

ORIGINAL RESEARCH PAPER

An Extended Poisson Family of Life Distribution: A Unified Approach in Competitive and Complementary Risks

Pedro L. Ramos^{ab*}, Dipak K. Dey^b, Francisco Louzada^a and Victor H. Lachos^b

^aInstitute of Mathematical Science and Computing, University of São Paulo, São Carlos, Brazil

^bDepartment of Statistics, University of Connecticut, Storrs, CT, USA

ARTICLE HISTORY

Compiled September 18, 2019

ABSTRACT

In this paper, we introduce a new approach to generate flexible parametric families of distributions. These models arise on competitive and complementary risks scenario, in which the lifetime associated with a particular risk is not observable; rather, we observe only the minimum/maximum lifetime value among all risks. The latent variables have a zero-truncated Poisson distribution. For the proposed family of distribution, the extra shape parameter has an important physical interpretation in the competing and complementary risks scenario. The mathematical properties and inferential procedures are discussed. The proposed approach is applied in some existing distributions in which it is fully illustrated by an important data set.

KEYWORDS

Extended Poisson family distribution; instantaneous failures; generalized Poisson distribution; zero-truncated Poisson distribution.

1. Introduction

There has been a renewed interest among researchers in presenting new class of distribution for describing these problems. For instance, Marshall and Olkin [17] introduced a new procedure for adding new parameters in common distributions. The authors showed that depending on the values of the new parameter, the new distribution may arrive from the minimum or maximum, where the latent variable follows a geometric distribution and each of components in risk came from a baseline distribution. Further, many special cases of this family were considered, see for example, Barreto-Souza et al. [6] and the references therein.

Another distribution that has been considered as latent variable is the zero-truncated Poisson (ZTP) distribution. Kus [10] derived the exponential-Poisson (EP) distribution by taking the minimum among the lifetimes, where the baseline is the exponential distribution and the latent variable has ZTP distribution. Cancho et al. [7] follow the opposite way and considered the maximum, the obtained model there is known as Poisson-exponential (PE) distribution. Lu and Shi [15] presented the Weibull-Poisson distribution as generalization of EP distribution. Barreto-Souza and

*Corresponding author. Email: pedrolramos@usp.br

Cribari-Neto [5] discussed another generalized exponential-Poisson distribution by inserting a power parameter in EP distribution. In fact, Tahir and Cordeiro [22] reviewed more than 20 already introduced distributions based on the ZTP distribution.

Although the procedures used to generate the new distributions seems to produce models with different forms, we discussed a unified approach to construct these distributions by extending one of the parameters to the negative space. More importantly this unified approach does not include any additional parameter. For instance, Kus [10] and Cancho et al. [7] distributions can be merged into one only by considering the shape parameter into the positive and negative space. This distribution can be defined as extended exponential-Poisson distribution and its shape parameter has an important interpretation in competitive and complementary risk (CCR) scenarios, representing the lifetime of the minimum (maximum) according to the values of the shape parameter (negative or positive). Indeed, CCR problems arise in several areas such as, biomedical studies, reliability and demography. In CCR problems the lifetime associated with a particular risk is not observable; rather, we observe only the minimum or the maximum lifetime of all risks [14].

We also noted that positive parameter space has been extended into negative space for the inverse Gaussian and Gompertz distribution earlier (see, for instance, Balka et al. [3, 4]). Whitmore [23] introduces the term "defective" to deal with these types of distributions. In this case, the defected distributions have improper survival functions and can be used to model the cure fraction of patients [20]. Here, the term defected is avoided since the obtained distributions have proper survival functions. Further, we discuss the application of our approach for some particular cases.

The remainder of this paper is organized as follows: Section 2 presents the genesis and some mathematical properties of our proposed family of distributions. Section 3 shows the maximum likelihood (ML) estimators in the presence of censorship and its properties. Section 4 presents the application of our proposed approach in some common distribution. Section 5 illustrates the obtained models to fit an airplane lifetime data. Some final comments are made in Section 6.

2. Genesis and Properties

In this section, we discuss the genesis and some properties of the proposed family of distributions.

2.1. *Competitive risks*

Let Y_i ($i = 1, 2, \dots$) denote the time-to-event due to the j -th competitive risks and N be a random variable with a ZTP distribution indexed by parameter ϕ , hereafter $ZTP(\phi)$, given by

$$P(N = n) = \frac{\phi^n}{n!(e^\phi - 1)}, \quad n \in \mathbb{N} \text{ and } \phi > 0.$$

