

# Integrable Models: from Dynamical Solutions to String Theory

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**Abstract** We review the status of integrable models from the point of view of their dynamics and integrability conditions. A few integrable models are discussed in detail. We comment on the use it is made of them in string theory. We also discuss the  $SO(6)$  symmetric Hamiltonian with  $SO(6)$  boundary. This work is especially prepared for the 70th anniversaries of André Swieca (in memoriam) and Roland Köberle.

**Keywords** Integrable systems · Yang–Baxter equations · Yang–Mills fields · String theory · Supersymmetry

## 1 Introduction

As a natural extension of quantum mechanics, relativistic quantum field theory (QFT) has demonstrated its predictive power in the calculation of processes in quantum electrodynamics. There are, however, conceptual and technical difficulties, since the local products of quantum fields, which are operator-valued distributions, are ill defined. This problem can only be resolved via the techniques of renormalization.

The general non-perturbative properties of quantum field theory were first extracted from a pertur

turbative setup by the so-called Lehmann–Symanzik–Zimmermann formalism. Next, dispersion relations were found and were used to obtain non-perturbative information. These developments were followed by the axiomatic approach, known as constructive QFT. An important consequence of this approach is the charge–parity–time (CPT) theorem connecting spin and statistics.

Nonetheless, dynamical calculations in QFT were, in the 1960s, restricted to perturbation theory. Therefore, calculations involving strong interactions were unreliable. Information about the bound-state spectrum were very poor and could only be obtained within crude approximate schemes. Thus, QFT fell into stagnation for many years. These difficulties provided a motivation for the S-matrix theory. But its predictive power turned out to be very small, since it was entirely based on kinematical principles, analyticity, and the bootstrap idea. An underlying dynamical framework was lacking. Nevertheless, analyticity in the complex angular momentum plane led to the important concept of duality. An explicit realization of these concepts by the Veneziano formula led to a new parallel development in the 1960s, the dual models. However, the predictions of the dual models for high-energy scattering processes were incorrect.

On the other hand, QFT explained very successfully the weak interactions. Moreover, symmetry principles had proven powerful in predicting the masses of strongly interacting particles without recourse to dynamical calculations. These facts led to a revival of QFT in the late 1960s. In the 1970s, much effort was spent on non-perturbative aspects. Quantum chromodynamics (QCD) was proposed as the fundamental theory of the strong interactions as a result of the successful perturbative explanation of high energy scattering as well as

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the success of the quark model. Nevertheless, reliable non-perturbative calculations were still lacking in four dimensions and were only available for specific models in two-dimensional space–time [1]. It was understood that the short distance singularities of quantum field theory play a key role in the dynamical structure of the theory. The experimental results on lepton–proton scattering at large momentum transfer required that a realistic theory of the strong interactions be asymptotically free.

The recourse to soluble or almost soluble models as a laboratory was a must for a dynamical understanding of QFT. The first soluble model was that describing a two-dimensional massless fermion with a current–current interaction formulated by Thirring in 1958 [2] as an example of a completely soluble quantum field theoretic model obeying the general principles of a QFT [3]. Subsequently, Schwinger [4, 5] obtained an exact solution of quantum electrodynamics in  $1+1$  dimensions,  $\text{QED}_2$ . A number of interesting properties, such as the nontrivial vacuum structure of this model, were understood only later [6] when it was found that there is a long-range Coulomb force for the charge sectors of the theory. This long-range force was interpreted as being responsible for the confinement of quarks [7, 8]. The problem of confinement and the related phenomenon of screening of charge quantum numbers in two dimensions have been studied by several authors [9–11] and have served as a basis for understanding important concepts in QFT. The surprisingly rich structure of two-dimensional quantum electrodynamics was found to describe several important features of the non-Abelian gauge theories, which were under investigation in the 1970s.

Several results of increasing importance followed. Two-dimensional classically integrable models were studied in great detail. Such models are characterized by the existence of an infinite number of conservation laws. If these conservation laws survive quantization, the corresponding S-matrices can be computed exactly [12–16]. Some of the results concerning classical integrability have also been generalized to higher dimensions [17–22] and used to understand QCD [23, 24].

Describing two-dimensional fermions in terms of bosons (bosonization) can lead to non-perturbative information. The building blocks of the procedure are the exponentials of the free bosonic fields. One obtains a fermion number which is connected to the infrared behavior of the massless scalar fields. One thus obtains a superselection rule [25], and the charged sectors appear in a natural way.

A particularly important class of two-dimensional integrable non-linear sigma models are those with a

geometrical origin [26–28], which share several properties with four-dimensional Yang–Mills theories [26–29]. Upon quantization, they exhibit dynamical mass generation and contain a long-range force [29] for simple gauge groups [30]. Such a long range force can be screened by dynamical fermions [31–35]. These properties make them appealing as toy models for the strong interactions [36, 37]. They are also very interesting mathematical objects, particularly important in the framework of string theory.

Furthermore, the study of these models has led to new developments in the study of quantum field theories in higher dimensions. High-energy scattering amplitudes involving fields with definite helicity or at high energy, in four-dimensional Quantum Chromodynamics, have a rather simple description, related to integrable models [23, 24]. In the former case, the scattering amplitudes are related to solutions of self-dual Yang–Mills equation, while in the latter case the interaction of external particles is described by the two-dimensional Heisenberg Hamiltonian of spin systems.

## 2 Exact S-Matrices and Yang–Baxter Equations

The most general invariance group of a non-trivial field theory in  $d > 2$  dimensions is the product of the Poincaré group and an internal symmetry [38] times supersymmetry [39]. The basic idea of the proof is that an infinite number of higher conservation laws implies that the momenta involved in the scattering process are individually conserved, so that the process merely consists in an exchange of quantum numbers. This would imply that the S-matrix does not depend analytically on the scattering momenta; in particular, the two-particle S-matrix would not depend analytically on the scattering angle.

In two-dimensional space–time, the situation is different. The scattering angle can only be zero or  $\pi$  and the clash with analyticity no longer exists. The constraints due to the conservation laws on the scattering process are very strong. The conservation of an infinite number of local charges implies conservation of the energy, momentum, and their powers: The higher conserved charges are higher-rank tensors  $Q_{\mu_1 \dots \mu_l}$ , transforming according to higher representations of the Lorentz group, commuting with one another and with the momentum [40].

The action of  $Q_{\mu_1 \dots \mu_n}$  on asymptotic states is severely restricted by Lorentz invariance. On a one-particle state, we have

$$Q^{\mu_1 \dots \mu_l} |P\rangle = P^{\mu_1} \dots P^{\mu_l} |P\rangle \quad . \quad (1)$$

Conservation of the higher charges thus imply

$$\sum_{i=1}^n P_i^{\mu_1} \cdots P_i^{\mu_l} = \sum_{i=1}^m P_i'^{\mu_1} \cdots P_i'^{\mu_l} \quad , \quad (2)$$

provided the corresponding scattering amplitudes do not vanish. Hence, there exists an infinite number of conservation laws that must be obeyed by the external momenta. Equations such as (2) can only be satisfied if  $n = m$ , i.e., if there is no particle production and the individual momenta are conserved. Thus, after a suitable rearrangement,  $P_i = P_i'$ , and the scattering only consists of time delays and exchange of quantum numbers [40–42] (we have ignored terms such as  $g^{\mu\nu} P^\rho |P^a\rangle$  since they are not essential. Notice also that since the mass operator commutes with the charge  $Q^{\mu\nu\rho}$ , there can be degeneracy).

## 2.1 Factorizable S-Matrix

Absence of particle production implies that the S-matrix is of the factorizable type, that is, the scattering S-matrix is given by the product of all possible two-particle scattering amplitudes [41, 42]. Furthermore, the two-particle processes are severely constrained by the so-called *factorization relations*.

In order to see this, one observes that intermediate multiparticle states, with the particles sufficiently separated, should satisfy the same selection rules as described above by (2). As a consequence, the S-matrix elements for  $N$ -particle scattering amplitudes can be expressed as a product of two-particle S-matrices.

