ON LINEAR KURZWEIL-HENSTOCK-INTEGRAL EQUATIONS

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Abstract

We give a Riemann-Stieltjes integral representation theorem for the elements of $L[\mathcal{C}([a,b],X),Y]$; this theorem completes a result of J.Batt and H.König [2]. We apply our result to obtain an integral representation for the elements of $L[\mathcal{K}([a,b],X),\mathcal{C}([c,d],Y)]$ where $\mathcal{K}([a,b],X)$ denotes the space of all functions $f:[a,b]\longrightarrow X$ that are integrable in the sense of Kurzweil-Henstock. This normed space is not complete but it is ultrabornological. This representation theorem is applied to study the corresponding linear Fredholm-Kurzweil and Volterra-Kurzweil integral equations

(F) $\quad x(t) - {}^{K}\!\int_a^b \alpha(t,s)\cdot x(s)ds = f(t)\ ,\ a\leq t\leq b,$ and

(V) $\quad x(t) - {}^{K}\!\int_a^t \alpha(t,s)\cdot x(s)ds = f(t)\ ,\ a\leq t\leq b.$

We prove that if for every $f\in\mathcal{K}([a,b],X)$ the equation (F) has one and only one solution $x_f\in\mathcal{K}([a,b],X)$ then the application $f\longmapsto x_f$ is bicontinuous and (F) has a resolvent that has a similar integral representation. For equation (V) we prove that for every $f\in\mathcal{K}([a,b],X)$ it has one and only one solution $x_f\in\mathcal{K}([a,b],X)$ and that the resolvent of (V) is given by the Neumann series.

1. The Riemann-Stieltjes integral and representation theorems

X,Y,Z always denote real normed spaces that are supposed to be complete unless otherwise stated. All results are still true for complex vector spaces. We denote by $L(X,Y)$ the Banach space of all linear continuous applications $F: X \longrightarrow Y$. We write $X' = L(X,\mathbb{R})$ and I_X or I denotes the identical application of X. $\mathcal{C}([a,b],X)$ or simply $\mathcal{C}$ denotes the Banach space of all functions $f:[a,b]\to X$ that are continuous and $\|f\| = \sup_{a\leq t\leq b}\|f(t)\|$. For a function $\alpha:[a,b]\longrightarrow L(X,Y)$ we denote by $SV_{[a,b]}[\alpha]$ or $SV[\alpha]$ the smallest element $c\in[0,\infty]$ such that for every division

$$t_0 = a < t_1 < t_2 < \cdots < t_n = b$$

of $[a,b]$ and for all $x_i \in X$ we have

$$\Big\| \sum_{i=1}^{n}\big[\alpha(t_i) - \alpha(t_{i-1})\big]\cdot x_i\Big\| \leq c \sup_{1\leq i\leq n}\|x_i\|$$

If $SV[\alpha]<\infty$ we say that α is a <u>function of bounded semivariation</u> (on $[a,b]$) and we write $\alpha\in SV([a,b],L(X,Y))$ or simply $\alpha\in SV$. If we have furthermore $\alpha(c) = 0$, where $c\in[a,b]$, we write

$$\alpha\in SV_c([a,b],L(X,Y))$$

For the following results and for further informations see [2] and [10], Chapter I; see also [12].

1.1 a - $SV_c([a,b],L(X,Y))$ is a Banach space when endowed with the norm $\alpha\longmapsto SV[\alpha]$.

b - For every $f\in\mathcal{C}([a,b],X)$ and $\alpha\in SV([a,b],L(X,Y))$ there exists the <u>Riemann-Stieltjes integral</u>

$$\int_a^b d\alpha(t)\cdot f(t) = \lim \sum_{i=1}^{n}\big[\alpha(t_i) - \alpha(t_{i-1})\big]\cdot f(\xi_i)\ ,\quad \xi_i\in[t_{i-1},t_i],$$

where lim denotes that we take $\sup_{1\leq i\leq n}|t_i - t_{i-1}|\longrightarrow 0$. We write

$$F_\alpha(f) = \int_a^b d\alpha(t)\cdot f(t)$$

and we have $F_\alpha\in L\big[\mathcal{C}([a,b],X),Y\big]$ and $\|F_\alpha\|\leq SV[\alpha]$.

c - If $\alpha\in SV([a,b],L(X,Z'))$ $\big[\alpha\in SV([a,b],L(X,Y))\big]$ then for every $t\in[a,b[$ there exists one and only one element $\alpha(t+)\in L(X,Z')$

$[\alpha(t\dot{+}) \in L(X,Y'')]$ such that for every $x \in X$ and $z \in Z$ $[z \in Y']$ we have

$$\lim_{\varepsilon \downarrow 0} \langle \alpha(t+\varepsilon)x, z\rangle = \langle \alpha(t\dot{+})x, z\rangle .$$

We define $\alpha^+(t) = \alpha(t\dot{+})$ if $a < t < b$ and $\alpha^+(a) = \alpha(a)$. α^+ is of bounded semivariation and we write

$$\alpha^+ \in SV^+(]a,b],L(X,Z'))$$

For every $f \in \mathcal{C}([a,b],X)$ we have $\int_a^b d\alpha^+(t)\cdot f(t) = \int_a^b d\alpha(t)\cdot f(t)$ and $\|F_\alpha\| = SV[\alpha^+]$.

