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# Covariant Poisson Brackets in Geometric Field Theory

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## Abstract

We establish a link between the multisymplectic and the covariant phase space approach to geometric field theory by showing how to derive the symplectic form on the latter, as introduced by Crnković-Witten and Zuckerman, from the multisymplectic form. The main result is that the Poisson bracket associated with this form, according to the standard rules of symplectic geometry, is precisely the covariant bracket due to Peierls and De Witt.

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# 1 Introduction

One of the most annoying flaws of the usual canonical formalism in field theory is its lack of manifest covariance, that is, its lack of explicit Lorentz invariance (in the context of special relativity) and more generally its lack of explicit invariance under space-time coordinate transformations (in the context of general relativity). Of course, this defect is built into the theory from the very beginning, since the usual canonical formalism represents the dynamical variables of classical field theory by functions on some spacelike hypersurface (Cauchy data) and provides differential equations for their time evolution off this hypersurface: thus it presupposes a splitting of space-time into space and time, in the form of a foliation of space-time into Cauchy surfaces. As a result, canonical quantization leads to models of quantum field theory whose covariance is far from obvious and in fact constitutes a formidable problem: as a well known example, we may quote the efforts necessary to check Lorentz invariance in (perturbative) quantum electrodynamics in the Coulomb gauge.

These and similar observations have over many decades nourished attempts to develop a fully covariant formulation of the canonical formalism in classical field theory, which would hopefully serve as a starting point for alternative methods of quantization. Among the many ideas that have been proposed in this direction, two have come to occupy a special role. One of these is the “covariant functional formalism”, based on the concept of “covariant phase space” which is defined as the (infinite-dimensional) space of solutions of the equations of motion. This approach was strongly advocated in the 1980’s by Crnković, Witten and Zuckerman [1–3] (see also [4]) who showed how to construct a symplectic structure on the covariant phase space of many important models of field theory (including gauge theories and general relativity), but the idea as such has a much longer history. The other has become known as the “multisymplectic formalism”, based on the concept of “multiphase space” which is a (finite-dimensional) space that can be defined locally by associating to each coordinate  $q^i$  not just one conjugate momentum  $p_i$  but  $n$  conjugate momenta  $p_i^\mu$  ( $\mu = 1, \dots, n$ ), where  $n$  is the dimension of the underlying space-time manifold. In coordinate form, this construction goes back to the classical work of De Donder and Weyl in the 1930’s [5, 6], whereas a global formulation was initiated in the 1970’s by a group of mathematical physicists, mainly in Poland [7–10] but also elsewhere [11, 12], and definitely established in the 1990’s [13, 14]; a detailed exposition, with lots of examples, can be found in the GIMmsy paper [15].

The two formalisms, although both fully covariant and directed towards the same ultimate goal, are quite different in nature; each of them has its own merits and drawbacks.

- The multisymplectic formalism is manifestly consistent with the basic principles of field theory, in particular that of locality, and is mathematically rigorous because it uses well established methods from calculus on finite-dimensional manifolds.

On the other hand, it does not seem to permit any obvious definition of the Poisson bracket between observables. Similarly, the introduction of  $n$  conjugate momenta for each coordinate obscures the usual duality between canonically conjugate variables (such as momenta and positions), which plays a fundamental role in all known methods of quantization. A definite solution to these problems has yet to be found.

- The covariant functional formalism fits neatly into the philosophy underlying the symplectic formalism in general; in particular, it admits a natural definition of the Poisson bracket (due to Peierls [16] and further elaborated by De Witt [17–19]) that preserves the duality between canonically conjugate variables. On the other hand, being of intrinsically global nature, it obscures the implementation of the principle of locality. Moreover, it cannot be considered to be mathematically rigorous since it is often restricted to the formal application of methods from calculus on manifolds, extrapolated to the infinite-dimensional case: transforming such formal results into mathematical theorems is a separate problem, often highly complex and difficult.

Unfortunately, the exact relation between the two approaches has never been made completely explicit in the literature. This is surprising since it should be clear from the previous remarks that they are to a certain extent complementary, so establishing a firm link between them will very likely be a source for new insights.

The present paper, based on the PhD thesis of the second author [20], is intended as a first step to overcome this deficiency. It is organized into two main sections. In Sect. 2, we briefly review some salient features of the multisymplectic approach to geometric field theory, focussing on the concepts needed to make contact with the covariant functional approach. In particular, this requires a digression on jet bundles of first and second order as well as on the definition of both extended and ordinary multiphase space as the twisted affine dual of the first order jet bundle and the twisted linear dual of the linear first order jet bundle, respectively: this will enable us to give a global definition of the space of solutions of the equations of motion, both in the Lagrangian and Hamiltonian formulation, in terms of a globally defined Euler - Lagrange operator  $\mathcal{E}$  and of a globally defined De Donder - Weyl operator  $\mathcal{D}$ , respectively. To describe the formal tangent space to this space of solutions at a given point, we also write down the linearization of each of these operators around a given solution. In Sect. 3, we apply these constructions to derive a general expression for the symplectic form  $\Omega$  on covariant phase space, à la Crnković-Witten-Zuckerman, in terms of the multisymplectic form  $\omega$  on extended multiphase space and/or the Cartan form  $\omega_{\mathcal{L}}$  obtained from it by pull-back via the Legendre transformation induced by a Lagrangian density  $\mathcal{L}$ . Finally, we show that the Poisson bracket associated with the form  $\Omega$ , according to the standard rules of symplectic geometry, suitably extended to this infinite-dimensional setting, is precisely the Peierls- De Witt bracket of classical field theory [16–19].

## 2 Multisymplectic Approach

### 2.1 Overview

The multisymplectic approach to geometric field theory, whose origins can be traced back to the early work of Hermann Weyl on the calculus of variations [6], is based on the idea of modifying the transition from the Lagrangian to the Hamiltonian framework by treating spatial derivatives and time derivatives of fields on an equal footing. Thus one associates to each field component  $\varphi^i$  not just its standard canonically conjugate momentum  $\pi_i^\mu$  but rather  $n$  conjugate momenta  $\pi_i^\mu$ , where  $n$  is the dimension of space-time. In a first order Lagrangian formalism, where one starts out from a Lagrangian  $L$  depending on the field and its first partial derivatives, these are obtained by a covariant analogue of the Legendre transformation

$$\pi_i^\mu = \frac{\partial L}{\partial \partial_\mu \varphi^i}. \quad (1)$$

This allows to rewrite the standard Euler - Lagrange equations of field theory

$$\partial_\mu \frac{\partial L}{\partial \partial_\mu \varphi^i} - \frac{\partial L}{\partial \varphi^i} = 0 \quad (2)$$

as a covariant first order system, the covariant Hamiltonian equations or De Donder - Weyl equations

$$\frac{\partial H}{\partial \pi_i^\mu} = \partial_\mu \varphi^i, \quad \frac{\partial H}{\partial \varphi^i} = -\partial_\mu \pi_i^\mu \quad (3)$$

where

$$H = \pi_i^\mu \partial_\mu \varphi^i - L \quad (4)$$

is the covariant Hamiltonian density or De Donder - Weyl Hamiltonian.

Multiphase space (ordinary as well as extended) is the geometric environment built by appropriately patching together local coordinate systems of the form  $(q^i, p_i^\mu)$  – instead of the canonically conjugate variables  $(q^i, p_i)$  of mechanics – together with space-time coordinates  $x^\mu$  and, in the extended version, a further energy type variable that we shall denote by  $p$  (without any index). The global construction of these multiphase spaces, however, has only gradually come to light; it is based on the following mathematical concepts.

- The collection of all fields in a given theory, defined over a fixed ( $n$ -dimensional orientable) space-time manifold  $M$ , is represented by the sections  $\varphi$  of a given fiber bundle  $E$  over  $M$ , with bundle projection  $\pi : E \rightarrow M$  and typical fiber  $Q$ . This bundle will be referred to as the *configuration bundle* of the theory since  $Q$  corresponds to the configuration space of possible field values.

- The collection of all fields together with their partial derivatives up to a certain order, say order  $r$ , is represented by the  $r$ -jets  $j^r\varphi \equiv (\varphi, \partial\varphi, \dots, \partial^r\varphi)$  of sections of  $E$ , which are themselves sections of the  $r^{\text{th}}$  order jet bundle  $J^rE$  of  $E$ , regarded as a fiber bundle over  $M$ . In this paper, we shall only need first order jet bundles, with one notable exception: the global formulation of the Euler - Lagrange equations requires introducing the second order jet bundle.
- Dualization – the concept needed to pass from the Lagrangian to the Hamiltonian framework via the Legendre transformation – comes in two variants, based on the fundamental observation that the first order jet bundle  $J^1E$  of  $E$  is an affine bundle over  $E$  whose difference vector bundle  $\tilde{J}^1E$  will be referred to as the linear jet bundle. Ordinary multiphase space is obtained as the twisted linear dual  $\tilde{J}^{1\otimes}E$  of  $\tilde{J}^1E$  while extended multiphase space is obtained as the twisted affine dual  $J^{1\otimes}E$  of  $J^1E$ , where the prefix “twisted” refers to the necessity of taking an additional tensor product with the bundle of  $n$ -forms on  $M$ .<sup>1</sup>
- The Lagrangian  $\mathcal{L}$  is a function on  $J^1E$  with values in the bundle of  $n$ -forms on  $M$  so that it may be integrated to provide an action functional which enters the variational principle. The De Donder - Weyl Hamiltonian  $\mathcal{H}$  is a section of  $J^{1\otimes}E$ , considered as an affine line bundle over  $\tilde{J}^{1\otimes}E$ .

Note that the formalism is set up so as to require no additional structure on the configuration bundle or on any other bundle constructed from it: all are merely fiber bundles over the space-time manifold  $M$ . Of course, additional structures do arise when one is dealing with special classes of fields (matter fields and the metric tensor in general relativity are sections of vector bundles, connections are sections of affine bundles, nonlinear fields such as those arising in the sigma model are sections of trivial fiber bundles with a fixed Riemannian metric on the fibers, etc.), but such additional structures depend on the kind of theory considered and thus are not universal. Finally, the restriction imposed on the order of the jet bundles considered reflects the fact that almost all known examples of field theories are governed by second order partial differential equations which can be derived from a Lagrangian that depends only on the fields and their partial derivatives of first order, which is why it is reasonable to develop the general theory on the basis of a first order formalism, as is done in mechanics [21,22].

## 2.2 The First Order Jet Bundle

The field theoretical analogue of the tangent bundle of mechanics is the *first order jet bundle*  $J^1E$  associated with the configuration bundle  $E$  over  $M$ . Given a point  $e$  in  $E$

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<sup>1</sup>We use an asterisk  $*$  to denote linear duals of vector spaces or bundles and a star  $\star$  to denote affine duals of affine spaces or bundles. These symbols are appropriately encircled to characterize twisted duals, as opposed to the ordinary duals defined in terms of linear or affine maps with values in  $\mathbb{R}$ .

with base point  $x = \pi(e)$  in  $M$ , the fiber  $J_e^1 E$  of  $J^1 E$  at  $e$  consists of all linear maps from the tangent space  $T_x M$  of the base space  $M$  at  $x$  to the tangent space  $T_e E$  of the total space  $E$  at  $e$  whose composition with the tangent map  $T_e \pi : T_e E \rightarrow T_x M$  to the projection  $\pi : E \rightarrow M$  gives the identity on  $T_x M$ :

$$J_e^1 E = \{ \gamma \in L(T_x M, T_e E) : T_e \pi \circ \gamma = \text{id}_{T_x M} \}. \quad (5)$$

Thus the elements of  $J_e^1 E$  are precisely the candidates for the tangent maps at  $x$  to (local) sections  $\varphi$  of the bundle  $E$  satisfying  $\varphi(x) = e$ . Obviously,  $J_e^1 E$  is an affine subspace of the vector space  $L(T_x M, T_e E)$  of all linear maps from  $T_x M$  to the tangent space  $T_e E$ , the corresponding difference vector space being the vector space of all linear maps from  $T_x M$  to the vertical subspace  $V_e E$ :

$$\tilde{J}_e^1 E = \{ \tilde{\gamma} \in L(T_x M, T_e E) : T_e \pi \circ \tilde{\gamma} = 0 \} = L(T_x M, V_e E) \cong T_x^* M \otimes V_e E. \quad (6)$$

The jet bundle  $J^1 E$  thus defined admits two different projections, namely the *target projection*  $\tau_E : J^1 E \rightarrow E$  and the *source projection*  $\sigma_E : J^1 E \rightarrow M$  which is simply its composition with the original bundle projection, that is,  $\sigma_E = \pi \circ \tau_E$ . The same goes for  $\tilde{J}^1 E$ , which we shall call the linearized first order jet bundle or simply *linear jet bundle* associated with the configuration bundle  $E$  over  $M$ .

The structure of  $J^1 E$  and of  $\tilde{J}^1 E$  as fiber bundles over  $M$  with respect to the source projection (in general without any additional structure), as well as that of  $J^1 E$  as an affine bundle and of  $\tilde{J}^1 E$  as a vector bundle over  $E$  with respect to the target projection, can most easily be seen in terms of local coordinates. Namely, local coordinates  $x^\mu$  for  $M$  and  $q^i$  for  $Q$ , together with a local trivialization of  $E$ , induce local coordinates  $(x^\mu, q^i)$  for  $E$  as well as local coordinates  $(x^\mu, q^i, q_\mu^i)$  for  $J^1 E \subset L(\pi^*(TM), TE)$  and  $(x^\mu, q^i, \tilde{q}_\mu^i)$  for  $\tilde{J}^1 E \subset L(\pi^*(TM), TE)$ . Moreover, local coordinate transformations  $x^\mu \rightarrow x'^\nu$  for  $M$  and  $q^i \rightarrow q'^j$  for  $Q$ , together with a change of local trivialization of  $E$ , correspond to a local coordinate transformation  $(x^\mu, q^i) \rightarrow (x'^\nu, q'^j)$  for  $E$  where

$$x'^\nu = x'^\nu(x^\mu), \quad q'^j = q'^j(x^\mu, q^i). \quad (7)$$

The induced local coordinate transformations  $(x^\mu, q^i, q_\mu^i) \rightarrow (x'^\nu, q'^j, q'_\mu^j)$  for  $J^1 E$  and  $(x^\mu, q^i, \tilde{q}_\mu^i) \rightarrow (x'^\nu, q'^j, \tilde{q}'_\nu^j)$  for  $\tilde{J}^1 E$  are then easily seen to be given by

$$q'_\nu^j = \frac{\partial x^\mu}{\partial x'^\nu} \frac{\partial q'^j}{\partial q^i} q_\mu^i + \frac{\partial x^\mu}{\partial x'^\nu} \frac{\partial q'^j}{\partial x^\mu}, \quad (8)$$

and

$$\tilde{q}'_\nu^j = \frac{\partial x^\mu}{\partial x'^\nu} \frac{\partial q'^j}{\partial q^i} \tilde{q}_\mu^i. \quad (9)$$

This makes it clear that  $J^1 E$  is an affine bundle over  $E$  with difference vector bundle

$$J^1 E = T^* M \otimes V E , \quad (10)$$

in accordance with eq. (6).<sup>2</sup>

That the (first order) jet bundle of a fiber bundle is the adequate arena to incorporate (first order) derivatives of fields becomes apparent by noting that a global section  $\varphi$  of  $E$  over  $M$  naturally induces a global section  $j^1 \varphi$  of  $J^1 E$  over  $M$  given by

$$j^1 \varphi(x) = T_x \varphi \in J^1_{\varphi(x)} E \quad \text{for } x \in M .$$

In the mathematical literature,  $j^1 \varphi$  is called the (first) prolongation of  $\varphi$ , but it would be more intuitive to simply call it the derivative of  $\varphi$  since in the local coordinates used above,

$$j^1 \varphi(x) = (x^\mu, \varphi^i(x), \partial_\mu \varphi^i(x)) ,$$

where  $\partial_\mu = \partial/\partial x^\mu$ ; this is symbolically summarized by writing  $j^1 \varphi \equiv (\varphi, \partial \varphi)$ .

