RT-MAT 2003-8

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Abril 2003

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Abstract

Starting from the Colombeau Generalized Functions, the sharp topologies and the notion of generalized points, we introduce a new kind of differential calculus (for functions between totally disconnected spaces). We also define here the notions of holomorphic generalized functions (in this new framework) and generalized manifold. Finally we give an answer to a question raised in [A-S].

1 INTRODUCTION

Since Colombeau introduced his definition of generalized functions there has been a great deal of developement in this field. The theory has proven to have many useful applications and gives new inside where the classical theory does not (see [O]). Scarpalezos, in [S1], [S2], introduced a natural topology in the algebra of Colombeau generalized functions which was a turning point in the field. Another turning point was the introduction of the important notion of generalized point value of a generalized function by M. Kunzinger and M. Oberguggenberger in [K-O].

Scarpalezos' topologies turned out to be exactly what was needed to boost the theory. Having this as a starting point Aragona and Juriaans were able to study many algebraic properties of the now topological ring of Colombeau generalized numbers $\overline{\boldsymbol{K}}$. A strong link was established between the topological and algebraic structure of $\overline{\boldsymbol{K}}$. With this [A-J] bridges algebra, analysis and topology in the field. An application of these algebraic results can be found in [A-S].

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*Research partially supported CNPq (Proc 300652/95-0)

1991 Mathematics Subject Classification: Primary 20C05 Secondary 16S34. Keywords and phrases: Colombeau algebra, generalized function, sharp topologies, generalized points, point values.

In this paper we propose a differential calculus which is based on the turning points mentioned above. We show that there are an open subset $\widetilde{\Omega}_c$ of $\overline{\boldsymbol{K}}$, a derivative D and an embedding $\iota : \mathcal{G}(\Omega) \to \mathcal{C}^1(\widetilde{\Omega}_c, \overline{\boldsymbol{K}})$ such that $D(\iota(f)) = \iota(f')$. We introduce the notion of holomorphic functions and analytic functions in a more general framework and also we define generalized manifolds. We notice that Oberguggenberger, Pilipovic and Scarpalezos independently introduced a notion of differential calculus based also on the turning points mentioned above (see [O-P-S]). However our seems to be more natural and suitable for applications.

The paper is organized as follows: In the second section we introduce our notion of differential calculus in $\overline{\boldsymbol{K}}^n$ and establish some of its basic properties. In the thirth section we recal some basic fact about $\widetilde{\Omega}_c$ and establish some results which will be needed in the other sections. In the fourth section, using the notion of generalized point value, we embed $\mathcal{G}(\Omega)$ into the algebra of $\mathcal{C}^\infty$ functions of $\widetilde{\Omega}_c$ into $\overline{\boldsymbol{K}}$. We prove that this embedding "commutes" with derivation and prove a kind of open mapping theorem. In the fifth section we introduced the notion of sub-linear, holomorphic and analytic functions. In particular for elements of $\mathcal{G}(\Omega)$ these notions are nearly equivalent. In the sixth section we introduce our notion of generalized manifold and starting from a classical manifold M we construct a generalized manifold M^*. Finally in the last section, using the open mapping theorem, we give an answer to a question raised by Aragona-Soares in [A-S].

More details of this differential calculus and its properties will appear elsewhere.

2 Differential Calculus in $\overline{\boldsymbol{K}}^n$

We equip $\overline{\boldsymbol{K}}$ with the topology given by Scarpalezos and $\overline{\boldsymbol{K}}^n$ with the product topology. The basic notation and some properties of the algebraic and the topological structures of $\overline{\boldsymbol{K}}$ can be founded in [A-J]. If $x = (x_1, \cdots, x_n) \in \overline{\boldsymbol{K}}^n$ we define $||x||_n := max\{||x_i|| : 1 \leq i \leq n\}$, where $||x_i||$ is defined in [A-J, Notation 1.5]. Frequently when $n > 1$ and ever if $n = 1$, the subscript n will be omitted of the above notation. If $r \in \boldsymbol{R}_+^*$ and $x_0 \in \overline{\boldsymbol{K}}^n$ then $B_r(x_0), B'_r(x_0)$ and $S_r(x_0)$ represent those element $x \in \overline{\boldsymbol{K}}^n$ for which $||x - x_0|| < r$, $||x - x_0|| \leq r$ and $||x - x_0|| = r$, respectively . For $r \in \boldsymbol{R}$ we denote by α_r the class of $\widehat{\alpha}_r(\varepsilon) := \varepsilon^r$. Note that $\alpha_r \in Inv(\overline{\boldsymbol{K}})$ for all $r \in \boldsymbol{R}$, and $\lim\limits_{r\to\infty} \alpha_r = 0$.

Lemma 2.1 *Let $U \subset \overline{\boldsymbol{K}}$ be an open subset, $f : U \to \overline{\boldsymbol{K}}$ and $x_0 \in U$. Then there exists at most one $z_0 \in \overline{\boldsymbol{K}}$ such that*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - z_0(x - x_0)}{\alpha_{-\log ||x - x_0||}} = 0 .$$

Proof. Suppose that $z_0, z_1 \in \overline{\boldsymbol{K}}$ are such that the above limit is zero for both. Then it follows that

$$\lim_{x \to x_0} \frac{(z_0 - z_1)(x - x_0)}{\alpha_{-\log ||x - x_0||}} = 0 .$$

In particular if we let $x_n := x_0 + \alpha_n$ then $\alpha_{-\log ||x_n - x_0||} = \alpha_n$ and from this we get that $0 = \lim_{n \to \infty} (z_1 - z_0) = 0$. ▮

The lemma tells us that the following definition is meaningfull.

Definition 2.2 *Given an open set $U \subset \overline{\boldsymbol{K}}$, $f : U \to \overline{\boldsymbol{K}}$ and $x_0 \in U$ we shall say that f is differentiable in x_0 if there exists $z_0 \in \overline{\boldsymbol{K}}$ such that*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - z_0(x - x_0)}{\alpha_{-\log ||x - x_0||}} = 0 .$$

In case f is differentiable in x_0 we shall write $D(f)(x_0) = z_0$ and call it the derivative of f at x_0. We shall say that f is differentiable if it is differentiable at each point of its domain.

Remark 2.3 *Note that if f is differentiable at x_0 then we may write*

$$f(x) - f(x_0) = D(f)(x_0)(x - x_0) + E(x)$$

with $\lim_{x \to x_0} \frac{E(x)}{\alpha_{-\log ||x - x_0||}} = 0$. Moreover $D(f)(x_0) = \lim_{n \to \infty} \frac{f(x_0 + \alpha_n) - f(x_o)}{\alpha_n}$.

Lemma 2.4 *Let $U \subset \overline{\boldsymbol{K}}$ be an open subset. If $f : U \to \overline{\boldsymbol{K}}$ is differentiable at x_0, then f is continous at this point.*

Proof. By the remark above we may write

$$f(x) - f(x_0) = D(f)(x_0)(x - x_0) + E(x)$$

with $\lim_{x \to x_0} \frac{E(x)}{\alpha_{-\log ||x - x_0||}} = 0$. Since $||\alpha_{-\log ||x - x_0||}|| = ||x - x_0||$ it follows that $\lim_{x \to x_0} E(x) = 0$ and hence the proof is completed. ▮

We now give an example of a non-constant function whose derivative vanishes everywhere. Hence a function is not determined by its derivative. This example also shows that the Mean Value Theorem is false in general.

Example 2.5 *Let* $f(x) := \alpha_{-2\log\|x\|}$, *if* $x \in \overline{\boldsymbol{K}}^{*}$ *and* $f(0) := 0$. *If* $x_0 \neq 0$ *then* f *is constant in the neighborhood* $S_{\|x_0\|}(0)$ *of* x_0 *and hence* $D(f)(x_0) = 0$. *So we only have to prove that* $D(f)(0) = 0$, *i.e.,* $\lim\limits_{x\to 0} \frac{f(x)}{\alpha_{-\log\|x\|}} = 0$. *But this is an obvious statement.*

A function will be called *almost constant* if it has vanishing derivative.

Using Remark 2.3 and the standard proofs of ordinary differential calculus we obtain the following result.

Proposition 2.6 *Let* $U \subset \overline{\boldsymbol{K}}$ *be an open subset. If* $f, g : U \to \overline{\boldsymbol{K}}$ *be functions differentiable, then*

1. fg *is differentiable and* $D(fg) = gD(f) + fD(g)$;
2. *if* $f(U)$ *is contained in the domain of* g *then* $D(g\circ f) = (D(g)\circ f)D(f)$;
3. $D(f \pm g) = D(f) \pm D(g)$;
4. *Where the value of* g *is unit we have that* $D(\frac{f}{g}) = \frac{gD(f) - fD(g)}{g^2}$.