1.2 (see [2], Satz 10 and [10], Theorem I.3.8)' - The mapping

$$\alpha \in SV_c^+(]a,b],L(X,Z')) \longmapsto F_\alpha \in L[\mathcal{C}([a,b],X),Z']$$

is an isometry (i.e. $\|F_\alpha\| = SV[\alpha]$) of the first Banach space onto the second.

1.3 (see [10], corol. I.3.9) - For every $F \in L[\mathcal{C}([a,b],X),Y]$ there exists one and only one $\alpha \in SV_c^+(]a,b],L(X,Y''))$ such that $F = F_\alpha$. (The mapping $F \longmapsto \alpha_F$ is not onto if $Y'' \neq Y$).

The next theorem completes 1.3 and characterizes the image of the mapping $F \longmapsto \alpha_F$.

If $\alpha \in SV_b^+(]a,b],L(X,Y''))$ we write $\alpha \in SV_b^{+\int}(]a,b],L(X,\dot{Y}))$ if we have $\alpha(a) \in L(X,\dot{Y})$ and $\int_a^t \alpha(s)x\,ds \in Y$ for all $t \in [a,b]$ and $x \in X$.

<u>Theorem 1.4</u> - The mapping

$$\alpha \in SV_b^{+\int}(]a,b],L(X,\dot{Y})) \longmapsto F_\alpha \in L[\mathcal{C}([a,b],X),Y]$$

where $F_\alpha f = \int_a^b d\alpha(t)\cdot f(t)$, is an isometry (i.e. $\|F_\alpha\| = SV[\alpha]$) of the first Banach space onto the second. We have $\int_a^t \alpha(s)x\,ds = -F_\alpha(g_{t,x})$ and $\alpha(a)x = -F_\alpha(\chi_{[a,b]}x)$ where for $t \in [a,b]$ and $x \in X$ we define

$$g_{t,x}(s) = \begin{cases} (s-a)x & \text{if } a \leq s \leq t \\ (t-a)x & \text{if } t \leq s \leq b \end{cases}$$

<u>Proof</u> - 1) For $F \in L[\mathcal{C}([a,b],X),Y]$ let α be the corresponding element by 1.3. We have $\alpha \in SV_b^{+\int}(]a,b],L(X,\dot{Y}))$: indeed, since $g_{t,x} \in \mathcal{C}([a,b],X)$ we have $F(g_{t,x}) \in Y$; but

$$F(g_{t,x}) = \int_a^b d\alpha(s)\cdot g_{t,x}(s) = -\int_a^b \alpha(s)\cdot dg_{t,x}(s) = -\int_a^t \alpha(s)x\,ds$$

where in the second equality we used the integration by parts formula for the Riemann-Stieltjes integral (with $\alpha(b) = 0$ and $g_{t,x}(a) = 0$). Analogously we have $F(\chi_{[a,b]}x) = -\alpha(a)x \in Y$.

2) On the other hand the functions $g_{t,x}$ and $\chi_{[a,b]}x$ ($t \in [a,b]$, $x \in X$) form a total subset of $\mathcal{C}([a,b],X)$; since $SV_b^{+\int}$ is a closed subspace of SV_b^+ it follows that the isometry is onto.

<u>Remark</u>: if $X = \mathbb{R}$ we have $\alpha(t+) = -\frac{d^+}{dt} F_\alpha(g_t) = -\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} F_\alpha(g_{t+\varepsilon} - g_t) = \frac{d^+}{dt}\int_a^t \alpha(s)\,ds$, $a \leq t < b$. If $\dim X = \infty$ the function α may not be differentiable (to the right or to the left) at any point. Example: take $X = Y = L_\infty([a,b])$ and $\alpha(t)x = \chi_{[t,b]}x$; see [11], p.14, Example 2. However for every $x \in X$ and $y \in Y$ we have $\langle \alpha(t),y\rangle = -\frac{d^+}{dt}\langle F_\alpha(g_{t,x}),y\rangle$ since the function $t \in [a,b] \longmapsto \langle F_\alpha(g_{t,x}),y\rangle = -\int_a^t \langle \alpha(s)x,y\rangle\,ds$ is differentiable to the right (the numerical function $t \in [a,b] \longmapsto \langle \alpha(t)x,y\rangle$ is of bounded variation and continuous to the right).