Now, let $X = \min \{Y_i\}_{i=1}^N$, where Y_i are independent of N and assumed to be independent and identically distributed according to a uniform distribution in the

interval (0,1). The conditional cumulative distribution function (cdf) of X is given by

$$F(x; N) = 1 - P(X > x; N) = 1 - (P(Y_1 > X))^N = 1 - (1 - x)^N.$$

Thus, the unconditional cdf of T is

$$F(x; \phi) = \sum_{n=1}^{\infty} \frac{\phi^n}{n!(e^\phi - 1)} - \sum_{n=1}^{\infty} \frac{\phi^n}{n!(e^\phi - 1)} (1 - (1 - x)^n) = \frac{1 - e^{-\phi x}}{1 - e^{-\phi}}.$$

Substituting $0 \leq x \leq 1$ by a generic cdf $F(x; \theta)$ we have that the pdf is given by

$$g(x; \theta, \phi) = \frac{\phi}{1 - e^{-\phi}} f(x; \theta) e^{-\phi F(x; \theta)}, \quad (1)$$

where $f(x; \theta)$ is the baseline distribution and $F(x; \theta)$ is the baseline cumulative function.

2.2. Complementary risks

Considering the competitive risk scenario, let Y_i ($i = 1, 2, \dots$) denote the time-to-event due to the j -th competitive risks and N follows a $ZTP(\lambda)$ distribution. Now, let $T = \max \{Y_i\}_{i=1}^N$ where Y_i are independent of N and assumed to be independent and identically distributed according to an uniform distribution in the interval (0,1). The conditional probability density function (pdf) of T given N is given by

$$f(t; N = n) = nt^{n-1}, t > 0 \quad n = 1, 2, \dots$$

Thus, the unconditional p.d.f. of T is given by

$$f(t; \lambda) = \sum_{n=1}^{\infty} nt^{n-1} \frac{\lambda^n e^{-\lambda}}{n!(1 - e^{-\lambda})} = \frac{\lambda e^{-\lambda}}{(1 - e^{-\lambda})} \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!} = \frac{\lambda e^{-\lambda - \lambda t}}{(1 - e^{-\lambda})}.$$

Substituting $0 \leq t \leq 1$ by a generic cdf $F(t; \theta)$ then the pdf is given by

$$g(t; \theta, \lambda) = \frac{\lambda}{1 - e^{-\lambda}} f(t; \theta) e^{-\lambda F(t; \theta)}, \quad (2)$$

where $f(t; \theta)$ is the pdf related to baseline distribution.

2.3. A unified approach

Note that, although (1) seems to differ from (2), if we assume that λ takes negative values, i.e., $\lambda = -\phi$, from (2) we have

$$f(t; \theta, \phi) = -\frac{\phi}{1 - e^{-\phi}} f(t; \theta) e^{\phi(1 - F(t; \theta))} = \frac{\phi}{1 - e^{-\phi}} f(t; \theta) e^{-\phi F(t; \theta)}, \quad (3)$$

which is the same family of distributions presented in (1) but considering $\phi < 0$. The relation in (3) holds since

$$-\frac{\phi}{1 - e^\phi} = \frac{\phi}{e^\phi - 1} = \frac{\phi}{e^\phi - 1} \frac{e^{-\phi}}{e^{-\phi}} = \frac{\phi e^{-\phi}}{1 - e^{-\phi}}.$$

Hence, both distributions can be unified in a simple form. Let $\mathbb{R}^* = \mathbb{R}/\{0\}$, if X has an extended Poisson-family of distributions then its cumulative distribution function is given by

$$G(t; \boldsymbol{\theta}, \lambda) = \frac{e^{-\lambda(1-F(t; \boldsymbol{\theta}))} - e^{-\lambda}}{1 - e^{-\lambda}} = \frac{e^{\lambda F(t; \boldsymbol{\theta})} - 1}{e^\lambda - 1}, \quad (4)$$

for all $t > 0$, where $\lambda \in \mathbb{R}^*$ is a shape and $\boldsymbol{\theta}$ is a vector of parameters related to the parametric baseline distribution. Additionally, the survival function $\bar{G}(\cdot)$ and the hazard function $h(\cdot)$ are given, respectively, by

$$\bar{G}(t; \boldsymbol{\theta}, \lambda) = \frac{1 - e^{-\lambda \bar{F}(t; \boldsymbol{\theta})}}{1 - e^{-\lambda}}, \quad \text{and} \quad h(t; \boldsymbol{\theta}, \lambda) = \frac{\lambda f(t; \boldsymbol{\theta})}{e^{\lambda \bar{F}(t; \boldsymbol{\theta})} - 1}. \quad (5)$$

The shape parameter of our class of models has an important interpretation in the competing and complementary risks scenario, for example, under the above assumptions if $\lambda < 0$ ($\lambda > 0$) then T represents the lifetime of the minimum (maximum) of Y_i . Moreover, as λ tends to zero, the new family of distribution converges to the baseline distribution (random). This family of distribution has an interesting property related to the quantile function, if the quantile function of the baseline distribution has closed-form expression the quantile function related to the composed distribution has also closed-form expression.