Considering the wave packet

$$\psi(x) = \int dp e^{-a(p-p_0)^2 + ip(x-x_0)} |p\rangle \quad (3)$$

the action of a higher (local) charge leads to

$$\begin{aligned} e^{icQ^{(n)}} \psi(x) &= \tilde{\psi}(x) \\ &= \int dp e^{-a(p-p_0)^2 + ip(x-x_0) + icp^n} |p\rangle, \end{aligned} \quad (4)$$

which is a wave packet now centered at the point  $\tilde{x}_0$ , given by  $\tilde{x}_0 = x_0 - nc p_0^{n-1}$ . The shift is proportional to a power of  $p_0$ ; hence, it grows with  $p_0$ .

It is not difficult to see that for a three-particle scattering  $(i, j, k)$ , a momentum-dependent shift implies

$$S_{ij}(\theta_{ij}) S_{ik}(\theta_{ik}) S_{jk}(\theta_{jk}) = S_{jk}(\theta_{jk}) S_{ik}(\theta_{ik}) S_{ij}(\theta_{ij}) \quad , \quad (5)$$

where  $\theta_{ij}$  is the rapidity defined by  $\theta_{ij} = \theta_i - \theta_j$ , with

$$p_i = m(\cosh \theta_i, \sinh \theta_i), \quad (6)$$

and where  $m$  is the mass of the fundamental particles. A second, purely algebraic interpretation of (5) is also possible. We consider the symbols  $\{A_i(\theta)\}$  to represent the set of particles. A given  $n$ -particle state is defined by the action of a product of these symbols on the vacuum, ordered according to their rapidities: The “in” states are identified with the products in order of decreasing rapidities, while the “out” states are arranged in the order of increasing rapidities. The commutation relations of the  $A$ ’s are defined in terms of the S-matrix, that is,

$$A(\theta_1) A'(\theta_2) = S_T(\theta_{12}) A'(\theta_2) A(\theta_1) + \cdots , \quad (7)$$

where  $S_T$  is the transition amplitude for  $AA' \rightarrow AA'$  and the dots represent other channels. There are different ways to consider the scattering of three particles and uniqueness of the result leads to (5).

The two-particle S-matrix in a factorizable two-dimensional theory is a function of the Mandelstam variable  $s$ . It is convenient to write the momenta  $p_i$  in terms of the rapidity variable  $\theta_i$  as defined in (6). The two-particle S-matrix elements depend on the difference of rapidities: They only depend on the variable  $s$ , related to  $\theta = \theta_i - \theta_j$  by

$$s = (p_i + p_j)^2 = m_i^2 + m_j^2 + 2m_i m_j \cosh \theta . \quad (8)$$

For equal masses, we have  $s = 2m^2(1 + \cosh \theta) = 4m^2(\cosh \frac{\theta}{2})^2$ .

In general, the two-particle amplitudes are analytic functions of  $s$ , with cuts along the real axis. The scattering amplitude has a cut for  $s \leq (m_1 - m_2)^2$  and for  $s \geq (m_1 + m_2)^2$ . The point  $s = (m_1 + m_2)^2$  corresponds to the two-particle threshold. The mapping (8) transforms the physical sheet in the  $s$ -plane into a strip  $0 < \Im s \theta < \pi$ . The scattering amplitude  $S(\theta)$  is real analytic and hence is real on the imaginary  $\theta$  axis. Moreover, on the real axis  $S(-\theta) = S^*(\theta)$ .

In the calculation of S-matrices in two dimensions, one first computes the so-called minimal S-matrix, which has a minimum number of zeros and poles on the physical sheet and grows slower than  $\exp \frac{p \cdot p'}{m^2}$  for large momenta. At this point, one requires that the S-matrix obeys unitarity and crossing [41, 42]. The first condition turns out to be a requirement on the modulus squared of the two-particle scattering amplitude, since there is no particle production [40].

In a relativistic theory, crossing corresponds to the substitution of an incoming particle of momentum  $p$

by an outgoing antiparticle with momentum  $-p$ . This is equivalent to the substitution  $s \rightarrow 4m^2 - s$  (or  $\theta \rightarrow i\pi - \theta$ ). In terms of equations, invariance under crossing implies

$$\begin{aligned} & \langle f'_1 f'_2 | S(P_1, P_2) | f_1, f_2 \rangle \\ &= \langle f'_1 \bar{f}_2 | S(P_1, -P_2) | f_1 \bar{f}'_2 \rangle \quad . \end{aligned} \quad (9)$$

Crossing symmetry leads to useful constraints on the scattering amplitudes and will be used frequently in order to fix the S-matrices.

We can summarize the whole program of computing exact S-matrices in the following steps [40, 43–46]:

1. Set up the factorization equations, either from the local conservation laws, such as in (5), or using the nonlocal conservation laws.
2. Impose crossing and unitarity.
3. Compute the minimal S-matrix, that is, the one obeying analyticity, having the minimum number of zeros or poles in the physical sheet, and growing asymptotically slower than  $\exp|\frac{p_1 p_2}{m^2}|$  for  $p_1, p_2 \rightarrow \infty$ .
4. Using qualitative information about the bound-state structure, introduce poles; resonances are supposed to be absent, since unstable particles do not exist for a factorizable S-matrix, due to the conservation of the number of particles.
5. Check the results by perturbation theory, or any other method available, as, e.g., semiclassical approximation, or  $1/N$  expansion.

As an example of factorizable S-matrix, we find those with symmetry groups  $U(N)$ .

Such a symmetry requirement implies that the particle-particle and particle-antiparticle scattering amplitudes are of the form

$$\begin{aligned} & \langle P_\gamma(\theta'_1) P_\delta(\theta'_2) out | P_\alpha(\theta_1) P_\beta(\theta_2) in \rangle \\ &= [u_1(\theta) \delta_{\alpha\gamma} \delta_{\beta\delta} + u_2(\theta) \delta_{\alpha\beta} \delta_{\gamma\delta} \delta(\theta_1 - \theta'_2) \delta(\theta_2 - \theta'_1) \\ & \pm [u_1(\theta) \delta_{\alpha\delta} \delta_{\beta\gamma} + u_2(\theta) \delta_{\alpha\beta} \delta_{\gamma\delta} \delta(\theta_1 - \theta'_2) \delta(\theta_2 - \theta'_1) \\ & \langle P_\gamma(\theta'_1) A_\delta(\theta'_2) out | P_\alpha(\theta_1) A_\beta(\theta_2) in \rangle \\ &= [t_1(\theta) \delta_{\alpha\gamma} \delta_{\beta\delta} + t_2(\theta) \delta_{\alpha\beta} \delta_{\gamma\delta}] \delta(\theta_1 - \theta'_1) \delta(\theta_2 - \theta'_2) \\ & \pm [r_1(\theta) \delta_{\alpha\gamma} \delta_{\beta\delta} + r_2(\theta) \delta_{\alpha\beta} \delta_{\gamma\delta}] \delta(\theta_1 - \theta'_2) \delta(\theta_2 - \theta'_1) \end{aligned}$$

where the  $t$ 's are transmission amplitudes and the  $r$ 's are the reflexion amplitudes.

Implementing the factorization equations, we find that the solutions fall into six classes as given below. The function  $f(\theta, \lambda)$  is a meromorphic function of  $\theta$ , for

$Re\lambda > 0$ ; it is uniquely defined by the requirement of being minimal. The only arbitrariness lies in the bound-state structure.