We define $\mathcal{C}_a([a,b],X) = \{f \in \mathcal{C}([a,b],X) \mid f(a) = 0\}$ and

$$SV_{\cdot b}^{+\int}([a,b],L(X,\dot{Y})) = \left\{\alpha \in SV_b^{+\int}(]a,b],L(X,\dot{Y})) \,\middle|\, \begin{matrix}\alpha(a+) = \alpha(a) \text{ instead} \\ \text{of } \alpha(a) \in L(X,Y)\end{matrix}\right\}.$$

<u>Theorem 1.5</u> - The mapping

$$\alpha \in SV_b^{+\int}([a,b],L(X,\dot{Y})) \longmapsto F_\alpha \in L[\mathcal{C}_a([a,b],X),Y]$$

is an isometry of the first Banach space onto the second.

Proof - 1) The mapping is one-to-one: indeed, if α is such that $F_\alpha f = 0$ for every $f \in \mathcal{C}_a$, then for all $t \in [a,b]$, $x \in X$ and $y \in Y$ we have $0 = \langle F_\alpha(g_{t,x}), y\rangle = -\int_a^t \langle \alpha(s)x, y\rangle\, ds$ hence $\alpha = 0$ since we have, by hypothesis, $\alpha(s\dot{+}) = \alpha(s)$ for every $s \in [a,b[$.

2) The mapping is onto: indeed, for $\alpha \in SV_b^{+\int}(]a,b], L(X,\dot{Y}))$ we define $\alpha_a(a) = \alpha(a\dot{+})$ and $\alpha_a(t) = \alpha(t)$ if $a < t \leq b$; we have $\alpha_a \in SV_b^{+\int}([a,b], L(X,\dot{Y}))$ (since $\alpha_a(a\dot{+}) = \alpha_a(a)$) and for every $f \in \mathcal{C}_a$ we have $F_{\alpha_a}(f) = F_\alpha(f)$.

3) The isometry follows from Theorem 1.4.

For a function $\alpha: (t,s) \in T \times S \longmapsto \alpha(t,s) \in Z$ we write

$$\alpha^t(s) = \alpha_s(t) = \alpha(t,s)$$

For a function $\alpha: [c,d] \times [a,b] \longrightarrow L(X,Y'')$ we consider the following properties:

For $a < s \leq b$ $(s = a)$ and $x \in X$ the function

$(\widetilde{\mathcal{C}}^\sigma)\,[(\widetilde{G}^\sigma); (\widetilde{L}^\sigma_\infty)]$: $t \in [c,d] \longmapsto \int_a^s \alpha(t,\sigma)x\,d\sigma \in Y$ $(t \in [c,d] \longmapsto \alpha(t,a)x \in Y)$ is continuous [regulated; bounded and measurable]

We write $\alpha \in \widetilde{\mathcal{C}}^\sigma \cdot SV_b^{+\int u}([c,d] \times]a,b], L(X,\dot{Y}))$ $[\widetilde{G}^\sigma \cdot SV_b^{+\int u} \ldots; \widetilde{L}^\sigma_\infty \cdot SV_b^{+\int u} \ldots]$ if besides $(\widetilde{\mathcal{C}}^\sigma)$ $[(\widetilde{G}^\sigma); (\widetilde{L}^\sigma_\infty)]$ α satisfies

$(SV_b^{+\int u})$: for every $t \in [c,d]$ we have $\alpha^t \in SV_b^{+\int}(]a,b], L(X,\dot{Y}))$ with $SV^u[\alpha] = \sup_{c \leq t \leq d} SV[\alpha^t] < \infty$.

In an analogous way we define $\widetilde{\mathcal{C}}^\sigma \cdot SV_b^{+\int u}([c,d] \times [a,b], L(X,\dot{Y}))$ (we take $\alpha^t \in SV_b^{+\int}([a,b], L(X,\dot{Y}))$) etc.

Theorem 1.6 - a) The mapping

$$\alpha \in \widetilde{\mathcal{C}}^\sigma \cdot SV_b^{+\int u}([c,d] \times]a,b], L(X,\dot{Y})) \longmapsto F_\alpha \in L[\mathcal{C}([a,b],X), \mathcal{C}([c,d],Y)]$$

where $(F_\alpha f)(t) = \int_a^b d_s\alpha(t,s)\cdot f(s)$, $c \leq t \leq d$, is an isometry (i.e. $\|F_\alpha\| = SV^u[\alpha]$) of the first Banach space onto the second. We have $\int_a^s \alpha(t,\sigma)x d\sigma = -F_\alpha[g_{s,x}](t)$, $a \leq s \leq b$, and $\alpha(t,a)x = -F_\alpha[\chi_{[a,b]}x](t)$.