The proposition tells us that our notion of derivation satisfies the usual properties of the derivation of ordinary differential calculus.

Definition 2.7 *Let* $U \subset \overline{\boldsymbol{R}}^{n}$ *be an open subset,* $f : U \to \overline{\boldsymbol{K}}$ *and* $x = (x_1, \cdots, x_n)$, $x_0 = (x_{01}, \cdots, x_{0n}) \in U$. *Let* $1 \leq i \leq n$ *and suppose that there is an element* a_i *of* $\overline{\boldsymbol{K}}$ *such that*

$$\lim_{h\to 0} \frac{f(x_1, \cdots, x_i + h, \cdots, x_n) - f(x_1, \cdots, x_i, \cdots, x_n) - a_i h}{\alpha_{-\log\|h\|}} = 0\,.$$

Then we shall define $\frac{\partial f}{\partial x_i}(x_0) := a_i$ *and call it the partial derivative of* f *with respect to* x_i *at* x_0. *We shall say that* f *is differentiable at* x_0 *if there exists* $a = (a_1, \cdots, a_n) \in \overline{\boldsymbol{K}}^{n}$ *such that*

$$\lim_{x\to x_0} \frac{f(x) - f(x_0) - \sum\limits_{1\leq i\leq n} a_i(x_i - x_{0i})}{\alpha_{-\log\|x - x_0\|}} = 0\,.$$

It is now standard to verify that all the known result of ordinary differential calculus hold also in our case. For example if f is differential at x_0 then it is continuos at x_0 and the a_i's in the definition of f be differentiable are exactly the partial derivatives at x_0. If $\boldsymbol{K} = \boldsymbol{R}$, the gradient of f at x_0 is defined by $\nabla f(x_0) := (\frac{\partial f}{\partial x_1}(x_0), \cdots, \cdots \frac{\partial f}{\partial x_n}(x_0))$

If U is an open subset of $\overline{\boldsymbol{R}}^n$ and $k \in \boldsymbol{N}$, we can define the set $C^k(U, \overline{\boldsymbol{K}}) := \{f : U \longrightarrow \overline{\boldsymbol{K}} \,|\, \partial^\alpha f \in C(U, \overline{\boldsymbol{K}})$ for all $\alpha \in \boldsymbol{N}^n$ such that $0 \leq |\alpha| \leq k\}$.

Definition 2.8 *Let $U \subset \overline{\boldsymbol{R}}^n$ be an open subset and $f : U \to \overline{\boldsymbol{R}}^m$. We may write $f = (f_1, \cdots, f_m)$ where each $f_i : U \to \overline{\boldsymbol{R}}$. We shall say that f is differentiable at $x_0 \in U$ if each f_i is differentiable at x_0.*

Remark 2.9 *It is easy to see that f is differentiable at x_0 if and only if there exists a $\overline{\boldsymbol{R}}$-linear map $T : \overline{\boldsymbol{R}}^n \to \overline{\boldsymbol{R}}^m$ such that*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - T(x - x_0)}{\alpha_{-\log\|x - x_0\|}} = 0\,.$$

The map T will be denoted by $Df(x_o)$.

3 Generalized Pointvalues

We start recalling the notion of generalized point value introduced by M. Kunzinger and M. Oberguggenberger (see [K-O]).

In what follows Ω denote a non-void open subset of $\boldsymbol{K}^n$, $\mathbf{I} :=]0,1] \subset \boldsymbol{R}$ and $\mathbf{I}_\eta :=]0, \eta[$ for $\eta \in \mathbf{I}$. Moreover, if $A \subset \mathbf{I}$ we denote by A^c the complement of A in $\mathbf{I}$ and by $\overline{A}$ the closure of A in $\overline{\mathbf{I}} := [0,1]$. Define

$$\Omega_M := \{(x_\varepsilon) \in \Omega^{\mathbf{I}} \,|\, \exists p > 0,\ \eta > 0 \text{ with } |x_\varepsilon| \leq \varepsilon^{-p} \text{ for all } \varepsilon \in \mathbf{I}_\eta\}$$

and that $(x_\varepsilon), (y_\varepsilon) \in \Omega_M$ are equivalent, $(x_\varepsilon) \sim (y_\varepsilon)$, if and only if given any $q > 0$, there exists $\eta > 0$ such that $|x_\varepsilon - y_\varepsilon| \leq \varepsilon^q$ for all $\varepsilon \in \mathbf{I}_\eta$. Let $\widetilde{\Omega} := \Omega_M / \sim$.

Note that if $\Omega = \boldsymbol{K}$ then $\widetilde{\Omega} = \overline{\boldsymbol{K}}$ and that $\widetilde{\boldsymbol{K}^n} = \overline{\boldsymbol{K}}^n$.

An element $x \in \Omega_M$ is said to be *compactly supported* if it has a representative (x_ε) and there exists K compact subset of Ω such that $x_\varepsilon \in K$ for ε sufficiently small. Define

$$\widetilde{\Omega}_c := \{x \in \widetilde{\Omega} | x \text{ is compactly supported}\}.$$

We can embed Ω into $\widetilde{\Omega}_c$ by the mapping $x \in \Omega \longmapsto cl(x_\varepsilon) \in \widetilde{\Omega}_c$, where $x_\varepsilon = x$ for all $\varepsilon \in \mathbf{I}$. Note that the image of Ω is a discrete subset of $\overline{\boldsymbol{K}}^n$.

Let $f \in \mathcal{G}(\Omega)$, $x \in \widetilde{\Omega}_c$ and $(x_\varepsilon), \widehat{f}$ be representatives of x and f respectively. Define $f(x) := cl(\varepsilon \in \mathbf{I} \longmapsto \widehat{f}(\varepsilon, x_\varepsilon)) \in \overline{\boldsymbol{K}}$. This is called the *generalized pointvalue of* f in x. In [K] it is shown that this definition does not depend on the chosen representatives. The following theorem tells us that $\widetilde{\Omega}_c$, and not Ω, is the natural domain of f.

Theorem 3.1 ([K] and [K-O]) *If $f \in \mathcal{G}(\Omega)$ then $f = 0$ if and only if $f(x) = 0$ for all $x \in \widetilde{\Omega}_c$.*

We now extend the definition of association in $\overline{\boldsymbol{K}}$ defined in [A-B, section 6] to $\overline{\boldsymbol{K}}^n$: $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n) \in \overline{\boldsymbol{K}}^n$ are *associated* in $\overline{\boldsymbol{K}}^n$ if and only if x_i and y_i is associated in $\overline{\boldsymbol{K}}$ for all $1 \leq i \leq n$. Using [A-J, Prop.2.15(b)] one has that all the elements of $B_1(0)^n \subset \overline{\boldsymbol{K}}^n$ are associated to 0.

Proposition 3.2 *The following assertions hold.*

1. $B_1(x)^n \subset \widetilde{\Omega}_c \subset B_1'(0)^n$, $\forall\, x \in \widetilde{\Omega}_c$;
2. *$\widetilde{\Omega}_c$ is an open subset of $\overline{\boldsymbol{K}}^n$;*
3. *if $x_0 \in \Omega$ and $V = \Omega \setminus \{x_0\}$, then $\widetilde{V}_c \subset \widetilde{\Omega}_c \setminus B_1(x_0)^n \subset S_1(x_0)^n$.*

Proof. Let $x \in \widetilde{\Omega}_c$ and take $y \in \overline{\boldsymbol{K}}^n$ such that $||y - x|| < 1$. Then $y - x \in B_1(0)^n$ and so $x - y$ is associated to 0, i.e., if (x_ε) and (y_ε) are representatives of x and y respectively then $\lim\limits_{\varepsilon \downarrow 0} |x_\varepsilon - y_\varepsilon| = 0$. So if K is an compact subset of Ω such that $x_\varepsilon \in K$ then, for ε sufficiently small, $y_\varepsilon \in K$ and thus $y \in \widetilde{\Omega}_c$. Note that (x_ε) is a bounded sequence and so $||x|| \leq 1$. This prove (1). Assertion (2) is a consequence of (1). For (3) take $x \in \widetilde{V}_c$ and let K be a compact subset of $\Omega \setminus \{x_0\}$ such that $x_\varepsilon \in K$ for ε sufficiently small. Then there is $r > 0$ such that $|x_\varepsilon - x_0| > r$ for ε sufficiently small. So $||x - x_0|| \geq 1$, i.e., $x \in \widetilde{\Omega}_c \setminus B_1(x_0)^n$. By (1) we obtain that $\widetilde{\Omega}_c \setminus B_1(x_0)^n \subset S_1(x_0)^n$. ∎

4 The Colombeau Differential Algebra $\iota(\mathcal{G}(\Omega))$

In this section we shall prove the main result of this paper. It contains our proposal of a new differential calculus. It is here also that we prove our "open mapping theorem".