Similarly, it can be shown that the linear jet bundle of a fiber bundle is the adequate arena to incorporate covariant derivatives of sections, with respect to an arbitrarily chosen connection.

Finally, let us discuss briefly the lifting, from  $E$  to  $J^1 E$ , of (local) bundle automorphisms and, passing to generators of one-parameter groups, of projectable vector fields. Let  $\Phi : E \rightarrow E$  be an automorphism of the fiber bundle  $E$  over  $M$  and  $\phi : M \rightarrow M$  the induced diffeomorphism of  $M$  such that the diagram

$$\begin{array}{ccc} & \Phi & \\ E & \longrightarrow & E \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{\phi} & M \end{array}$$

commutes. This can be lifted to an automorphism of the jet bundle  $J^1 E$ , as an affine bundle over  $E$ , by defining  $J^1 \Phi : J^1 E \rightarrow J^1 E$  as follows: given a point  $e$  in  $E$  with base point  $x = \pi(e)$  in  $M$  and a 1-jet  $\gamma \in J^1_e E$ , define the 1-jet  $J^1 \Phi(\gamma) \in J^1_{\Phi(e)} E$  by

$$J^1 \Phi(\gamma) = T_e \Phi \circ \gamma \circ (T_x \phi)^{-1} . \quad (11)$$

<sup>2</sup>Given any vector bundle  $V$  over  $M$ , such as  $TM$ ,  $T^* M$  or any of their exterior powers, one can consider it as a vector bundle over  $E$  by forming its pull-back  $\pi^* V$ . In order not to overload the notation, we shall here and in what follows suppress the symbol  $\pi^*$ .

Obviously, this formula defines a linear map from  $L(T_x M, T_e E)$  to  $L(T_{\phi(x)} M, T_{\Phi(e)} E)$  that restricts to an affine map from  $J_e^1 E$  to  $J_{\Phi(e)}^1 E$ :

$$\begin{aligned} T_{\Phi(e)} \pi \circ J^1 \Phi(\gamma) &= T_{\Phi(e)} \pi \circ T_e \Phi \circ \gamma \circ (T_x \phi)^{-1} \\ &= T_e (\pi \circ \Phi) \circ \gamma \circ (T_x \phi)^{-1} \\ &= T_e (\phi \circ \pi) \circ \gamma \circ (T_x \phi)^{-1} \\ &= T_x \phi \circ T_e \pi \circ \gamma \circ (T_x \phi)^{-1} \\ &= T_x \phi \circ \text{id}_{T_x M} \circ (T_x \phi)^{-1} \\ &= \text{id}_{T_{\phi(x)} M}. \end{aligned}$$

In particular, the diagram

$$\begin{array}{ccc} J^1 E & \xrightarrow{J^1 \Phi} & J^1 E \\ \tau \downarrow & & \downarrow \tau \\ E & \xrightarrow{\Phi} & E \end{array}$$

commutes, justifying to call  $J^1 \Phi$  the *first prolongation* of  $\Phi$ . This construction can be generalized to any fiber bundle map  $\Phi : E \rightarrow F$  over a (local) diffeomorphism  $\phi : M \rightarrow N$ , giving an affine bundle map  $J^1 \Phi : J^1 E \rightarrow J^1 F$  over  $\Phi : E \rightarrow F$ .

Passing to the description of the infinitesimal situation, let us consider a projectable vector field  $V$  on  $E$ , whose flow is a one-parameter group of (local) automorphisms of  $E$  that can be lifted to a one-parameter group of local automorphisms of  $J^1 E$ , generated by a projectable vector field  $J^1 V$  on  $J^1 E$ : this is then defined to be the prolongation of  $V$ . Thus

$$V = \frac{\partial \Phi_\lambda}{\partial \lambda} \Big|_{\lambda=0} \implies J^1 V = \frac{\partial (J^1 \Phi_\lambda)}{\partial \lambda} \Big|_{\lambda=0}.$$

In local coordinates as before, we can write

$$V = V^\mu \frac{\partial}{\partial x^\mu} + V^i \frac{\partial}{\partial q^i} \quad (12)$$

where  $V^\mu = V^\mu(x^\nu)$  and  $V^i = V^i(x^\nu, q^j)$ , and since the lifting of bundle automorphisms is described by the transformation law (8), differentiation with respect to  $\lambda$  gives

$$J^1 V = V^\mu \frac{\partial}{\partial x^\mu} + V^i \frac{\partial}{\partial q^i} + \left( \frac{\partial V^i}{\partial q^k} q_\mu^k - \frac{\partial V^\mu}{\partial x^\mu} q_\kappa^i + \frac{\partial V^i}{\partial x^\mu} \right) \frac{\partial}{\partial q_\mu^i}. \quad (13)$$

## 2.3 Duality

The next problem to be addressed is how to define an adequate notion of dual for  $J^1 E$ . The necessary background information from the theory of affine spaces and of affine

bundles (including the definition of the affine dual of an affine space and of the transpose of an affine map between affine spaces) is summarized in the Appendix. Briefly, the rules state that if  $A$  is an affine space of dimension  $k$  over  $\mathbb{R}$ , its dual  $A^*$  is the space  $A(A, \mathbb{R})$  of affine maps from  $A$  to  $\mathbb{R}$ , which is a vector space of dimension  $k+1$ . Thus the *affine dual*  $J^{1*}E$  of  $J^1E$  and the *linear dual*  $\tilde{J}^{1*}E$  of  $\tilde{J}^1E$  are obtained by taking their fiber over any point  $e$  in  $E$  to be the vector space

$$J_e^{1*}E = \{ z_e : J_e^1E \rightarrow \mathbb{R} \text{ affine} \} \quad (14)$$

and

$$\tilde{J}_e^{1*}E = \{ \tilde{z}_e : \tilde{J}_e^1E \rightarrow \mathbb{R} \text{ linear} \} \quad (15)$$

respectively. However, as mentioned before, the multiphase spaces of field theory are defined with an additional twist, replacing the real line by the one-dimensional space of volume forms on the base manifold  $M$  at the appropriate point. In other words, the *twisted affine dual*

$$J^{1\otimes}E = J^{1*}E \otimes \bigwedge^n T^*M \quad (16)$$

of  $J^1E$  and the *twisted linear dual*

$$\tilde{J}^{1\otimes}E = \tilde{J}^{1*}E \otimes \bigwedge^n T^*M \quad (17)$$

of  $\tilde{J}^1E$  are defined by taking their fiber over any point  $e$  in  $E$  with base point  $x = \pi(e)$  in  $M$  to be the vector space

$$J_e^{1\otimes}E = \{ z_e : J_e^1E \rightarrow \bigwedge^n T_x^*M \text{ affine} \} \quad (18)$$

and

$$\tilde{J}_e^{1\otimes}E = \{ \tilde{z}_e : \tilde{J}_e^1E \rightarrow \bigwedge^n T_x^*M \text{ linear} \} \quad (19)$$

respectively.<sup>2</sup> As in the case of the jet bundle and the linear jet bundle, all these duals admit two different projections, namely a *target projection* onto  $E$  and a *source projection* onto  $M$  which is simply its composition with the original projection  $\pi$ .

Using local coordinates as before, it is easily shown that all these duals are fiber bundles over  $M$  with respect to the source projection (in general without any additional structure) and are vector bundles over  $E$  with respect to the target projection. Namely, introducing local coordinates  $(x^\mu, q^i)$  for  $E$  together with the induced local coordinates  $(x^\mu, q^i, q_\mu^i)$  for  $J^1E$  and  $(x^\mu, q^i, \tilde{q}_\mu^i)$  for  $\tilde{J}^1E$  as before, we obtain local coordinates  $(x^\mu, q^i, p_i^\mu, p)$  both for  $J^{1*}E$  and for  $J^{1\otimes}E$  as well as local coordinates  $(x^\mu, q^i, p_i^\mu)$  both for  $\tilde{J}^{1*}E$  and for  $\tilde{J}^{1\otimes}E$ , respectively. These are defined by requiring the dual pairing between a point in  $J^{1*}E$  or  $J^{1\otimes}E$  with coordinates  $(x^\mu, q^i, p_i^\mu, p)$  and a point in  $J^1E$  with coordinates  $(x^\mu, q^i, q_\mu^i)$  to be given by

$$p_i^\mu q_\mu^i + p \quad (20)$$

in the ordinary (untwisted) case and by

$$(p_i^\mu q_\mu^i + p) d^n x \quad (21)$$

in the twisted case, whereas the dual pairing between a point in  $\tilde{J}^{1*}E$  or in  $\tilde{J}^{1\otimes}E$  with coordinates  $(x^\mu, q^i, p_i^\mu)$  and a point in  $\tilde{J}^1E$  with coordinates  $(x^\mu, q^i, \tilde{q}_\mu^i)$  is given by

$$p_i^\mu \tilde{q}_\mu^i \quad (22)$$

in the ordinary (untwisted) case and by

$$p_i^\mu \tilde{q}_\mu^i d^n x \quad (23)$$

in the twisted case. Moreover, a local coordinate transformation  $(x^\mu, q^i) \rightarrow (x''^\nu, q'^j)$  for  $E$  as in eq. (7) induces local coordinate transformations for  $J^1E$  and for  $\tilde{J}^1E$  as in eqs (8) and (9) which in turn induce local coordinate transformations  $(x^\mu, q^i, p_i^\mu, p) \rightarrow (x''^\nu, q'^j, p_j''^\nu, p')$  both for  $J^{1*}E$  and for  $J^{1\otimes}E$  as well as local coordinate transformations  $(x^\mu, q^i, p_i^\mu) \rightarrow (x''^\nu, q'^j, p_j''^\nu)$  both for  $\tilde{J}^{1*}E$  and for  $\tilde{J}^{1\otimes}E$ : these are given by

$$p_j''^\nu = \frac{\partial x''^\nu}{\partial x^\mu} \frac{\partial q^i}{\partial q'^j} p_i^\mu \quad , \quad p' = p - \frac{\partial q'^j}{\partial x^\mu} \frac{\partial q^i}{\partial q'^j} p_i^\mu \quad (24)$$

in the ordinary (untwisted) case and

$$p_j''^\nu = \det\left(\frac{\partial x}{\partial x'}\right) \frac{\partial x''^\nu}{\partial x^\mu} \frac{\partial q^i}{\partial q'^j} p_i^\mu \quad , \quad p' = \det\left(\frac{\partial x}{\partial x'}\right) \left(p - \frac{\partial q'^j}{\partial x^\mu} \frac{\partial q^i}{\partial q'^j} p_i^\mu\right) \quad (25)$$

in the twisted case.

Finally, it is worth noting that the affine duals  $J^{1*}E$  and  $J^{1\otimes}E$  of  $J^1E$  contain line subbundles  $J_c^{1*}E$  and  $J_c^{1\otimes}E$  whose fiber over any point  $e$  in  $E$  with base point  $x = \pi(e)$  in  $M$  consists of the constant (rather than affine) maps from  $J_e^1E$  to  $\mathbb{R}$  and to  $\bigwedge^n T_x^*M$ , respectively, and the corresponding quotient vector bundles over  $E$  can be naturally identified with the respective linear duals  $\tilde{J}^{1*}E$  and  $\tilde{J}^{1\otimes}E$  of  $\tilde{J}^1E$ , i.e., we have

$$J^{1*}E/J_c^{1*}E \cong \tilde{J}^{1*}E \quad (26)$$

and

$$J^{1\otimes}E/J_c^{1\otimes}E \cong \tilde{J}^{1\otimes}E \quad (27)$$

respectively. This shows that, in both cases, the corresponding projection onto the quotient amounts to “forgetting the additional energy variable” since it takes a point with coordinates  $(x^\mu, q^i, p_i^\mu, p)$  to the point with coordinates  $(x^\mu, q^i, p_i^\mu)$ ; it will be denoted by  $\eta$  and is easily seen to turn  $J^{1*}E$  and  $J^{1\otimes}E$  into affine line bundles over  $\tilde{J}^{1*}E$  and over  $\tilde{J}^{1\otimes}E$ , respectively.

## 2.4 The Second Order Jet Bundle

For an appropriate global formulation of the standard Euler - Lagrange equations of field theory, which are second order partial differential equations, it is useful to introduce the *second order jet bundle*  $J^2E$  associated with the configuration bundle  $E$  over  $M$ . It can be defined either directly, as is usually done, or by invoking an iterative procedure, which is the method we shall follow here. Starting out from the first order jet bundle  $J^1E$  of  $E$ , regarded as a fiber bundle over  $M$ , we consider its first order jet bundle  $J^1J^1E$  and define, in a first step, the *semiholonomic second order jet bundle*  $\bar{J}^2E$  of  $E$  to be the subbundle of  $J^1J^1E$  given by

$$\bar{J}^2E = \{ \kappa \in J^1J^1E : \tau_{J^1E}(\kappa) = J^1\tau_E(\kappa) \} \quad (28)$$

where  $\tau_{J^1E} : J^1J^1E \rightarrow J^1E$  is the target projection of  $J^1J^1E$  while  $J^1\tau_E : J^1J^1E \rightarrow J^1E$  is the prolongation of the target projection  $\tau_E : J^1E \rightarrow E$  of  $J^1E$ , considered as a map of fiber bundles over  $M$ . As will become clear below,  $\bar{J}^2E$  is an affine bundle over  $E$ , with difference vector bundle  $(T^*M \oplus (T^*M \otimes T^*M)) \otimes VE$ . Therefore, using the construction of the affine quotient of an affine space (bundle) by a vector subspace (subbundle) of its difference vector space (bundle), as explained in the Appendix, we may complete the construction by observing that since  $T^*M \otimes T^*M$  contains  $\bigwedge^2 T^*M$  as a vector subbundle (and hence so does  $T^*M \oplus (T^*M \otimes T^*M)$ ), it is possible to define the second order jet bundle  $J^2E$  of  $E$  as the quotient

$$J^2E = \bar{J}^2E / \bigwedge^2 T^*M \otimes VE. \quad (29)$$

Once again,  $J^2E$  is an affine bundle over  $E$ , with difference vector bundle

$$\bar{J}^2E = (T^*M \oplus \bigvee^2 T^*M) \otimes VE. \quad (30)$$

These assertions can be proved by introducing local coordinates  $(x^\mu, q^i)$  for  $E$  together with the induced local coordinates  $(x^\mu, q^i, q_\mu^i)$  for  $J^1E$  as before to first define induced local coordinates  $(x^\mu, q^i, q_\mu^i, r_\mu^i, q_{\mu\rho}^i)$  for  $J^1J^1E$ . Simple calculations then show that the points of  $\bar{J}^2E$  are characterized by the condition  $q_\mu^i = r_\mu^i$  and the points of  $J^2E$  by the additional condition  $q_{\mu\rho}^i = q_{\rho\mu}^i$ . Moreover, a local coordinate transformation  $(x^\mu, q^i) \rightarrow (x'^\nu, q'^j)$  for  $E$  as in eq. (7) induces a local coordinate transformation for  $J^1E$  as in eq. (8) which in turn induces a local coordinate transformation for  $J^1J^1E$ , given by eq. (8) together with

$$r'^j_\nu = \frac{\partial x^\mu}{\partial x'^\nu} \frac{\partial q'^j}{\partial q^i} r^i_\mu + \frac{\partial x^\mu}{\partial x'^\nu} \frac{\partial q'^j}{\partial x^\mu}, \quad (31)$$

$$q'^j_{\nu\sigma} = \frac{\partial x^\rho}{\partial x'^\sigma} \frac{\partial q'^j_\nu}{\partial q^i} q^i_{\mu\rho} + \frac{\partial x^\rho}{\partial x'^\sigma} \frac{\partial q'^j_\nu}{\partial x^\rho}. \quad (32)$$

In particular, eqs (8) and (31) show that  $q_\mu^i = r_\mu^i$  implies  $q'^j_\nu = r'^j_\nu$ , as required by the global, coordinate independent nature of the definition of  $\bar{J}^2E$  as a subbundle of