We have analogous results for the mappings

b) $\alpha \in \tilde{G}^\sigma\cdot SV_b^{+\int u}([c,d]\times]a,b],L(X,\dot{Y})) \longmapsto F_\alpha \in L[\mathcal{C}([a,b],X),G([c,d],Y)]$

c) $\alpha \in \tilde{L}_\infty^\sigma\cdot SV_b^{+\int u}([c,d]\times]a,b],L(X,\dot{Y})) \longmapsto F_\alpha \in L[\mathcal{C}([a,b],X),L_\infty([c,d],Y)]$

d) $\alpha \in \tilde{\mathcal{C}}^\sigma\cdot SV_b^{+\int u}([c,d]\times[a,b],L(X,\dot{Y})) \longmapsto F_\alpha \in L[\mathcal{C}_a([a,b],X), \mathcal{C}([c,d],Y)]$

e) $\alpha \in \tilde{G}^\sigma\cdot SV_b^{+\int u}([c,d]\times[a,b],L(X,\dot{Y})) \longmapsto F_\alpha \in L[\mathcal{C}_a([a,b],X),G([c,d],Y)]$

f) $\alpha \in \tilde{L}_\infty^\sigma\cdot SV_b^{+\int u}([c,d]\times[a,b],L(X,\dot{Y})) \longmapsto F_\alpha \in L[\mathcal{C}_a([a,b],X),L_\infty([c,d],Y)]$

The proof follows the steps of the proof of Theorem I.5.10 of [11] (and the remark that follows it) p.49 to 52: the properties of F_α follow from the Theorem of Helly (see loc. cit.). Reciprocally given F for every $t \in [c,d]$ the continuous mapping $f \in \mathcal{C}([a,b],X) \longmapsto (Ff)(t) \in Y$ $[f \in \mathcal{C}_a \longmapsto (Ff)(t) \in Y]$ is, by Theorem 1.4 [Theorem 1.5], represented by an $\alpha^t \in SV_b^{+\int}(]a,b],L(X,\dot{Y}))$ $[\alpha^t \in SV_b^{+\int}([a,b],L(X,\dot{Y}))]$ etc.

2. The Kurzweil-Henstock integral and representation theorems.

We say that a function $f:[a,b] \longrightarrow X$ is integrable in the sense of Kurzweil-Henstock (we write $f \in \mathcal{K}([a,b],X)$ or $f \in \mathcal{K}$) and that $J \in X$ is its Kurzweil-Henstock integral (we write ${}^K\int_a^b f(t)dt = J$) if for every $\varepsilon > 0$ there exists a function $\delta:[a,b] \longrightarrow]0,\infty[$ (called gauge function) such that for every tagged division (ξ_i,t_i) of $[a,b]$ (i.e. a division $t_0 = a < t_1 < t_2 < \cdots < t_n = b$ of $[a,b]$ with points, tags, $\xi_i \in [t_{i-1},t_i]$) that is δ-fine, i.e.

$$\xi_i \in [t_{i-1},t_i] \subset]\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)[,$$

we have $\|J - \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1})\| < \varepsilon$.

It can be proved that for every gauge δ there exist always δ-fine tagged divisions of $[a,b]$ and that J is unique, see [15]. The integral is linear, additive with respect to subintervals. However the existence of ${}^K\!\int_a^b f(t)dt$ does not imply the existence of ${}^K\!\int_a^b \|f(t)\|dt$ even in the numerical case ($X = \mathbb{R}$), see [13], 1.7 and 1.8. $\mathcal{K}([a,b],X)$ encompasses the functions $f:[a,b] \longrightarrow X$ that are integrable in the sense of Bochner-Lebesgue or in the sense of Riemann (proper or improper), see [13], 1.8 and 1.6. We recall that in the numerical case $\mathcal{K}([a,b])$ coincides with the spaces of Denjoy or Perron integrable functions.

In $\mathcal{K}([a,b],X)$ we define the Alexiewicz seminorm (see [1] or [4])

$$\|f\|_A = \sup_{a \leq t \leq b} \left\|{}^K\!\int_a^t f(s)\,ds\right\| .$$

2.1 (see [15] or [20]) - For $f \in \mathcal{K}([a,b],X)$ we have $\tilde{f} \in \mathcal{C}_a([a,b],X)$ where $\tilde{f}(t) = {}^K\!\int_a^t f(s)\,ds$, $a \leq t \leq b$.

We say that $f,g \in \mathcal{K}$ are <u>equivalent</u> if $\tilde{f} = \tilde{g}$. If $\dim X = \infty$ this does not necessarely imply that $f = g$ a.e. Example: for $X = \ell_2([a,b])$ and $f(t) = e_t$, where $e_t(s) = \begin{cases} 1 & \text{if } s = t \\ 0 & \text{if } s \neq t \end{cases}$, we have $\tilde{f} = 0$; see [10], p.26 or [11] p.12. We denote by $K([a,b],X)$ the space of the equivalence classes of $\mathcal{K}([a,b],X)$. Endowed with the norm $\| \ \|_A$, $K([a,b],X)$ is not complete and $K([a,b],X)$ allows no natural Banach space or Frechet space topology: see [13], Theorem 2.7. However Sargent [19] and Thomson [21] proved that $K([a,b],X)$ is barreled and Gilioli extended this result:

<u>Theorem 2.2</u> ([8]) - $K([a,b],X)$ is ultrabornological.