Proposition 2.1. *Let $F^{-1}(p; \boldsymbol{\theta})$ be the quantile function of $F(t; \boldsymbol{\theta})$, if $F^{-1}(p; \boldsymbol{\theta})$ has closed-form then $G^{-1}(p; \boldsymbol{\theta}, \lambda)$ also has closed form and $G^{-1}(p; \boldsymbol{\theta}, \lambda) = F^{-1}(q(p, \lambda); \boldsymbol{\theta})$ where*

$$q(p, \lambda) = \frac{\log((e^\lambda - 1)p + 1)}{\lambda}. \quad (6)$$

Proof. We have that $G(t; \boldsymbol{\theta}, \lambda) = p \Leftrightarrow G^{-1}(p; \boldsymbol{\theta}, \lambda) = t$ by definition. But

$$\begin{aligned} G(t; \boldsymbol{\theta}, \lambda) = p &\Leftrightarrow \frac{e^{\lambda F(t; \boldsymbol{\theta})} - 1}{e^\lambda - 1} = p \Leftrightarrow e^{\lambda F(t; \boldsymbol{\theta})} = (e^\lambda - 1)p + 1 \Leftrightarrow \\ F(t; \boldsymbol{\theta}) &= \frac{\log((e^\lambda - 1)p + 1)}{\lambda} \Leftrightarrow t = F^{-1}(q(p, \lambda); \boldsymbol{\theta}), \end{aligned}$$

where $q(p, \lambda)$ is given by (6). Therefore $G^{-1}(p; \boldsymbol{\theta}, \lambda) = F^{-1}(q(p, \lambda); \boldsymbol{\theta})$. \square

2.4. Presence of instantaneous failures

When data to be modeled has the presence of instantaneous failures (inliers), standard distributions may not be suitable. For instance, in lifetime testing of electronic devices the occurrence of fail at time 0 may be observed due to inferior quality or construction

problem. Another example is in weather forecasts where the occurrence of dry periods without the presence of precipitation is very common, standard models such as Gamma, Weibull, Lognormal cannot be used. Although for our family of distribution the parametric baseline distributions are greater than zero, some of the compound models may allow the occurrence of zero value.

Proposition 2.2. *Let $h_F(t; \boldsymbol{\theta})$ be the baseline hazard function related to the cdf $F(t; \boldsymbol{\theta})$, then if $0 < h_F(0; \boldsymbol{\theta}) < \infty$, we have $g(0; \boldsymbol{\theta}, \lambda) > 0$.*

Proof. Note that

$$\begin{aligned} \lim_{t \rightarrow 0} g(t; \boldsymbol{\theta}, \lambda) &= \lim_{t \rightarrow 0} \frac{\lambda}{1 - e^{-\lambda}} f(t; \boldsymbol{\theta}) e^{-\lambda(1 - F(t; \boldsymbol{\theta}))} \\ &= \lim_{t \rightarrow 0} \frac{\lambda}{1 - e^{-\lambda}} h_F(t; \boldsymbol{\theta}) \bar{F}(t; \boldsymbol{\theta}) e^{-\lambda(\bar{F}(t; \boldsymbol{\theta}))} \\ &= \frac{\lambda}{e^\lambda - 1} h_F(0; \boldsymbol{\theta}). \end{aligned} \quad (7)$$

Since $\lambda(e^\lambda - 1)^{-1} > 0, \forall \lambda \in \mathbb{R}^*$, then for $0 < h_F(0; \boldsymbol{\theta}) < \infty$, we have $g(0; \boldsymbol{\theta}, \lambda) > 0$. \square

Therefore, depending on the behavior of the baseline hazard function, some of the resulting models will be capable to accommodate data with zero value.

3. Inference

In statistical inference, different procedures can be considered in order to obtain the parameter estimates of particular distributions [21]. The ML estimators are usually considered due to its attractive limiting properties such as consistency, asymptotic normality and efficiency [18].

Let T_i be the lifetime of i th component with censoring time C_i , which are assumed to be independent of T_i s and its distribution does not depend on the parameters, the data set is represented by $\mathcal{D} = (t_i, \delta_i)$, where $t_i = \min(T_i, C_i)$ and $\delta_i = I(T_i \leq C_i)$. The random censoring scheme has as special cases the type I and II censoring mechanism. The likelihood function for $\boldsymbol{\theta}$ is given by

$$\begin{aligned} L(\boldsymbol{\theta}, \lambda; \mathcal{D}) &= \prod_{i=1}^n g(t_i | \boldsymbol{\theta}, \lambda)^{\delta_i} \bar{G}(t_i | \boldsymbol{\theta}, \lambda)^{1 - \delta_i} \\ &= \left(\frac{\lambda}{1 - e^{-\lambda}} \right)^n \prod_{i=1}^n f(t_i | \boldsymbol{\theta})^{\delta_i} e^{-\lambda \delta_i \bar{F}(t_i | \boldsymbol{\theta})} \left(\frac{1 - e^{-\lambda \bar{F}(t_i | \boldsymbol{\theta})}}{\lambda} \right)^{1 - \delta_i}. \end{aligned}$$