$J^1 J^1 E$ , while eq. (32) can be further evaluated by differentiating eq. (8) with respect to  $q_\mu^i$  and  $x^\rho$ , which leads to

$$\begin{aligned} q'_{\nu\sigma}^{ij} &= \frac{\partial x^\mu}{\partial x'^\nu} \frac{\partial x^\rho}{\partial x'^\sigma} \frac{\partial q'^j}{\partial q^i} q_{\mu\rho}^i \\ &+ \left( \frac{\partial x^\mu}{\partial x'^\nu} \frac{\partial x^\rho}{\partial x'^\sigma} \frac{\partial^2 q'^j}{\partial x^\rho \partial q^i} - \frac{\partial x^\kappa}{\partial x'^\nu} \frac{\partial x^\rho}{\partial x'^\sigma} \frac{\partial x^\mu}{\partial x'^\lambda} \frac{\partial^2 x'^\lambda}{\partial x^\rho \partial x^\kappa} \frac{\partial q'^j}{\partial q^i} \right) q_\mu^i \\ &+ \left( \frac{\partial x^\mu}{\partial x'^\nu} \frac{\partial x^\rho}{\partial x'^\sigma} \frac{\partial^2 q'^j}{\partial x^\rho \partial x^\mu} - \frac{\partial x^\kappa}{\partial x'^\nu} \frac{\partial x^\rho}{\partial x'^\sigma} \frac{\partial x^\mu}{\partial x'^\lambda} \frac{\partial^2 x'^\lambda}{\partial x^\rho \partial x^\kappa} \frac{\partial q'^j}{\partial x^\mu} \right). \end{aligned} \quad (33)$$

This is also the induced local coordinate transformation for  $\bar{J}^2 E$ , whereas that for  $J^2 E$  is obtained by symmetrization:

$$\begin{aligned} q'_{\nu\sigma}^{ij} &= \frac{\partial x^\mu}{\partial x'^\nu} \frac{\partial x^\rho}{\partial x'^\sigma} \frac{\partial q'^j}{\partial q^i} q_{\mu\rho}^i \\ &+ \left( \frac{1}{2} \left( \frac{\partial x^\mu}{\partial x'^\nu} \frac{\partial x^\rho}{\partial x'^\sigma} + \frac{\partial x^\mu}{\partial x'^\sigma} \frac{\partial x^\rho}{\partial x'^\nu} \right) \frac{\partial^2 q'^j}{\partial x^\rho \partial q^i} - \frac{\partial x^\kappa}{\partial x'^\nu} \frac{\partial x^\rho}{\partial x'^\sigma} \frac{\partial x^\mu}{\partial x'^\lambda} \frac{\partial^2 x'^\lambda}{\partial x^\rho \partial x^\kappa} \frac{\partial q'^j}{\partial q^i} \right) q_\mu^i \\ &+ \left( \frac{\partial x^\mu}{\partial x'^\nu} \frac{\partial x^\rho}{\partial x'^\sigma} \frac{\partial^2 q'^j}{\partial x^\rho \partial x^\mu} - \frac{\partial x^\kappa}{\partial x'^\nu} \frac{\partial x^\rho}{\partial x'^\sigma} \frac{\partial x^\mu}{\partial x'^\lambda} \frac{\partial^2 x'^\lambda}{\partial x^\rho \partial x^\kappa} \frac{\partial q'^j}{\partial x^\mu} \right). \end{aligned} \quad (34)$$

Both formulas indicate that  $\bar{J}^2 E$  and  $J^2 E$  are indeed affine bundles over  $E$ , with difference vector bundles as stated above.

The equivalence between the definition of the second order jet bundle given here and the traditional one is obtained observing that the iterated jet  $j^1 j^1 \varphi$  of a (local) section  $\varphi$  of  $E$  assume values not only in  $\bar{J}^2 E$  but even in  $J^2 E$ , due to the Schwarz rule. Therefore, second order jets in the traditional sense, that is, classes of (local) sections where the equivalence relation is the equality between the Taylor expansion up to second order, are in one-to-one correspondence with these iterated jets of (local) sections. Moreover, a global section  $\varphi$  of  $E$  over  $M$  naturally induces a global section  $j^2 \varphi$  of  $J^2 E$  over  $M$  such that in the local coordinates used above

$$j^2 \varphi(x) = (x^\mu, \varphi^i(x), \partial_\mu \varphi^i(x), \partial_\mu \partial_\nu \varphi^i(x)).$$

where  $\partial_\mu = \partial/\partial x^\mu$ ; this is symbolically summarized by writing  $j^2 \varphi = (\varphi, \partial \varphi, \partial^2 \varphi)$ .

## 2.5 The Legendre Transformation

A Lagrangian field theory is defined by its configuration bundle  $E$  over  $M$  and its Lagrangian density or simply *Lagrangian*, which in the present first order formalism is a map of fiber bundles over  $E$ :

$$\mathcal{L} : J^1 E \longrightarrow \bigwedge^n T^* M. \quad (35)$$

The requirement that  $\mathcal{L}$  should take values in the volume forms rather than the functions on space-time is imposed to guarantee that the *action* functional  $S : \Gamma(E) \rightarrow \mathbb{R}$  given by

$$S[\varphi] = \int_M \mathcal{L}(\varphi, \partial\varphi) \quad \text{for } \varphi \in \Gamma(E) \quad (36)$$

be well-defined and independent of the choice of additional structures, such as a space-time metric.<sup>3</sup> Such a Lagrangian gives rise to a *Legendre transformation*, which comes in two variants: as a map

$$\tilde{\mathbf{F}}\mathcal{L} : J^1 E \longrightarrow \tilde{J}^{1 \otimes E} \quad (37)$$

or as a map

$$\mathbf{F}\mathcal{L} : J^1 E \longrightarrow J^{1 \otimes E} \quad (38)$$

of fiber bundles over  $E$ . For any point  $\gamma$  in  $J_e^1 E$ , the latter is defined as the usual fiber derivative of  $\mathcal{L}$  at  $\gamma$ , which is the linear map from  $\tilde{J}_e^1 E$  to  $\bigwedge^n T_x^* M$  given by

$$\tilde{\mathbf{F}}\mathcal{L}(\gamma) \cdot \vec{\kappa} = \frac{d}{d\lambda} \mathcal{L}(\gamma + \lambda \vec{\kappa}) \Big|_{\lambda=0} \quad \text{for } \vec{\kappa} \in \tilde{J}_e^1 E, \quad (39)$$

whereas the former encodes the entire Taylor expansion, up to first order, of  $\mathcal{L}$  around  $\gamma$  along the fibers, which is the affine map from  $J_e^1 E$  to  $\bigwedge^n T_x^* M$  given by

$$\mathbf{F}\mathcal{L}(\gamma) \cdot \kappa = \mathcal{L}(\gamma) + \frac{d}{d\lambda} \mathcal{L}(\gamma + \lambda(\kappa - \gamma)) \Big|_{\lambda=0} \quad \text{for } \kappa \in J_e^1 E. \quad (40)$$

Of course,  $\tilde{\mathbf{F}}\mathcal{L}$  is just the linear part of  $\mathbf{F}\mathcal{L}$ , that is, its composition with the bundle projection  $\eta$  from extended to ordinary multiphase space:  $\tilde{\mathbf{F}}\mathcal{L} = \eta \circ \mathbf{F}\mathcal{L}$ . In local coordinates as before,  $\mathbf{F}\mathcal{L}$  is given by

$$p_i^\mu = \frac{\partial L}{\partial q_i^\mu}, \quad p = L - \frac{\partial L}{\partial q_i^\mu} q_\mu^i \quad (41)$$

where  $\mathcal{L} = L d^n x$ . Finally, if  $\mathcal{L}$  is supposed to be *hyperregular*, which by definition means that  $\tilde{\mathbf{F}}\mathcal{L}$  should be a global diffeomorphism, then one can define the De Donder-Weyl Hamiltonian  $\mathcal{H}$  to be the section of  $J^{1 \otimes E}$  over  $\tilde{J}^{1 \otimes E}$  given by

$$\mathcal{H} = \mathbf{F}\mathcal{L} \circ (\tilde{\mathbf{F}}\mathcal{L})^{-1}. \quad (42)$$

In local coordinates as before, this leads to

$$H = p_i^\mu q_\mu^i - L \quad (43)$$

where  $\mathcal{L} = L d^n x$  and  $\mathcal{H} = -H d^n x$ , as stipulated in eq. (4).

Conversely, the covariant Hamiltonian formulation of a field theory that can be described in terms of a configuration bundle  $E$  over  $M$  is defined by its Hamiltonian

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<sup>3</sup>Strictly speaking, the integration should be restricted to compact subsets of space-time, which leads to an entire family of action functionals.

density or simply *Hamiltonian*, in the spirit of De Donder and Weyl, which in global terms is a section of extended multiphase space  $\tilde{J}^{1\otimes E}$  as an affine line bundle over ordinary multiphase space  $J_e^{1\otimes E}$ :

$$\mathcal{H} : \tilde{J}^{1\otimes E} \longrightarrow J_e^{1\otimes E}. \quad (44)$$

Such a Hamiltonian gives rise to an *inverse Legendre transformation*, which is a map

$$F\mathcal{H} : \tilde{J}^{1\otimes E} \longrightarrow J_e^{1\otimes E} \quad (45)$$

of fiber bundles over  $E$  defined as follows. For any point  $\tilde{z}$  in  $\tilde{J}_e^{1\otimes E}$ , the usual fiber derivative of  $\mathcal{H}$  at  $\tilde{z}$  is a linear map from  $\tilde{J}_e^{1\otimes E}$  to  $J_e^{1\otimes E}$  which when composed with the projection  $\eta$  from  $J_e^{1\otimes E}$  to  $J_e^{1\otimes E}$  gives the identity on  $J_e^{1\otimes E}$  (since  $\mathcal{H}$  is a section): such linear maps form an affine subspace of the vector space of all linear maps from  $\tilde{J}_e^{1\otimes E}$  to  $J_e^{1\otimes E}$  that can be naturally identified with the original affine space  $J_e^{1\otimes E}$ , as explained in the Appendix. In local coordinates as before,  $F\mathcal{H}$  is given by

$$q_\mu^i = \frac{\partial H}{\partial p_i^\mu} \quad (46)$$

where  $\mathcal{H} = -H d^n x$ . Finally, if  $\mathcal{H}$  is supposed to be *hyperregular*, which by definition means that  $F\mathcal{H}$  should be a global diffeomorphism, then one can define the Lagrangian  $\mathcal{L}$  to be given by

$$\mathcal{L}(\gamma) = (\mathcal{H} \circ (F\mathcal{H})^{-1})(\gamma) \cdot \gamma. \quad (47)$$

In local coordinates as before, this leads to

$$L = p_i^\mu q_\mu^i - H \quad (48)$$

where  $\mathcal{L} = L d^n x$  and  $\mathcal{H} = -H d^n x$ .

Thus in the hyperregular case, the two processes are inverse to each other and allow one to pass freely between the Lagrangian and the Hamiltonian formulation. Of course, this is no longer true for field theories with local symmetries, in particular gauge theories, which require additional conceptual input.

At any rate, it has become apparent that even in the regular case, the full power of the multiphase space approach to geometric field theory can only be explored if one uses the ordinary and extended multiphase spaces in conjunction.

## 2.6 Canonical Forms

The distinguished role played by the extended multiphase space is due to the fact that it carries a naturally defined multisymplectic form  $\omega$ , derived from an equally naturally defined multicanonical form  $\theta$  by exterior differentiation: it is this property that turns

it into the field theoretical analogue of the cotangent bundle of mechanics.<sup>4</sup> Global constructions are given in the literature [13–15], so we shall content ourselves with stating that in local coordinates  $(x^\mu, q^i, p_i^\mu, p)$  as before,  $\theta$  takes the form

$$\theta = p_i^\mu dq^i \wedge d^n x_\mu + p d^n x, \quad (49)$$

so  $\omega = -d\theta$  becomes

$$\omega = dq^i \wedge dp_i^\mu \wedge d^n x_\mu - dp \wedge d^n x. \quad (50)$$

Given a Lagrangian  $\mathcal{L}$ , we can use the associated Legendre transformation  $\mathbb{FL}$  to pull back  $\theta$  and  $\omega$  and thus define the *Poincaré-Cartan forms*  $\theta_{\mathcal{L}}$  and  $\omega_{\mathcal{L}}$  on  $J^1 E$  associated with the Lagrangian  $\mathcal{L}$ :

$$\theta_{\mathcal{L}} = (\mathbb{FL})^* \theta, \quad \omega_{\mathcal{L}} = (\mathbb{FL})^* \omega. \quad (51)$$

Similarly, given a Hamiltonian  $\mathcal{H}$ , we can use it to pull back  $\theta$  and  $\omega$  and thus define the *De Donder-Weyl forms*  $\theta_{\mathcal{H}}$  and  $\omega_{\mathcal{H}}$  on  $\tilde{J}^{1\otimes} E$  associated with the Hamiltonian  $\mathcal{H}$ :

$$\theta_{\mathcal{H}} = \mathcal{H}^* \theta, \quad \omega_{\mathcal{H}} = \mathcal{H}^* \omega. \quad (52)$$

Of course,  $\omega_{\mathcal{L}} = -d\theta_{\mathcal{L}}$  and  $\omega_{\mathcal{H}} = -d\theta_{\mathcal{H}}$ ; moreover, supposing that  $\mathcal{H} \circ \tilde{\mathbb{FL}} = \mathbb{FL}$ , we have

$$\theta_{\mathcal{L}} = (\tilde{\mathbb{FL}})^* \theta_{\mathcal{H}}, \quad \omega_{\mathcal{L}} = (\tilde{\mathbb{FL}})^* \omega_{\mathcal{H}}. \quad (53)$$

In local coordinates as before, eq. (49) implies that

$$\theta_{\mathcal{L}} = \frac{\partial L}{\partial q_\mu^i} dq^i \wedge d^n x_\mu + \left( L - \frac{\partial L}{\partial q_\mu^i} q_\mu^i \right) d^n x, \quad (54)$$

$$\theta_{\mathcal{H}} = p_i^\mu dq^i \wedge d^n x_\mu - H d^n x. \quad (55)$$

It is useful to note that the forms  $\theta_{\mathcal{L}}$  and  $\theta_{\mathcal{H}}$  allow us to give a very simple definition of the action functional: it is given by pull-back and integration over space-time.<sup>4</sup> Thus in the Lagrangian framework, the action associated with a section  $\varphi$  of  $E$  over  $M$  is obtained by taking the pull-back of  $\theta_{\mathcal{L}}$  with its derivative which is a section  $(\varphi, \partial\varphi)$  of  $J^1 E$  over  $M$ ,

$$S[\varphi] = \int_M (\varphi, \partial\varphi)^* \theta_{\mathcal{L}} \quad \text{for } \varphi \in \Gamma(E), \quad (56)$$

whereas in the Hamiltonian framework, the action associated with a section  $(\varphi, \pi)$  of  $\tilde{J}^{1\otimes} E$  over  $M$  is simply

$$S[\varphi, \pi] = \int_M (\varphi, \pi)^* \theta_{\mathcal{H}} \quad \text{for } (\varphi, \pi) \in \Gamma(\tilde{J}^{1\otimes} E). \quad (57)$$

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<sup>4</sup>Note that this statement fails if one uses the ordinary duals instead of the twisted ones.

In both cases, it can be shown that the stationary points of the action are precisely the solutions of the corresponding Euler - Lagrange and De Donder - Weyl equations, respectively. It is therefore no surprise that these equations can be formulated globally through the vanishing of certain (in general nonlinear) differential operators  $\mathcal{E}$  and  $\mathcal{D}$  defined solely in terms of the forms  $\omega_{\mathcal{L}}$  and  $\omega_{\mathcal{G}}$ , respectively. However, an explicit construction in the spirit of global analysis [23, 24] does not seem to be readily available, although there do exist various attempts that go a long way in the right direction; see, e.g., [25] for the Lagrangian case and [13] for the Hamiltonian case.