It is easy to define the Kurzweil integral for "intervals" $[a,b]\subset\mathbb{R}^n$ and in this case too $K([a,b],X)$ is barreled, see [16], and ultrabornological, see [9].

2.3 - The mapping $f\in K([a,b],X) \longmapsto \tilde{f}\in \mathcal{C}_a([a,b],X)$ is an isometry (i.e. $\|\tilde{f}\| = \|f\|_A$) of the first space onto a dense subspace of the second.

<u>Theorem 2.4</u> - For every $\alpha\in SV([a,b],L(X,Y))$ and $f\in K([a,b],X)$ we have $\alpha\cdot f\in K([a,b],Y)$ and

$$^K\!\int_a^b \alpha(t)\cdot f(t)dt = \int_a^b \alpha(t)\cdot d\tilde{f}(t) = \alpha(b)\cdot\tilde{f}(b) - \alpha(a)\cdot\tilde{f}(a) - \int_a^b d\alpha(t)\cdot\tilde{f}(t)$$

and $\quad \left\|^K\!\int_a^b \alpha(t)\cdot f(t)\,dt\right\| \leq \left[\|\alpha(a)\| + SV[\alpha]\right]\|f\|_A$.

Our proof of this integration by parts formula for the Kurzweil-Henstock integral (see [13], 1.15 and 1.16) is algebraic and much simpler then the analytic proofs given in [5], [3] , [7] and [17].

<u>Theorem 2.5</u> - The mapping

$$\alpha\in SV_b^{+J}([a,b],L(X,Y)) \longmapsto H_\alpha\in L\left[K([a,b],X),Y\right]$$

where $H_\alpha f = {}^K\!\int_a^b \alpha(s)\cdot f(s)\,ds$, is an isometry (i.e. $\|H_\alpha\| = SV[\alpha]$) of the first Banach space onto the second. We have

$$H_\alpha(\chi_{[a,t]}x) = \int_a^t \alpha(s)x\,ds\ ,\quad x\in X,\ t\in[a,b].$$

<u>Proof</u>: by Theorem 2.4 we have $\|H_\alpha\|\leq SV[\alpha]$. On the other hand given $H\in L\left[K([a,b],X),Y\right]$ and $f\in K([a,b],X)$ we define $\hat{H}(\tilde{f}) = -H(f)$ and by 2.3 $\hat{H}$ has a unique continuous extension to $\mathcal{C}_a([a,b],X)$. If $\alpha\in SV_b^{+J}([a,b],L(X,Y))$ represents $\hat{H}$ (see Theorem 1.5) we have

$$H(f) = -\hat{H}(\tilde{f}) = -\int_a^b d\alpha(s)\cdot\tilde{f}(s) = {}^K\!\int_a^b \alpha(s)\cdot f(s)\,ds$$

where we applied Theorem 2.4 in the last equality. The isometry follows from $\|H\| = \|\hat{H}\| = SV[\alpha]$.

Remark: as we mentioned in the remark that follows Theorem 1.4, from $H_\alpha(\chi_{[a,t]}x) = \int_a^t \alpha(s)x\, ds$ we cannot deduce that $\alpha(t+)x = \frac{d^+}{dt} H_\alpha(\chi_{[a,t]}x)$.

From 2.1, Theorems 1.6 and 2.5 follows

Theorem 2.6 - The mapping

$$\alpha \in \tilde{\mathcal{C}}^{\sigma}\cdot SV_b^{+\int u}([c,d]\times[a,b], L(X,\dot{Y})) \longmapsto H_\alpha \in L[K([a,b],X), \mathcal{C}([c,d],Y)]$$

where $(H_\alpha f)(t) = {}^K\!\int_a^b \alpha(t,s)\cdot f(s)\, ds$, is an isometry (i.e. $\|H_\alpha\| = SV^u[\alpha]$) of the first Banach space onto the second. We have

$$\int_a^{s} {}'\alpha(t,\sigma)x\, d\sigma = [H_\alpha(\chi_{[a,s]}x)](t), \quad s \in [a,b],\ t \in [c,d],\ x \in X.$$

3. Linear Kurzweil-Henstock integral equations

Theorem 3.1 - Let E be a normed space and F a Banach space such that $F \subset E$ with continuous immersion; let $H \in L(E,F)$ be such that for every $f \in E$ the equation $x - Hx = f$ has one and only one solution $x_f \in E$. Then

a) the mapping $f \in E \longmapsto x_f \in E$ is bicontinuous;

b) if $I + H + H^2 + H^3 + \dots$ $(= (I-H)^{-1})$ is convergent in $L(F)$ then the convergence of this Neumann series takes also place in $L(E)$.