In what follows $\mathcal{G}(\Omega)$ will be endowed with the sharp topology([S1], [S2]) and $\mathcal{C}^1(\widetilde{\Omega}_c, \overline{\boldsymbol{K}})$ with the point topology, i.e., $f_n \to f$ if and only if $f_n(x) \to f(x)$ for all $x \in \widetilde{\Omega}_c$. From Theorem 3.1 it follows that the natural map

$$\iota = \iota_\Omega : f \in \mathcal{G}(\Omega) \longmapsto \iota(f) \in \mathcal{C}^1(\widetilde{\Omega}_c, \overline{\boldsymbol{K}}) ,$$

where $\iota(f)(x) := f(x)$ for every $x \in \widetilde{\Omega}_c$, is injective. We can now state the main result of this paper.

Theorem 4.1 (Embedding Theorem) *Let Ω be an open subset of $\boldsymbol{R}^n$. The embedding $\iota : \mathcal{G}(\Omega) \to \mathcal{C}^1(\widetilde{\Omega}_c, \overline{\boldsymbol{K}})$ is an injective homomorphism of $\overline{\boldsymbol{K}}$-algebras. Moreover, ι is continuous and $\iota(\frac{\partial f}{\partial x_i}) = \frac{\partial(\iota(f))}{\partial x_i}$ for all $f \in \mathcal{G}(\Omega)$ and $1 \leq i \leq n$.*

Proof. The homomorphism of $\overline{\boldsymbol{K}}$-algebras is obvious and that it is injective is implied by Theorem 3.1. Hence we only have to prove continuty at zero. For this let $f_n \to 0$ in $\mathcal{G}(\Omega)$, $x \in \widetilde{\Omega}_c$ and $\widehat{f}_n, (x_\varepsilon)$ representatives of these elements. Choose an exhaustion (Ω_m) of Ω and fixe m_0 so that $x_\varepsilon \in \Omega_{m_0}$ for ε sufficientely small. Applying [A-J, Prop.1.12] it follows at once that $\iota(f_n)(x) \to 0$.

To prove the second part we first suppose that $n = 1$. Let $x_0 \in \widetilde{\Omega}_c$ whose support is contained in a compact subset $K \subset \Omega$. We claim that $\iota(f)$ is differentiable at x_0 and that $D(\iota(f))(x_0) = \iota(f')(x_0)$. In fact, choose $x \in \widetilde{\Omega}_c$ with $||x - x_0|| < 1$ and let $\widehat{f}, (x_\varepsilon)$ and $(x_{0\varepsilon})$ be representatives of f, x and x_0, respectively. Since $\lim\limits_{\varepsilon \downarrow 0}(x_\epsilon - x_{0\varepsilon}) = 0$ and $\widehat{f}$ is moderate there is, from Taylor's formula, $C := cl(C_\varepsilon) \in \overline{\boldsymbol{K}}$ such that for sufficiently small ε one has $|\widehat{f}(\varepsilon, x_\varepsilon) - \widehat{f}(\varepsilon, x_{0\varepsilon}) - \widehat{f}'(\varepsilon, x_{0\varepsilon})(x_\varepsilon - x_{0\varepsilon})| \leq C_\varepsilon |x_\varepsilon - x_{0\varepsilon}|^2$. Then, by [A-J, Corollary 1.6], $||\iota(f)(x) - \iota(f)(x_0) - \iota(f')(x_0)(x - x_0)|| \leq ||C||\,||x - x_0||^2$ and hence $||\frac{1}{\alpha_{-log||x-x_0||}}||\,||f(x) - f(x_0) - \iota(f)(x_0)(x - x_0)|| \leq C||x - x_0||$. This proves the existence of partial derivatives. To prove differentiabillity in the general case it is enough to repeat, for $\overline{\boldsymbol{K}}^n$ equiped with the product topology, the same prove given above for the case $n = 1$. ∎

Definition 4.2 *Let $x \in \overline{\boldsymbol{K}}$ and let $\widehat{x}$ be one of its representatives. Then the function $\epsilon \in \mathbf{I} \mapsto |\widehat{x}(\epsilon)|$ gives rise to an element $|x| \in \overline{\boldsymbol{K}}$ which depend only on x and which is called the module of x.*

For $x = (x_1, ..., x_n) \in \widetilde{\Omega}_c$ define $[x]_2 := [\sum\limits_{i=1}^{n} |x_i|^2]^{\frac{1}{2}}$. It is easy to show(see [A-J-O-S]) that for a given $x \in \overline{\boldsymbol{R}}$ such that $x \geq 0$(i.e. x has a representative

$\widehat{x}$ such that $\widehat{x}(\varepsilon) \geq 0$ for all $\varepsilon \in \mathbf{I}$) there is a unique $y \in \overline{\boldsymbol{R}}$, $y \geq 0$ verifying $y^2 = x$, which is denoted by $x^{1/2}$ or $\sqrt{x}$. Note that $[x]_2$ is induced by the following $\overline{\boldsymbol{K}}$-bilinear map $\langle x|y\rangle := \sum_{i=1}^{n} x_i \overline{y_i}$ from $\overline{\boldsymbol{K}}^2$ to $\overline{\boldsymbol{K}}$.

The following result can be proved easily using the classical analog.

Lemma 4.3 (Generalized Cauchy-Schwarz inequality)*Let $x, y \in \overline{\boldsymbol{K}}$. Then $|\langle x|y\rangle| \leq [x]_2[y]_2$.*

Proposition 4.4 *Let Ω be an open subset of $\boldsymbol{R}^n$ and $f \in \mathcal{G}(\Omega)$. The following assertions hold.*

1. *$\iota(\mathcal{G}(\Omega)) \subset C^\infty(\widetilde{\Omega}_c, \overline{\boldsymbol{K}})$ and $\iota(\mathcal{G}(\Omega)) \neq C^\infty(\widetilde{\Omega}_c, \overline{\boldsymbol{K}})$.*
2. *If Ω is connected, then given $x, y \in \widetilde{\Omega}_c$ there is $c \in \widetilde{\Omega}_c$ such that $\iota(f)(x) - \iota(f)(y) = \langle \nabla\iota(f)(c)|x-y\rangle$ and $|\iota(f)(x) - \iota(f)(y)| \leq [\nabla\iota(f)(c)]_2[x-y]_2$.*
3. *If Ω is connected and $D(\iota(f)) = 0$ then f is constant.*
4. *$\iota(f)\,|_{B_1(0)}$ is a Lipschitz function.*
5. *If $K \subset \Omega$ is a compact subset and $\widetilde{K} = \{x \in \Omega_M | supp(x) \subset K\}$, then $\iota(f)|_{\widetilde{K}}$ is bounded.*
6. *If $n = 1$, $m \in \boldsymbol{N}^*$ and Ω is connected, then given x, $y \in \widetilde{\Omega}_c$ there is $z \in \widetilde{\Omega}_c$ such that*

$$\iota(f)(x) = \sum_{0 \leq j \leq m} \frac{\iota(f^{(j)})(y)(x-y)^j}{j!} + \frac{\iota(f^{(m+1)})(z)(x-y)^{m+1}}{(m+1)!}$$

and there are (x_ε), (y_ε) and (z_ε) representatives of x, y and z respectively whith $x_\varepsilon \leq z_\varepsilon \leq y_\varepsilon$.