- Class I

$$\begin{aligned} r_1(\theta) &= 0, & t_1(\theta) &= 1, & u_1(\theta) &= 1, \\ r_2(\theta) &= 0, & t_2(\theta) &= 0, & u_2(\theta) &= 0. \end{aligned}$$

- Class II

$$\begin{aligned} r_1(\theta) &= 0, & t_1(\theta) &= f(\theta, \lambda), & u_1(\theta) &= t_1(i\pi - \theta), \\ r_2(\theta) &= 0, & t_2(\theta) &= \frac{i\pi\lambda}{\theta - i\pi} t_1(\theta), \\ u_2(\theta) &= -\frac{i\pi\lambda}{\theta} u_1(\theta). \end{aligned}$$

- Class III

$$\begin{aligned} r_1(\theta) &= -\frac{i\pi\lambda}{\theta} t_1(\theta), & t_1(\theta) &= f(\theta, \lambda), \\ & f(i\pi - \theta, \lambda), & u_1(\theta) &= t_1(\theta), \\ r_2(\theta) &= \frac{i\pi\lambda}{\theta - i\pi} t_1(\theta), & t_2(\theta) &= r_2(\theta), & u_2(\theta) &= r_1(\theta). \end{aligned}$$

- Class IV

$$\begin{aligned} r_1(\theta) &= -\frac{i\pi\lambda}{\theta} t_1(\theta), \\ t_1(\theta) &= f(\theta, \lambda) f(i\pi - \theta, \lambda) i \tanh \frac{1}{2} \left( \theta + \frac{1}{2} i\pi \right), \\ u_1(\theta) &= -t_1(\theta), \\ r_2(\theta) &= \frac{i\pi\lambda}{\theta - i\pi} t_1(\theta), & t_2(\theta) &= r_2(\theta), & u_2(\theta) &= r_1(\theta). \end{aligned}$$

- Class V

$$\begin{aligned} r_1(\theta) &= \prod_{k=-\infty}^{\infty} \frac{f(\theta, k/2\mu i)}{f(\theta, k/2\mu i + \frac{1}{2})}, & t_1(\theta) &= 0, & u_1(\theta) &= 0, \\ r_2(\theta) &= \frac{\sin \mu(i\pi - \theta)}{\sin \mu\theta} r_1(\theta), \\ t_2(\theta) &= r_2(\theta), & u_2(\theta) &= r_1(\theta). \end{aligned}$$

- Class VI

$$\begin{aligned} r_1(\theta) &= \prod_{k=-\infty}^{\infty} \frac{f(\theta, k/2\mu i)}{f(\theta, k/2\mu i + \frac{1}{2})}, & t_1(\theta) &= 0, & u_1(\theta) &= 0, \\ r_2(\theta) &= \frac{\sin \mu(i\pi - \theta)}{\sin \mu\theta} r_1(\theta), & t_2(\theta) &= e^{i\mu(i\pi - \theta)} r_2(\theta), \\ u_2(\theta) &= e^{i\mu\theta} r_1(\theta). \end{aligned}$$

From these classes, we see that for a  $U(N)$  symmetry the solution of the factorization equations is not unique. In the case of  $CP^{N-1}$  and chiral models, the solution will be found to be of class II; to obtain it, we shall use

the non-local conservation laws. The solutions belonging to class III correspond to an  $O(N)$  symmetry.

The chiral fermion field in two dimensions is a  $SU(N)(1) \otimes \tilde{U}(1)$  multiplet of fermions. The Lagrangian is defined by

$$\mathcal{L} = i\bar{\psi}_i \not{\partial} \psi_i + \frac{1}{2}g[(\bar{\psi}_i \psi_i)^2 - (\bar{\psi}_i \gamma_5 \psi_i)^2], \quad (10)$$

where summation over the  $SU(N)$  index  $i$  is understood. The Lagrangian (10) again defines an integrable model. The Noether current associated with the  $U(N)$  symmetry is given by

$$j_{ij}^\mu = i\bar{\psi}_j \gamma^\mu \psi_i, \quad (11)$$

Use of the equation of motion and Fierz transformation shows that it satisfies

$$\partial_\mu j_\nu - \partial_\nu j_\mu + \frac{1}{2}g[j_\mu, j_\nu] = 0. \quad (12)$$

This relation shows the integrability of the model and implies the existence of a non-local conserved charge of the usual form.

A possible candidate to exact S-matrix describing the scattering of elementary fermions is that of class II in the table. A strong indication of this fact should come with the  $1/N$  expansion. However, there is a massless field in the theory if we try to obtain perturbation naively. In two-dimensional space-time, this can lead to infrared divergencies very difficult to deal with. The solution of such a problem was given by two independent papers. We quickly review them here.

*Cancellation of Infrared Singularities* In order to obtain the  $1/N$  expansion of this model, we have to reformulate it. The theory can be reduced to a quadratic form in  $\psi$  at the expense of two auxiliary fields,

$$\mathcal{L} = i\bar{\psi} \not{\partial} \psi - \frac{1}{2g}(\sigma^2 + \pi^2) + \bar{\psi}(\sigma + i\pi \gamma_5)\psi. \quad (13)$$

However, the  $1/N$  expansion of the model using the above Lagrangian cannot be performed, due to serious infrared (IR) problems [47]: We find a massless pole in the  $\pi$  propagator. It plays the role of the problematic massless Goldstone boson [1]. We now rewrite the fields in terms of  $\sigma + i\pi = \rho e^{i\phi}$ , leading to the Lagrangian

$$\mathcal{L} = i\bar{\psi} \not{\partial} \psi - \frac{1}{2g}\rho^2 + \rho\bar{\psi} e^{i\phi} \gamma_5 \psi. \quad (14)$$

We now discuss the quantum theory associated with the above classical Lagrangian. The most pedestrian approach consists in the extensive use of the bosonization formulae. The fermionic fields are bosonized in terms of an  $N$ -plet  $\varphi_i$ . The situation is analogous to the

massive Thirring model and one obtains the equivalent bosonic Lagrangian

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^N (\partial_\mu \varphi_i)^2 - \frac{1}{2g}\rho^2 + \frac{\mu}{2\pi}\rho \sum_{i=1}^N \cos(\phi + \varphi_i \sqrt{4\pi}). \quad (15)$$

It is now convenient to use “fermionization” formulae in order to rewrite (15) in terms of new fermion fields  $\tilde{\psi}_i$  [48, 49], by making the identifications

$$\begin{aligned} i\bar{\tilde{\psi}}_i \not{\partial} \tilde{\psi}_i &= \frac{1}{2} \left[ \partial_\mu \left( \varphi_i + \frac{\phi}{\sqrt{4\pi}} \right) \right]^2, \\ \bar{\tilde{\psi}}_i \tilde{\psi}_i &= \frac{\mu}{2\pi} \cos(\phi + \varphi_i \sqrt{4\pi}), \\ \bar{\tilde{\psi}}_i \gamma_\mu \tilde{\psi}_i &= -\frac{1}{\sqrt{\pi}} \epsilon_{\mu\nu} \partial^\nu \left( \varphi_i + \frac{\phi}{\sqrt{4\pi}} \right). \end{aligned} \quad (16)$$

The Lagrangian (15) then takes the form [49]

$$\begin{aligned} \mathcal{L} = \bar{\tilde{\psi}}_i \not{\partial} \tilde{\psi}_i - \frac{1}{2g}\rho^2 + \rho \bar{\tilde{\psi}}_i \tilde{\psi}_i + \frac{1}{2} \partial_\mu \phi \bar{\tilde{\psi}}_i \gamma^\mu \gamma_5 \tilde{\psi}_i \\ + \frac{N}{8\pi} (\partial_\mu \phi)^2. \end{aligned} \quad (17)$$

The important point is that the massless field  $\phi$  interacts only via its derivative, thus implying the absence of infrared problems in the correlation functions of  $\tilde{\psi}$ . We can obtain the same Lagrangian (17) by computing the fermionic determinant associated with the Lagrangian (14) (see [1]).

## 2.2 The $\frac{1}{N}$ Expansion

The large  $N$  expansion of the model defined by the Lagrangian (17) can be explicitly performed [49]. The propagator of the  $\rho$  field is exactly the same as that obtained for the  $\tilde{\sigma}$  field in the  $O(N)$  case. The zeroth order contribution to the  $\rho$ -propagator is thus given by

$$\tilde{\Gamma}(p) = -\frac{i}{2\pi} \frac{\theta}{\tanh \frac{\theta}{2}}, \quad (18)$$

where  $\theta$  is defined by  $p^2 = -4m^2 \sinh^2 \frac{\theta}{2}$ .

Since only  $\partial_\mu \phi$  occurs in (17), we just need the two point function of  $A_\mu = \sqrt{N} \epsilon_{\mu\nu} \partial^\nu \phi$ , given by

$$\tilde{\Gamma}_{\mu\nu}(p) = \frac{1}{2\pi} \theta \tanh \frac{\theta}{2} (g_{\mu\nu} p^2 - p_\mu p_\nu), \quad (19)$$

where  $p^2 = -4m^2 \sinh^2 \frac{\theta}{2}$ .