Proof - a): 1) For every $g \in F$ the equation $y - Hy = g$ has one and only one solution $y_g \in F$ and the mapping $g \in F \longmapsto y_g \in F$ is bicontinuous. Indeed: since $F \subset E$ the equation $y - Hy = g$ with $g \in F$ has by hypothesis one and only one solution $y_g \in E$; but since $HE \subset F$ we have $Hy_g \in F$ hence $y_g = Hy_g + g \in F$. On the other hand the mapping $y \in F \longmapsto g = y - Hy \in F$ is a continuous bijection, hence, by the closed graph theorem, its inverse $g \in F \longmapsto y_g \in F$ is continuous.

2) The equation $x - Hx = f$ with $x, f \in E$, is equivalent to the equation $y - Hy = g$ with $g = Hf$ and $y = x - f$. The mapping $f \in E \longmapsto g = Hf \in E$ is continuous; by 1) the mapping $Hf \in F \longmapsto y_{Hf} \in F$ is also continuous; since the mapping $y_{Hf} \in F \longmapsto y_{Hf} \in E$ is also continuous it follows that the composed mapping $f \in E \longmapsto y_{Hf} \in E$ is also continuous, hence so is the mapping $f \in E \longmapsto y_{Hf} + f \in E$. The result follows from $y_{Hf} = x_f - f$, i.e., $x_f = y_{Hf} + f$.

b) Let $(I - H)^{-1} = I + H + H^2 + H^3 + \dots$ be convergent in $L(F)$. Since $H \in L(E,F)$ and $F \subset E$ is continuous, it follows that the series is also convergent in $L(E)$. It is immediate that if a Neumann series is convergent in some $L(Z)$ (Z not necessarely complete) then it converges to $(I - H)^{-1}$.

<u>Theorem 3.2</u> - Take $H \in L[K([a,b],X), \mathcal{C}([a,b],X)]$ and α the corresponding kernel by Theorem 2.6 (i.e., $H = H_\alpha$). We suppose that H is such that for every $f \in K([a,b],X)$ the equation $x - Hx = f$, i.e. the linear Fredholm-Kurzweil integral equation

$$x(t) - {}^K\!\!\int_a^b \alpha(t,s)\cdot x(s)\, ds = f(t), \quad a \leq t \leq b$$

has one and only one solution $x_f \in K([a,b],X)$. Then there exists one and only one kernel

$$\rho \in \tilde{\mathcal{C}}^{\sigma}\cdot SV_b^{+Ju}([a,b]\times[a,b], L(X,\dot{X}))$$

such that for every $f \in K([a,b],X)$ we have

$$x_f(t) = f(t) - {}^K\!\!\int_a^b \rho(t,s)\cdot f(s)\, ds\ , \quad a \leq t \leq b.$$

Proof: if we apply Theorem 3.1 with $E = K([a,b],X)$ and $F = \mathcal{C}([a,b],X)$ it follows that $(I - H)^{-1} \in L[K([a,b],X)]$. If we define $I - R = (I - H)^{-1}$ it follows that $R = -H(I - H)^{-1} \in L[K([a,b],X),\mathcal{C}([a,b],X)]$ is represented by a kernel $\rho \in \widetilde{\mathcal{C}}^{\sigma} \cdot SV_b^{+\int u}$.

Remarks - 1) If $H \in L[K([a,b],X),\mathcal{C}([a,b],X)]$ satisfies the hypothesis of Th.3.2 then we still have an analogous result if we replace $K([a,b],X)$ by any dense subspace that contains $\mathcal{C}([a,b],X)$. More generally, if E is a dense subspace of $K([a,b],X)$ that contains $\mathcal{C}([a,b],X)$ and $H \in L[E,\mathcal{C}([a,b],X)]$ is such that for every $f \in E$ the equation $x - Hx = f$ has one and only one solution $x_f \in E$ then Theorem 3.2 remains true if we replace $K([a,b],X)$ by E. Let us mention as example $R_2([a,b],X)$, the subspace formed by the (equivalence classes of) functions $f: [a,b] \longrightarrow X$ that are integrable in the improper sense of Riemann with a finite numbers of singularities (points of improper integration) in $[a,b]$. This space too is ultrabornological: see [8], p.359, Example 4.7, F_{11}^{α} with $\alpha = 1$. The same is also true for the other spaces of functions of this Example 4.7.

2) We still have analogous results if in Theorem 3.2 and in Remark 1 we replace $\mathcal{C}([a,b],X)$ by $G([a,b],X)$ or by $L_\infty([a,b],X)$.