The log-likelihood function is given by

$$\begin{aligned} l(\boldsymbol{\theta}, \lambda; \mathcal{D}) &= n \log \left(\frac{\lambda}{1 - e^{-\lambda}} \right) + \sum_{i=1}^n \delta_i \log f(t_i | \boldsymbol{\theta}) + \sum_{i=1}^n (1 - \delta_i) \log \left(\frac{1 - e^{-\lambda \bar{F}(t_i | \boldsymbol{\theta})}}{\lambda} \right) \\ &\quad - \lambda \sum_{i=1}^n \delta_i \bar{F}(t_i | \boldsymbol{\theta}). \end{aligned}$$

Under the assumption that the likelihood function is differentiable at $\boldsymbol{\theta}$ and λ . From the partial derivatives of the the log-likelihood function, the likelihood equations are

$$\frac{\partial l(\boldsymbol{\theta}, \lambda; \mathcal{D})}{\partial \lambda} = \frac{n}{\lambda} + \frac{n}{1 - e^\lambda} + \sum_{i=1}^n \frac{(1 - \delta_i) \left(\lambda \bar{F}(t_i | \boldsymbol{\theta}) - e^{\lambda \bar{F}(t_i | \boldsymbol{\theta})} + 1 \right)}{\lambda (e^{\lambda \bar{F}(t_i | \boldsymbol{\theta})} - 1)} - \sum_{i=1}^n \delta_i \bar{F}(t_i | \boldsymbol{\theta}),$$

$$\frac{\partial l(\boldsymbol{\theta}, \lambda; \mathcal{D})}{\partial \boldsymbol{\theta}} = \sum_{i=1}^n \delta_i \frac{\partial \log f(t_i | \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} + \sum_{i=1}^n \frac{\lambda(1 - \delta_i)}{e^{\lambda \bar{F}(t_i | \boldsymbol{\theta})} - 1} \frac{\partial \bar{F}(t_i | \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} - \lambda \sum_{i=1}^n \delta_i \frac{\partial \bar{F}(t_i | \boldsymbol{\theta})}{\partial \boldsymbol{\theta}}.$$

Setting the partial derivatives equal to zero, the solutions provide the ML estimates. In many cases, numerical methods such as Newton-Rapshon are required to find the solution of these nonlinear systems.

Under mild conditions the ML estimators of $\boldsymbol{\theta}, \lambda$ have an asymptotically Normal joint distribution given by

$$(\hat{\boldsymbol{\theta}}, \hat{\lambda}) \sim N[(\boldsymbol{\theta}, \lambda), H^{-1}(\boldsymbol{\theta}, \lambda)], \text{ as } n \rightarrow \infty, \quad (8)$$

where $H(\boldsymbol{\theta}, \lambda)$ is the observed information matrix where the elements are given by

$$\begin{aligned} H_{\lambda, \lambda}(\boldsymbol{\theta}, \lambda) &= \frac{n}{\lambda^2} - \frac{ne^\lambda}{(e^\lambda - 1)^2} + \sum_{i=1}^n (1 - \delta_i) \frac{\bar{F}(t_i | \boldsymbol{\theta}) e^{\lambda \bar{F}(t_i | \boldsymbol{\theta})}}{\lambda^2 (e^{\lambda \bar{F}(t_i | \boldsymbol{\theta})} - 1)^2} - \sum_{i=1}^n \frac{(1 - \delta_i)}{t_i^2} \\ &\quad + \sum_{i=1}^n \delta_i \bar{F}(t_i | \boldsymbol{\theta}), \\ H_{\lambda, \boldsymbol{\theta}}(\boldsymbol{\theta}, \lambda) &= - \sum_{i=1}^n \delta_i \frac{\partial \bar{F}(t_i | \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} + \sum_{i=1}^n \frac{(1 - \delta_i) \left(\lambda \bar{F}(t_i | \boldsymbol{\theta}) e^{\lambda \bar{F}(t_i | \boldsymbol{\theta})} - e^{\lambda \bar{F}(t_i | \boldsymbol{\theta})} + 1 \right)}{(e^{\lambda \bar{F}(t_i | \boldsymbol{\theta})} - 1)^2} \frac{\partial \bar{F}(t_i | \boldsymbol{\theta})}{\partial \boldsymbol{\theta}}, \\ H_{\boldsymbol{\theta}, \boldsymbol{\theta}}(\boldsymbol{\theta}, \lambda) &= \sum_{i=1}^n \frac{\lambda(1 - \delta_i)}{(e^{\lambda \bar{F}(t_i | \boldsymbol{\theta})} - 1)^2} \left(\lambda e^{\lambda \bar{F}(t_i | \boldsymbol{\theta})} \left(\frac{\partial \bar{F}(t_i | \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right)^2 - (e^{\lambda \bar{F}(t_i | \boldsymbol{\theta})} - 1) \frac{\partial^2 \bar{F}(t_i | \boldsymbol{\theta})}{\partial^2 \boldsymbol{\theta}} \right) \\ &\quad - \sum_{i=1}^n \delta_i \frac{\partial^2 \log f(t_i | \boldsymbol{\theta})}{\partial^2 \boldsymbol{\theta}} + \lambda \sum_{i=1}^n \delta_i \frac{\partial^2 \bar{F}(t_i | \boldsymbol{\theta})}{\partial^2 \boldsymbol{\theta}}. \end{aligned}$$