The amplitudes for particle scattering are all free from IR divergencies and may be computed without difficulty. We can compute the two-particle scattering

amplitude in the lowest order [1]. The lowest-order contributions to  $u_1(\theta)$  lead to

$$u_1(\theta) = 1 + \frac{i\pi}{N} \coth\left(\frac{\theta}{2}\right). \quad (20)$$

Moreover, the backward fermion antifermion scattering vanishes, confirming the S-matrix belonging to the class II defined before.

**Operator Formulation** This model may also be studied in the operator formalism, which leads to the  $1/N$  expansion, and a correct understanding of the relation between the “candidate” Goldstone boson and chiral symmetry.

Since the fields  $\psi_i$  lie in the fundamental representation of  $U(N)$ , we have the bosonic representation [50, 51]

$$\begin{aligned} \psi_i(x) = K_i \left(\frac{\mu}{2\pi}\right)^{\frac{1}{2}} e^{-i\frac{\pi}{4}\gamma_5} :e^{i\sqrt{\frac{\pi}{N}}[\gamma_5\chi(x) + \int_{x^1}^{\infty} dy^1 \dot{\chi}_i(x^0, y^1)]}: \\ :e^{-i\sqrt{\pi}[\gamma_5\chi_i(x) + \int_{x^1}^{\infty} dy^1 \dot{\chi}_i(x^0, y^1)]}: \end{aligned} \quad (21)$$

with  $i = 1, \dots, N$ . Since the  $\chi_i$ 's are  $SU(N)$  valued, they are not independent but satisfy

$$\sum_{i=1}^N \chi_i(x) = 0. \quad (22)$$

The field  $\chi$  is the potential of the conserved  $U(1)$  current. Its zero-mass character will ensure that the  $U(1)$  symmetry is not spontaneously broken.

In the above,  $K_i$  is a Klein factor, necessary to enforce the correct anticommutation relations among different  $\psi_i$ 's. Due to the  $U(1) \times \tilde{U}(1)$  symmetry, the divergence and the curl of the  $U(1)$  current vanish, so that the field  $\chi(x)$  is massless. Therefore, the fermion fields contain the so-called infraparticles [52], and we need to extract them in order to arrive at the physical fields of the theory. They are given by

$$\hat{\psi}_i(x) = K_i \sqrt{\frac{\mu}{2\pi}} e^{i\sqrt{\pi}\{\gamma_5\chi_i(x) + \int_{x^1}^{\infty} dy^1 \dot{\chi}_i(x^0, y^1)\}}. \quad (23)$$

The  $\psi$  fields (23) will be found to correspond to the field  $\tilde{\psi}_i$  in (17). These fields no longer carry  $U(1) \times \tilde{U}(1)$  charge and transform as a representation of  $SU(N)$ . The constraint (22) implies

$$\hat{\psi}_i^\dagger \sim \frac{1}{(n-1)!} \epsilon_{i_1 \dots i_{n-1}} \hat{\psi}_{i_1} \dots \hat{\psi}_{i_{n-1}}, \quad (24)$$

where on the right-hand side a suitable redefinition of the Klein factor and the normal product prescription

is required. Equation 24 states that the antifermions of the chiral Gross–Neveu model can be viewed as a bound state of  $N-1$  fermions. We use this fact to determine the S-matrix and its pole structure.

Asymptotically, one expects  $\hat{\psi}$  to describe massive particles, so that one should have [50, 51]

$$\hat{\psi}(vt, t) \rightarrow \frac{1}{\sqrt{|t|}} \{e^{-im\gamma^{-1}t} \hat{a}(m\gamma v) + e^{im\gamma^{-1}t} \hat{b}^\dagger(m\gamma v)\}, \quad (25)$$

$$\text{where } \gamma = \frac{1}{\sqrt{1-v^2}}.$$

The fields  $\hat{\psi}_i$  carry spin  $s = \frac{1}{2}(1-1/N)$ ,

$$\hat{\psi}(x, t) \hat{\psi}(y, t) = e^{2\pi i s \epsilon(x-y)} \hat{\psi}(y, t) \hat{\psi}(x, t), \quad (26)$$

implying an unusual *statistics* for the creation and annihilation operators defined in (25)

$$\hat{a}^\dagger(p) \hat{a}^\dagger(p') = e^{2\pi i s \epsilon(p-p')} \hat{a}^\dagger(p') \hat{a}^\dagger(p). \quad (27)$$

Since no scattering theory is known for particles with the above statistics, it is necessary to replace the field  $\hat{\psi}$  by another field  $\psi'$  with a well-defined statistics. This is achieved by introducing in (21) free massless scalar and pseudoscalar fields  $B$  and  $A$ , quantized with metric opposite to that of  $\chi(x)$ , in such a way that the divergent infrared behavior of  $\psi$  induced by  $\chi(x)$  is compensated, without affecting the statistics. We define [50, 51]

$$\psi'_i(x) = e^{i\sqrt{\frac{\pi}{N}}[\gamma^5 A(x) + B(x)]} \psi_i(x). \quad (28)$$

Correspondingly, the operators  $a^\dagger, a, b^\dagger, b$  are related to  $\hat{a}^\dagger, \hat{a}, \hat{b}^\dagger, \hat{b}$  by

$$\begin{aligned} \hat{a}_{\text{in}}^\dagger(p) &= a_{\text{in}}^\dagger(p) e^{2\pi i(s-\frac{1}{2}) \int_p^\infty N_{\text{in}}(p') dp'}, \\ \hat{a}_{\text{out}}^\dagger(p) &= a_{\text{out}}^\dagger(p) e^{2\pi i(s-\frac{1}{2}) \int_\infty^p N_{\text{out}}(p') dp'}. \end{aligned} \quad (29)$$

where  $N_{\text{inout}}$  are the corresponding particle number operators.

Since we expect  $\psi'_i(x)$  to be a local field describing massive degrees of freedom, we should have in the far past and future [53]

$$\psi'(vt, t) \rightarrow \frac{1}{\sqrt{|t|}} \{e^{-im\gamma^{-1}t} a_{\text{outin}}(m\gamma v) + e^{im\gamma^{-1}t} b_{\text{outin}}^\dagger(m\gamma v)\}. \quad (30)$$

Substitution of  $\psi$  in terms of  $\psi'$  in (10) leads formally to the Lagrangian

$$\begin{aligned} \mathcal{L} = & i\bar{\psi}'_i \not{\partial} \psi'_i + \frac{1}{2}g[(\bar{\psi}'_i \psi'_i)^2 - (\bar{\psi}'_i \gamma_5 \psi'_i)^2] - \frac{1}{2}(\partial_\mu A)^2 \\ & - \frac{1}{2}(\partial_\mu B)^2 + \frac{\alpha}{\sqrt{N}}\bar{\psi}' \gamma^5 \gamma^\mu \psi' \partial_\mu A \\ & - \frac{\beta}{\sqrt{N}}\bar{\psi}' \gamma^\mu \psi' \partial_\mu B, \end{aligned} \quad (31)$$

where we allowed for general couplings  $\alpha$  and  $\beta$ , which after renormalization should reduce to  $\sqrt{\pi}$  as the renormalized value. We will come back to this point after obtaining the  $1/N$  expansion, which we consider next.

The effective action obtained from the Lagrangian (31) after introduction of the auxiliary fields  $\sigma$  and  $\pi$  (compare with (13)) is given by

$$\begin{aligned} S_{\text{eff}} = & -iN \text{tr} \ln \left\{ i \not{\partial} + \sigma + i\pi \gamma_5 + \frac{\alpha}{\sqrt{N}}\gamma^5 \not{\partial} A - \frac{\beta}{\sqrt{\pi}} \not{\partial} B \right\} \\ & - \frac{1}{2g} \int d^2x (\sigma^2 + \pi^2) - \frac{1}{2} \int d^2x [(\partial_\mu A)^2 + (\partial_\mu B)^2]. \end{aligned} \quad (32)$$

The field  $\sigma$  is found to have a non-vanishing vacuum expectation value  $\langle \sigma \rangle = -m$ , so that it is convenient to write

$$\sigma = -m + \frac{\tilde{\sigma}}{\sqrt{N}}, \quad \text{and} \quad \pi = \frac{\tilde{\pi}}{\sqrt{N}}. \quad (33)$$

The second-order contribution to the effective action can be computed, and the  $1/N$  expansion turns out to be well-defined.