7. *If $n = 1$, $m \in \boldsymbol{N}^*$ and $x \in \widetilde{\Omega}_c$, then*

$$\lim_{||x-y||\to 0} \frac{1}{(\alpha_{-log\,||x-y||})^n} \left[\iota(f)(x) - \sum_{1 \leq j \leq m} \frac{\iota(f^{(j)})(y)(x-y)^j}{j!}\right] = 0.$$

Proof. Theorem 4.1 together with Example 2.5 gives us 1. The second part of 2. follows from Lemma 4.3. Assertion 3. follows from 2. The others assertions are proved with similar arguments to the proof of Theorem 4.1 using the Taylor's formula with appropriate order in each case. ∎

Proposition 4.5 *Let* $f \in \mathcal{C}^\infty(\Omega) \subset \mathcal{G}(\Omega)$, $U := \iota(f)(\widetilde{\Omega}_c)$ *and* $V := f(\Omega)$. *Then*

1. $U \subset \widetilde{V}_c := \{x \in \boldsymbol{K}_M | supp(x) \subset V\}$ *and* $\iota(f)$ *is a bounded function;*
2. *if* f *is an open mapping, then* $U = \widetilde{V}_c$ *and* U *is an open subset of* $\widetilde{\boldsymbol{K}}_c$.

Proof. It is immediate that $U \subset \widetilde{V}_c$ and hence, by Proposition 3.2(1), $\iota(f)$ is a bounded function. To prove the second statement let $z \in \widetilde{V}_c$, (z_ε) a representative of z and K a compact subset of V such that $z_\varepsilon \in K$ for ε sufficiently small. As f is an open mapping there exists a compact subset $L \subset \Omega$ such that $K \subset f(L)$ and hence there is an element $x \in \widetilde{\Omega}_c$, whose support is contained in L, such that $f(x_\varepsilon) = z_\varepsilon$. By Proposition 3.2(2), U is open. ∎

Notice that we actually do not really need that f is an open mapping, what we actually need is that there exists an exaustion (Ω_n) of relatively compact sets of Ω such that $(f(\Omega_n))$ is an exaustion of $Im(f)$.

The following corollary is what we call The Open Mapping Theorem.

Corollary 4.6 (Open Mapping Theorem) *Let* $f \in \mathcal{C}^\infty(\Omega)$ *be an open mapping. Then for every open subset* $W \subset \Omega$ *we have that* $\iota(f)(\widetilde{W}_c)$ *is open.*

Proposition 4.7 *Let* Ω *be connected,* $f \in \mathcal{G}(\Omega)$ *and suppose that* $Im(\iota(f))$ *is a discrete set. Then* f *is constant.*

Proof. Since $Im(\iota(f))$ is discrete it follows that $0 = (\iota(f))' = \iota(f')$ and so $f' = 0$. Hence the result follows. ∎

We finish this section with a proposition about the composition of functions. To this end we shall need some definitions and results that are an easy adaptation of [A-B, 7.1-7.3] and [F].

If Ω is an open subset of $\boldsymbol{R}^n$ and F is a $\boldsymbol{K}$-vector space of dimension m endowed with a norm $|.|_F$ we can easily define $\mathcal{E}_M[\Omega; F]$ and $\mathcal{N}[\Omega; F]$ in the same way as the case $F = \boldsymbol{K}$. Clearly $\mathcal{N}[\Omega; F]$ is a sub-vector space of $\mathcal{E}_M[\Omega; F]$ and then, the $\boldsymbol{K}$-vector space of generalized mappings from Ω into F is defined by $\mathcal{G}(\Omega; F) = \mathcal{E}_M[\Omega; F]/\mathcal{N}[\Omega; F]$. We can prove that $\mathcal{G}(\Omega; F) = \oplus_{1 \leq \nu \leq m} \mathcal{G}(\Omega; \boldsymbol{K}) cl(\widehat{\xi}_\nu)$ where $(\xi_\nu)_{1 \leq \nu \leq m}$ is a $\boldsymbol{K}$-basis of F and $\widehat{\xi}_\nu : (\varepsilon, x) \in I \times \Omega \longmapsto \xi_\nu$. The mapping

$$\phi_{\Omega,m} : f = \sum_{1 \leq \nu \leq m} cl(\widehat{e}_\nu) \in \mathcal{G}(\Omega; \boldsymbol{K}^m) \longmapsto (f_1, ..., f_m) \in \mathcal{G}(\Omega)^m$$

is an isomorphism, where $(e_\nu)_{1\leq\nu\leq m}$ is the canonical $\boldsymbol{K}$-basis of $\boldsymbol{K}^m$. The isomorphism $\phi_{\Omega,m}$ transfers the product sharp topology τ_Ω^m of $\mathcal{G}(\Omega)^m$ over $\mathcal{G}(\Omega;\boldsymbol{K}^m)$ getting a topology $\tau_{\Omega,m} := \phi_{\Omega,m}^{-1}(\tau_\Omega^m)$ which will be called the sharp topology on $\mathcal{G}(\Omega;\boldsymbol{K}^m)$. In what follows we always will consider $\mathcal{G}(\Omega;\boldsymbol{K}^m)$ endowed with this topology $\tau_{\Omega,m}$. Denote by $\mathcal{C}_s^\infty(\widetilde{\Omega}_c,\overline{\boldsymbol{K}})$ the $\overline{\boldsymbol{K}}$-algebra $\mathcal{C}^\infty(\widetilde{\Omega}_c,\overline{\boldsymbol{K}})$ endowed with the topology of uniform convergence on the finite subsets of $\widetilde{\Omega}_c$(or, equivalently, the topology induced on $\mathcal{C}^\infty(\widetilde{\Omega}_c,\overline{\boldsymbol{K}})$ by the product space $\overline{\boldsymbol{K}}^{\widetilde{\Omega}_c}$). The mapping $\iota : \mathcal{G}(\Omega) \longrightarrow \mathcal{C}_s^\infty(\widetilde{\Omega}_c,\overline{\boldsymbol{K}})$ is, by Theorem 4.1, a continuos injective homomorphism of $\overline{\boldsymbol{K}}$-algebras. Hence (see [A, ch 1, sec 3]) there exists an unique continuos $\overline{\boldsymbol{K}}$-linear map ι_Ω^m making the following diagram commutative:

$$\begin{tikzcd}
\mathcal{G}(\Omega) \arrow[r, "\iota_\Omega"] & \mathcal{C}_s^\infty(\widetilde{\Omega}_c,\overline{\boldsymbol{K}}) \\
\mathcal{G}(\Omega)^m \arrow[u] \arrow[r, "\iota_\Omega^m"] & \mathcal{C}_s^\infty(\widetilde{\Omega}_c,\overline{\boldsymbol{K}})^m \arrow[u]
\end{tikzcd}$$

where the vertical arrow denotes the natural projections. Moreover ι_Ω^m is injective. If $\pi_\nu : \overline{\boldsymbol{K}}^m \longrightarrow \overline{\boldsymbol{K}}$ denotes de ν-th projection $(1 \leq \nu \leq m)$, the natural map $\varphi_\Omega : f \in \mathcal{C}_s^\infty(\widetilde{\Omega}_c,\overline{\boldsymbol{K}^m}) \longrightarrow (\pi_1 \circ f, ..., \pi_m \circ f) \in \mathcal{C}_s^\infty(\widetilde{\Omega}_c,\overline{\boldsymbol{K}}))^m$ is $\overline{\boldsymbol{K}}$-linear homeomorphism and the linear map

$$j_\Omega^m : \mathcal{G}(\Omega;\boldsymbol{K}^m) \longrightarrow \mathcal{C}_s^\infty(\widetilde{\Omega}_c,\overline{\boldsymbol{K}^m})$$

defined by $j_\Omega^m := \varphi_\Omega^{-1} \circ \iota_\Omega^m \circ \phi_{\Omega,m}$ is continuos and injective.

Let $\Omega' \subset \boldsymbol{R}^m$ be an open set. We say that $f \in \mathcal{G}(\Omega;\boldsymbol{R}^m)$ is valued in Ω' if there exists a representative $\widehat{f} \in \mathcal{E}_M[\Omega;\boldsymbol{R}^m]$ of f which the following property

$$\forall K \subset\subset \Omega, \exists K' \subset\subset \Omega', \exists \eta \in \mathbf{I} \text{ such that } \widehat{f}(\mathbf{I}_\eta \times K) \subset K'.$$

It is easily seen that all representatives of f will have the same property. We define the subset of $\mathcal{G}(\Omega;\boldsymbol{R}^m)$:

$$\mathcal{G}_*(\Omega;\Omega') := \{f \in \mathcal{G}(\Omega;\boldsymbol{R}^m) \mid f \text{ is valued in } \Omega'\}.$$

Note that $\mathcal{G}_*(\Omega;\Omega') \subset \mathcal{F}(\widetilde{\Omega}_c;\widetilde{\Omega'}_c)$ and that $\mathcal{G}_*(\Omega;\boldsymbol{R}^m) \subset \mathcal{G}(\Omega;\boldsymbol{R}^m)$ and are differents(indeed, it is enough to choose an inclusion $\boldsymbol{i} : \mathcal{D}'(\Omega) \hookrightarrow \mathcal{G}(\Omega;\boldsymbol{R})$ and remark that $\boldsymbol{i}(\delta) \notin \mathcal{G}_*(\Omega;\boldsymbol{R})$). It can be proved that (see [F, Prop. 1.1.20]) if $f \in \mathcal{G}(\Omega;\boldsymbol{R}^m)$, $\Omega = \cap_{\lambda\in\Lambda}\Omega_\lambda$, where Ω_λ is an open set for each $\lambda \in \Lambda$, and $f_\lambda := f|_{\Omega_\lambda} \in \mathcal{G}_*(\Omega_\lambda;\Omega')$ for every $\lambda \in \Lambda$, then $f \in \mathcal{G}_*(\Omega;\Omega')$.