4. Application

In this section, we applied our proposed methodology for some common distributions.

4.1. Exponential distribution

Let Z_1, \dots, Z_N be a non-negative random sample with an exponential distribution where its cdf is given by $F(z; \beta) = 1 - e^{-\beta z}$, $\beta > 0$. Then, using (4) it follows that

$$g(t; \lambda, \beta) = \frac{\lambda \beta e^{-\beta t - \lambda e^{-\beta t}}}{1 - e^{-\lambda}}, \quad (9)$$

where $\lambda \in \mathbb{R}^*$ is the shape parameter. Although the p.d.f. has the same form as presented by Cancho et al. [7], by extending the shape parameter into \mathbb{R}^* we unify the PE distribution with the EP distribution [10] without adding an additional parameter. More importantly, the shape parameter of this extended exponential Poisson (EEP) distribution has a biological interpretation in terms of CCR problems, i.e., if $\lambda < 0$ ($\lambda > 0$) the activation mechanism is the minimum (maximum). Adamidis et al. [2] discussed a similar interpretation for the shape parameter of the extended exponential geometric distribution which is a unification of the exponential geometric distribution (minimum) [1] and the complementary exponential geometric distribution (maximum) [13].

The hazard function of EEP distribution is $h(t) = \lambda \beta e^{-\beta t - \lambda e^{-\beta t}} (1 - e^{-\lambda e^{-\beta t}})^{-1}$. Figure 1 gives examples of different shapes for the hazard function.

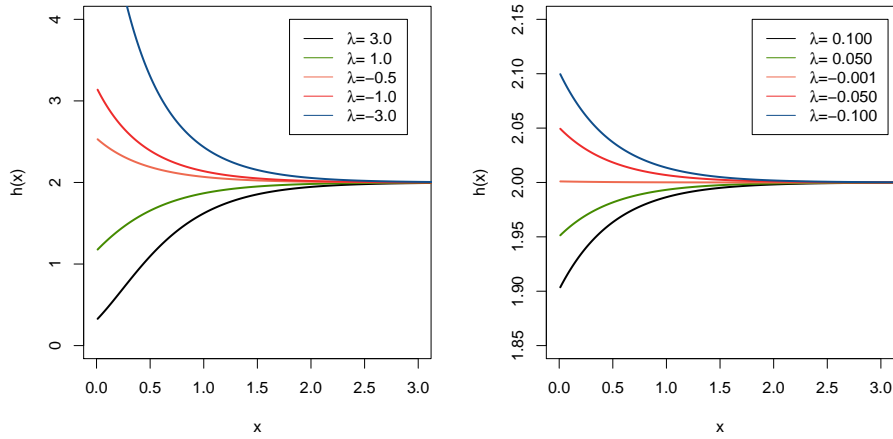


Figure 1. Hazard function shapes for EEP distribution considering different values of λ and $\beta = 2$.

Since the exponential distribution has quantile function in closed-form, then using the Proposition 2.1 the quantile function of the EEP distribution is given by $F^{-1}(q(p, \lambda), \beta) = -\beta^{-1} \log(1 - q(p, \lambda))$, where $q(p, \lambda)$ is given in (6). Additionally the hazard function of the exponential distribution is given by $h_F(0, \beta) = \beta$ and $0 < h_F(0, \beta) < \infty$. Then, we have $g(0; \lambda, \beta) = \beta \lambda (1 - e^{-\lambda})^{-1} \geq 0$, i.e., the EEP distribution also allow the occurrence of zero value.

