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On Przymusiński's theorem

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Abstract-Przymusiński's theorem ($[2]$, pg 23) is generalized for a larger class of topological spaces.

1) Notations and definitions

AC denotes Axiom of Choice, CH denotes Continuum Hypothesis and MA denotes Martin's Axiom ($[2]$, pg 23).

Definition 1 - A partial order is countably closed if every descending sequence has a lower bound.

Definition 2 - A partial order is well-met if any two compatible elements have an inf.

Definition 3 - A partial order is $\aleph_1$-linked if it is the union of $\aleph_1$ pairwise compatible subsets.

Baumgartner's Axiom - Let $(P, \leq)$ be a partial order such that

(1) P is countably closed

(2) P is well-met

(3) P is $\aleph_1$-linked

then if $K < 2^{\aleph_1}$ and $\{D_\alpha\}_{\alpha < K}$ are dense subsets of P, then there is $G \subseteq P$ such that

(4) $p \geq q \in G \longrightarrow p \in G$

(5) if $p, q \in G$, there is an $r \in G$, $r \leq p$ and $r \leq q$,

(6) $G \bigcap D_\alpha \neq \emptyset$ for all $\alpha < K$

Baumgartner's Axiom is denoted by BA.

2) Main Theorems

<u>Theorem 1</u> - Let (X,τ) be a Lindelof, regular, topological space and $|X| < C$ satisfying the following:

a) $\tau \times \tau$ is finer than a metrizable second countable topology τ_o on $X \times X$. Let d be a compatible metric.

b) Every point $(x,y) \in X \times X$ has a countable fundamental system of neighborhoods in the $\tau \times \tau$ topology which are closed in the τ_o topology.

Under (MA + $\neg$ CH) $(X \times X, \tau \times \tau)$ is normal.

<u>Proof</u> - Given $U \subset X \times X$, $\overline{U}^{\tau_o}$ denotes the closure of U in the τ_o topology.

For $(x,y) \in X \times X$ and $n \in \omega$ define an open neighborhood $U_n(x,y) \subset B_{1/n}(x,y)$, being $B_{1/n}(x,y)$ the closed ball with center (x,y), radius $1/n$ in the metric space $(X \times X, d)$.

Furthermore $(U_n(x,y))_{n < \omega}$ is a fundamental system of neighborhoods of (x,y) in $(X \times X, \tau \times \tau)$. Let H and K be two closed subsets in the $\tau \times \tau$ topology and $H \cap K = \emptyset$.

For $p \in H$, let $J_p = \{n \in \omega \mid \overline{U_n(p)}^{\tau_o} \cap K = \emptyset\}$

For $p \in K$, let $J_q = \{n \in \omega \mid \overline{U_n(q)}^{\tau_o} \cap H = \emptyset\}$

Let P be the set of all functions defined from a finite subset of $H \sqcup K$ into ω such that $p \in \text{dom } f \cap H$ and $q \in \text{dom } f \cap K$ imply that $f(p) \in J_p$, $f(q) \in J_q$ and $B_{1/f(p)}(p) \cap B_{1/f(q)}(q) = \emptyset$.

Define $f \leq g$ in P provided that $\text{dom } f \subset \text{dom } g$ and $g|_{\text{dom } f} = f$.

$(P, \leq)$ is a partially ordered set. For $p \in H \sqcup K$ define

$$X_p = \{f \in P \mid p \in \text{dom } f\}$$

Each X_p is open and dense in $(P, \leq)$. If we prove that $(P, \leq)$ is ccc, MA will imply that there exists a family $(f_p)_{p \in H \sqcup K}$ of comparable terms of P. Define $f(p) = f_p(p)$ for each $p \in H \sqcup K$.

Then $U = \bigsqcup_{p \in H} U_{f(p)}(p)$ and $V = \bigsqcup_{q \in K} U_{f(q)}(q)$ are disjoint open

subsets containing H and K respectively. Thus $(X \times X, \tau \times \tau)$ is

normal. The next step is to prove that $(P, \leq)$ is ccc.

Let Z_0 be an uncountable subset of P. There exists a par (i,j) and

an uncountable subset Z_1 of Z_0 such that $f \in Z_1$ implies that:

$$H^f = \text{dom } f \sqcap H \quad \text{has i terms}$$

$$K^f = \text{dom } f \sqcap K \quad \text{has j terms}$$

$H \sqcup K$ can be well ordered (AC) such that every term of H is

strictly smaller than any term of K. In that way $H^f \sqcup K^f$ can be

expressed like: $p_1^f, p_2^f, \ldots, p_i^f, p_{i+1}^f, \ldots, p_{i+j}^f$. Now the image of f

can be seen as a term of ω^{i+j}.

There exists an $(i+j)$-uple of members of ω and an uncountable subset

Z_2 of Z_1 such that f and g in Z_2 imply

$$f(p_k^f) = g(p_k^g) \quad 1 \leq k \leq i+j$$

Consider now $(V_n)_{n \in \omega}$ a denumerable basis for the τ_0 topology. For

each $f \in Z_2$ and $n \in (1, \ldots, i)$, $m \in (i+1, \ldots, i+j)$ there exists a

pair (h,k) such that $p_n^f \in V_h$ and $p_m^f \in V_k$ and if $p \in V_h$ and $q \in V_k$

then

$$\overline{U_{f(p_n^f)}(p)}^{\tau_0} \sqcap U_{f(p_m^f)}(q) = \emptyset \quad *$$

It is easy to prove that being $f(p_n^f) = n'$ and $f(p_m^f) = m'$ it follows

that

$$B_{1/n'}(p_n^f) \sqcap B_{1/m'}(p_m^f) = \emptyset \quad \text{and so}$$

$$d(p_n^f, p_m^f) > \sup\{1/n', 1/m'\}$$

This is still true if we take the closure of both sets. **

If can be associated to each $f \in Z$ a subfamily having $i + j$ terms

of the basis $(V_n)_{n < \omega}$.