## 2.7 Euler - Lagrange and De Donder - Weyl Operator

**Theorem 1** *Given a Lagrangian density as in eq. (35) above, define the corresponding Euler - Lagrange operator to be the map*

$$\mathcal{E} : J^2 E \longrightarrow V^* E \otimes \bigwedge^n T^* M \quad (58)$$

of fiber bundles over  $J^1 E$ <sup>5</sup> that associates to each 2-jet  $(\varphi, \partial\varphi, \partial^2\varphi)$  of (local) sections  $\varphi$  of  $E$  over  $M$  and each vertical vector field  $V$  on  $E$  the  $n$ -form on  $M$  given by

$$\mathcal{E}(\varphi, \partial\varphi, \partial^2\varphi) \cdot V = (\varphi, \partial\varphi)^* (i_{J^1 V} \omega_{\mathcal{L}}) . \quad (59)$$

Then for any section  $\varphi$  of  $E$ ,  $\mathcal{E}(\varphi, \partial\varphi, \partial^2\varphi)$  is the zero section if only if  $\varphi$  satisfies the Euler - Lagrange equations associated to  $\mathcal{L}$ .

**PROOF:** Let  $V$  be a vertical vector field on  $E$ , with local coordinate expression

$$V = V^i \frac{\partial}{\partial q^i}$$

(cf. eq. (12)), and let  $J^1 V$  be its lifting to  $J^1 E$ , with local coordinate expression

$$J^1 V = V^i \frac{\partial}{\partial q^i} + \left( \frac{\partial V^i}{\partial q^k} q_\mu^k + \frac{\partial V^i}{\partial x^\mu} \right) \frac{\partial}{\partial q_\mu^i}$$

(cf. eq. (13)). Applying the exterior derivative to eq. (54), contracting with  $J^1 V$  and then pulling back with  $(\varphi, \partial\varphi)$  gives, after some calculation,

$$\begin{aligned} (\varphi, \partial\varphi)^* (i_{J^1 V} \omega_{\mathcal{L}}) &= \frac{\partial^2 L}{\partial x^\mu \partial q_\mu^i} (\varphi, \partial\varphi) V^i(\varphi) d^n x + \frac{\partial^2 L}{\partial q^i \partial q_\mu^i} (\varphi, \partial\varphi) V^i(\varphi) \partial_\mu \varphi^i d^n x \\ &\quad + \frac{\partial^2 L}{\partial q_\nu^i \partial q_\mu^i} (\varphi, \partial\varphi) V^i(\varphi) \partial_\mu \partial_\nu \varphi^i d^n x - \frac{\partial L}{\partial q^i} (\varphi, \partial\varphi) V^i(\varphi) d^n x \\ &= \left( \partial_\mu \left( \frac{\partial L}{\partial q_\mu^i} (\varphi, \partial\varphi) \right) - \frac{\partial L}{\partial q^i} (\varphi, \partial\varphi) \right) V^i(\varphi) d^n x , \end{aligned}$$

<sup>5</sup>Again, we suppress the symbols indicating the pull-back of bundles from  $E$  or  $M$  to  $J^1 E$ .

where it is to be noted that the terms depending on the derivatives of  $V$  have dropped out. This leads to the following explicit formula for  $\mathcal{E}$ :

$$\mathcal{E}(\varphi, \partial\varphi, \partial^2\varphi) = \left( \partial_\mu \left( \frac{\partial L}{\partial q_\mu^i}(\varphi, \partial\varphi) \right) - \frac{\partial L}{\partial q^i}(\varphi, \partial\varphi) \right) dq^i \otimes d^n x . \quad (60)$$

In particular, it is clear that  $\mathcal{E}$  depends on  $\varphi$  only through the point values of  $\varphi$  and its partial derivatives up to second order, which concludes the proof.  $\square$

**Theorem 2** *Given a Hamiltonian density as in eq. (44) above, define the corresponding De Donder - Weyl operator to be the map*

$$\mathcal{D} : J^1(\tilde{J}^{1\otimes}E) \longrightarrow V^*(\tilde{J}^{1\otimes}E) \otimes \wedge^n T^*M \quad (61)$$

of fiber bundles over  $\tilde{J}^{1\otimes}E$ <sup>6</sup> that associates to each 1-jet  $(\varphi, \pi, \partial\varphi, \partial\pi)$  of (local) sections  $(\varphi, \pi)$  of  $\tilde{J}^{1\otimes}E$  over  $M$  and each vertical vector field  $V$  on  $\tilde{J}^{1\otimes}E$  the  $n$ -form on  $M$  given by

$$\mathcal{D}(\varphi, \pi, \partial\varphi, \partial\pi) \cdot V = (\varphi, \pi)^* (i_V \omega_{\mathcal{H}}) . \quad (62)$$

Then for any section  $(\varphi, \pi)$  of  $\tilde{J}^{1\otimes}E$ ,  $\mathcal{D}(\varphi, \pi, \partial\varphi, \partial\pi)$  is the zero section if and only if  $(\varphi, \pi)$  satisfies the De Donder - Weyl equations associated to  $\mathcal{H}$ .

PROOF: Let  $V$  be a vertical vector field on  $\tilde{J}^{1\otimes}E$ , with local coordinate expression

$$V = V^i \frac{\partial}{\partial q^i} + V_i^\mu \frac{\partial}{\partial p_i^\mu} .$$

Applying the exterior derivative to eq. (55), contracting with  $V$  and then pulling back with  $(\varphi, \pi)$  gives, after a short calculation,

$$\begin{aligned} (\varphi, \pi)^* (i_V \omega_{\mathcal{H}}) &= \partial_\mu \pi_i^\mu V^i(\varphi, \pi) d^n x + \frac{\partial H}{\partial q^i}(\varphi, \pi) V^i(\varphi, \pi) d^n x \\ &\quad - \partial_\mu \varphi^i V_i^\mu(\varphi, \pi) d^n x + \frac{\partial H}{\partial p_i^\mu}(\varphi, \pi) V_i^\mu(\varphi, \pi) d^n x . \end{aligned}$$

This leads to the following explicit formula for  $\mathcal{D}$ :

$$\begin{aligned} \mathcal{D}(\varphi, \pi, \partial\varphi, \partial\pi) &= \left( \frac{\partial H}{\partial q^i}(\varphi, \pi) - \partial_\mu \pi_i^\mu \right) dq^i \otimes d^n x \\ &\quad + \left( \frac{\partial H}{\partial p_i^\mu}(\varphi, \pi) + \partial_\mu \varphi^i \right) dp_i^\mu \otimes d^n x . \end{aligned} \quad (63)$$

In particular, it is clear that  $\mathcal{D}$  depends on  $(\varphi, \pi)$  only through the point values of  $\varphi$  and  $\pi$  and their partial derivatives up to first order, which concludes the proof.  $\square$

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<sup>6</sup>Again, we suppress the symbols indicating the pull-back of bundles from  $E$  or  $M$  to  $\tilde{J}^{1\otimes}E$ .

## 2.8 Jacobi Operators

In order to make contact with the functional formalism to be discussed in the next section, we must also derive explicit expressions for the linearization of the Euler-Lagrange operator and the De Donder-Weyl operator around a given solution of the equations of motion. This leads to *linear* differential operators between vector bundles over  $M$  that we shall refer to as *Jacobi operators*, generalizing the familiar derivation of the Jacobi equation by linearizing the geodesic equation.

In its Lagrangian version, the *Jacobi operator* is a second order differential operator

$$\mathcal{J}_L[\varphi] : \Gamma(V_\varphi) \longrightarrow \Gamma(V_\varphi^\otimes), \quad (64)$$

where  $V_\varphi = \varphi^*(VE)$  and  $V_\varphi^\otimes = \varphi^*(V^*E) \otimes \Lambda^n T^*M$ , obtained by linearizing the Euler-Lagrange operator  $\mathcal{E}$  around a given solution  $\varphi$  of the equations of motion. Similarly, in its Hamiltonian version, the *Jacobi operator* is a first order differential operator

$$\mathcal{J}_H[\varphi, \pi] : \Gamma(V_{(\varphi, \pi)}) \longrightarrow \Gamma(V_{(\varphi, \pi)}^\otimes), \quad (65)$$

where  $V_{(\varphi, \pi)} = (\varphi, \pi)^*(V(\mathcal{J}^{1\otimes}E))$  and  $V_{(\varphi, \pi)}^\otimes = (\varphi, \pi)^*(V^*(\mathcal{J}^{1\otimes}E)) \otimes \Lambda^n T^*M$ , obtained by linearizing the De Donder-Weyl operator  $\mathcal{D}$  around a given solution  $(\varphi, \pi)$  of the equations of motion. (Thus in both cases, the vector bundles involved are obtained by pulling back the appropriate vertical bundle and its twisted dual with the solution of the nonlinear equation around which the linearization is performed.) To obtain explicit expressions, consider an arbitrary variation  $\varphi_\lambda$  around  $\varphi$  and evaluate  $\mathcal{E}(\varphi_\lambda, \partial\varphi_\lambda, \partial^2\varphi_\lambda)$  which, for each  $\lambda$ , is a section of  $V_\varphi^\otimes$ , observing that since  $\varphi = \varphi_\lambda|_{\lambda=0}$  is a solution,  $\mathcal{E}(\varphi_\lambda, \partial\varphi_\lambda, \partial^2\varphi_\lambda)|_{\lambda=0}$  is the zero section of  $V_\varphi^\otimes$ , and setting

$$\delta\varphi = \frac{\partial}{\partial\lambda} \varphi_\lambda \Big|_{\lambda=0}. \quad (66)$$

Noting that in local coordinates, the value of  $\mathcal{E}(\varphi_\lambda, \partial\varphi_\lambda, \partial^2\varphi_\lambda)$  at a point  $x$  in  $M$  with coordinates  $x^\mu$  has coordinates  $(x^\mu, \varphi_\lambda^i(x), \mathcal{E}(\varphi_\lambda, \partial\varphi_\lambda, \partial^2\varphi_\lambda)_i(x))$  where the last piece is the coefficient of  $dq^i \otimes d^n x$  in eq. (60), we get by differentiation with respect to  $\lambda$

$$\begin{aligned} & \frac{\partial}{\partial\lambda} \mathcal{E}(\varphi_\lambda, \partial\varphi_\lambda, \partial^2\varphi_\lambda) \Big|_{\lambda=0} \\ &= \delta\varphi^i \frac{\partial}{\partial q^i} \\ & \quad + \left( \partial_\mu \left( \frac{\partial^2 L}{\partial q^j \partial q_\mu^i}(\varphi, \partial\varphi) \delta\varphi^j + \frac{\partial^2 L}{\partial q_\nu^j \partial q_\mu^i}(\varphi, \partial\varphi) \partial_\nu \delta\varphi^j \right) \right. \\ & \quad \left. - \frac{\partial^2 L}{\partial q^j \partial q^i}(\varphi, \partial\varphi) \delta\varphi^j - \frac{\partial^2 L}{\partial q_\nu^j \partial q^i}(\varphi, \partial\varphi) \partial_\nu \delta\varphi^j \right) dq^i \otimes d^n x. \end{aligned}$$

Similarly, consider an arbitrary variation  $(\varphi_\lambda, \pi_\lambda)$  around  $(\varphi, \pi)$  and evaluate  $\mathcal{D}(\varphi_\lambda, \pi_\lambda, \partial\varphi_\lambda, \partial\pi_\lambda)$  which, for each  $\lambda$ , is a section of  $V_{(\varphi_\lambda, \pi_\lambda)}^*$ , observing that since  $(\varphi, \pi) = (\varphi_\lambda, \pi_\lambda)|_{\lambda=0}$  is a solution,  $\mathcal{D}(\varphi_\lambda, \pi_\lambda, \partial\varphi_\lambda, \partial\pi_\lambda)|_{\lambda=0}$  is the zero section of  $V_{(\varphi, \pi)}^*$ , and setting

$$(\delta\varphi, \delta\pi) = \frac{\partial}{\partial\lambda} (\varphi_\lambda, \pi_\lambda) \Big|_{\lambda=0}. \quad (67)$$

Again, noting that in local coordinates, the value of  $\mathcal{D}(\varphi_\lambda, \pi_\lambda, \partial\varphi_\lambda, \partial\pi_\lambda)$  at a point  $x$  in  $M$  with coordinates  $x^\mu$  has coordinates  $(x^\mu, \varphi_\lambda^i(x), (\pi_\lambda)_i^\mu(x), \mathcal{D}(\varphi_\lambda, \pi_\lambda, \partial\varphi_\lambda, \partial\pi_\lambda)_i(x), \mathcal{D}(\varphi_\lambda, \pi_\lambda, \partial\varphi_\lambda, \partial\pi_\lambda)_\mu^i(x))$  where the last two pieces are the coefficients of  $dq^i \otimes d^n x$  and of  $dp_i^\mu \otimes d^n x$  in eq. (63), we get by differentiation with respect to  $\lambda$

$$\begin{aligned} & \frac{\partial}{\partial\lambda} \mathcal{D}(\varphi_\lambda, \pi_\lambda, \partial\varphi_\lambda, \partial\pi_\lambda) \Big|_{\lambda=0} \\ &= \delta\varphi^i \frac{\partial}{\partial q^i} + \delta\pi_i^\mu \frac{\partial}{\partial p_i^\mu} \\ &+ \left( \frac{\partial^2 H}{\partial q^j \partial q^i}(\varphi, \pi) \delta\varphi^j + \frac{\partial^2 H}{\partial p_j^\nu \partial q^i}(\varphi, \pi) \delta\pi_j^\nu - \partial_\mu \delta\pi_i^\mu \right) dq^i \otimes d^n x \\ &+ \left( \frac{\partial^2 H}{\partial q^j \partial p_i^\mu}(\varphi, \pi) \delta\varphi^j + \frac{\partial^2 H}{\partial p_j^\nu \partial p_i^\mu}(\varphi, \pi) \delta\pi_j^\nu + \partial_\mu \delta\varphi^i \right) dp_i^\mu \otimes d^n x. \end{aligned}$$

In order to show how to extract the Jacobi operators from these expressions, by means of a globally defined prescription, we apply the following construction [11]. Let  $F$  be a fiber bundle over  $M$ , with bundle projection  $\pi_{F,M} : F \rightarrow M$ , and  $W$  be a vector bundle over  $F$  with bundle projection  $\pi_{W,F} : W \rightarrow F$ , which is then also a fiber bundle (but not necessarily a vector bundle) over  $M$  with respect to the composite bundle projection  $\pi_{W,M} = \pi_{F,M} \circ \pi_{W,F} : W \rightarrow M$ . Thus  $W$  admits two different kinds of vertical bundles,  $V_F W$  and  $V_M W$ , with fibers defined by  $(V_F)_w W = \ker T_w \pi_{W,F}$  and  $(V_M)_w W = \ker T_w \pi_{W,M}$  for  $w \in W$ ; obviously, the former is contained in the latter as a vector subbundle. Moreover, since  $W$  is supposed to be a vector bundle over  $F$ , there is a canonical isomorphism  $V_F W \cong \pi_{W,F}^* W$ . On the other hand, consider the vertical bundle  $V_F$  of  $F$  which can be pulled back to  $W$  to obtain a vector bundle  $\pi_{W,F}^*(V_F)$  over  $W$ , with fibers defined by  $(\pi_{W,F}^*(V_F))_w = V_f F = \ker T_f \pi_{F,M}$  for  $w \in W$  with  $f = \pi_{W,F} w$ . Note also that the tangent map to the bundle projection  $\pi_{W,F}$ , which by definition has kernel  $V_F W$ , maps  $V_M W$  onto  $V_F$ , so we have the following exact sequence of vector bundles over  $W$ :

$$0 \longrightarrow V_F W \cong \pi_{W,F}^* W \longrightarrow V_M W \longrightarrow \pi_{W,F}^*(V_F) \longrightarrow 0.$$

The crucial observation is now that this exact sequence admits a canonical splitting over the zero section  $0 : F \rightarrow W$ , given simply by its tangent map. Indeed, its tangent map  $T_f 0 : T_f F \rightarrow T_{0(f)} W$  at any point  $f \in F$  takes the vertical subspace  $V_F$  to the  $M$ -vertical subspace  $(V_M)_{0(f)} W$  and so restricts to a vertical tangent map

$V_f 0 : V_f F \rightarrow (V_M)_{0(f)} W$  whose composition with the restriction of the tangent map  $T_{0(f)} \pi_{W,F} : T_{0(f)} W \rightarrow T_f F$  to  $(V_M)_{0(f)} W$  gives the identity on  $V_f F$ . Thus the image of  $V_f 0$  is a subspace of  $(V_M)_{0(f)} W$  that is complementary to the subspace  $(V_F)_{0(f)} W = W_f$  and provides a surjective linear map  $\sigma_f : (V_M)_{0(f)} W \rightarrow W_f$  of which it is the kernel. At the level of bundles, this corresponds to a surjective vector bundle homomorphism  $\sigma : V_M W|_0 \rightarrow W$ .