The maximum likelihood estimators were discussed earlier by Kus [10] and Cancho et al. [7]. Let t_1, \dots, t_n be a random sample of size n from EPE distribution, the likelihood function is given by

$$L(\lambda, \beta; \mathbf{t}) = \left(\frac{\lambda \beta}{1 - e^{-\lambda}} \right)^n \exp \left\{ -\beta \sum_{i=1}^n t_i - \lambda \sum_{i=1}^n e^{-\beta t_i} \right\}. \quad (10)$$

Cancho et al. [7] presented the following log-likelihood function

$$\ell(\lambda, \beta; \mathbf{t}) = n \log(\lambda\beta) - \beta \sum_{i=1}^n t_i - \lambda \sum_{i=1}^n e^{-\beta t_i} - n \log(1 - e^{-\lambda}).$$

Some care must be taken with the EEP distribution, for instance, λ can take negative values then $n \log(\lambda)$ may not be computed. This problem is easily overcome by considering the fact that

$$\left(\frac{\lambda}{1 - e^{-\lambda}}\right)^n > 0 \Leftrightarrow n \log\left(\frac{\lambda}{1 - e^{-\lambda}}\right) \in \mathbb{R}, \quad \forall \lambda \in \mathbb{R}^*.$$

Therefore the log-likelihood function of (10) is given as

$$\ell(\lambda, \beta; \mathbf{t}) = n \log\left(\frac{\lambda}{1 - e^{-\lambda}}\right) + n \log(\beta) - \beta \sum_{i=1}^n t_i - \lambda \sum_{i=1}^n e^{-\beta t_i}. \quad (11)$$

4.2. Weibull distribution

Now, let Z_1, \dots, Z_N be a non-negative random sample with cdf given by $F(z; \alpha, \beta) = 1 - e^{-\beta t^\alpha}$ where $\beta > 0$ and $\alpha > 0$. From (4) we have

$$g(t; \lambda, \alpha, \beta) = \frac{\alpha \lambda \beta t^{\alpha-1} e^{-\beta t^\alpha - \lambda e^{-\beta t^\alpha}}}{1 - e^{-\lambda}}, \quad (12)$$

where $\lambda \in \mathbb{R}^*$. Hemmati et al. [8] discussed a particular case of (12) when $\lambda < 0$ (minimum) and named as Weibull-Poisson (WP) distribution. It is worth mentioning that Lu and Shi [15] independently developed the same distribution and named as WP distribution. Since, our new model also includes the maximum activation mechanism we could be named as extended Weibull-Poisson (EWP) distribution. The hazard function of EWP distribution is $h(t; \lambda, \alpha, \beta) = \lambda \beta t^{\alpha-1} e^{-\beta t^\alpha - \lambda e^{-\beta t^\alpha}} (1 - e^{-\lambda e^{-\beta t^\alpha}})^{-1}$. Figure 2 gives examples of different shapes for the hazard function.

Although Lu and Shi [15] had shown that WP distribution has increasing and decreasing hazard rate, the extended version also has decreasing-increasing-decreasing and unimodal hazard shape without adding an extra parameter. The Weibull distribution has quantile function in closed-form then using the Proposition 2.1 the quantile function of the EWP distribution is given by $F^{-1}(q(p, \lambda), \beta, \alpha) = (-\log(1 - q(p, \lambda)) / \beta)^{1/\alpha}$. Additionally, the hazard function of Weibull distribution is $h_F(t, \alpha, \beta) = \alpha \beta t^{\alpha-1}$ and $0 < h_F(0, \alpha, \beta) < \infty$ if and only if $\alpha = 1$. Then, the EWP distribution only allow the occurrence of zero value when its reduces to the EEP distribution.

4.3. Exponentiated exponential-Poisson distribution

A generalization of exponential-Poisson distribution was proposed by Barreto-Souza and Cribari-Neto [5] known as generalized exponential-Poisson (GEP) distribution. A

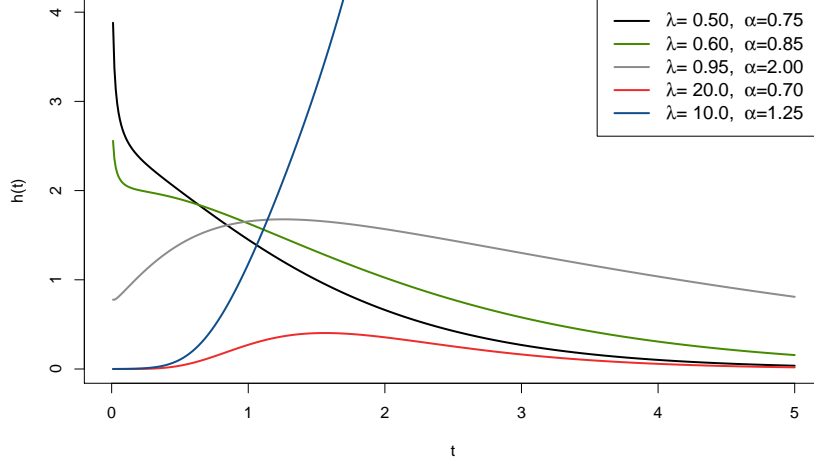


Figure 2. Hazard function shapes for EWP distribution for $\beta = 2$ and considering different values of λ and α .