Moreover (see [F, Lema 1.1.22]) if $f \in \mathcal{G}_*(\Omega;\Omega')$, $g \in \mathcal{G}(\Omega';\boldsymbol{R}^p)$, $\widehat{f}$ and $\widehat{g}$ are representatives of f and g respectively and $\mathcal{K} = (K_\nu)_{\nu\in N}$ is an exhaustive sequence of compact sets of Ω, then there exists a sequence $(\eta_\nu)_{\nu\in N}$ in $\mathbf{I}$ such that $\eta_\nu > \eta_{\nu+1}$ and $\widehat{f}(\mathbf{I}_{\eta_\nu} \times K_\nu) \subset\subset \Omega'$, for all $\nu \in \boldsymbol{N}$. For each $\nu \in \boldsymbol{N}$, the map $\widehat{h}_\nu : \mathbf{I} \times \overset{\circ}{K}_\nu \longrightarrow \boldsymbol{R}^p$ defined by

$$h_\nu(\varepsilon,x) := \begin{cases} \widehat{g}(\varepsilon,\widehat{f}(\varepsilon,x)) & ,if \quad (\varepsilon,x) \in \mathbf{I}_{\eta_\nu} \times \overset{\circ}{K}_\nu \\ \widehat{g}(\varepsilon,\widehat{f}(\eta_\nu/2,x)) & ,if \quad (\varepsilon,x) \in \mathbf{I}^c_{\eta_\nu} \times \overset{\circ}{K}_\nu \end{cases}$$

is moderate, i.e, $\widehat{h}_\nu \in \mathcal{E}_M[\overset{\circ}{K}_\nu;\boldsymbol{R}^p]$. If we set $h_\nu := cl(\widehat{h}_\nu) \in \mathcal{G}(\overset{\circ}{K}_\nu;\boldsymbol{R}^p)$ ($\nu \in \boldsymbol{N}$), then there exist an unique element $g \circ f$ belong to $\mathcal{G}(\Omega;\boldsymbol{R}^p)$ such that $(g\circ f)|_{\overset{\circ}{K}_\nu} = h_\nu$ for all $\nu \in \boldsymbol{N}$ and $g \circ f$ is well defined. This generalized map $g \circ f$ is called the composition of f with g. If $f \in \mathcal{C}^\infty(\Omega;\boldsymbol{R}^m)$ satisfies $f(\Omega) \subset \Omega'$ then clearly we have $f \in \mathcal{G}_*(\Omega;\Omega')$ and a representative of $g\circ f$ is the map $\widehat{g}\circ(1_{\mathbf{I}} \times f) : (\varepsilon,x) \in \mathbf{I}\times\Omega \longmapsto g(\varepsilon,f(x)) \in \boldsymbol{R}^p$. If in addition we assume that $g \in \mathcal{C}^\infty(\Omega;\boldsymbol{R}^p)$, then $g \circ f \in \mathcal{C}^\infty(\Omega;\boldsymbol{R}^p) \subset \mathcal{G}(\Omega;\boldsymbol{R}^p)$, coincides in the classical and in the generalized sense.

The result below, whose proof is not difficult, avoids the transference of information from $\mathcal{G}(\Sigma;\boldsymbol{R}^l)$ to $\mathcal{C}^\infty(\widetilde{\Sigma}_c;\overline{\boldsymbol{R}}^l)$ by using the map j^l_Σ.

Proposition 4.8 *Let $\Omega \subset \boldsymbol{R}^n$ and $\Omega' \subset \boldsymbol{R}^m$ be open sets, $f \in \mathcal{G}(\Omega;\boldsymbol{R}^m)$ and $g \in \mathcal{G}(\Omega';\boldsymbol{R}^p)$, then:*

1. *$f \in \mathcal{G}_*(\Omega;\Omega')$ if and only if $(j^m_\Omega(f))(\widetilde{\Omega}_c) \subset \widetilde{\Omega'}_c$ and, in this case, $j^p_\Omega(g\circ f) = j^p_{\Omega'}(g) \circ j^m_\Omega(f)$*
2. *$j^n_\Omega(1_\Omega) = 1_{\widetilde{\Omega}_c}$[here $1_A(x) := x \,\forall x \in A$, hence $1_\Omega \in \mathcal{C}^\infty(\Omega;\boldsymbol{R}^n) \subset \mathcal{G}_*(\Omega;\Omega)$ and therefore $j^m_\Omega(1_\Omega) \in \mathcal{C}^\infty(\widetilde{\Omega}_c;\overline{\boldsymbol{R}}^n)$].*
3. *If $n = m$, $f \in \mathcal{C}^\infty(\Omega;\boldsymbol{R}^n)$ is a $\mathcal{C}^\infty$-diffeomorphism and $f(\Omega) \subset \Omega'$ then $j^n_\Omega(f)$ is a $\mathcal{C}^\infty$-diffeomorphism of $\widetilde{\Omega}_c$ over $j^n_\Omega(f)(\widetilde{\Omega}_c)$.*

Proof. Note that 2. is trivial and that 3. follows at once from 1. and 2., hence it is enough to shows 1. It is not hard to see that $f \in \mathcal{G}_*(\Omega;\Omega')$ if and only if $(j^m_\Omega(f))(\widetilde{\Omega}_c) \subset \widetilde{\Omega'}_c$. Note that if $f \in \mathcal{G}_*(\Omega;\Omega')$, $\xi = cl[(\xi_\varepsilon)] \in \widetilde{\Omega}_c$ and $\widehat{g}$ is any representative of g then, a representative of $g(f(\xi))$ is the map

$$u : \varepsilon \in \mathbf{I} \longmapsto \widehat{g}(\varepsilon,\widehat{f}(\varepsilon,\xi_\varepsilon)) \in \boldsymbol{R}^p.$$

On the other hand, with the above notations on composition, if we choose $\nu \in \boldsymbol{N}$ such that $K \subset K_\nu$, a representative of $(g\circ f)(\xi)$ is the map

$$v : \varepsilon \in \mathbf{J} := \mathbf{I}_{\eta_\nu} \cap \mathbf{I}_\sigma \longmapsto h_\nu(\varepsilon,\xi_\varepsilon) = \widehat{g}(\varepsilon,\widehat{f}(\varepsilon,\xi_\varepsilon)) \in \boldsymbol{R}^p \; (v = 0 \text{ on } \mathbf{J}^c)$$

hence $(g \circ f)(\xi) = cl(v) = cl(u) = g(f(\xi))$, and so
$[j^p_{\Omega'}(g) \circ j^m_{\Omega}(f)](\xi) = j^p_{\Omega'}(g)(f(\xi)) = g(f(\xi)) = (g \circ f)(\xi) = j^p_{\Omega}(g \circ f)(\xi)$. ∎

The Proposition tells us that in this new setting composition of generalized functions obey the classing laws. This once again shows the consistancy of our proposal.

5 Holomorphic and Analytic Functions

In this section we shall define the notions of holomorphic functions and analytic functions. Here $\boldsymbol{K}$ shall always stand for $\boldsymbol{C}$, Ω denotes a non-void open set of $\boldsymbol{C}$, $\mathcal{H}(\Omega) = \{f \in C^1(\Omega; \boldsymbol{C}) : \overline{\partial} f = 0\}$ and $\mathcal{HG}(\Omega) = \{f \in \mathcal{G}(\Omega; \boldsymbol{C}) : \overline{\partial} f = 0\}$. It is obvious that $\overline{\boldsymbol{C}} = \overline{\boldsymbol{R}} + i\overline{\boldsymbol{R}}$, where $i^2 = -1$. So we considere $\overline{\boldsymbol{C}}$ to be $\overline{\boldsymbol{R}}$-isomorphic to $\overline{\boldsymbol{R}}^2$. If $z = x + iy$, with $x, y \in \overline{\boldsymbol{R}}$ then define the following operators:

$$\frac{\partial}{\partial z} := \frac{1}{2}\{\partial_x - i\partial_y\}, \quad \frac{\partial}{\partial \overline{z}} := \frac{1}{2}\{\partial_x + i\partial_y\}.$$

Our Embedding theorem of the last section gives us an Embedding theorem in the complex case in the obvious way. We also have an Open Mapping theorem in this case.