Applying this construction to the situation at hand, take  $F = E$  in the Lagrangian case and  $F = \tilde{J}^{1 \otimes E}$  in the Hamiltonian case, setting  $W = V^*(F) \otimes \bigwedge^n T^* M$  in both cases. The fact that the Euler-Lagrange or De Donder-Weyl operator is being linearized around a solution  $\varphi$  or  $(\varphi, \pi)$  of the equations of motion then means that we are evaluating its derivative, which a priori takes the variation  $\delta\varphi$  or  $(\delta\varphi, \delta\pi)$  to a vector field on  $W$  along  $M$  which is vertical with respect to the projection of  $W$  onto  $M$ , precisely over the zero section, so we can apply the operator  $\sigma$  just introduced to project it down to a section of  $W$  over  $M$  itself. This operation completes the definition of the Jacobi operators, namely

$$\mathcal{J}_L[\varphi] \cdot \delta\varphi = \sigma \left( \frac{\partial}{\partial \lambda} \mathcal{E}(\varphi_\lambda, \partial\varphi_\lambda, \partial^2 \varphi_\lambda) \Big|_{\lambda=0} \right), \quad (68)$$

and

$$\mathcal{J}_H[\varphi, \pi] \cdot (\delta\varphi, \delta\pi) = \sigma \left( \frac{\partial}{\partial \lambda} \mathcal{D}(\varphi_\lambda, \pi_\lambda, \partial\varphi_\lambda, \partial\pi_\lambda) \Big|_{\lambda=0} \right). \quad (69)$$

The local coordinate expressions are the ones derived above, that is,

$$\begin{aligned} \mathcal{J}_L[\varphi] \cdot \delta\varphi &= \left\{ \frac{\partial^2 L}{\partial q_\nu^j \partial q_\mu^i} (\varphi, \partial\varphi) \partial_\mu \partial_\nu \delta\varphi^j \right. \\ &\quad + \left( \partial_\mu \left( \frac{\partial^2 L}{\partial q_\nu^j \partial q_\mu^i} (\varphi, \partial\varphi) \right) + \left( \frac{\partial^2 L}{\partial q_\nu^j \partial q_\nu^i} - \frac{\partial^2 L}{\partial q_\nu^i \partial q_\nu^j} \right) (\varphi, \partial\varphi) \right) \partial_\nu \delta\varphi^j \quad (70) \\ &\quad \left. + \left( \partial_\mu \left( \frac{\partial^2 L}{\partial q_\nu^j \partial q_\mu^i} (\varphi, \partial\varphi) \right) - \frac{\partial^2 L}{\partial q_\nu^j \partial q_\nu^i} (\varphi, \partial\varphi) \right) \delta\varphi^j \right\} dq^i \otimes d^n x, \end{aligned}$$

and

$$\begin{aligned} \mathcal{J}_H[\varphi, \pi] \cdot (\delta\varphi, \delta\pi) &= \left( \frac{\partial^2 H}{\partial q^j \partial q^i} (\varphi, \pi) \delta\varphi^j + \frac{\partial^2 H}{\partial p_j^\nu \partial q^i} (\varphi, \pi) \delta\pi_j^\nu - \partial_\mu \delta\pi_i^\mu \right) dq^i \otimes d^n x \quad (71) \\ &\quad + \left( \frac{\partial^2 H}{\partial q^j \partial p_i^\mu} (\varphi, \pi) \delta\varphi^j + \frac{\partial^2 H}{\partial p_j^\nu \partial p_i^\mu} (\varphi, \pi) \delta\pi_j^\nu + \partial_\mu \delta\varphi^i \right) dp_i^\mu \otimes d^n x. \end{aligned}$$

### 3 Functional Approach

Let us begin by recalling the definition of the Poisson bracket between functions on a symplectic manifold with symplectic form  $\omega$ . First, one associates to each (smooth) function  $f$  a (smooth) Hamiltonian vector field  $X_f$ , uniquely determined by the condition

$$i_{X_f}\omega = df. \quad (72)$$

Then the Poisson bracket of two functions  $f$  and  $g$  is defined to be the function  $\{f, g\}$  given by

$$\{f, g\} = -i_{X_f}i_{X_g}\omega = df(X_g) = -dg(X_f). \quad (73)$$

The goal of this section is to show that formally, the same construction applied to covariant phase space links the Witten symplectic form to the Peierls bracket.

#### 3.1 Covariant Phase Space

In contrast to the traditional non-covariant Hamiltonian formalism of field theory, where phase space is a “space” of Cauchy data, *covariant phase space*, denoted here by  $\mathcal{S}$ , is the “space” of solutions of the equations of motion, or field equations. Of course, one cannot expect these two interpretations of phase space to be equivalent in complete generality, since it is well known that, for nonlinear equations, time evolution of regular Cauchy data may lead to solutions that, within finite time, develop some kind of singularity. An even more elementary prerequisite is that the underlying space-time manifold  $M$  must admit at least some Cauchy surface  $\Sigma$ : this means that  $M$  should be globally hyperbolic.

Thus our basic assumption for the remainder of this paper will be that the underlying space-time manifold  $M$  should be *globally hyperbolic*. Globally hyperbolic space-times are the natural arena for the mathematical theory of hyperbolic (systems of) partial differential equations, in which the Cauchy problem is well posed. There are by now various and apparently quite different definitions of the concept of a globally hyperbolic space-time, but they have ultimately turned out to be all equivalent; see Chapter 8 of [26] for an extensive discussion. For our purposes, the most convenient one is that  $M$  admits a global time function whose level surfaces provide a foliation of  $M$  into Cauchy surfaces, providing a global diffeomorphism  $M \cong \mathbb{R} \times \Sigma$ . As an immediate corollary, we can define the concept of a (closed/open) *time slice* in  $M$ : it is a (closed/open) subset of  $M$  which under such a global diffeomorphism corresponds to a subset of the form  $I \times \Sigma$  where  $I$  is a (closed/open) interval in  $\mathbb{R}$ .

In the Lagrangian as well as in the Hamiltonian approach to field theory, the equations of motion are derived from a variational principle, that is, their solutions are the stationary points of a certain functional  $S$  called the *action* and defined on a space of sections of an appropriate fiber bundle over space-time which is usually referred to

as the *space of field configurations* of the theory and will in what follows be denoted by  $\mathcal{C}$ . More concretely,  $\mathcal{C}$  is the space  $\Gamma(F)$  of smooth sections  $\phi$  of a fiber bundle  $F$  over  $M$ : in the Lagrangian approach,  $F$  is the configuration bundle  $E$ , whereas in the Hamiltonian approach,  $F$  is the multiphase space  $\mathcal{J}^1 E$ , regarded as a fiber bundle over  $M$ .

Formally, we shall as usual think of  $\mathcal{C}$  as being a manifold (which is of course infinite-dimensional). As such, it has at each of its points  $\phi$  a tangent space  $T_\phi \mathcal{C}$  that can be defined formally as a space of smooth sections, with appropriate support properties, of the vector bundle  $V_\phi = \phi^*(VF)$  over  $M$ , i.e.,  $T_\phi \mathcal{C} \subset \Gamma^\infty(V_\phi)$ . The cotangent space  $T_\phi^* \mathcal{C}$  will then be the space of distributional sections, with dual support properties, of the vector bundle  $V_\phi^* = \phi^*(V^*F) \otimes \Lambda^n T^* M$  over  $M$ , i.e.,  $T_\phi^* \mathcal{C} \subset \Gamma^{-\infty}(V_\phi^*)$ . It contains as a subspace the corresponding space of smooth sections, where the pairing between a smooth section of  $V_\phi$  and a smooth section of  $V_\phi^*$  (with appropriate support conditions) is given by contraction and integration of the resulting form over  $M$ . Similarly, the second tensor power  $T_\phi^* \mathcal{C} \otimes T_\phi^* \mathcal{C}$  of  $T_\phi^* \mathcal{C}$  can be thought of as the space of distributional sections, again with dual support properties, of the second exterior tensor power<sup>7</sup>  $V_\phi^* \boxtimes V_\phi^*$  of  $V_\phi^*$ ; it contains as a subspace the corresponding space of smooth sections, where the pairing between a pair of smooth sections of  $V_\phi$  and a smooth section of  $V_\phi^* \boxtimes V_\phi^*$  (with appropriate support conditions) is given by contraction and integration of the resulting form over  $M \times M$ .

Regarding the support conditions to be imposed, the first two options that come to mind would be to require that either the elements of  $T_\phi \mathcal{C}$  or the elements of  $T_\phi^* \mathcal{C}$  should have compact support, which would imply that the support of the elements of the corresponding dual,  $T_\phi^* \mathcal{C}$  or  $T_\phi \mathcal{C}$ , could be left completely arbitrary:

$$\text{OPTION 1 : } T_\phi \mathcal{C} = \Gamma^\infty(V_\phi) , \quad T_\phi^* \mathcal{C} = \Gamma_c^{-\infty}(V_\phi^*) \quad (74)$$

$$\text{OPTION 2 : } T_\phi \mathcal{C} = \Gamma_c^\infty(V_\phi) , \quad T_\phi^* \mathcal{C} = \Gamma^{-\infty}(V_\phi^*) \quad (75)$$

There is a third option that makes use of the assumption that  $M$  is globally hyperbolic. To formulate it, we introduce the following terminology. A section of a vector bundle over  $M$  is said to have *spatially compact support* if the intersection between its support and any (closed) time slice in  $M$  is compact, and it is said to have *temporally compact support* if its support is contained in some time slice. Then we require the elements of  $T_\phi \mathcal{C}$  to have spatially compact support and the elements of  $T_\phi^* \mathcal{C}$  to have temporally compact support:

$$\text{OPTION 3 : } T_\phi \mathcal{C} = \Gamma_{sc}^\infty(V_\phi) , \quad T_\phi^* \mathcal{C} = \Gamma_{tc}^{-\infty}(V_\phi^*) \quad (76)$$

Obviously, for each of these three options, the two spaces listed above are naturally dual to each other.<sup>8</sup>

<sup>7</sup>If  $V$  and  $W$  are vector bundles over  $M$ ,  $V \boxtimes W$  is defined to be the vector bundle over  $M \times M$  with fibers given by  $(V \boxtimes W)_{(x,y)} = V_x \otimes W_y$ , for all  $x, y \in M$ .

<sup>8</sup>Here and in what follows, the symbols  $\Gamma_c$ ,  $\Gamma_{sc}$  and  $\Gamma_{tc}$  indicate spaces of sections of compact, spatially compact and temporally compact support, respectively.

These constructions can be applied to elucidate the nature of functional derivatives of functionals on  $\mathcal{C}$ , such as the action. Namely, given a (formally smooth) functional  $A : \mathcal{C} \rightarrow \mathbb{R}$ , its functional derivative at a point  $\phi$  is the distributional section  $(\delta A / \delta \phi)[\phi]$  of  $V_\phi^*$  that, according to

$$A'[\phi] \cdot \delta \phi = \int_M d^n x \frac{\delta A}{\delta \phi}[\phi](x) \cdot \delta \phi(x) , \quad (77)$$

represents the directional derivative of  $A$  at  $\phi$  along  $\delta \phi$  – a smooth section of  $V_\phi$  with appropriate support properties, defined by the requirement that for any one-parameter family of sections  $\phi_\lambda$  of  $F$  such that  $\phi_\lambda|_{\lambda=0} = \phi$ ,

$$A'[\phi] \cdot \delta \phi = \frac{d}{d\lambda} A[\phi_\lambda] \Big|_{\lambda=0} \quad \text{if} \quad \delta \phi = \frac{\partial}{\partial \lambda} \phi_\lambda \Big|_{\lambda=0} .$$

In local coordinates, we have in the Lagrangian framework

$$\frac{\delta A}{\delta \phi}[\phi](x) = \frac{\delta A}{\delta \varphi^i}[\varphi](x) dq^i \otimes d^n x ,$$

whereas in the Hamiltonian framework

$$\frac{\delta A}{\delta \phi}[\phi](x) = \frac{\delta A}{\delta \varphi^i}[\varphi, \pi](x) dq^i \otimes d^n x + \frac{\delta A}{\delta \pi^\mu}[\varphi, \pi](x) dp_i^\mu \otimes d^n x .$$

Similarly, the second functional derivative of  $A$  at  $\phi$  is the distributional section  $(\delta^2 A / \delta \phi^2)[\phi]$  of  $V_\phi^* \boxtimes V_\phi^*$  that, according to

$$A''[\phi] \cdot (\delta \phi_1, \delta \phi_2) = \int_M d^n x \int_M d^n y \frac{\delta^2 A}{\delta \phi^2}[\phi](x, y) \cdot (\delta \phi_1(x), \delta \phi_2(y)) , \quad (78)$$

represents the Hessian of  $A$  at  $\phi$  evaluated on  $\delta \phi_1$  and  $\delta \phi_2$  – two smooth sections of  $V_\phi$  with appropriate support properties, defined by the requirement that for any two-parameter family of sections  $\phi_{\lambda_1, \lambda_2}$  of  $F$  such that  $\phi_{\lambda_1, \lambda_2}|_{\lambda_1, \lambda_2=0} = \phi$ ,

$$A''[\phi] \cdot (\delta \phi_1, \delta \phi_2) = \frac{\partial^2}{\partial \lambda_1 \partial \lambda_2} A[\phi_{\lambda_1, \lambda_2}] \Big|_{\lambda_1, \lambda_2=0}$$

if

$$\delta \phi_1 = \frac{\partial}{\partial \lambda_1} \phi_{\lambda_1, \lambda_2} \Big|_{\lambda_1, \lambda_2=0} , \quad \delta \phi_2 = \frac{\partial}{\partial \lambda_2} \phi_{\lambda_1, \lambda_2} \Big|_{\lambda_1, \lambda_2=0} .$$

In local coordinates, we have in the Lagrangian framework

$$\frac{\delta^2 A}{\delta \phi^2}[\phi](x, y) = \frac{\delta^2 A}{\delta \varphi^i \delta \varphi^j}[\varphi](x, y) (dq^i \otimes d^n x) \otimes (dq^j \otimes d^n y) ,$$

whereas in the Hamiltonian framework

$$\begin{aligned}
 \frac{\delta^2 A}{\delta \phi^2}[\phi](x, y) &= \frac{\delta^2 A}{\delta \varphi^i \delta \varphi^j}[\varphi, \pi](x, y) (dq^i \otimes d^n x) \otimes (dq^j \otimes d^n y) \\
 &+ \frac{\delta^2 A}{\delta \varphi^i \delta \pi_j^\nu}[\varphi, \pi](x, y) (dq^i \otimes d^n x) \otimes (dp_j^\nu \otimes d^n y) \\
 &+ \frac{\delta^2 A}{\delta \pi_i^\mu \delta \varphi^j}[\varphi, \pi](x, y) (dp_i^\mu \otimes d^n x) \otimes (dq^j \otimes d^n y) \\
 &+ \frac{\delta^2 A}{\delta \pi_i^\mu \delta \pi_j^\nu}[\varphi, \pi](x, y) (dp_i^\mu \otimes d^n x) \otimes (dp_j^\nu \otimes d^n y) .
 \end{aligned}$$

Of course, for the integrals in eqs (77) and (78) to make sense, even when interpreted in the sense of pairing distributions with test functions, we must make some assumption about support properties, which leads us back to the options stated in eqs (74)-(76). Option 1: when  $A$  is *arbitrary*, we have to restrict the sections  $\delta\phi$ ,  $\delta\phi_1$ ,  $\delta\phi_2$  of  $V_\phi$  considered above to have compact support (which can be achieved if the sections  $\phi_\lambda$ ,  $\phi_{\lambda_1, \lambda_2}$  of  $F$  are supposed to be independent of the parameters outside a compact subset). Option 2: when  $A$  is *local*, which we understand to mean that its functional dependence on the fields is non-trivial only within a compact region, or equivalently, that its functional derivative  $\delta A/\delta\phi$  at each  $\phi$  has compact support, the sections  $\delta\phi$ ,  $\delta\phi_1$ ,  $\delta\phi_2$  of  $V_\phi$  considered above may be allowed to have arbitrary support; this is the case for local observables defined as integrals of local densities over compact regions of space-time and, in particular, over compact regions within a Cauchy surface  $\Sigma$  (energy, momentum, angular momentum, charges etc. within a finite volume). Option 3: when  $A$  is *local in time*, which we understand to mean that its functional dependence on the fields is non-trivial only within a time slice, or equivalently, that its functional derivative  $\delta A/\delta\phi$  at each  $\phi$  has temporally compact support, we have to restrict the sections  $\delta\phi$ ,  $\delta\phi_1$ ,  $\delta\phi_2$  of  $V_\phi$  considered above to have spatially compact support (which can be achieved if the sections  $\phi_\lambda$ ,  $\phi_{\lambda_1, \lambda_2}$  are supposed to be independent of the parameters outside a spatially compact subset); this is the case for global observables defined as integrals of local densities over time slices and, in particular, over a Cauchy surface  $\Sigma$  (total energy, total momentum, total angular momentum, total charges etc.).