Lemma 3.3 - For $\alpha \in \widetilde{\mathcal{C}}^{\sigma} \cdot SV_b^{+\int u}([a,b] \times [a,b], L(X,\dot{Y}))$ and $f \in \mathcal{K}([a,b],X)$ the function $t \in [a,b] \longmapsto {}^{K}\!\int_a^t \alpha(t,s) \cdot f(s)\, ds \in Y$ is continuous.

Proof: since the step functions are dense in $\mathcal{K}([a,b],X)$ it is enough to prove that for every $x \in X$ the function

$$t \in [a,b] \longmapsto {}^{K}\!\int_a^t \alpha(t,s)x\, ds \in Y$$

is continuous: for $\varepsilon > 0$ we have

$$\int_a^{t+\varepsilon} \alpha(t+\varepsilon,s)x\, ds = \int_a^t \alpha(t+\varepsilon,s)x\, ds + \int_t^{t+\varepsilon} \alpha(t+\varepsilon,s)x\, ds$$

and the first integral on the right tends to $\int_a^t \alpha(t,s)x\, ds$ (by $(\tilde{\mathcal{C}}^\sigma)$) and the norm of the second integral is $\leq \varepsilon \|\alpha\|_\infty \|x\|$ hence the continuity to the right. The continuity to the left is proved in an analogous way.

Remark: in an analogous way one proves that for $\alpha \in \tilde{G}^\sigma \cdot SV_b^{+\int u}$ $[\tilde{L}_\infty^\sigma \cdot SV_b^{+\int u}]$ the integral of Lemma 3.3 is a regulated [bounded measurable] function of $t \in [a,b]$.

We say that an operator $H \in L[\mathcal{K}([a,b],X), \mathcal{C}([a,b],Y)]$ is causal if for every $f \in \mathcal{K}$ and every $t \in [a,b]$ we have

$$f_{|[a,t]} = 0 \quad \text{implies} \quad (Hf)_{|[a,t]} = 0$$

Lemma 3.3 says that for $\alpha \in \tilde{\mathcal{C}}^\sigma \cdot SV_b^{+\int u}$ with $\alpha(t,s) = 0$ for $s > t$ (we write $\alpha \in \tilde{\mathcal{C}}_0^\sigma \cdot SV^{+\int u}([a,b]\times[a,b], L(X,\mathring{X}))$) the operator $H = H_\alpha$ is causal. Reciprocally every causal operator $H \in L[\mathcal{K}, \mathcal{C}]$ is of the form $H = H_\alpha$ with $\alpha \in \tilde{\mathcal{C}}_0^\sigma \cdot SV^{+\int u}$: indeed, if H_α represents H then for every $t \in [a,b]$ and $x \in X$ we have $0 = [H_\alpha(\chi_{[t,b]}x)](t) = \int_t^b \alpha(t,s)x\,ds$ hence the result (Cf. the Remark that follows the proof of Theorem 1.4: the function $s \in [a,b] \longmapsto \langle \alpha(t,s)x, y \rangle \in \mathbb{R}$ is of bounded variation continuous to the right). Hence we proved the

Theorem 3.4 - The mapping

$$\alpha \in \tilde{\mathcal{C}}_0^\sigma \cdot SV^{+\int u}([a,b]\times[a,b], L(X,\mathring{Y})) \longmapsto H_\alpha \in L[K([a,b],X), \mathcal{C}([a,b],Y)]$$

is an isometry onto the subspace of causal operators.

Remark: we have the corresponding results, with analogous proofs if we replace $\mathcal{C}([a,b],Y)$ by $G([a,b],Y)$ or by $L_\infty([a,b],Y)$.

Let us now consider the linear Volterra-Kurzweil integral equation

$$(V)\qquad x(t) - {}^{K}\!\!\int_a^t \alpha(t,s)\cdot x(s)\,ds = f(t)\ ,\quad a \leq t \leq b$$

with $x,f \in \mathcal{K}([a,b],X)$ and $\alpha \in \widetilde{\mathcal{C}}_0^{\sigma}\cdot SV^{+\int u}([a,b]\times[a,b],L(X,\dot{X}))$ i.e. H_α is a causal operator.

<u>Theorem 3.5</u> - For every $f \in \mathcal{K}([a,b],X)$ the equation (V) has one and only one solution $x_f \in \mathcal{K}([a,b],X)$ and the bijection $f \longmapsto x_f$ too is causal and may be written as

$$(\rho)\qquad x_f(t) = f(t) - {}^{K}\!\!\int_a^t \rho(t,s)\cdot f(s)\,ds\ ,\quad a \leq t \leq b$$

where $\rho \in \widetilde{\mathcal{C}}_0^{\sigma}\cdot SV^{+\int u}([a,b]\times[a,b],L(X,\dot{X}))$. We have the Neumann series $I - H_\rho = I + H_\alpha + H_\alpha^{\,2} + H_\alpha^{\,3} + \dots$ that is convergent in $L[K([a,b],X)]$.