We fix the parameters  $\alpha$  and  $\beta$  in (31) by requiring that the IR divergencies cancel. We expect to obtain for the non-renormalized values,  $\alpha = \infty$  (corresponding to  $\alpha_{\text{ren}} = \sqrt{\pi}$  and  $\beta = \beta_{\text{ren}} = \sqrt{\pi}$ , since  $B$  couples to a conserved current).

We have thus verified in the  $N \rightarrow \infty$  limit that both Lagrangians (17) and (31) lead to the same result. In the limit  $\alpha \rightarrow \infty$ , the renormalized coupling  $\alpha_{\text{ren}}$  indeed turns out to be  $\sqrt{\pi}$ , as one reads off from the four-point function, and the pole in the  $\pi$ -propagator vanishes. To summarize, we conclude that part of the field (21) which carries chirality decouples [48–51] from the physical spectrum and the remaining part describes an  $SU(N)$  multiplet with a well-defined factorizable S-matrix.

### 2.3 Quantization of Non-local Charge

The discussion of the existence and conservation of a non-local charge in the quantum chiral Gross–Neveu

model follows exactly the same pattern as in the  $O(N)$  invariant model. No anomaly exists in this case. It is not difficult to see that the action of the charges on asymptotic states is given in this case by

$$\begin{aligned} Q^{ab} |\theta_1 i; \theta_2 j\rangle &= |\theta_1 k; \theta_2 l\rangle \left[ -(I^{ac})^{ik} (I^{cb})^{jl} + \frac{N}{i\pi} \theta_1 (I^{ab})^{ik} \delta^{jl} \right. \\ &\quad \left. + \frac{N}{i\pi} \theta_2 (I^{ab})^{jl} \delta^{ik} \right], \\ \langle \theta_1 i; \theta_2 j | Q^{ab} &= \langle \theta_1 k; \theta_2 l | \left[ -(I^{ac})^{ki} (I^{cb})^{lj} + \frac{N}{i\pi} \theta_1 (I^{ab})^{ki} \delta^{lj} \right. \\ &\quad \left. + \frac{N}{i\pi} \theta_2 (I^{ab})^{lj} \delta^{ik} \right], \\ Q^{ab} |\theta_1 i; \bar{\theta}_2 j\rangle &= |\theta_1 k; \bar{\theta}_2 l\rangle \left[ -(I^{ac})^{ik} (I^{cb})^{jl} + \frac{N}{i\pi} \theta_1 (I^{ab})^{ik} \delta^{jl} \right. \\ &\quad \left. - \frac{N}{i\pi} \bar{\theta}_2 (I^{ab})^{lj} \delta^{ik} \right], \\ \langle \theta_1 i; \bar{\theta}_2 j | Q^{ab} &= \langle \theta_1 k; \bar{\theta}_2 l | \left[ -(I^{ac})^{ki} (I^{cb})^{lj} + \frac{N}{i\pi} \theta_1 (I^{ab})^{ki} \delta^{lj} \right. \\ &\quad \left. - \frac{N}{i\pi} \bar{\theta}_2 (I^{ab})^{jl} \delta^{ik} \right], \end{aligned}$$

where  $I^{ab}$  are the  $SU(N)$  generators [1]. Conservation of the charge leads to the factorization equations and to the exact S-matrix of the problem.

**First Conclusions and Physical Interpretation** The Gross–Neveu models are simple but physically rich models. The semi-classical analysis, both in the  $O(N)$  and in the  $SU(N) \times U(1) \times \tilde{U}(1)$ -symmetric cases, reveals that the models have a rich bound-state structure [54].

The chiral Gross–Neveu model is particularly interesting, due to the chiral symmetry breaking issue. Since, as we saw in the previous section, a mass term is dynamically generated for the fermion, one could be led to conclude that the chiral symmetry is broken, which is prohibited in two-dimensional space–time. This problem has been discussed at length by several authors [48, 49]. The interesting outcome is that the chirality carrying field decouples from the theory (28). In the operator language, this is realized by the factorization of the auxiliary fields  $A$  and  $B$ . The physical fermions, as given by either (23) or (28), though exhibiting a non-vanishing mass gap, are chiral singlets. This physical picture is carried over to the supersymmetric  $CP^{N-1}$  model and reflects the fact that antiparticles are bound states of particles in both the Gross–Neveu model and in the supersymmetric  $CP^{N-1}$  model. This permits the computation of the S-matrix for these two cases [55].

### 3 The Exact Solutions of Classes of Integrable Models and String Theories

Large  $N$  Yang–Mills theory has been frequently studied since the first seminal paper by 't Hooft [56]. Some time ago, it has been discovered that there is a large  $N$  limit in  $\mathcal{N} = 4$  supersymmetric Yang–Mills which corresponds to type IIB string theory. More recently, we learned from [57] how to get the spectrum from the gauge theory counterpart. The fact that the spectrum is related to the hamiltonian of an integrable model [58] is an outstanding achievement. The integrable model is obtained from the matrix describing the anomalous dimensions of certain classes of fields in super-Yang–Mills theory in the field theory counterpart.

The procedure is obtained from the renormalization group equation

$$\left\{ \mu \frac{\partial}{\partial \mu} + \gamma \right\} \Gamma = 0 \quad (34)$$

where  $\Gamma$  describes the correlator of the fields under study and  $\mu$  is a renormalization group parameter. As it turns out,  $\gamma$  describes a matrix-valued Hamiltonian whose indices describe the different fields in the correlator, and its diagonalization amounts to a solution of an integrable model.

Such a statement is a very nontrivial fact about some field theories relating them in a very remarkable fashion. Indeed, the existence of integrable structures in gauge theories, at classical as well as quantum level, in two- and four-dimensional space–time has been suspected long ago in different setups [31–35, 59–64] and a huge amount of more recent literature concerning integrable structures in string related theories have appeared [24, 57, 65].

Here we discuss boundaries in open spin chains with  $SO(6)$  symmetry and their corresponding interpretation in super-Yang–Mills theory with four supercharges. Furthermore, the spin chain with static boundary conditions has a more general parameter space, which may suggest a larger class of operators whose one-loop anomalous dimension matrix corresponds to an integrable spin chain. Here the most general  $SO(6)$  open spin chain Hamiltonian will be proposed using integrability requirements.

The anti-de Sitter/conformal field theory (AdS/CFT) conjecture relates two very different theories in two very different settings; this is why at first the conjecture seems so surprising and interesting. In one side of the conjecture, we have a quantum theory of gravity in an asymptotic AdS space and the other we have a

conformal quantum field theory in the boundary of the AdS space, which is the standard Minkowski space. The claim is that for every observable in one side of the conjecture, there is a corresponding observable in the other side of it. Gauge invariant single trace operators in the quantum field theory side corresponds to physical states in the quantum gravity side. And correlation functions (there is no S-matrix in a CFT) in the quantum field theory are calculated using quantum gravity states with appropriate boundary conditions.

The possible objects to compare in both sides are not limited to states and correlation functions. There is a very large amount of evidence for this conjecture, and we refer to [66] for the most important ones. The best known example is the case of type IIB string theory in  $AdS_5 \times S^5$  space which is dual to  $N = 4$  SYM theory in four dimensions [67]. This case is particularly interesting since it preserves all possible supersymmetries in ten dimensions and has the largest possible symmetry algebra in four dimensions.

The issue that prevents a better understanding of this conjecture is that the sigma models describing the dynamics of the string in such backgrounds is a complicated CFT. Although these sigma models appear to be integrable [68] (they have an infinite number of conserved charges), no one was able to use the integrable structure to make any non-trivial computation. There are many questions regarding this problem, for example, integrable field theories in  $d = 2$  usually have a mass gap, but in the case at hand, there is no S-matrix. At least in the first order of perturbation theory it was shown that there is no particle production, a property of integrable field theories. On the other hand, there has been much progress in the super-Yang–Mills side of the conjecture.