Finally, covariant phase space  $\mathcal{S}$  is defined to be the subset of  $\mathcal{C}$  consisting of the critical points of the action:

$$\mathcal{S} = \{\phi \in \mathcal{C} / S'[\phi] = 0\} . \quad (79)$$

Formally, we can think of  $\mathcal{S}$  as being a submanifold of  $\mathcal{C}$  whose tangent space at any point  $\phi$  of  $\mathcal{S}$  will be the subspace  $T_\phi \mathcal{S}$  of the tangent space  $T_\phi \mathcal{C}$  consisting of the solutions of the linearized equations of motion (where “linearized” means “linearized around the solution  $\phi$  of the full equations of motion”), which are precisely the sections of  $V_\phi$  belonging to the kernel of the corresponding Jacobi operator  $\mathcal{J}[\phi] : \Gamma(V_\phi) \longrightarrow \Gamma(V_\phi^\otimes)$ :

$$T_\phi \mathcal{S} = \ker \mathcal{J}[\phi] . \quad (80)$$

### 3.2 Symplectic Structure

Our next goal is to justify the term “covariant phase space” attributed to  $\mathcal{S}$  by showing that, formally,  $\mathcal{S}$  carries a naturally defined symplectic form  $\Omega$ , derived from an equally naturally defined canonical form  $\Theta$  by formal exterior differentiation. According to Crnković, Witten and Zuckerman [1–3] (see also [4]), the symplectic form  $\Omega$  can be obtained by integration of a “symplectic current”, which is a closed  $(n-1)$ -form on space-time, over an arbitrary spacelike hypersurface  $\Sigma$ . Here, we show that this “symplectic current” can be derived directly from the multisymplectic form  $\omega$  or, more explicitly, from the Poincaré–Cartan form  $\omega_L$  in the Lagrangian approach and the De Donder–Weyl form  $\omega_{\mathcal{H}}$  in the Hamiltonian approach.

We begin with the definition of  $\Theta$  and  $\Omega$  in terms of  $\theta$  and  $\omega$ , which is achieved by a mixture of contraction and pull-back: given a point  $\phi$  in  $\mathcal{C}$  (a smooth section  $\phi$  of  $F$ ) and smooth sections  $\delta\phi$ ,  $\delta\phi_1$ ,  $\delta\phi_2$  of  $V_\phi$ , insert  $\delta\phi$  into the first of the  $n$  arguments of  $\theta$  or  $\delta\phi_1$  and  $\delta\phi_2$  into the first two of the  $n+1$  arguments of  $\omega$  and apply the definition of the pull-back with  $\phi$  (which amounts to composition with the derivatives  $\partial\phi$  of  $\phi$ ) to the remaining  $n-1$  arguments to obtain  $(n-1)$ -forms on space-time which are integrated over  $\Sigma$ . Note that these integrals exist if we assume that  $\delta\phi$  and either  $\delta\phi_1$  or  $\delta\phi_2$  have spatially compact support, since this will intersect  $\Sigma$  in a compact subset.

Explicitly, in the Lagrangian framework, we have

$$\Theta_\phi(\delta\phi) = \int_\Sigma (\varphi, \partial\varphi)^* \theta_L(\delta\varphi, \partial\delta\varphi) \quad (81)$$

and

$$\Omega_\phi(\delta\phi_1, \delta\phi_2) = \int_\Sigma (\varphi, \partial\varphi)^* \omega_L(\delta\varphi_1, \partial\delta\varphi_1, \delta\varphi_2, \partial\delta\varphi_2) \quad (82)$$

where the notation is the same as that employed in eq. (56):  $\phi = \varphi$  is a section of  $E$  over  $M$  and  $j^1\varphi = (\varphi, \partial\varphi)$  is its (first) prolongation or derivative, a section of  $J^1E$  over  $M$ , while  $\delta\phi = \delta\varphi$ ,  $\delta\phi_1 = \delta\varphi_1$ ,  $\delta\phi_2 = \delta\varphi_2$  are variations of  $\phi = \varphi$ , all sections of  $VE$  over  $M$ , and  $\delta j^1\varphi = (\delta\varphi, \partial\delta\varphi)$ ,  $\delta j^1\varphi_1 = (\delta\varphi_1, \partial\delta\varphi_1)$ ,  $\delta j^1\varphi_2 = (\delta\varphi_2, \partial\delta\varphi_2)$  are the induced variations of  $j^1\varphi = (\varphi, \partial\varphi)$ , all sections of  $V(J^1E) \cong J^1(VE)$  over  $M$ . In local coordinates,

$$\delta\varphi = \frac{\partial}{\partial\lambda} \varphi_\lambda \Big|_{\lambda=0} = \delta\varphi^i \frac{\partial}{\partial q^i}$$

and

$$\delta j^1\varphi = \frac{\partial}{\partial\lambda} j^1\varphi_\lambda \Big|_{\lambda=0} = \delta\varphi^i \frac{\partial}{\partial q^i} + \partial_\mu \delta\varphi^i \frac{\partial}{\partial q_\mu^i}$$

whereas  $\theta_L$  is given by eq. (54) and  $\omega_L$  by

$$\begin{aligned} \omega_L = & \left( \frac{\partial^2 L}{\partial q^i \partial q_\mu^j} dq^i \wedge dq^j + \frac{\partial^2 L}{\partial q_\nu^j \partial q_\mu^i} dq^i \wedge dq_\nu^j \right) \wedge d^n x_\mu \\ & + \frac{\partial^2 L}{\partial x^\mu \partial q_\mu^i} dq^i \wedge d^n x - d \left( L - \frac{\partial L}{\partial q_\mu^i} q_\mu^i \right) \wedge d^n x. \end{aligned}$$

(The exterior derivative in the last term could be worked out explicitly, but we shall not need this expression because the last two terms vanish under contraction with two vertical vectors.) Then

$$\Theta_\phi(\delta\phi) = \int_\Sigma d\Sigma_\mu \frac{\partial L}{\partial q_\mu^i}(\varphi, \partial\varphi) \delta\varphi^i \quad (83)$$

and

$$\Omega_\phi(\delta\phi_1, \delta\phi_2) = \int_\Sigma d\Sigma_\mu J_\phi^\mu(\delta\phi_1, \delta\phi_2) \quad (84)$$

with the “symplectic current”  $J$  given by

$$\begin{aligned} J_\phi^\mu(\delta\phi_1, \delta\phi_2) = & \frac{\partial^2 L}{\partial q^j \partial q_\mu^i}(\varphi, \partial\varphi) (\delta\varphi_1^i \delta\varphi_2^j - \delta\varphi_2^i \delta\varphi_1^j) \\ & + \frac{\partial^2 L}{\partial q_\nu^j \partial q_\mu^i}(\varphi, \partial\varphi) (\delta\varphi_1^i \partial_\nu \delta\varphi_2^j - \delta\varphi_2^i \partial_\nu \delta\varphi_1^j), \end{aligned} \quad (85)$$

or equivalently

$$\begin{aligned} J_\phi^\mu(\delta\phi_1, \delta\phi_2) = & - \left( \frac{\partial^2 L}{\partial q^j \partial q_\mu^i}(\varphi, \partial\varphi) \delta\varphi_1^j + \frac{\partial^2 L}{\partial q_\nu^j \partial q_\mu^i}(\varphi, \partial\varphi) \partial_\nu \delta\varphi_1^j \right) \delta\varphi_2^i \\ & + \left( \frac{\partial^2 L}{\partial q^j \partial q_\mu^i}(\varphi, \partial\varphi) \delta\varphi_2^j + \frac{\partial^2 L}{\partial q_\nu^j \partial q_\mu^i}(\varphi, \partial\varphi) \partial_\nu \delta\varphi_2^j \right) \delta\varphi_1^i. \end{aligned} \quad (86)$$

The same results can be obtained even more directly in the Hamiltonian framework, in which we have

$$\Theta_\phi(\delta\phi) = \int_\Sigma (\varphi, \pi)^* \theta_{\mathcal{H}}(\delta\varphi, \delta\pi) \quad (87)$$

and

$$\Omega_\phi(\delta\phi_1, \delta\phi_2) = \int_\Sigma (\varphi, \pi)^* \omega_{\mathcal{H}}(\delta\varphi_1, \delta\pi_1, \delta\varphi_2, \delta\pi_2) \quad (88)$$

where the notation is the same as that employed in eq. (57):  $\phi = (\varphi, \pi)$  is a section of  $\tilde{J}^1 \otimes E$  over  $M$  while  $\delta\phi = (\delta\varphi, \delta\pi)$ ,  $\delta\phi_1 = (\delta\varphi_1, \delta\pi_1)$ ,  $\delta\phi_2 = (\delta\varphi_2, \delta\pi_2)$  are variations of  $\phi = (\varphi, \pi)$ , all sections of  $V(\tilde{J}^1 \otimes E)$  over  $M$ . In local coordinates,

$$\delta\varphi = \frac{\partial}{\partial \lambda} \varphi_\lambda \Big|_{\lambda=0} = \delta\varphi^i \frac{\partial}{\partial q^i}, \quad \delta\pi = \frac{\partial}{\partial \lambda} \pi_\lambda \Big|_{\lambda=0} = \delta\pi^\mu_i \frac{\partial}{\partial p_\mu^i}$$

whereas  $\theta_{\mathcal{H}}$  is given by eq. (55) and  $\omega_{\mathcal{H}}$  by

$$\omega_{\mathcal{H}} = dq^i \wedge dp_i^\mu \wedge d^n x_\mu - dH \wedge d^n x$$

(The exterior derivative in the last term could be worked out explicitly, but we shall not need this expression because the last term vanishes under contraction with two vertical vectors.) Then

$$\Theta_\phi(\delta\phi) = \int_\Sigma d\Sigma_\mu \pi_i^\mu \delta\varphi^i \quad (89)$$

and

$$\Omega_\phi(\delta\phi_1, \delta\phi_2) = \int_{\Sigma} d\Sigma_\mu J_\phi^\mu(\delta\phi_1, \delta\phi_2) \quad (90)$$

with the “symplectic current”  $J$  given by

$$J_\phi^\mu(\delta\phi_1, \delta\phi_2) = \delta\varphi_1^i \delta\pi_{2,i}^\mu - \delta\varphi_2^i \delta\pi_{1,i}^\mu. \quad (91)$$

Incidentally, these formulas show that, just like in mechanics, the canonical form  $\Theta$  and the symplectic form  $\Omega$  do not depend on the choice of the Hamiltonian  $\mathcal{H}$ .

Another important result, duly emphasized in the literature [1–4], is the fact that on covariant phase space  $\mathcal{S}$ , the symplectic form  $\Omega$  does not depend on the choice of the hypersurface  $\Sigma$  used in its definition, since for any solution  $\phi$  of the equations of motion and any two solutions  $\delta\phi_1, \delta\phi_2$  of the linearized equations of motion, the “symplectic current”  $J_\phi(\delta\phi_1, \delta\phi_2)$  is a closed form on space-time. To prove this, assume that  $\phi$  is a point in  $\mathcal{S}$  and observe that a tangent vector  $\delta\phi$  in  $T_\phi\mathcal{C}$  belongs to the subspace  $T_\phi\mathcal{S}$  if and only if  $\delta\phi$ , as a section of  $V_\phi$ , satisfies the pertinent Jacobi equation, which reads

$$\begin{aligned} \partial_\mu \left( \frac{\partial^2 L}{\partial q^j \partial q_\mu^i}(\varphi, \partial\varphi) \delta\varphi^j + \frac{\partial^2 L}{\partial q_\nu^j \partial q_\mu^i}(\varphi, \partial\varphi) \partial_\nu \delta\varphi^j \right) \\ = \frac{\partial^2 L}{\partial q^j \partial q^i}(\varphi, \partial\varphi) \delta\varphi^j + \frac{\partial^2 L}{\partial q^i \partial q^j}(\varphi, \partial\varphi) \partial_\nu \delta\varphi^j \end{aligned} \quad (92)$$

in the Lagrangian framework and

$$\begin{aligned} \partial_\mu \delta\pi_i^\mu &= \frac{\partial^2 H}{\partial q^j \partial q^i}(\varphi, \pi) \delta\varphi^j + \frac{\partial^2 H}{\partial p_\nu^j \partial q^i}(\varphi, \pi) \delta\pi_j^\nu \\ \partial_\mu \delta\varphi^i &= -\frac{\partial^2 H}{\partial q^j \partial p_\mu^i}(\varphi, \pi) \delta\varphi^j - \frac{\partial^2 H}{\partial p_\nu^j \partial p_\mu^i}(\varphi, \pi) \delta\pi_j^\nu \end{aligned} \quad (93)$$

in the Hamiltonian framework. Thus if  $\delta\phi_1$  and  $\delta\phi_2$  both satisfy the Jacobi equation, we have

$$\begin{aligned} \partial_\mu J_\phi^\mu(\delta\phi_1, \delta\phi_2) &= -\partial_\mu \left( \frac{\partial^2 L}{\partial q^j \partial q_\mu^i}(\varphi, \partial\varphi) \delta\varphi_1^j + \frac{\partial^2 L}{\partial q_\nu^j \partial q_\mu^i}(\varphi, \partial\varphi) \partial_\nu \delta\varphi_1^j \right) \delta\varphi_2^i \\ &\quad - \left( \frac{\partial^2 L}{\partial q^j \partial q_\mu^i}(\varphi, \partial\varphi) \delta\varphi_1^j + \frac{\partial^2 L}{\partial q_\nu^j \partial q_\mu^i}(\varphi, \partial\varphi) \partial_\nu \delta\varphi_1^j \right) \partial_\mu \delta\varphi_2^i \\ &\quad + \partial_\mu \left( \frac{\partial^2 L}{\partial q^j \partial q_\mu^i}(\varphi, \partial\varphi) \delta\varphi_2^j + \frac{\partial^2 L}{\partial q_\nu^j \partial q_\mu^i}(\varphi, \partial\varphi) \partial_\nu \delta\varphi_2^j \right) \delta\varphi_1^i \\ &\quad + \left( \frac{\partial^2 L}{\partial q^j \partial q_\mu^i}(\varphi, \partial\varphi) \delta\varphi_2^j + \frac{\partial^2 L}{\partial q_\nu^j \partial q_\mu^i}(\varphi, \partial\varphi) \partial_\nu \delta\varphi_2^j \right) \partial_\mu \delta\varphi_1^i \end{aligned}$$

