and that between the umbilic points $p_1 = (x_0,0,z_0)$ and $p_2 = (-x_0,0,z_0)$ is given by $S_2 = \int_{-a^2}^{-b^2} \frac{\sqrt[4]{-u}}{\sqrt{h(u)}}du$.

It is obvious that the ellipse Σ is the union of four umbilic points and the four principal umbilical separatrices for the principal foliations. So $\Sigma\backslash\{p_1, p_2, p_3, p_4\}$ is a transversal section of both geometric mean curvature foliations. The differential equation of the geometric mean curvature lines in the principal chart (u, v) is given by $\sqrt{eE}du^2 - \sqrt{gG}dv^2 = 0$, which is equivalent to $(\sqrt[4]{eE}du)^2 = (\sqrt[4]{gG}dv)^2$, which amounts to $d\sigma_1 = \pm d\sigma_2$. Therefore near the umbilic point p_1 the geometric mean curvature lines with a geometric mean curvature umbilic separatrix contained in the region $\{y > 0\}$ define a the transition map $\sigma_+ : \Sigma \to \Sigma$ which is an isometry, reversing the orientation, with $\sigma_+(p_1) = p_1$. This follows because in the principal chart (u, v) this map is defined by $\sigma_+ : \{u = -b^2\} \to \{v = -b^2\}$ which satisfies the differential equation $\frac{d\sigma_2}{d\sigma_1} = -1$. By analytic continuation it results that σ_+ is an orientation reversing isometry, with two fixed points $\{p_1,\ p_3\}$. The geometric reflection σ_-, defined in the region $y < 0$ have the two umbilics $\{p_2,\ p_4\}$ as fixed points.

So on the ellipse parametrized by arclength defined by σ_i, the Poincaré return map $\pi_1 : \Sigma \to \Sigma$ (composition of two isometries σ_+ and σ_-) is a rotation with rotation number given by $\frac{S_2}{S_1}$.

Analogously for the other geometric mean curvature foliation, with the Poincaré return map given by $\pi_2 = \tau_+ \circ \tau_-$, where τ_+ and τ_- are two isometries having respectively $\{p_2, p_4\}$ and $\{p_1, p_3\}$ as fixed points. □

7. Geometric mean curvature structural stability

In this section the results of sections 3, 4 and 5 are put together to provide sufficient conditions for geometric mean curvature stability, outlined below.

THEOREM 4.— *The set of immersions $\mathcal{G}_i(\mathbb{M}^2), i = 1,\ 2$ which satisfy conditions i), ... , v) below are i-C^s-mean curvature structurally stable and $\mathcal{G}_i, i = 1,\ 2$ is open in $\mathcal{M}^{r,s}(\mathbb{M}^2)$, $r \geqslant s \geqslant 6$.*

i) The parabolic curve is regular : $\mathcal{K} = 0$ implies $d\mathcal{K} \neq 0$ and the tangential singularities are saddles and nodes.

ii) The umbilic points are of type G_i, $i = 1,\ 2,\ 3$.

iii) The geometric mean curvature cycles of $\mathbb{G}_{\alpha,i}$ are hyperbolic.

iv) The geometric mean curvature foliations $\mathbb{G}_{\alpha,i}$ has no separatrix connections. This means that there is no geometric mean curvature line joing two umbilic or tangential parabolic singularities and being separatrices at both ends. See propositions 2 and 5.

v) The limit set of every leaf of $\mathbb{G}_{\alpha,i}$ is a parabolic point, umbilic point or a geometric mean curvature cycle.

Proof. — The openness of $\mathcal{G}_i(\mathbb{M}^2)$ follows from the local structure of the geometric mean curvature lines near the umbilic points G_i, $i = 1, 2, 3$, near the geometric mean curvature cycles and by the absence of umbilic geometric mean curvature separatrix connections and the absence of recurrences. The equivalence can be performed by the method of canonical regions and their continuation as was done in [12], [14] for principal lines, and in [9], for asymptotic lines. □

Notice that Theorem 4 can be reformulated so as to give the mean geometric stability of the configuration rather than that of the separate foliations. To this end it is necessary to consider the folded extended lines, that is to consider the line of one foliation that arrive at the parabolic set at a given transversal point as continuing through the line of the other foliation leaving the parabolic set at this point, in a sort of "billiard". This gives raise to the extended folded cycles and separatrices that must be preserved by the homeomorphism mapping simultaneously the two foliations.

Therefore the third, fourth and fifth hypotheses above should be modified as follows:

iii') the extended folded periodic cycles should be hyperbolic,

iv') the extended folded separatrices should be disjoint,

v') the limit set of extended lines should be umbilic points, parabolic singularities and extended folded cycles.

The class of immersions which verify the extended five conditions i), ii), iii'), iv'), v') of a compact and oriented manifold $\mathbb{M}^2$ will be denoted by $\mathcal{G}(\mathbb{M}^2)$.

This procedure has been adopted by the authors in the case of asymptotic lines by the suspension operation in order to pass from the foliations to the configuration and properly formulate the stability results. See [9].

Remark 5. — In the space of convex immersions $\mathcal{M}_c^{r,s}(\mathbb{S}^2)$ ($\mathcal{K}_\alpha > 0$), the sets $\mathcal{G}(\mathbb{S}^2)$ and $\mathcal{G}_1(\mathbb{S}^2) \cap \mathcal{G}_2(\mathbb{S}^2)$ coincide.

The genericity result involving the five conditions above is formulated now.