#### 3.1 $N = 4$ Supersymmetric Yang–Mills Theory

In four dimensions, there is only one field theory with 16 supercharges that do not contain gravity:  $N = 4$  supersymmetric Yang–Mills theory with coupling constant  $g$  and gauge group  $SU(N)$ . Other gauge groups are allowed, but will not be considered here. This theory is unique up to the choice of the gauge group and coupling constant. Its field content is the gauge field  $A^\mu$ , four fermions in the fundamental representation of  $SU(4)$  (the R-symmetry group)  $\psi_\alpha^A$ , where  $A$  is an  $SU(4)$  index and  $\alpha$  is a spinor index and there are six scalars in the antisymmetric representation of  $SU(4)$   $\phi^{AB}$ . The scalars can also be seen as vectors of  $SO(6)$  and the fermions as spinors of the same group. We can use the gamma matrices  $\gamma_{AB}^I$  to transform one representation into the other.

The Lagrangian of this theory ignoring terms with fermions is

$$S = \int d^4x \text{Tr} \left[ \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} D_\mu \phi^{AB} D^\mu \phi_{AB} + \frac{1}{4} g^2 [\phi^I, \phi^J] [\phi_I, \phi_J] + \dots \right] \quad (35)$$

The underlying symmetry group of this theory is very large. The classical conformal invariance is not broken in the quantum theory. The conformal transformations together with the super-Poincaré group form the algebra  $\text{PSU}(2, 2|4)$ , with 30 bosonic (including the R-symmetry generators) and 32 fermionic generators. Among all this symmetries, a especial one is the scale symmetry, generated by the dilatation operator  $\mathcal{D}$ , to be defined later on.

### 3.2 Single Trace Operators

One class of interesting observables in this theory are the gauge invariant single trace operators. The most obvious example is

$$\mathcal{O}_F = \text{Tr}(F^{\mu\nu} F_{\mu\nu}). \quad (36)$$

In the AdS/CFT correspondence, this operator couples to the dilatation. Therefore, it corresponds to a change in the coupling constant. This is an example of a chiral operator as well, since it is annihilated by half of the supercharges (half of the supersymmetry generators and half of the superconformal transformations). This operator is a descendant of

$$\mathcal{O}_\phi = \text{Tr}(\phi^{IJ} \phi^{IJ}), \quad (37)$$

where  $\{IJ\}$  means symmetric traceless combination. This means it can be obtained from the above expression by means of the action of some supercharges. All chiral single trace operators with only two fields can be obtained from the one above from the action of the supercharges. More generally, *all* chiral operators are obtained from

$$\mathcal{O}_n = \text{Tr}(\phi^{I_1} \phi^{I_2} \dots \phi^{I_{n-1}} \phi^{I_n}). \quad (38)$$

In summary, all chiral operators in  $N = 4$  super-Yang–Mills are related to massless states in the corresponding string theory. The spectrum of these operators is easy to compute, since they are protected by quantum corrections. The dimensions are the classical ones, which can be easily computed from the classical action. A much more difficult problem, which has not yet been completely solved, is the computation of the dimension of *any* gauge invariant single trace operator. The most important progress on this problem is the conjecture

that the dimension of any gauge invariant operator is an eigenvalue of the Hamiltonian of some integrable spin chain. The simplest example of a nonchiral operator is the Konishi operator

$$\mathcal{O}_K = \text{Tr}(\phi^I \phi^I). \quad (39)$$

Its one-loop anomalous dimension can be computed using standard methods and does not vanish.

### 3.3 Dilatation Operator and Spin Chain Hamiltonian

In field theory, the dilatation operator  $\mathcal{D}$  gives the conformal dimension (classical plus anomalous dimension) upon commutation, by means of the expression

$$[\mathcal{D}, \mathcal{O}] = \Delta_{\mathcal{O}} \mathcal{O}, \quad (40)$$

whenever we have a diagonal base,  $\Delta$  being the conformal dimension. The more general situation is

$$[\mathcal{D}, \mathcal{O}_i] = \Delta_{ij} \mathcal{O}_j, \quad (41)$$

where  $\Delta_{ij}$  is the matrix of anomalous dimensions.

In a CFT, this knowledge allows one to compute any two point function, since the latter is fixed, in the simple case of scalar operators, to be

$$\langle \mathcal{O}_i(x) \mathcal{O}_j(y) \rangle = \frac{\delta_{ij}}{|x - y|^{2\Delta_i}}. \quad (42)$$

Thus, knowing the conformal dimensions is a small step toward a solution of the full quantum field theory. Three-point functions can also be obtained, but more knowledge is necessary.

### 3.4 The $\text{SO}(6)$ Spin Chain

The problem of studding the full  $\text{PSU}(2, 2|4)$  spin chain is too broad for our proposes here. Thus, we shall review the results of the one-loop anomalous dimension and the spin chain for the  $\text{SO}(6)$  sector, which is closed at one loop. We refer to [69].

Our interest lies in operators of the form

$$\mathcal{O}_n = \sigma_{I_1 I_2 \dots I_n} \text{Tr}(\phi^{I_1} \phi^{I_2} \dots \phi^{I_n}), \quad (43)$$

where  $\sigma_{I_1 I_2 \dots I_n}$  are constant polarizations. At one-loop level, these operators do not mix with other types, and we can use only the first line of (35) to perform computations. Thus, supersymmetry is not directly responsible for integrability at least at one-loop level. Note that we are not imposing any condition on the above operator.

Although  $N = 4$  super-Yang–Mills is a finite theory, some renormalization has to be done. We only need a

wave function renormalization, what is responsible for the change of the classical dimension. We define

$$\Gamma_{\mathcal{O}} = \Lambda \frac{\partial Z_{\mathcal{O}}}{\partial \Lambda}, \quad (44)$$

in the simplest case. The task of computing  $\Gamma_{\mathcal{O}}$  using (35) for the operators (43) at one-loop level has been explained in [69]. The matrix of anomalous dimensions is given by

$$\Gamma_{\mathcal{O}} = \lambda \sum_{l=1}^L (K_{l,l+1} - 2P_{l,l+1} + 2), \quad (45)$$

where  $K_{l,l+1}$  is the trace operator and  $P_{l,l+1}$  is the permutation operator. This matrix was identified with the integrable Hamiltonian of an SO(6) spin chain.

Using the Bethe ansatz [70] (see also [71]) to find the eigenvalues of this Hamiltonian, one finds

$$\gamma_{\mathcal{O}} = \lambda \sum_{i=1}^n \frac{1}{x_i + 1/4}, \quad (46)$$

where  $n$  is the number of particle-like excitations and  $x_i$  are the rapidity parameters. We shall see that the addition of boundaries does not change these eigenvalues, although it will put restriction on possible operators and will change the Bethe equations.

### 3.5 Solutions with Boundaries

We now discuss how boundaries may appear in the spin chain and in the gauge invariant operators. The single trace operators in the Yang–Mills theory are dual to closed string states. Open strings will appear in the conjecture if there are D-branes in the theory. The D-brane states, or giant gravitons, are represented by determinant operators

$$\mathcal{O}_{GG} = \det(Z), \quad (47)$$

where  $Z = \phi_1 + i\phi_6$  is the highest weight state in SO(6) representation and the determinant is in the adjoint representation of  $SU(N)$ . We shall attach an open string to such state. We remove one  $Z$  in the determinant above and replace it with a string of operators. To be more explicit, the determinant is of the form

$$\mathcal{O}_{GG} = \epsilon^{j_1 \dots j_N} \epsilon_{i_1 \dots i_N} Z_{j_1}^{i_1} \dots Z^{i_N \dots j_N}, \quad (48)$$

and we attach the “open string”  $(\psi_1 \dots \psi_L)_{j_1}^i$  to the giant graviton as

$$\mathcal{O}_o = \epsilon^{j_1 \dots j_N} \epsilon_{i_1 \dots i_N} Z_{j_1}^{i_1} \dots Z^{i_{N-1} \dots j_{N-1}} (\psi_1 \dots \psi_L)_{j_N}^{i_N}, \quad (49)$$

where  $\psi_a$  is one of the other scalar fields. Berenstein and Vázquez have shown that the anomalous dimension matrix for operators of this type corresponds to

the Hamiltonian of an open spin chain with static boundary conditions. They analyzed the behavior of wave functions of this Hamiltonian, and it was shown that the boundary conditions for elementary excitations satisfy Dirichlet boundary conditions. In this section, the most general SO(6)-invariant spin chain with open static boundary conditions will be derived.