$$\begin{aligned}
&= - \left( \frac{\partial^2 L}{\partial q^i \partial q^i}(\varphi, \partial\varphi) \delta\varphi_1^i + \frac{\partial^2 L}{\partial q^i \partial q_\nu^j}(\varphi, \partial\varphi) \partial_\nu \delta\varphi_1^i \right) \delta\varphi_2^i \\
&\quad - \left( \frac{\partial^2 L}{\partial q^i \partial q_\mu^i}(\varphi, \partial\varphi) \delta\varphi_1^i + \frac{\partial^2 L}{\partial q_\nu^j \partial q_\mu^i}(\varphi, \partial\varphi) \partial_\nu \delta\varphi_1^i \right) \partial_\mu \delta\varphi_2^i \\
&\quad + \left( \frac{\partial^2 L}{\partial q^j \partial q^i}(\varphi, \partial\varphi) \delta\varphi_2^j + \frac{\partial^2 L}{\partial q^i \partial q_\nu^j}(\varphi, \partial\varphi) \partial_\nu \delta\varphi_2^j \right) \delta\varphi_1^i \\
&\quad + \left( \frac{\partial^2 L}{\partial q^j \partial q_\mu^i}(\varphi, \partial\varphi) \delta\varphi_2^j + \frac{\partial^2 L}{\partial q_\nu^j \partial q_\mu^i}(\varphi, \partial\varphi) \partial_\nu \delta\varphi_2^j \right) \partial_\mu \delta\varphi_1^i
\end{aligned}$$

in the Lagrangian framework and

$$\begin{aligned}
\partial_\mu J_\phi^\mu(\delta\varphi_1, \delta\varphi_2) &= \partial_\mu \delta\varphi_1^i \delta\pi_{2,i}^\mu + \delta\varphi_1^i \partial_\mu \delta\pi_{2,i}^\mu - \partial_\mu \delta\varphi_2^i \delta\pi_{1,i}^\mu - \delta\varphi_2^i \partial_\mu \delta\pi_{1,i}^\mu \\
&= - \left( \frac{\partial^2 H}{\partial q^j \partial p_i^\mu}(\varphi, \pi) \delta\varphi_1^j + \frac{\partial^2 H}{\partial p_\nu^j \partial p_i^\mu}(\varphi, \pi) \delta\pi_{1,j}^\nu \right) \delta\pi_{2,i}^\mu \\
&\quad + \delta\varphi_1^i \left( \frac{\partial^2 H}{\partial q^j \partial q^i}(\varphi, \pi) \delta\varphi_2^j + \frac{\partial^2 H}{\partial p_\nu^j \partial q^i}(\varphi, \pi) \delta\pi_{2,j}^\nu \right) \\
&\quad + \left( \frac{\partial^2 H}{\partial q^j \partial p_i^\mu}(\varphi, \pi) \delta\varphi_2^j + \frac{\partial^2 H}{\partial p_\nu^j \partial p_i^\mu}(\varphi, \pi) \delta\pi_{2,j}^\nu \right) \delta\pi_{1,i}^\mu \\
&\quad - \delta\varphi_2^i \left( \frac{\partial^2 H}{\partial q^j \partial q^i}(\varphi, \pi) \delta\varphi_1^j + \frac{\partial^2 H}{\partial p_\nu^j \partial q^i}(\varphi, \pi) \delta\pi_{1,j}^\nu \right)
\end{aligned}$$

in the Hamiltonian framework: obviously, both of these expressions vanish.

Of course, independence of the choice of hypersurface holds only for  $\Omega$  but not for  $\Theta$ . In fact, if  $M_{1,2}$  is a region of space-time whose boundary is the disjoint union of two hypersurfaces  $\Sigma_1$  and  $\Sigma_2$ , then  $\Omega_{\Sigma_2} = \Omega_{\Sigma_1}$  but

$$\Theta_{\Sigma_2} - \Theta_{\Sigma_1} = \delta S_{M_{1,2}} \quad (94)$$

where  $S_{M_{1,2}}$  is the action calculated by integration over  $M_{1,2}$  and  $\delta$  is the functional exterior derivative, or variational derivative, on  $\mathcal{S}$ .

### 3.3 Poisson Bracket

Given a relativistic field theory with a regular first-order Lagrangian, one expects each of the corresponding Jacobi operators  $\mathcal{J}[\varphi]$  ( $\varphi \in \mathcal{S}$ ) to form a hyperbolic system of second-order partial differential operators. A typical example is provided by the sigma model, where  $E$  is a trivial product bundle  $M \times Q$ , with a given Lorentzian metric  $g$  on the base manifold  $M$ , as usual, and a given Riemannian metric  $h$  on the typical fiber  $Q$ . Its Lagrangian is

$$L = \frac{1}{2} \sqrt{|g|} g^{\mu\nu} h_{ij} q_\mu^i q_\nu^j,$$

so that the coefficients of the highest degree terms of the Jacobi operator  $\mathcal{J}[\varphi]$  are

$$\frac{\partial^2 L}{\partial q_\nu^j \partial q_\mu^i} (\varphi, \partial \varphi) = \frac{1}{2} \sqrt{|g|} g^{\mu\nu} h_{ij}(\varphi) ,$$

which clearly exhibits the hyperbolic nature of the resulting linearized field equations.

A general feature of hyperbolic systems of linear partial differential equations is the possibility to guarantee existence and uniqueness of various types of Green functions. In the present context, what we need is existence and uniqueness of the retarded Green function  $G_\phi^-$ , the advanced Green function  $G_\phi^+$  and the causal Green function  $G_\phi$  for the Jacobi operator  $\mathcal{J}[\phi]$ , for each  $\phi \in \mathcal{S}$ . By definition, the first two are solutions of the inhomogeneous Jacobi equations

$$\mathcal{J}_x[\phi] G_\phi^\pm(x, y) = \delta(x, y) , \quad \mathcal{J}_y[\phi] G_\phi^\pm(x, y) = \delta(x, y) , \quad (95)$$

or more explicitly,

$$\mathcal{J}_x[\phi]_{km} G_\phi^{\pm ml}(x, y) = \delta_k^l \delta(x, y) , \quad \mathcal{J}_y[\phi]_{km} G_\phi^{\pm lm}(x, y) = \delta_k^l \delta(x, y) , \quad (96)$$

where  $\mathcal{J}_z[\phi]$  denotes the Jacobi operator with respect to the variable  $z$ , characterized by the following support condition: for any  $x, y \in M$ ,  $G_\phi^-(x, y) = 0$  when  $x \notin J^+(y)$  and  $G_\phi^+(x, y) = 0$  when  $x \notin J^-(y)$ , where  $J^+(y)$  and  $J^-(y)$  are the future cone and the past cone of  $y$ , respectively. The causal Green function, also called the propagator, is then simply their difference:

$$G_\phi = G_\phi^- - G_\phi^+ . \quad (97)$$

Obviously, it satisfies the homogeneous Jacobi equations

$$\mathcal{J}_x[\phi] G_\phi(x, y) = 0 , \quad \mathcal{J}_y[\phi] G_\phi(x, y) = 0 . \quad (98)$$

Note that the symmetry of the Jacobi operator  $\mathcal{J}[\phi]$ , stemming from the fact that it represents the second variational derivative of the action, forces these Green functions to be symmetric, except for the exchange between the retarded and the advanced function:

$$G_\phi^{\pm lk}(y, x) = G_\phi^{\mp kl}(x, y) , \quad G_\phi^{lk}(y, x) = G_\phi^{kl}(x, y) . \quad (99)$$

It should be pointed out that existence and uniqueness of these Green functions cannot be guaranteed in complete generality: this requires not only that  $M$  be globally hyperbolic but also that the linearized field equations should form a hyperbolic system. Here, we shall simply assume this to be the case and proceed from there; further comments on the question will be deferred to the end of the section.

Our next step will be to study solutions  $\delta_A \phi$  of the general inhomogeneous Jacobi equation

$$\mathcal{J}[\phi](\delta_A^\pm \phi) = \frac{\delta A}{\delta \phi}[\phi] \quad (100)$$

for smooth functionals  $A$  on covariant phase space which are (at least) local in time. To eliminate the ambiguity in this equation stemming from the fact that its rhs does not really belong to the image space of the Jacobi operator  $\mathcal{J}[\phi]$  but rather to a quotient space thereof (an inclusion of the form  $T_\phi \mathcal{S} \subset T_\phi \mathcal{C}$  induces a natural projection from  $T_\phi^* \mathcal{C}$  to  $T_\phi^* \mathcal{S}$ ), it is necessary to first of all extend the given functional  $A$  on  $\mathcal{S}$  to a functional  $\tilde{A}$  on  $\mathcal{C}$  of the same type (smooth and local in time), whose functional derivative  $\delta \tilde{A} / \delta \phi$  will be a one-form on  $\mathcal{C}$ . Now performing a convolution with the retarded or advanced Green functions introduced above, we can construct formal vector fields over  $\mathcal{S}$  which to each solution  $\phi \in \mathcal{S}$  of the field equations associate (distributional) sections  $\delta_{\tilde{A}}^\pm \phi$  of  $V_\phi$  defined by

$$\delta_{\tilde{A}}^\pm \phi^k(x) = \int_M d^n y \, G_\phi^{\pm kl}(x, y) \frac{\delta \tilde{A}}{\delta \phi^l}[\phi](y) , \quad (101)$$

satisfying the inhomogeneous Jacobi equation

$$\mathcal{J}[\phi](\delta_{\tilde{A}}^\pm \phi) = \frac{\delta \tilde{A}}{\delta \phi}[\phi] . \quad (102)$$

Similarly, convolution with the causal Green function leads to a formal vector field over  $\mathcal{S}$  which to each solution  $\phi \in \mathcal{S}$  of the field equations associates a (distributional) section  $\delta_{\tilde{A}} \phi$  of  $V_\phi$  defined by

$$\delta_{\tilde{A}} \phi^k(x) = \int_M d^n y \, G_\phi^{kl}(x, y) \frac{\delta \tilde{A}}{\delta \phi^l}[\phi](y) , \quad (103)$$

satisfying the homogeneous Jacobi equation

$$\mathcal{J}[\phi](\delta_{\tilde{A}} \phi) = 0 , \quad (104)$$

since

$$\delta_{\tilde{A}} \phi = \delta_{\tilde{A}}^- \phi - \delta_{\tilde{A}}^+ \phi . \quad (105)$$

Note that the convolutions in eqs (101) and (103) exist due to our support assumptions on  $\delta \tilde{A} / \delta \phi$  and to the support properties of the Green functions  $G_\phi^\pm$  and  $G_\phi$ .

According to eq. (104), the prescription of associating to each solution  $\phi \in \mathcal{S}$  of the field equations the section  $\delta_{\tilde{A}} \phi$  of  $V_\phi$  defines a formal vector field on  $\mathcal{S}$  which is tangent to  $\mathcal{S}$ . (It becomes more than just a formal vector field if  $\tilde{A}$  is such that  $\delta_{\tilde{A}} \phi$  belongs to  $T_\phi \mathcal{S}$ , which requires it to be not just a distributional section but a smooth section of  $V_\phi$  and to satisfy appropriate support properties; we shall come back to this point later on.) The main statement about this formal vector field, to be proved below, is that (a) it does not depend on the choice of the extension  $\tilde{A}$  of  $A$ , so we may simply denote it by  $\delta_A \phi$ , and (b) that it is formally the Hamiltonian vector field associated to  $A$  with respect to the symplectic form  $\Omega$  discussed in the previous subsection. More explicitly, we claim that for any solution  $\phi \in \mathcal{S}$  of the field equations and any smooth section  $\delta \phi$

of  $V_\phi$  with spatially compact support which is a solution of the linearized field equations, we have

$$\Omega_\phi(\delta_A \phi, \delta \phi) = \int_M d^n x \frac{\delta \tilde{A}}{\delta \phi}[\phi](x) \cdot \delta \phi(x) . \quad (106)$$

Note that under the assumptions stated, both sides of this equation make sense although we have originally defined  $\Omega_\phi(\delta \phi_1, \delta \phi_2)$  only in the case where both  $\delta \phi_1$  and  $\delta \phi_2$  are smooth; the extension of this definition, given in the previous subsection, to the case where one of them is a distribution is straightforward.

To prove this key statement, let us begin by recalling that the symplectic form  $\Omega$  and the symplectic current  $J$  of the previous subsection are really defined on  $\mathcal{C}$  and not only on  $\mathcal{S}$  – the only difference is that on  $\mathcal{C}$ ,  $\Omega$  is only a presymplectic form so that  $J$  should be more appropriately called the presymplectic current and that  $J$  on  $\mathcal{C}$  is no longer be conserved so that  $\Omega$  on  $\mathcal{C}$  will depend on the choice of the hypersurface  $\Sigma$ . At any rate, we can almost literally repeat the calculation performed at the end of the previous subsection, either in the Lagrangian or in the Hamiltonian formulation, to show that for any solution  $\phi \in \mathcal{S}$  of the field equations and any smooth section  $\delta \phi$  of  $V_\phi$  with spatially compact support, we have

$$\partial_\mu J_\phi^\mu(\delta_A^\pm \phi, \delta \phi) = (\mathcal{J}[\phi]_{kl} \delta_A^\pm \phi^l) \delta \phi^k - (\mathcal{J}[\phi]_{kl} \delta \phi^l) \delta_A^\pm \phi^k , \quad (107)$$

so that if  $\delta \phi$  is a solution of the linearized field equations,

$$\partial_\mu J_\phi^\mu(\delta_A^\pm \phi, \delta \phi) = \frac{\delta \tilde{A}}{\delta \phi}[\phi] \cdot \delta \phi . \quad (108)$$

Now since, by assumption, the support of  $\delta \tilde{A}/\delta \phi$  is contained in some time slice, we can choose two Cauchy surfaces  $\Sigma_-$  to the past and  $\Sigma_+$  to the future of this time slice and, using that  $\delta \phi$  has spatially compact support, integrate eq. (108) over the time slice  $S_-$  between  $\Sigma_-$  and  $\Sigma$  and similarly over the time slice  $S_+$  between  $\Sigma$  and  $\Sigma_+$ . Applying Stokes' theorem, this gives

$$\begin{aligned} \int_\Sigma d\sigma_\mu(x) J_\phi^\mu(\delta_A^- \phi, \delta \phi)(x) &= \int_{\Sigma_-} d\sigma_\mu(x) J_\phi^\mu(\delta_A^- \phi, \delta \phi)(x) + \int_{S_-} d^n x \frac{\delta \tilde{A}}{\delta \phi}[\phi](x) \cdot \delta \phi(x) , \\ \int_\Sigma d\sigma_\mu(x) J_\phi^\mu(\delta_A^+ \phi, \delta \phi)(x) &= \int_{\Sigma_+} d\sigma_\mu(x) J_\phi^\mu(\delta_A^+ \phi, \delta \phi)(x) - \int_{S_+} d^n x \frac{\delta \tilde{A}}{\delta \phi}[\phi](x) \cdot \delta \phi(x) . \end{aligned}$$

But the support conditions on  $G_\phi^\pm$ , together with the fact that the support of  $(\delta \tilde{A}/\delta \phi)[\phi]$  lies to the future of  $\Sigma_-$  and to the past of  $\Sigma_+$ , imply that  $\delta_A^- \phi$  vanishes on  $\Sigma_-$  and similarly that  $\delta_A^+ \phi$  vanishes on  $\Sigma_+$ , so the first term on the rhs of each of these equations is zero. Thus taking their difference and inserting eq. (105), we get

$$\int_\Sigma d\sigma_\mu(x) J_\phi^\mu(\delta_A \phi, \delta \phi)(x) = \int_{S_-} d^n x \frac{\delta \tilde{A}}{\delta \phi}[\phi](x) \cdot \delta \phi(x) + \int_{S_+} d^n x \frac{\delta \tilde{A}}{\delta \phi}[\phi](x) \cdot \delta \phi(x) ,$$

and since  $(\delta\tilde{A}/\delta\phi)[\phi]$  vanishes outside  $S_- \cup S_+$ ,

$$\Omega_\phi(\delta_{\tilde{A}}\phi, \delta\phi) = \int_M d^n x \frac{\delta\tilde{A}}{\delta\phi}[\phi](x) \cdot \delta\phi(x) . \quad (109)$$

Finally, observe that since  $\delta\phi$  is supposed to be a solution of the linearized field equations (and hence tangent to  $\mathfrak{S}$ ), the rhs of this equation does not depend on the choice of the extension  $\tilde{A}$  of  $A$ . Therefore,  $\delta_{\tilde{A}}\phi$  will not depend on this choice either provided the symplectic form  $\Omega_\phi$  is weakly non-degenerate. Now using the space-time split of  $M$  over  $\Sigma$  provided by the tangent vector field  $\partial_t$  of some global time function  $t$  on  $M$  or its dual  $dt$ , and identifying solutions  $\delta\phi$  of the linearized field equations with their Cauchy data on  $\Sigma$ ,<sup>9</sup> it can be seen by direct inspection, either of eqs (84) and (85) in the Lagrangian formalism or of eqs (90) and (91) in the Hamiltonian formalism, that the expression  $\Omega_\phi(\delta\phi_1, \delta\phi_2)$  can only be zero for all  $\delta\phi_2$  if  $\delta\phi_1$  vanishes, as soon as we require the Lagrangian  $L$  to be regular in time derivatives, that is, to satisfy

$$\det \frac{\partial^2 L}{\partial q_0^i \partial q_0^j} \neq 0 , \quad (110)$$

or equivalently, the Hamiltonian to be regular in timelike conjugate momenta, that is, to satisfy

$$\det \frac{\partial^2 H}{\partial p_i^0 \partial p_j^0} \neq 0 . \quad (111)$$

Moreover, it can be shown that this statement will remain true if  $\delta\phi_1$  is allowed to be a distributional solution of the linearized field equations with arbitrary support, as long as  $\delta\phi_2$  runs through the space of smooth solutions of the linearized field equations with spatially compact support.