Theorem 5.1 *If $f \in \mathcal{H}(\Omega)$ is non-constant, then $(\iota(f))(\widetilde{W}_c)$ is an open subset for all open set $W \subset \Omega$.*

Proof. This follows at once from Proposition 4.5. ∎

Definition 5.2 *Let $f \in \mathcal{G}(\Omega)$. We shall say that f is* **sub-linear** *in Ω if there exists a representative $\widehat{f}$ of f with the following property:*

> *for all $x \in \widetilde{\Omega}_c$ there are (x_ε) representative of x, $k \in \boldsymbol{R}$, $(\eta_n)_{n \in N}$ sequence in $\mathbf{I}$ and $(c_n)_{n \in N}$ and $(p_n)_{n \in N}$ sequences in $\boldsymbol{R}$ such that $\lim\limits_{n \to \infty} (p_n + kn) = \infty$ and $|\widehat{f}^{(n)}(\varepsilon, x_\varepsilon)| \leq c_n \varepsilon^{p_n}$ for all $\varepsilon \in \mathbf{I}_{\eta_n}$ and $n \in \boldsymbol{N}$.*

Notice that the definition does not depend on the representative of f. It is immediate to verify that the set of all sub-linear functions of Ω is a $\boldsymbol{K}$-algebra of $\mathcal{G}(\Omega)$.

Example 5.3 *If* $f \in \mathcal{C}^\infty(\Omega) \subset \mathcal{G}(\Omega)$ *then* f *is sub-linear.*

Definition 5.4 *Let* $U \subset \overline{\boldsymbol{K}}$ *be an open subset and* $z_0 \in U$. *We say that* $f \in \overline{\boldsymbol{K}}^U$ *is analytic in* z_0 *if there exist a sequence* $(a_n)_{n\in N}$ *in* $\overline{\boldsymbol{K}}$ *and a series of the form* $\sum\limits_{n\geq 0} a_n(z-z_0)^n$ *which converges in a neighborhood of* z_0 *and such that* $f(z) = \sum\limits_{n\geq 0} a_n(z-z_0)^n$ *in this neighborhood. We say that* f *is analytic and we write* $f \in \mathcal{AG}(U)$ *if* f *is analytic in* z_0 *for all* $z_0 \in U$.

In the proof of the results below we use the following fact, which holds for general complete ultra-metrics abelian groups G but here we restrict our attention to the case $G = \overline{\boldsymbol{K}}$: *If* $(b_n)_{n\in N}$ *is a sequence in* $\overline{\boldsymbol{K}}$, *then* $\sum\limits_{n\geq 0} b_n$ *converges if and only if* $\lim\limits_{n\to\infty} ||b_n|| = 0$.

Theorem 5.5 *Let* $r > 0$, $z_0 \in \overline{\boldsymbol{K}}$ *and* $f(z) = \sum\limits_{n\geq 0} a_n(z-z_0)^n \in \mathcal{AG}(B_r(z_0))$. *Then* f *is differentiable and* $f'(z) = \sum\limits_{n\geq 1} na_n(z-z_0)^{n-1}$.

Proof. Let $z \in B_r(z_0)$ and $s > 0$ such that $||z - z_0|| < s < r$. As $\sum\limits_{n\geq 0} a_n(\alpha_{-\log s})^n$ converges we have, for $w \in B_s(z_0)$, that

$$\lim_{n\to\infty} ||na_n(w-z_0)^{n-1}|| \leq \lim_{n\to\infty} ||a_n|| s^{n-1} = \lim_{n\to\infty} ||a_n(\alpha_{-\log s})^n|| s^{-1} = 0$$

and thus $\sum\limits_{n\geq 1} na_n(w-z_0)^{n-1}$ converges uniformly on $B_s(z_0)$. Hence

$$\begin{aligned} f'(z) &= \lim_{w\to z} \frac{f(w)-f(z)}{\alpha_{-\log||w-z||}} = \lim_{w\to z}\lim_{m\to\infty} \sum_{1\leq n\leq m} a_n \frac{[(w-z_0)^n - (z-z_0)^n]}{\alpha_{-\log||w-z||}} \\ &\overset{(*)}{=} \lim_{m\to\infty}\lim_{w\to z} \sum_{1\leq n\leq m} a_n \frac{[(w-z_0)^n - (z-z_0)^n]}{\alpha_{-\log||w-z||}} \\ &= \lim_{m\to\infty} \sum_{1\leq n\leq m} a_n n(z-z_0)^{n-1} = \sum_{n\geq 1} na_n(z-z_0)^{n-1}. \end{aligned}$$

Note that $(\varphi_m)_{m\in N}$, where $\varphi_m(w) := \sum\limits_{1\leq n\leq m} a_n \frac{[(w-z_0)^n - (z-z_0)^n]}{\alpha_{-\log||w-z||}}$ is a Cauchy sequence in $B_s(z_0)$ and hence the change of the order of the limits in (*), follows from a classical result of Osgood, which clearly holds for function with values in $\overline{\boldsymbol{K}}$, since $\overline{\boldsymbol{K}}$ is complete. ∎

Corollary 5.6 *Let* $r > 0$, $z_0 \in \overline{\boldsymbol{K}}$ *and* $f(z) = \sum\limits_{n\geq 0} a_n(z-z_0)^n$ *for* $z \in B_r(z_0)$. *Then* $f \in C^\infty(B_r(z_0);\overline{\boldsymbol{K}})$ *and for* $k \in \boldsymbol{N}^*$ *one has* $f^{(k)}(z) = \sum\limits_{n\geq k} n(n-1)...(n-k+1)a_n(z-z_0)^{n-k}$. *In particular* $k!a_k = f^{(k)}(z_0)$.

Define in the obvious way, the set $\mathrm{HG}(U)$ of all holomorphic generalized functions on a non-void open subset $U \subset \overline{\boldsymbol{C}}$. Theorem 5.5 shows that $\mathcal{AG}(U) \subset \mathrm{HG}(U)$. Is $\mathrm{HG}(U) \subset \mathcal{AG}(U)$, i.e., does the classical Goursat Theorem extend to this new framework ? We do not have yet an answer for this question.

Theorem 5.7 *Let* $f \in \mathcal{G}(\Omega)$. *The following assertions hold.*

1. *If* $\iota(f)$ *is analytic, then* f *is sub-linear.*
2. *If* $f \in \mathcal{HG}(\Omega)$ *and* f *is sub-linear, then* $\iota(f)$ *is analytic and for all* $z_0 \in \widetilde{\Omega}_c$ *there is* $r \in]0,1[$ *such that for all* $z \in B_r(z_0)$ *one has* $\iota(f)(z) = \sum\limits_{n\geq 0} \frac{\iota(f)^{(n)}(z_0)}{n!}(z-z_0)^n$ *and this series converges uniformly in* $B_r(z_0)$. *Moreover* $\frac{\partial}{\partial \bar{z}}(\iota(f)) = 0$.

Proof. Let $z_0 \in \widetilde{\Omega}_c$ and suppose that $\iota(f)(z) = \sum\limits_{n\geq 0} a_n(z-z_0)^n$ for all $z \in B_R(z_0)$ with $R > 0$, where $n!a_n = \iota(f)^{(n)}(z_0)$ [see Corollary 5.6]. Let $r > 0$ such that $e^{-r} < R$ and $z = z_0 + \alpha_r$. As $||z - z_0|| < R$ one has that the series $\sum\limits_{n\geq 0} a_n(z-z_0)^n$ converges and so $0 = \lim\limits_{n\to\infty} ||a_n(z-z_0)^n|| = \lim\limits_{n\to\infty} e^{-V(\widehat{a_n})-nr}$ where $\widehat{a}_n$ is a representative of a_n. Let $\widehat{f}$ be a representative of f and $(z_{0\varepsilon})$ one of z_0. Take $k = r$, $c_n = 1$ and $p_n = V(\widehat{a}_n) - 1$ if $V(\widehat{a}_n) \in \boldsymbol{R}$ and $p_n = n$ if $V(\widehat{a}_n) = \infty$; then $\lim\limits_{n\to\infty}(p_n + nk) = \infty$ and $|\widehat{f}^{(n)}(\varepsilon, z_{0\varepsilon})| = n!|\widehat{a}_n| \leq c_n\varepsilon^{p_n}$ for ε sufficiently small.