random variable T with GEP has the p.d.f. given by

$$f(t; \phi, \beta, \alpha) = \frac{\alpha \phi \beta}{(1 - e^{-\phi})^\alpha} \left(1 - e^{-\phi + \phi e^{-\beta t}}\right)^{\alpha-1} e^{-\phi - \beta t + \phi e^{-\beta t}}, \quad (13)$$

where $t, \phi, \beta, \alpha > 0$. Note that including a power parameter in (4), we have

$$F(t; \theta, \lambda, \alpha) = \left(\frac{e^{-\lambda(1-F(t; \theta))} - e^{-\lambda}}{1 - e^{-\lambda}} \right)^\alpha, \quad (14)$$

where $\alpha > 0$. Following the same procedure described in Section 4.1, a generalized extended exponential-Poisson (GE2P) has p.d.f. given by

$$g(t; \beta, \lambda, \alpha) = \frac{\alpha \lambda \beta}{1 - e^{-\lambda}} \left(\frac{e^{-\lambda e^{-\beta t}} - e^{-\lambda}}{1 - e^{-\lambda}} \right)^{\alpha-1} e^{-\beta t - \lambda e^{-\beta t}}, \quad (15)$$

where $t, \beta, \alpha > 0$ and $\lambda \in \mathbb{R}^*$. The p.d.f. (15) is the same as (14) when $\lambda < 0$. The current form of the p.d.f. (15) is important since $1/(1 - e^{-\lambda})^\alpha$ may not be defined when $\lambda < 0$. The survival function of the GE2P distribution is given by

$$\bar{G}(t; \beta, \lambda, \alpha) = 1 - \left(\frac{e^{-\lambda e^{-\beta t}} - e^{-\lambda}}{1 - e^{-\lambda}} \right)^\alpha.$$

The hazard function is obtained from $h(t, \lambda, \beta, \alpha) = g(t, \beta, \lambda, \alpha) / \bar{G}(t, \beta, \lambda, \alpha)$. Barreto-Souza and Cribari-Neto [5] proved that the hazard function has decreasing, increasing or unimodal shape (for $\lambda < 0$). Although we have the same number of parameters, when $\lambda > 0$ the hazard function of the GE2P can have bathtub shape (see Figure 3).

Therefore, this simple extension of the GEP distribution has the hazard function with decreasing, increasing, bathtub or unimodal shape. Note that, the EGEP distribution is a double compounded distribution in which we applied firstly our approach

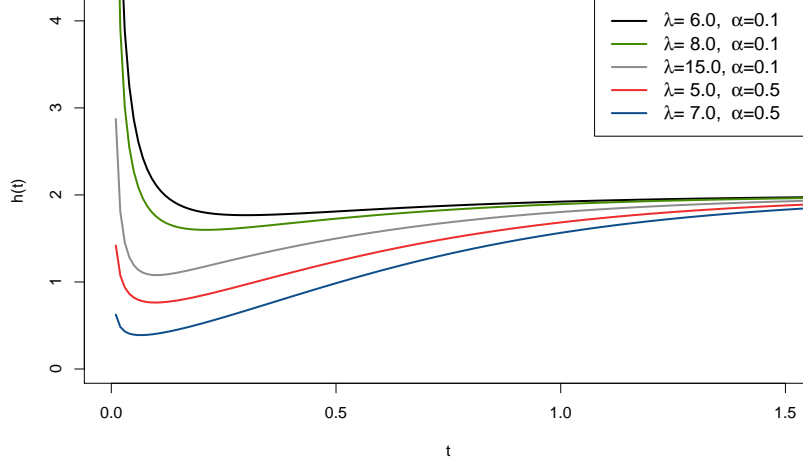


Figure 3. Hazard function shapes for GE2P distribution for $\beta = 2$ and considering different values of λ and α .

in the exponential distribution and secondly the Lehmann [11] approach. Therefore, since EEP distribution allow the occurrence of zero value the EGEP distribution has the same property. The quantile function of the EGEP distribution is given by $F^{-1}(q(p, \lambda), \beta, \alpha) = -\beta^{-1} \log(1 - q(p, \lambda)^{1/\alpha})$ where $q(p, \lambda)$ is given in (6).