We start with some definitions. The SO(6) invariant rational  $R$  matrix is given by [70]

$$R(\theta) = \left(1 - \frac{\theta}{2}\right) I + \theta \left(\frac{\theta}{2} - 1\right) P + \frac{\theta}{2} K, \quad (50)$$

which satisfy the permuted Yang–Baxter equation

$$R_{12}(\theta) R_{23}(\theta + \theta') R_{12}(\theta') = R_{23}(\theta') R_{12}(\theta + \theta') R_{23}(\theta). \quad (51)$$

These operators are explicitly represented by

$$I = \sum_{i,j=1}^6 \hat{e}_{ii} \otimes \hat{e}_{jj}, \quad P = \sum_{i,j=1}^6 \hat{e}_{ij} \otimes \hat{e}_{ji}, \quad K = \sum_{i,j=1}^6 \hat{e}_{i'j} \otimes \hat{e}_{i'j}, \quad (52)$$

where  $i' = 7 - i$  and  $(\hat{e}_{ij})_{\alpha\beta} = \delta_{i\alpha} \delta_{j\beta}$  are standard  $6 \times 6$  Weyl matrices.

Using the S-matrix language, we can define  $S(\theta) = PR(\theta)$ , in order to recover the Yang–Baxter equation (5) from the R-matrix equation (51).

Following Sklyanin [72], it turns out that an integrable SO(6) open spin chain can be obtained from the double-row transfer matrix defined as the following trace over the  $6 \times 6$  auxiliary space  $\mathcal{A}$ :

$$T(\theta) = \text{tr} (K_{\mathcal{A}}^+(\theta) \hat{R}_{\mathcal{A}L}(\theta) \dots \hat{R}_{\mathcal{A}1}(\theta) K_{\mathcal{A}}^-(\theta) \hat{R}_{\mathcal{A}1}(\theta) \dots \hat{R}_{\mathcal{A}L}(\theta)). \quad (53)$$

While the operator  $R_{\mathcal{A}j}(u)$  determines the dynamics of the bulk, the  $6 \times 6$  matrices  $K_{\mathcal{A}}^{\pm}(u)$  describe the interactions at the ends of the open chain. Moreover, compatibility with the bulk integrability demands these matrices to satisfy the reflection equation, which for  $K_{\mathcal{A}}^-(u)$  reads

$$\begin{aligned} R_{12}(\theta - \mu) K_1^-(\theta) R_{12}(\theta + \mu) K_1^-(\mu) \\ = K_1^-(\mu) R_{12}(\theta + \mu) K_1^-(\theta) R_{12}(\theta - \mu). \end{aligned} \quad (54)$$

while a dual equation should also hold for the matrix  $K^+(u)$ . Here  $K_1^-(u) = K^-(u) \otimes I$  and  $K_2^-(u) = I \otimes K^-(u)$ .

The solutions of the reflection equation (54) for the SO(6)  $R$ -matrix (50) were derived in [73]. Here we will consider only the particular solution,

$$K^-(\theta) = \text{diag}(k_{11}^-(\theta), \dots, k_{66}^-(\theta)) \quad (55)$$

where

$$k_{11}^-(\theta) = 1$$

$$\begin{aligned} k_{22}^-(\theta) &= \dots = k_{55}^-(\theta) = -\frac{p_- \theta - 1}{p_- \theta + 1} \\ k_{66}^-(\theta) &= \frac{p_- \theta - 1}{p_- \theta + 1} \frac{p_-(\theta + 1) - 1}{p_-(\theta - 1) + 1} \end{aligned} \quad (56)$$

where  $p_-$  is a free parameter.

The diagonal matrix  $K^+(\theta)$  is obtained from crossing symmetry  $\theta \rightarrow -\theta + 2$ . It turns out that the matrix elements of  $K^+(\theta)$  are given by

$$k_{11}^+(\theta) = 1 \quad (57)$$

$$k_{22}^+(\theta) = \dots = k_{55}^+(\theta) = -\frac{p_+(-\theta + 2) - 1}{p_+(-\theta + 2) + 1} \quad (58)$$

$$k_{66}^+(\theta) = \frac{p_+(-\theta + 2) - 1}{p_+(-\theta + 2) + 1} \frac{p_+(-\theta + 3) - 1}{p_+(-\theta - 1) + 1} \quad (59)$$

in (56) and  $p_+$  is a second free parameter.

Associated to the double-row transfer matrix (53), we find the following open spin chain Hamiltonian which is proportional to the first-order expansion of  $T(\theta)$  in the spectral parameter [72].

$$\begin{aligned} \mathcal{H} = & -\sum_{i=1}^{L-1} P_{i,i+1} + \frac{1}{2} \sum_{i=1}^{L-1} E_{i,i+1} + \frac{1}{2} \frac{d(K^-(\theta))}{d\theta} \Big|_{\theta=0} \\ & + \frac{\text{tr}(K^+(0) H_{L,0})}{\text{tr}(K^+(0))} \end{aligned} \quad (60)$$

$$\text{where } H_{i,i+1} = -P_{i,i+1} + \frac{1}{2} E_{i,i+1}.$$

In order to obtain the spectrum of (60) in a non-perturbative way, we proceed with the exact diagonalization of the double-row operator (53). Since the  $K$ -matrices considered here are diagonal, this problem can be tackled by means of the boundary algebraic Bethe ansatz in the lines of [74].

## 4 Conclusions

There is a vast literature about the relation between four-dimensional gauge theories and two-dimensional integrable models. First arose the relation between Yang–Mills theory and two-dimensional sigma models; later, a few papers appeared implying an important relation of four-dimensional QCD at high energies and

spin systems in two dimensions, and a third group, more recently, about the relation of large number of colors QCD, string theories, and integrable models, which is the basic concern of the present paper [59–64]. While the first group of relations points into general coincidences and parallels between the two classes of models, the latter two classes of relations are definite identifications of four-dimensional physical operators and correlators with their two-dimensional counterparts.

In the second case above, the large  $N$  scattering in four-dimensional QCD at high energies in the leading logarithm approximation is described by a nearest neighbor Hamiltonian equivalent to that of the Heisenberg spin chain. Such properties have been discovered in the framework of a Feynman diagrammatic expansion [75]. Later, it has been argued that (3+1)-dimensional coordinates can be split into fast (with large Fourier transform) and slow variables, and Lorentz contraction in the direction of the motion of the fast particles rendered the corresponding field strength to the form of a shock wave nonvanishing only in the direction of a hyperplane passing through the trajectory of the particle.

Here the problem is even more sophisticated, relying on further properties of the string/field theory duality. The field theory correlators of some operators have anomalous dimension matrices corresponding to integrable model Hamiltonians. The latter have not only familiar structures but also display further interesting properties concerning deformation and especially perturbations by boundary operators. Such boundary operators can be understood, in the string theory counterpart, as perturbing branes. In our problem, these are actually zero branes, namely point particle operators which do not break the original symmetries of the problem.

We also presented the most general SO(6) spin chain with open static boundary conditions. We expected that this type of spin chain can be associated with the one-loop anomalous dimension matrix of giant graviton operators in SYM theory [76]. The Hamiltonian found in the present paper is more general than the one found previously in the literature in the sense that it has more general boundary conditions. It would be interesting to have an interpretation of these boundary conditions in terms of giant graviton and D-branes in the AdS/CFT duality.

We have established a perspective relating work performed in the 1970s and 1980s to modern developments in string theory. The fact that today several pieces of information from the dynamical knowledge of two-dimensional field theory is used in the search of structure in string and superstring theories as well

as super-Yang-Mills model shows that the models discussed in this paper are not only relevant from the point of view of a theoretical laboratory but as standard tools in the search for realistic field theories. That is the case of integrable models in the structure of Yang-Mills fields as well as string theory. The Bethe ansatz solutions are used to obtain the structure of anomalous dimensions, and further, integrable structures in two dimensional can also be used in order to achieve knowledge about the structure of analogous structures in the very important  $AdS \otimes S_5$  space in string theory. We are also sure that much of the dynamical structure of two-dimensional models, such as that discussed in the framework of the chiral fermion model, has not been fully used as an interesting full-fledged dynamical model.