In other words, the previous construction does not apply directly to degenerate systems such as gauge theories: these require a separate treatment.

Having established eq. (106), it is now easy to write down the Poisson bracket of two functionals  $A$  and  $B$  on  $\mathfrak{S}$ : it is, in complete analogy with eq. (73), given by

$$\{A, B\}[\phi] = \frac{\delta A}{\delta\phi^k}[\phi] \delta_B \phi^k = - \frac{\delta B}{\delta\phi^k}[\phi] \delta_A \phi^k . \quad (112)$$

Inserting eq. (103) and omitting the tildes, we arrive at the main conclusion of this paper.

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<sup>9</sup>Explicitly, in the Lagrangian formalism, the Cauchy data for  $\delta\varphi$  on  $M$  are  $\delta\varphi$  and  $\delta\dot{\varphi}$  on  $\Sigma$ , whereas in the Hamiltonian formalism, the Cauchy data for  $(\delta\varphi, \delta\pi)$  on  $M$  are  $\delta\varphi$  and  $\delta\pi^0$  on  $\Sigma$ .

**Theorem 3** *The Poisson bracket associated with the symplectic form  $\Omega$  on covariant phase space as defined by Crnković, Witten and Zuckerman, according to the standard prescription of symplectic geometry, suitably adapted to the infinite-dimensional setting encountered in this context, is precisely the field theoretical bracket first proposed by Peierls and brought into a more geometric form by De Witt:*

$$\{A, B\}[\phi] = \int_M d^n x \int_M d^n y \frac{\delta A}{\delta \phi^k}[\phi](x) G_\phi^{kl}(x, y) \frac{\delta B}{\delta \phi^l}[\phi](y). \quad (113)$$

Of course, for the expressions in eqs (112) and (113) to exist, it is not sufficient to require  $A$  and/or  $B$  to be local in time. In fact, if we want to use conditions that (a) are sufficient to guarantee existence of this Poisson bracket without making use of specific regularity and support properties of the propagator, (b) are the same for  $A$  and  $B$  and (c) are reproduced under the Poisson bracket, we are forced to impose quite rigid assumptions: the functionals under consideration must be assumed to be both regular and local, in the sense that their functional derivative at any point  $\phi$  of  $\mathcal{S}$  must be a smooth section of  $V_\phi^{\otimes}$  of compact support (this will force the corresponding Hamiltonian vector field to be a smooth section of  $V_\phi$  of spatially compact support).

On the other hand, it must be pointed out that this Poisson bracket, which we might call the Peierls - De Witt bracket, has all the structural properties expected from a good Poisson bracket: bilinearity, antisymmetry, validity of the Jacobi identity and validity of the Leibniz rule with respect to plain and ordinary multiplication of functionals. This can be seen directly by noting that the first two properties and the Leibniz rule are obvious, while the Jacobi identity expresses the propagator identity for the causal Green function. But it is of course much simpler to argue that all these properties follow immediately from the above theorem, in combination with standard results of symplectic geometry. Moreover, the Peierls - De Witt bracket trivially satisfies the fundamental axiom of field theoretic locality: functionals localized in spacelike separated regions commute. All this suggests that the Peierls - De Witt bracket is the correct classical limit of the commutator of quantum field theory. Therefore, it ought to play an outstanding role in any attempt at quantizing classical field theories through algebraic methods, a popular example of which is deformation quantization.

The basic complication inherent in the algebraic structure provided by the Peierls - De Witt bracket is that it is inherently dynamical: the bracket between two functionals depends on the underlying dynamics. This could not be otherwise. In fact, it is the price to be paid for being able to extend the canonical commutation relations of classical field theory, representing a non-dynamical equal-time Poisson bracket, to a covariant Poisson bracket. The dynamical nature of covariant Poisson brackets is simplified (but still not trivial) for free field theories, where the equations of motion are linear, implying that the Jacobi operator  $\mathcal{J}[\phi]$  and its causal Green function  $G_\phi$  do not depend on the background solution  $\phi$ .

Finally, we would like to remark that the main mathematical condition to be imposed in order for the constructions presented here to work is that linearization of the

field equations around any solution  $\phi$  should provide a hyperbolic system of partial differential equations on  $M$ , for which existence and uniqueness of the Green functions  $G_\phi^\pm$  and  $G_\phi$  can be guaranteed. There are various definitions of the concept of a hyperbolic system that can be found in the literature, but the most appropriate one seems to be that of regular hyperbolicity, proposed by Christodoulou [27–29] in the context of Lagrangian systems, according to which the matrix

$$u^\mu u^\nu \frac{\partial^2 L}{\partial q_\mu^i \partial q_\nu^j}$$

should (in our sign convention for the metric tensor) be positive definite for timelike vectors  $u$  and negative definite for spacelike vectors  $u$ : a typical example is provided by the sigma model as discussed at the beginning of this subsection. What is missing is to translate this condition into the Hamiltonian formalism and to compare it with other definitions of hyperbolicity for first order systems, such as the traditional one of Friedrichs.

Summarizing the main result of this paper, we have shown that in the context of regular hyperbolic Lagrangian or covariant Hamiltonian field theories defined on a globally hyperbolic space-time, the Peierls–De Witt bracket is identical with the canonical Poisson bracket associated with the standard symplectic form on covariant phase space.

## Appendix: Affine Spaces and Duality

In this appendix, we collect some basic facts of linear algebra for affine spaces which are needed in this paper but which do not seem to be readily available in the literature.

A (nonempty) set  $A$  is said to be an *affine space* modelled on a vector space  $V$  if there is given a map

$$\begin{aligned} + : A \times V &\longrightarrow A \\ (a, v) &\longmapsto a + v \end{aligned} \tag{114}$$

satisfying the following two conditions:

- $a + (u + v) = (a + u) + v$  for all  $a \in A$  and all  $u, v \in V$ .
- Given  $a, b \in A$ , there exists a unique  $v \in V$  such that  $a = b + v$ .

Elements of  $A$  are called points and elements of  $V$  are called vectors, so the map (114) can be viewed as a transitive and fixed point free action of  $V$  (as an Abelian group) on  $A$ , associating to any point and any vector a new point called their sum. Correspondingly, the vector  $v$  whose uniqueness and existence is postulated in the second condition is often denoted by  $a - b$  and called the difference of the points  $a$  and  $b$ .

For every affine space  $A$ , the vector space on which it is modelled is determined uniquely up to isomorphism and will usually be denoted by  $\vec{A}$ .

A map  $f : A \rightarrow B$  between affine spaces  $A$  and  $B$  is said to be *affine* if there exists a point  $a \in A$  such that the map  $\vec{f}_a : \vec{A} \rightarrow \vec{B}$  defined by

$$\vec{f}_a(v) = f(a + v) - f(a) \quad (115)$$

is linear, that is,  $\vec{f}_a \in L(\vec{A}, \vec{B})$ . It is easily seen that this condition does not depend on the choice of the reference point: in fact, if the map  $\vec{f}_a$  is linear for some choice of  $a$ , then the maps  $\vec{f}_{a'}$  are all equal as  $a'$  varies through  $A$ , so it makes sense to speak of the *linear part*  $\vec{f}$  of an affine map  $f$ . Denoting the set of all affine maps from  $A$  to  $B$  by  $A(A, B)$ , we thus have a projection

$$\begin{aligned} l : A(A, B) &\longrightarrow L(\vec{A}, \vec{B}) \\ f &\longmapsto \vec{f} \end{aligned} . \quad (116)$$

This construction is particularly important in the special case where  $B$  is itself a vector space, rather than just an affine space. Given an affine space  $A$  and a vector space  $W$ , the set  $A(A, W)$  of affine maps from  $A$  to  $W$  is easily seen to be a vector space: in fact it is simply a linear subspace of the vector space  $\text{Map}(A, W)$  of all maps from  $A$  to  $W$ . Moreover, the projection

$$\begin{aligned} l : A(A, W) &\longrightarrow L(\vec{A}, W) \\ f &\longmapsto \vec{f} \end{aligned} . \quad (117)$$

is a linear map whose kernel consists of the constant maps from  $A$  to  $W$ . Identifying these with the elements of  $W$  itself, we obtain a natural isomorphism

$$A(A, W)/W \cong L(\vec{A}, W) , \quad (118)$$

or equivalently, an exact sequence of vector spaces, as follows:

$$0 \longrightarrow W \longrightarrow A(A, W) \xrightarrow{l} L(\vec{A}, W) \longrightarrow 0 . \quad (119)$$

In the general case, one shows that given two affine spaces  $A$  and  $B$ , the set  $A(A, B)$  of affine maps from  $A$  to  $B$  is again an affine space, such that  $\overline{A(A, B)} = A(A, \vec{B})$ , and that the projection (116) is an affine map.

Concerning dimensions, we may choose a reference point  $o$  in  $A$  which provides not only an isomorphism between  $A$  and  $\vec{A}$  but also a splitting of the exact sequence (119) and hence an isomorphism between  $A(A, W)$  and  $W \oplus L(\vec{A}, W)$ , to show that

$$\dim A(A, W) = \dim W + \dim L(\vec{A}, W) . \quad (120)$$

Choosing  $W$  to be the real line  $\mathbb{R}$ , we obtain the *affine dual*  $A^*$  of an affine space  $A$ :

$$A^* = A(A, \mathbb{R}) . \quad (121)$$

Observe that this is not only an affine space but even a vector space which, according to eq. (119), is a one-dimensional extension of the linear dual  $\tilde{A}^*$  of the model space  $\tilde{A}$  by  $\mathbb{R}$ , that is, we have the following exact sequence of vector spaces:

$$0 \longrightarrow \mathbb{R} \longrightarrow A^* \xrightarrow{l} \tilde{A}^* \longrightarrow 0 . \quad (122)$$

In particular, according to eq. (120), its dimension equals 1 plus the dimension of the original affine space:

$$\dim A^* = \dim A + 1 . \quad (123)$$

More generally, we may replace the real line  $\mathbb{R}$  by a (fixed but arbitrary) one-dimensional real vector space  $R$  (which is of course isomorphic but in general not canonically isomorphic to  $\mathbb{R}$ ) to define the *twisted affine dual*  $A^\otimes$  of an affine space  $A$ :

$$A^\otimes = A(A, R) . \quad (124)$$

Again, this is not only an affine space but even a vector space which, according to eq. (119), is a one-dimensional extension of the linear dual  $\tilde{A}^*$  of the model space  $\tilde{A}$  by  $R$ , that is, we have the following exact sequence of vector spaces:

$$0 \longrightarrow R \longrightarrow A^\otimes \xrightarrow{l} \tilde{A}^* \longrightarrow 0 . \quad (125)$$

Obviously, the dimension is unchanged:

$$\dim A^\otimes = \dim A + 1 . \quad (126)$$

Moreover, we have the following canonical isomorphism of vector spaces

$$A^\otimes \cong A^* \otimes R , \quad (127)$$

and more generally, for any vector space  $W$ ,

$$A(A, W) \cong A^* \otimes W . \quad (128)$$

Regarding the splittings of the exact sequence (125), we note the following fact which is used in the construction of the inverse Legendre transformation: these splittings form an affine space modelled on the bidual  $\tilde{A}^{**}$  of  $\tilde{A}$ , which in finite dimensions can be identified with  $\tilde{A}$  itself.

The concept of duality applies not only to spaces but also to maps between spaces: given an affine map  $f : A \rightarrow B$  between affine spaces  $A$  and  $B$ , the formula

$$(f^*(b^*))(a) = b^*(f(a)) \quad \text{for } b^* \in B^*, a \in A \quad (129)$$

yields a linear map  $f^* : B^* \rightarrow A^*$  between their affine duals  $B^*$  and  $A^*$ . As a result, the operation of taking the affine dual can be regarded as a (contravariant) functor from the category of affine spaces to the category of vector spaces. This functor is compatible with the usual (contravariant) functor of taking linear duals within the category of vector spaces in the sense that the following diagram commutes:

$$\begin{array}{ccc} B^* & \xrightarrow{f^*} & A^* \\ \downarrow & & \downarrow \\ \tilde{B}^* & \xrightarrow{\tilde{f}^*} & \tilde{A}^* \end{array} \quad (130)$$

Finally, we also need the construction of quotients in the affine category. These are defined by dividing out not affine subspaces but rather linear subspaces of the model space. In fact, given an affine space  $A$  and a linear subspace  $V$  of its model space  $\tilde{A}$ , we can declare two points  $a$  and  $a'$  of  $A$  to be *equivalent* modulo  $V$  if  $a - a' \in V$ . Obviously, this relation is reflexive, symmetric and transitive, and hence is an equivalence relation dividing  $A$  into equivalence classes; the set of equivalence classes is as usual denoted by  $A/V$ . It is then easy to see that there is a unique affine structure on  $A/V$  turning  $A/V$  into an affine space such that  $\overrightarrow{A/V} = \tilde{A}/V$  and such that the natural projection

$$\begin{array}{ccc} \rho : & A & \longrightarrow A/V \\ & a & \mapsto [a] \end{array} \quad (131)$$

is an affine map. Moreover, this construction satisfies the standard factorization property: given two affine spaces  $A$  and  $B$ , two linear subspaces  $V$  of  $\tilde{A}$  and  $W$  of  $\tilde{B}$  and an affine map  $f : A \rightarrow B$  whose linear part  $\tilde{f} : \tilde{A} \rightarrow \tilde{B}$  maps  $V$  into  $W$ , there exists a unique affine map  $[f] : A/V \rightarrow B/W$  such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \rho \downarrow & & \downarrow \rho \\ A/V & \xrightarrow{[f]} & B/W \end{array} \quad (132)$$

commutes.

Concluding this appendix, we would like to point out that all the concepts introduced above can be extended naturally from the purely algebraic setting to that of fiber bundles. For example, affine bundles are fiber bundles modelled on an affine space whose transition functions (with respect to a suitably chosen atlas) are affine maps. Moreover, functors such as the affine dual or the construction of quotients are smooth (see [30] for a definition of the concept of smooth functors in a similar context) and therefore extend naturally to bundles (over a fixed base manifold  $M$ ). This means that any affine bundle  $A$  over  $M$  has a naturally affine dual, which is a vector bundle  $A^*$  over  $M$ , and that given any vector subbundle  $V$  of the difference vector bundle  $\tilde{A}$  of an affine bundle  $A$  over  $M$ , we can form the quotient affine bundle  $A/V$  over  $M$ .

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