THEOREM 5. — *The sets $\mathcal{G}_i$, $i = 1, 2$ are dense in $\mathcal{M}^{r,2}(\mathbb{M}^2)$, $r \geqslant 6$.*

In the space $\mathcal{M}_c^{r,2}(\mathbb{S}^2)$ the set $\mathcal{G}(\mathbb{S}^2)$ is dense.

The main ingredients for the proof of this theorem are the Lifting and Stabilization Lemmas, essential for the achievement of condition five. The conceptual background for this approach goes back to the works of Peixoto and Pugh.

The elimination of non-trivial recurrences – the so called "Closing Lemma Problem"– as a step to achieve condition v) is by far the most difficult of the steps in the proof. See the book of Palis and Melo, [16], for a presentation of these ideas in the case of vector fields on surfaces.

The proof of theorem above will be postponed to a forthcoming paper [10]. It involves technical details that are closer to the proofs of genericity theorems given by Gutierrez and Sotomayor, [13], [14], for principal curvature lines and by Garcia and Sotomayor, [7], for arithmetic mean curvature lines.

Bibliography

[1] ANDRONOV (A.) and LEONTOVICH (E.) et al. — Theory of Bifurcations of Dynamic Systems on a Plane, Jerusalem, Israel Program of Scientific Translations, 1973.

[2] ANOSOV (D.V.). — Geodesic Flows on Closed Riemannian Manifolds of Negative Curvature, Proc. Steklov Institute of Mathematics, Amer. Math. Soc. Transl., **90**, 1967 and 1969.

[3] BANCHOFF (T.), GAFFNEY (T.) and MCCRORY (C.). — Cusps of Gauss Maps, Pitman Research Notes in Math., London, **55** (1982), 1-130.

[4] BRUCE (B.) and FIDAL (D.). — On binary differential equations and umbilic points, Proc. Royal Soc. Edinburgh, **111A** (1989), 147-168.

[5] DARBOUX (G.). — Leçons sur la Théorie des Surfaces, vol. IV. Sur la forme des lignes de courbure dans la voisinage d'un ombilic, Note 07, Paris:Gauthier Villars, 1896.

[6] GARCIA (R.) and SOTOMAYOR (J.). — Structural stability of parabolic points and periodic asymptotic lines, Matemática Contemporânea, **12** (1997), 83-102.

[7] GARCIA (R.) and SOTOMAYOR (J.). — Structurally stable configurations of lines of mean curvature and umbilic points on surfaces immersed in $\mathbb{R}^3$, Publ. Matemátiques., **45:2** (2001), 431-466.

[8] GARCIA (R.) and SOTOMAYOR (J.). — Lines of Harmonic Mean Curvature on surfaces immersed in $\mathbb{R}^3$, Pré-Publication du Laboratoire de Topologie, Université de Bourgogne, **294** (2002), 1-27.

[9] GARCIA (R.), GUTIERREZ (C.) and SOTOMAYOR (J.). — Structural stability of asymptotic lines on surfaces immersed in $\mathbb{R}^3$, Bull. Sciences Math., **123** (1999), pp. 599-622.

[10] GARCIA (R.) and SOTOMAYOR (J.). — Mean curvature lines on surfaces immersed in $\mathbb{R}^3$. In preparation.

[11] GUÍÑEZ (V.). — Positive quadratic differential forms and foliations with singularities on surfaces, Trans. Amer. Math. Soc., **309:2** (1988), pp. 477-502.

[12] GUTIERREZ (C.) and SOTOMAYOR (J.). — Structural Stable Configurations of Lines of Principal Curvature, Asterisque, **98-99** (1982), 185-215.

[13] GUTIERREZ (C.) and SOTOMAYOR (J.). — An Approximation Theorem for Immersions with Structurally Stable Configurations of Lines of Principal Curvature, Lect. Notes in Math., **1007**, 1983.

[14] GUTIERREZ (C.) and SOTOMAYOR (J.). — Lines of Curvature and Umbilic Points on Surfaces, 18^{th} Brazilian Math. Colloquium, Rio de Janeiro, IMPA, 1991. Reprinted as Structurally Configurations of Lines of Curvature and Umbilic Points on Surfaces, Lima, Monografias del IMCA, 1998.

[15] GUTIERREZ (C.) and SOTOMAYOR (J.). — Lines of Curvature, Umbilical Points and Carathéodory Conjecture, Resenhas IME-USP, **03** (1998), 291-322.

[16] MELO (W.) and PALIS (J.). — Geometric Theory of Dynamical Systems, New York, Springer Verlag, 1982.

[17] OCCHIPINTI (R.). — *Sur un double système de lignes d'une surface.* L'enseignement mathématique (1914), 38-44.

[18] OGURA (K.). — *On the T-System on a Surface,* Tohoku Math. Journal, **09** (1916), 87-101.

[19] PALIS (J.) and TAKENS (F.). — *Topological equivalence of normally hyperbolic dynamical systems,* Topology, **16** (1977), 335-345.

[20] PEIXOTO (M.). — *Structural Stability on two-dimensional manifolds*, Topology, **1** (1962), 101-120.

[21] ROUSSARIE (R.). — Bifurcations of Planar Vector Fields and Hilbert's Sixteen Problem, Progress in Mathematics, 164, Birkhaüser Verlag, Basel, 1988.

[22] SPIVAK (M.). — Introduction to Comprehensive Differential Geometry, Vol. III Berkeley, Publish or Perish, 1980.

[23] SANSONE (G.) and CONTI (R.). — Equazioni Differenziali non Lineari, Edizioni Cremonese, Roma, 1956.

[24] STRUIK (D.). — Lectures on Classical Differential Geometry, Addison Wesley Pub. Co., Reprinted by Dover Publications, Inc., 1988.