To prove the second assertion let $(K_\nu)_{\nu\in \boldsymbol{N}}$ an exhaustive sequence of compact subsets of Ω such that $\overset{\circ}{K}_\nu$ is a C^∞-strictly pseudoconvex domain for each $\nu \in \boldsymbol{N}$. By [A-S, Lemma 1.1] there is a representative $\widehat{f}_\nu$ of f such that $\widehat{f}_\nu(\varepsilon,.) \in \mathcal{H}(\overset{\circ}{K}_\nu)$ for all $\varepsilon \in \mathbf{I}$ and $\nu \in \boldsymbol{N}$. Take $z_0 \in \widetilde{\Omega}_c$. As f is sub-linear there are $(\eta_n)_{n\in \boldsymbol{N}}$, a sequence in $\mathbf{I}$, $k \in \boldsymbol{R}$, $(z_{0\varepsilon})$, a representative of z_0, and $(c_n)_{n\in \boldsymbol{N}}$, $(p_n)_{n\in \boldsymbol{N}}$, sequences in $\boldsymbol{R}$, such that $\lim\limits_{n\to\infty}(p_n + kn) = \infty$ and $|\widehat{f}^{(n)}(\varepsilon, x_\varepsilon)| \leq c_n \varepsilon^{p_n}$ for all $\varepsilon \in \mathbf{I}_{\eta_n}$ and $n \in \boldsymbol{N}$. So if $0 < r < e^{-|k|}$ one has $\lim\limits_{n\to\infty} ||\frac{\iota(f)^{(n)}(z_0)}{n!}(z - z_0)^n|| \leq \lim\limits_{n\to\infty} e^{-p_n}e^{-kn} = 0$ for all $z \in B_r(z_0)$, and hence the series $\sum\limits_{n\geq 0} \dfrac{\iota(f)^{(n)}(z_0)}{n!}(z - z_0)^n$ converges uniformly in $B_r(z_0)$. Let $\nu \in \boldsymbol{N}$ and $s > 0$ such that $z_{0\varepsilon} \in K_\nu \subset \overset{\circ}{K}_{\nu+1}$ and $B'_s(z_{0\varepsilon}) \subset \overset{\circ}{K}_{\nu+1}$ for all $\varepsilon \in \mathbf{I}$. Since all the elements of $B_1(0)$ are associated to 0 and $\widehat{f}_\nu(\varepsilon, x) = \sum\limits_{n\geq 0} \dfrac{\widehat{f}_\nu^{(n)}(\varepsilon, z_{0\varepsilon})}{n!}(x - z_{0\varepsilon})^n$, for all $x \in B'_s(z_{0\varepsilon})$ and ε sufficiently small, we conclude that $\iota(f)(z) = \sum\limits_{n\geq 0} \dfrac{\iota(f)^{(n)}(z_0)}{n!}(z - z_0)^n$, for all $z \in B_r(z_0)$. Moreover as $\frac{\partial}{\partial \bar{z}}(\iota(f)) = \iota(\frac{\partial}{\partial \bar{z}} f)$ one has that $\frac{\partial}{\partial \bar{z}}(\iota(f)) = 0$. ▮

Note that in the proof of the above theorem we actually obtain a lower bound for the radius of convergence in each point.

Corollary 5.8 *Let $f \in \mathcal{HG}(\Omega)$. Then $\iota(f)$ is analytic if and only if f is sub-linear.*

6 Generalized Manifolds

Since, seemingly, there is no natural way to define generalized functions on classical C^∞-manifolds, we introduce here, as an alternative, the concept of *generalized manifold*. On these structures we can define generalized functions and do differential calculus.

Definition 6.1 *Let X be a set. A generalized atlas of class C^∞ and dimension N on X is a family $\mathcal{A} = ((\mathcal{U}_\lambda, u_\lambda))_{\lambda\in\Lambda}$ verifying the following conditions:*

1. $\emptyset \neq \mathcal{U}_\lambda \subset X$, $\forall\, \lambda \in \Lambda$ *and* $X = \cup_{\lambda\in\Lambda}\mathcal{U}_\lambda$*;*
2. $u_\lambda : \mathcal{U}_\lambda \longrightarrow u_\lambda(\mathcal{U}_\lambda)$ *is a bijective map from* $\mathcal{U}_\lambda$ *onto an open subset* $u_\lambda(\mathcal{U}_\lambda)$ *of* $\overline{\boldsymbol{R}}^N$ *for each* $\lambda \in \Lambda$ *and for every pair* $\lambda, \mu \in \Lambda$*, the set* $u_\lambda(\mathcal{U}_\lambda \cap \mathcal{U}_\mu)$ *is an open subset of* $\overline{\boldsymbol{R}}^N$*;*

3. *the map* $u_\mu \circ u_\lambda^{-1} : u_\lambda(\mathcal{U}_\lambda \cap \mathcal{U}_\mu) \longrightarrow u_\mu(\mathcal{U}_\lambda \cap \mathcal{U}_\mu)$ *is a* C^∞*-diffeomorphism for each* $(\lambda, \mu) \in \Lambda \times \Lambda$.

A generalized differentiable manifold of class C^∞ *and dimension* N *is a pair* $(X, \mathcal{A})$ *where* X *is a nonvoid set and* $\mathcal{A}$ *is a generalized atlas of class* C^∞ *and dimension* N *on* X.

In what follows we abbreviate the long names introduced in **Definition 6.1** by G-atlas of dimension N and G-manifold of dimension N(or simply G-atlas and G-manifold). There are a number of basic facts and terminology about classical C^∞-manifolds which extend easily to our case. Assume that X and $\mathcal{A}$ are as in Definition 6.1, then every pair $(\mathcal{U}_\lambda, u_\lambda)$ is called a chart of $\mathcal{A}$ and, in the sequel, an(abusive) statement of the type $(A, \varphi) \in \mathcal{A}$ means that there is $\lambda \in \Lambda$ such that $(A, \varphi) = (\mathcal{U}_\lambda, u_\lambda)$. If $\mathcal{U} \subset X$ and $u : \mathcal{U} \longrightarrow u(\mathcal{U})$ is a bijective map from $\mathcal{U}$ onto an open subset $u(\mathcal{U})$ of $\overline{\mathbb{R}}^N$, we say that $(\mathcal{U}, u)$ is compatible with $\mathcal{A}$ if the map $u_\lambda \circ u^{-1} : u(\mathcal{U} \cap \mathcal{U}_\lambda) \longrightarrow u_\lambda(\mathcal{U} \cap \mathcal{U}_\lambda)$ is a C^∞-diffeomorfhism $\forall\, \lambda \in \Lambda$. It follows that, for a given G-atlas $\mathcal{A}$ of dimension N on X, there exists an unique maximal G-atlas $\mathcal{A}^*$ of dimension N on X, defined by:

$$(\mathcal{U}, u) \in \mathcal{A}^* \Longleftrightarrow (\mathcal{U}, u) \in \mathcal{A} \text{ or } (\mathcal{U}, u) \text{ is compatible with } \mathcal{A}.$$

Next we show that there is a natural way of associating to each classical C^∞-manifold M of dimension N a G-manifold M^* of dimension N.

Lemma 6.2 *Let* X *be a set,* $\emptyset \neq A \subset X$ *and* $\psi : A \longrightarrow \psi(A) \subset \mathbb{R}^N$ *a bijective map. Then there exist a set* $A^*(\psi)$*, a natural map* $i_{A,\psi}$ *from* A *to* $A^*(\psi)$ *and a bijective map* $\psi^* : A^*(\psi) \longrightarrow \widetilde{\psi(A)}_c$ *making the following diagram commutative*

$$\begin{tikzcd}
A \arrow[r, "\psi"] \arrow[d, "i_{A,\psi}"'] & \psi(A) \arrow[d] \\
A^*(\psi) \arrow[r, "\psi^*"] & \widetilde{\psi(A)}_c
\end{tikzcd}$$

where the second vertical arrow is the natural imersion of $\psi(A)$ *into* $\widetilde{\psi(A)}_c$.