There exists a choice of i + j terms of the basis such that there is an uncountable subset Z_3 of Z_2 whose elements have the same h(n) and k(m) of *.

Then for f and g in Z_3.

$$\overline{U_{f(p_n^f)}(p_n^f)}^{\tau_0} \sqcap \overline{U_{g(p_m^g)}(p_m^g)}^{\tau_0} = \emptyset$$

(this is by **).

In that way all the terms of Z_3 are compatible (impossible!) . ■

The next theorem gives a result depending on BA + CH. So it cannot be seeing as a generalization of the first one.

<u>Notation</u> - Given X a topological space, U a family of subsets of X and A any subset of X, (A; U) denotes the following subset of X:

$$(A; \, U) = \bigsqcup \{U \in U \mid U \sqcap A \neq \emptyset\}.$$

<u>Theorem 2</u> - (BA + CH) - Let $(X \times X, \tau_0)$ be a nondiscrete topological space such every G_δ-subset is open and that has a basis $B = \bigsqcup_{i < \omega_1} B_i$, $|B_i| \leq \aleph_0$ and each B_i is an open disjoint covering of $X \times X$ and B_j refines B_i if j > i. [1]. Let τ be another topology an $X \times X$ regular concerning to τ_0. Then $(X \times X, \tau)$ is a normal topological space.

<u>Proof</u> - Before start the proof it will be stablished the following notation: $B_j(p)$ is the term of B_j to which p belongs. Now let A and B be two subsets of $X \times X$ closed in the τ topology, $A \sqcap B = \emptyset$. Let $P = \{(G,H) \mid G$ and H are open in τ, $\overline{G}^{\tau_0} \sqcap B = \overline{H}^{\tau_0} \sqcap A = \emptyset$ and exists $k \in \omega_1$ such that $(G; B_k) \sqcap (H; B_k) = \emptyset\}$.

P is partially ordered by the relation $\leq$ being $(G,H) \leq (G',H')$ if and only if $G \subset G'$ and $H \subset H'$.

It has to be verifyed that $(P,\leq)$ has the conditions of BA.

a) $(P,\leq)$ is countably closed.

Let $(G_i,H_i)_{i \in \omega}$, $(G_i,H_i) \in P$, be a decreasing sequence.

Let G and H be defined as follows:

$$G = \bigsqcup_{i < \omega} G_i \qquad H = \bigsqcup_{i < \omega} H_i$$

Then $\overline{G}^T o = \bigsqcup_{i < \omega} \overline{G}_i^T o$ and $\overline{H}^T o = \bigsqcup_{i < \omega} \overline{H}_i^T o$.

It is easy to prove that $\overline{G}^T o \sqcap B = \overline{H}^T o \sqcap A = \emptyset$.

b) $(P,\leq)$ is well-met.

Let (G,H) and (G',H') be two compatible terms of P, then $(G \sqcup G' , H \sqcup H')$ is the inf.

c) $(P,\leq)$ is $\aleph_1$-linked.

For each $(G,H) \in P$ it is possible to associate $k \in \omega_1$ such that

$$(G; B_k) \sqcap (H; B_k) = \emptyset \quad \text{and subsets}$$

$$G_o = \bigsqcup \{u \in B_k \ / \ u \sqcap H = \emptyset\} \supset G$$

$$H_o = \bigsqcup \{u \in B_k \ / \ u \sqcap G = \emptyset\} \supset H$$

and so $G_o \sqcap H_o = \emptyset$.

For any $p \in G_o$ and $q \in H_o$ $B_{k+1}(p) \sqcap B_{k+1}(q) = \emptyset$.

So $(G_o; B_{k+1}) \sqcap (H_o; B_{k+1}) = \emptyset$.

It is possible to define:

$$P_k = \{(G',H') \in P \mid G' \subset G_o \text{ and } H' \subset H_o \text{ and}$$
$$(G', B_{k+1}) \sqcap (H', B_{k+1}) = \emptyset\}.$$

Any two terms of P_k are compatible and

$$P = \bigsqcup_{k < \omega_1} P_k$$

For each $p \in X \times X$ define

$$D_p = \{(G,H) \in P \mid p \in A \sqcap G\}$$

$$E_p = \{(G,H) \in P \mid p \in B \sqcap H\}$$

D_p and E_p are both open and dense in P.

BA quarantees the existence of a subfamily E of

$\{D_p\}_{p \in X \times X} \sqcup \{E_p\}_{p \in X \times X}$ such that every two terms of E are compatible.

Now it is enough to consider

$$U = \bigsqcup_{(G,H) \in E} G \quad \text{and} \quad V = \bigsqcup_{(G,H) \in E} H \; ; \text{ it follows that}$$

$U \supset A$, $V \supset B$, U and V are open in the topology τ and $U \sqcap V = \emptyset$. This completes the proof.

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After the conclusion of this paper the author has noticed that K. Alster and T. Przymusiński have proved a similar result to theorem 1. (Fund. Math. 91(1976) 124-130).

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