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## References

1. E. Abdalla, M.C.B. Abdalla, K.D. Rothe, *Non-perturbative Methods in Two-Dimensional Quantum Field Theory*. (World Scientific, Singapore, 1991)
2. W. Thirring, Ann. Phys. **3**, 91 (1958)
3. B. Klaiber, in *Lectures in Theoretical Physics*, Boulder 1967. (Gordon and Breach, New York, 1968)
4. J. Schwinger, Phys. Rev. **128**, 2425 (1962)
5. J. Schwinger, Phys. Rev. Lett. **3**, 296 (1959)
6. J. Lowenstein, J.A. Swieca, Ann. Phys. **68**, 172 (1971)
7. A. Casher, J. Kogut, L. Susskind, Phys. Rev. Lett. **31**, 792 (1973)
8. A. Casher, J. Kogut, L. Susskind, Phys. Rev. **D10**, 732 (1974)
9. H.J. Rothe, K.D. Rothe, J. A. Swieca, Phys. Rev. **D15**, 1675 (1977)
10. E. Abdalla, R. Mohayaee, A. Zadra, Int. J. Mod. Phys. **A12** (1997) 4539
11. E. Abdalla, R. Banerjee, Phys. Rev. Lett. **80** (1998) 238.
12. A.B. Zamolodchikov, Al. B. Zamolodchikov, Ann. Phys. **120**, 253 (1979)
13. M. Lüscher, Nuclear Phys. **B135**, 1 (1978)
14. M. Karowski, Phys. Rep. **C49**, 229 (1979)
15. E. Abdalla, Lec. Notes Phys. **226**, 140 (1984)
16. N. Sanchez, H.J. de Vega (ed.), H.J. de Vega, Phys. Lett. **B87**, 233 (1979)
17. M.K. Prasad, A. Sinha, L.L. Chau Wang, Phys. Rev. Lett. **43**, 750 (1979)
18. M.K. Prasad, A. Sinha, L.L. Chau Wang, Phys. Lett. **B87**, 237 (1979)
19. E. Witten, Phys. Lett. **77B**, 394 (1978)
20. E. Witten, Nucl. Phys. **B266**, 245 (1986)
21. E. Abdalla, M. Forger, M. Jacques, Nucl. Phys. **B307**, 198 (1988)
22. J. Harnad, J. Hurtubise, M. Légaré, S. Shnider, Nucl. Phys. **B256**, (1985) 609
23. L.D. Faddeev, G.P. Korchemsky, Phys. Lett. **B 342**, 311 (1995)
24. E. Abdalla, M.C.B. Abdalla, Phys. Rep. **265**, 253 (1996)
25. J.A. Swieca, Fortschr. Phys. **25**, 303 (1977)
26. H. Eichenherr, Nucl. Phys. **B146**, 215 (1978)
27. H. Eichenherr, Nucl. Phys. E **B155**, 544 (1979)
28. H. Eichenherr, M. Forger, Nucl. Phys. **B155**, 381 (1979)
29. A. D'Adda, P. di Vecchia, M. Lüscher, Nucl. Phys. **B146**, 63 (1978)
30. E. Abdalla, M. Forger, M. Gomes, Nucl. Phys. **B210**, 181 (1982)
31. E. Abdalla, M.C.B. Abdalla, M. Gomes, Phys. Rev. **D23**, 1800 (1981)
32. E. Abdalla, M.C.B. Abdalla, M. Gomes, Phys. Rev. **D25**, 452 (1982)
33. E. Abdalla, M.C.B. Abdalla, M. Gomes, Phys. Rev. **D27**, 825 (1983)
34. E. Witten, Y.Y. Goldschmidt, Phys. Lett. **91B**, 392 (1980)
35. V. Kurak, R. Köberle, Phys. Rev. **D36**, 627 (1987)
36. A.A. Migdal, Soviet Phys. JETP **42**, 413742 (1976)
37. A. M. Polyakov, Phys. Lett. **59B**, 79 (1975)
38. S. Coleman, J. Mandula, Phys. Rev. **159**, 1251 (1967)
39. R. Haag, J. Lopuszanski, M. Sohnius, Nucl. Phys. **B88**, 257 (1975)
40. A.B. Zamolodchikov, Al.B. Zamolodchikov, Ann. Phys. **120**, 253 (1979)
41. D. Iagolnitzer, Phys. Rev. **D18**, 1275 (1978)
42. D. Iagolnitzer, *The S Matrix*. (North Holland, Amsterdam, 1978)
43. M. Karowski, Phys. Rep. **49**, 229 (1979)
44. H.J. de Vega, Phys. Lett. **87B**, 233 (1979)
45. H.J. de Vega, H. Eichenherr, J.M. Maillet, Commun. Math. Phys. **92**, 507 (1984)
46. H.J. de Vega, H. Eichenherr, J.M. Maillet, Nucl. Phys. **B240**, 377 (1984)
47. B. Berg, P. Weisz, Nucl. Phys. **B145**, 205 (1978)
48. E. Witten, Nucl. Phys. **B145**, 110 (1978)
49. E. Abdalla, B. Berg, P. Weisz, Nucl. Phys. **B157**, 387 (1979)
50. R. Köberle, V. Kurak, J.A. Swieca, Phys. Rev. **D20**, 897 (1979),
51. R. Köberle, V. Kurak, J.A. Swieca, Phys. Rev. E **D20**, 2638 (1979)
52. B. Schroer, Fortschr. Phys. **11**, 1 (1963)
53. H. Araki, R. Haag, Commun. Math. Phys. **4**, 77 (1967)
54. A.B. Zamolodchikov, Al.B. Zamolodchikov, Phys. Lett. **B72**, 481 (1978)
55. E. Abdalla, A. Lima-Santos, Phys. Rev. **D29**, 1851 (1984)
56. Gerard 't Hooft, Nuclear Phys. **B75**, 461 (1974)
57. D.E. Berenstein, J.M. Maldacena, H.S. Nastase, JHEP **0204**, 13 (2002)
58. N. Beisert, Phys. Rept. **405**, 1. [arXiv:hep-th/0407277](https://arxiv.org/abs/hep-th/0407277) (2005)
59. E. Abdalla, M. Forger, M. Gomes, Nucl. Phys. **B210**, 181 (1982)
60. E. Abdalla, M.C.B. Abdalla, A. Lima-Santos, Phys. Lett. **B140**, 71 (1984)
61. E. Abdalla, M.C.B. Abdalla, A. Lima-Santos, Phys. Lett. E **B146**, 457 (1984)
62. E. Abdalla, M. Forger, M. Jacques, Nucl. Phys. **B307**, 198 (1988)
63. A. D'Adda, P. Di Vecchia, M. Lüscher, Nucl. Phys. **B152**, 125 (1979)
64. A.M. Polyakov Phys. Lett. **B72**, 224 (1977)
65. O. DeWolfe, N. Mann, JHEP **0404**, 35 (2004)
66. O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri, Y. Oz, Phys. Rept. **323**, 183 (2000)

67. B.C. Vallilo, JHEP **0403**, 37 (2004)
68. E. Abdalla, Lec. Notes Phys. **226**, 140 (1984)
69. J.A. Minahan, K. Zarembo, JHEP **0303**, 13 (2003)
70. N.Y. Reshetikhin, Theor. Math. Phys. **63**, 555 (1985)
71. A. Lima-Santos, J. Stat. Mech. 0607 P003, [arXiv:nlin.SI/0602003](https://arxiv.org/abs/nlin/0602003) (2006)
72. E.K. Sklyanin, J. Phys. **A21**, 2375 (1988)
73. A. Lima-Santos, R. Malara, Nucl. Phys. **B675**, 661. [arXiv:nlin.SI/0307046](https://arxiv.org/abs/nlin/0307046) (2003)
74. G.L. Li, K.J. Shi, J. Stat. Mech. 0701, 18. [arXiv:hep-th/0611127](https://arxiv.org/abs/hep-th/0611127) (2007)
75. H. Cheng, T.T. Wu, *Expanding Protons: Scattering at High Energies* (MIT, Boston, 1987)
76. D. Berenstein, S.E. Vazquez, JHEP **0506**, 59 (2005)