<u>Proof</u>: we define $g(t) = {}^{K}\!\!\int_a^t \alpha(t,s)\cdot f(s)\,ds$ and $y = x - f$. The equation (V) is equivalent to the equation

$$(V_1)\qquad y(t) - {}^{K}\!\!\int_a^t \alpha(t,s)\cdot y(s)\,ds = g(t)\ ,\quad a \leq t \leq b$$

with $g,y \in \mathcal{C}([a,b],X)$ (since $H_\alpha \in L[K,\mathcal{C}]$) and hence $g = H_\alpha f$ and $y = g + H_\alpha y$. If we define $K(t,s)x_0 = -\int_s^t \alpha(t,\sigma)x_0\,d\sigma$ for $x_0 \in X$ and $a \leq s \leq t \leq b$ then the equation (V_1) is equivalent to the linear Volterra-Stieltjes integral equation

$$(V_2)\qquad y(t) - \int_a^t d_sK(t,s)\cdot y(s) = g(t)\ ,\quad a \leq t \leq b$$

(with $g = H_\alpha f$ and $y = x - f$). In order to prove that for every $g \in \mathcal{C}([a,b],X)$ the equation (V_2) has one and only one solution $y_g \in \mathcal{C}([a,b],X)$ (and that the operator $g \longmapsto y_g$ is causal) by Theorem 3.8 of [12] it is enough to prove that

1) For every $t \in [a,b[$ the operator $I - K(t^+,t)$ is invertible (in $L(X)$).

2) There exists a division $s_0 = a < s_1 < s_2 < \cdots < s_n = b$ of $[a,b]$ such that

$$\sup_{s_{i-1} \leq t \leq s_i} SV_{[s_{i-1},t]}[K^t] < 1 \quad \text{for } i = 1,2,\cdots,n$$

The proof of 1) is immediate since $K(t^+,t) = 0$:

$$\|K(t+\varepsilon,t)x_0\| = \left\|\int_t^{t+\varepsilon} \alpha(t+\varepsilon,\sigma)x_0 \, d\sigma\right\| \leq \varepsilon \|\alpha\|_\infty \|x_0\| \leq \varepsilon SV^u[\alpha] \cdot \|x_0\| .$$

2) follows from

$$SV_{[s_{i-1},t]}[K^t] \leq V_{[s_{i-1},t]}[K^t] \leq (t-s_{i-1}) \|\alpha\|_\infty \leq (t-s_{i-1}) SV^u[\alpha]$$

where $V_{[s_{i-1},t]}[K^t]$ denotes the variation of K^t in the interval $[s_{i-1},t]$.

The assertion on the Neumann series of the resolvent of (V_2) in $L[\mathcal{C}([a,b],X)]$ follows from Theorem 3.9 of [12]. From Theorem 3.1 follows the same result in $L[K([a,b],X)]$ and since F_K is causal, where $(F_K g)(t) = \int_a^t d_s K(t,s) \cdot g(s) = (H_\alpha g)(t)$, $a \leq t \leq b$, the same is true for $(F_K)^n$ and hence for the operator represented by the Neumann series, i.e., for $(I - F_K)^{-1}$. The same result for H_α follows from $F_K = H_\alpha$ (and from $I - H_\rho = (I - H_\alpha)^{-1}$).

<u>Remarks</u> - 1) We have still an analogous result if we replace $K([a,b],X)$ by $R_2([a,b],X)$ or the other spaces mentioned in Remark 1 that follows Theorem 3.2.

2) Analogously we may replace $\mathcal{C}([a,b],X)$ by $G([a,b],X)$ or by $L_\infty([a,b],X)$.

For $\alpha : [a,b] \longrightarrow L(X,Y)$ and $f : [a,b] \longrightarrow X$ it is easy to define the Kurzweil-Stieltjes integral ${}^K\!\int_a^b d\alpha(t) \cdot f(t)$, a generalization of the Ward or Perron-Stieltjes numerical integral, see [21].

If $\alpha \in SV([a,b],L(X,Y))$ we define

$$K_\alpha([a,b],X) = \left\{ f \in X^{[a,b]} \mid \exists\ {}^K\!\!\int_a^b d\alpha(t)\cdot f(t) \in Y \right\}$$

and the seminorm $\|f\|_\alpha = \sup_{a \leq t \leq b} \left\| {}^K\!\!\int_a^t d\alpha(s)\cdot f(s) \right\|$. Without some restriction on α, $K_\alpha([a,b],X)$ may not be ultrabornological (example of Luiz Fichmann). Elsewhere we will study the Fredholm-Kurzweil-Stieltjes integral equations and the Volterra-Kurzweil-Stieltjes integral equations.

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