Proof. On the set $A_M(\psi) := \{(x_\varepsilon) \in A^{\mathbf{I}} \mid cl[(\psi(x_\varepsilon))] \in \widetilde{\psi(A)}_c\}$ we consider the equivalence relation $\approx$ defined by

$$(x_\varepsilon) \approx (y_\varepsilon) \Longleftrightarrow (\psi(x_\varepsilon) - \psi(y_\varepsilon)) \in \mathcal{N}(\mathbb{R})^N .$$

If we denote by $kl[(x_\varepsilon)]$ the equivalence class of $(x_\varepsilon) \in A_M(\psi)$ modulo $\approx$, it is clear that, for given (x_ε) and (y_ε) in $A_M(\psi)$, we have:

$$kl[(x_\varepsilon)] = kl[(y_\varepsilon)] \Longleftrightarrow cl[(\psi(x_\varepsilon))] = cl[(\psi(y_\varepsilon))]\,.$$

So we can define $A^*(\psi) := A_M(\psi)/\approx$ and

$$\psi^* : kl[(x_\varepsilon)] \in A^*(\psi) \longrightarrow cl[(\psi(x_\varepsilon))] \in \widetilde{\psi(A)}_c\,.$$

If for each $a \in A$ we denote by (a) the constant family (a_ε) defined by $a_\varepsilon := a$, for all $\varepsilon \in \mathbf{I}$, then we have $(a) \in A_M(\psi)$ and hence we get the map $i_{A,\psi} : a \in A \longrightarrow kl[(a)] \in A^*(\psi)$. Now, the commutativity of the diagram and the bijectivity of ψ^* follows at once. ∎

We can now give our example of a non-trivial G-manifold associated to a classical one.

Theorem 6.3 *Let $(M, \mathcal{A})$ be a $\mathcal{C}^\infty$-differentiable manifold of dimension N where $\mathcal{A} = ((\mathcal{U}_\lambda, \varphi_\lambda))_{\lambda\in\Lambda}$ and, with the notations of Lemma 6.2, set $M^* := \cup_{\lambda\in\Lambda} U^*_\lambda(\varphi_\lambda)$ and $\mathcal{A}^* := ((U^*_\lambda(\varphi_\lambda), \varphi^*_\lambda))_{\lambda\in\Lambda}$. Then $(M^*, \mathcal{A}^*)$ is a G-manifold of dimension N.*

Proof. In the proof we abbreviate the notation $U^*_\lambda(\varphi_\lambda)$ by U^*_λ. It is enough to show that $(M^*, \mathcal{A}^*)$ satisfies the conditions 6.1.1, 6.1.2 and 6.1.3, being clair that 6.1.1 and the first statement of 6.1.2 holds. Therefore, will be enough to verifies the following satements; for any $\lambda, \mu \in \Lambda$ given:

(I.) $U^*_\lambda \cap U^*_\mu \neq \emptyset \Rightarrow U_\lambda \cap U_\mu \neq \emptyset$,
(II.) $\varphi^*_\lambda(U^*_\lambda \cap U^*_\mu) = (\varphi_\lambda(\widetilde{U_\lambda \cap U_\mu}))_c$,
(III.) $\varphi^*_\mu \circ {\varphi^*_\lambda}^{-1} = j^m_{\varphi_\lambda(U_\lambda\cap U_\mu)}(\varphi_\mu \circ \varphi_\lambda^{-1})$.

Indeed, (I.) implies that (II.) make sense. From (II.) we get the last satement of 6.1.2. Since the domains of both members of (III.) are the two sets in (II.), this statement is a necessary condition for (III.) to be true. Finally, since $\varphi_\mu \circ \varphi_\lambda^{-1}$ is a $\mathcal{C}^\infty$-diffeomorphism from $\varphi_\lambda(U_\lambda \cap U_\mu)$ onto $\varphi_\mu(U_\lambda \cap U_\mu)$, the statement (III.) together Proposition 4.8(3) implies that $\varphi^*_\mu \circ {\varphi^*_\lambda}^{-1}$ is a $\mathcal{C}^\infty$-diffeomorphism from $\varphi^*_\lambda(U^*_\lambda \cap U^*_\mu)$ onto $\varphi^*_\mu(U^*_\lambda \cap U^*_\mu)$, which shows that 6.1.3 holds. Now, the verification of the above satements (I.), (II.) and (III.) follows from elementary and rather tedious set theory. ∎

We call $(M^*, \mathcal{A}^*)$ the G-manifold of dimension N associated to the $\mathcal{C}^\infty$-differentiable manifold $(M, \mathcal{A})$ of dimension N.

7 Applications

In [A-S], using [A-J, Prop. 2.5], the non-existence of solutions for a certain first order linear partial differential equation was proved. There it is asked whether this result could be generalized to any linear operator with constant coeficients. Our next result generalizes and gives an answer to this question. We make use of the Open Mapping Theorem in a simple but interesting way.

Theorem 7.1 *Let* Ω *be a connected open subset of* $\mathbb{C}^n$, $f \in \mathcal{H}(\Omega)$ *non-constant,* $L = \sum_{1 \leq k \leq m} a_k \frac{\partial}{\partial z_k}$ *a linear differential operator with constant coefficients* $a_1, ..., a_m \in \overline{\boldsymbol{K}}$ *and* J *the ideal generated by* $\{a_1, ..., a_m\}$. *If there exists* $u \in \mathcal{G}(\Omega)$ *such that* $L(u) = f$ *then* $J = \overline{\boldsymbol{K}}$.

Proof. Suppose that J is a proper ideal of $\overline{\boldsymbol{K}}$. If there is u be such that $L(u) = f$, then $Im(\iota(f)) \subset J$. By [A-J, Prop. 2.3], J is a rare subset of $\overline{\boldsymbol{K}}$ and hence $Im(\iota(f))$ would not be open. This contradicts Theorem 5.1. ∎

In the classical theory if $L = \sum_{1 \leq k \leq m} a_k D^k$ is a operator with constant coeficients, then a solution of $L(u) = 0$ is of the form $f(z) = exp(\lambda z)$ where $\sum_{1 \leq k \leq m} a_k \lambda^k = 0$. Let $f := exp \in \mathcal{C}^\infty(\boldsymbol{K})$ and $g := \iota(f) \in \iota(\mathcal{G}(\boldsymbol{K}))$. Then $g(z) = \sum_{n \in N} \frac{1}{n!} z^n$, for $z \in B_1(0)$(see Theorem 5.7) and $g(B_1(0)) \subset B_1(1) \subset Inv(\overline{\boldsymbol{K}})$[A-J, Corollary 2.10]. Consider $a_k \in \overline{\boldsymbol{K}}$, $1 \leq k \leq m$ and look for a solution of $L(u) = 0$ of the form $h(z) = g(\lambda z)$. Since $g(\lambda z)$ is a unit, we must have that $\sum_{1 \leq k \leq m} a_k \lambda^k = 0$ and so we must solve this characteristic equation. Our next result shows that this classical way does not work in this generalized case. We will use some results and notation that are in [A-J, sec. 4].

Proposition 7.2 *Let* $A \subset \mathrm{I}$ *such that* $0 \in \overline{A} \cap \overline{A^c}$ *and* L *the differential operator defined by* $L = \chi_A D^2 + \chi_{A^c} Id$. *Then the solutions of* $L(u) = 0$ *are all of the form* $\chi_A f$ *with* $D^2 f = 0$ *and* $\chi_A \lambda^2 + \chi_{A^c} \neq 0$ *for all* $\lambda \in \overline{\boldsymbol{K}}$.

Proof. If $D^2 f = 0$ then it is immediate that $L(\chi_A f) = 0$. Let f be such that $L(f) = 0$. As $\chi_A D^2 f + \chi_{A^c} f = 0$ we have that $\chi_A[\chi_A D^2 f + \chi_{A^c} f] = 0$. Thus $\chi_A D^2 f = 0$ and $\chi_{A^c} f = 0$. As $1 = \chi_A + \chi_{A^c}$ we conclude that $f = \chi_A f$

and $D^2 f = 0$. Note that if $\chi_A \lambda^2 + \chi_{A^c} = 0$, then $\chi_{A^c}[\chi_A \lambda^2 + \chi_{A^c}] = 0$. So $\chi_{A^c} = 0$ and this is a contradiction. ∎

The Embedding Theorem tells us that in $\iota(\mathcal{G}(\Omega))$ functions are indeed determined by their derivatives. The example given in the second section does not belong to $Im(\iota)$. So we cannot expect a general uniqueness when solving equations.

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