4.4. Generalized extreme value distribution

The Generalized extreme value (GEV) distribution plays an important role in extreme value theory for modeling rare events. The GEV distribution [9] has as special cases the Gumbel, Fréchet and Weibull distribution, its cdf is given by

$$F(t|\sigma, \mu, \xi) = \begin{cases} \exp \left\{ -[1 + \xi(t - \mu)/\sigma]_+^{-1/\xi} \right\}, & \xi \neq 0 \\ \exp \left\{ -\exp[-(t - \mu)/\sigma] \right\}, & \xi = 0, \end{cases} \quad (16)$$

where $\sigma > 0$, $\mu, \xi \in \mathbb{R}$ and $t_+ = \max(t, 0)$. The extended generalized extreme value Poisson (EGEVP) distribution has the cdf given by

$$G(t; \lambda, \sigma, \mu, \xi) = \frac{e^{\lambda e^{-r(t; \sigma, \mu, \xi)}} - 1}{e^\lambda - 1}, \quad (17)$$

where

$$r(t; \sigma, \mu, \xi) = \begin{cases} [1 + \xi(t - \mu)/\sigma]_+^{-1/\xi}, & \xi \neq 0 \\ \exp[-(t - \mu)/\sigma], & \xi = 0. \end{cases}$$

The pdf of the EGEVP distribution has the following form

$$g(t; \lambda, \sigma, \mu, \xi) = \frac{\lambda r(t; \sigma, \mu, \xi)^{\xi+1}}{(1 - e^{-\lambda})\sigma} \exp \left\{ -\lambda(1 - e^{-r(t; \sigma, \mu, \xi)}) - r(t; \sigma, \mu, \xi) \right\}. \quad (18)$$

The EGEVP distribution (18) has as special cases the EEP distribution, EWP distribution, extended Gumbel-Poisson distribution, extended Fréchet-Poisson distribution,

to list a few. The quantile function has closed-form and is given by

$$F^{-1}(q(p, \lambda), \sigma, \mu, \xi) = \begin{cases} \mu + \frac{\sigma}{\xi} \left([-\log(q(p, \lambda))]^{-\xi} - 1 \right), & \xi \neq 0 \\ \mu - \sigma \log(-\log(q(p, \lambda))), & \xi = 0. \end{cases} \quad (19)$$

The hazard function is given by

$$h(t; \theta, \lambda) = \frac{\lambda r(t; \sigma, \mu, \xi)^{\xi+1} e^{-r(t; \sigma, \mu, \xi)}}{\sigma (e^{\lambda(1 - \exp\{-r(t; \sigma, \mu, \xi)\})} - 1)}. \quad (20)$$

Figure 4 gives examples of different shapes for the hazard function in which allows us to fit data with increasing, decreasing and unimodal hazard rate.

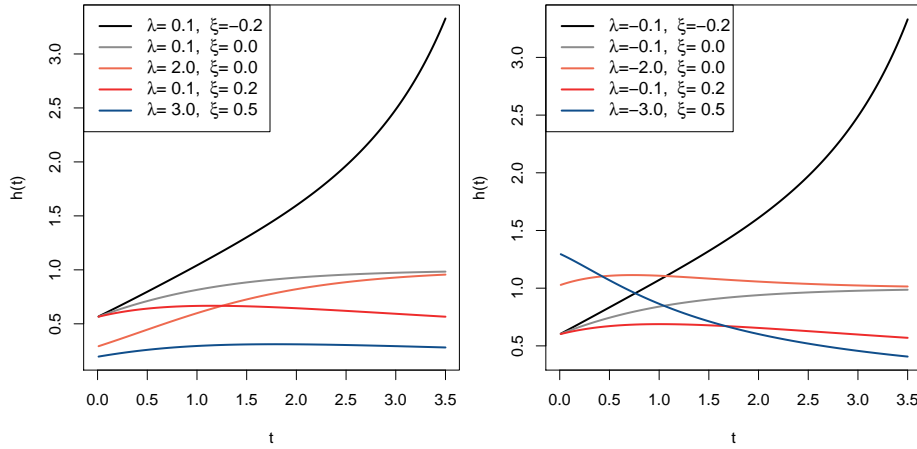


Figure 4. Hazard function shapes for EGEVP distribution for $\mu = 0, \sigma = 1$ and considering different values of λ and ξ .

4.5. Other compound models based on the Poisson distribution

Many distributions have already been proposed using the minimum or maximum activation mechanism. Tahir and Cordeiro [22] presented an interesting discussion about compounding different distributions. They reviewed some already introduced distributions based on the ZTP distribution such as: modified EP, exponentiated EP, Beta-Weibull Poisson, complementary modified Weibull-Poisson, complementary exponentiated Weibull-Poisson, Log-logistic generalized Weibull Poisson, Lai-modified Weibull-Poisson, Exponentiated Lomax-Poisson, complementary Poisson-Lomax, Lindley-Poisson, complementary extended Lindley-Poisson, Poisson Birnbaum-Saunders, exponentiated-Burr XII Poisson, complementary Burr III Poisson, complementary failure rate Poisson and the complementary exponentiated power Lindley-Poisson distribution (see [22] and references therein). Our approach can be applied in any of these distributions unifying the minimum/maximum without including an extra parameter.

This approach can be applied in various models used in other areas. For instance, Macera et al. [16] introduced a new model for recurrent event data characterized by a baseline rate function based on the exponential-Poisson distribution. Following Zhao and Zhou [24], the authors presented a rate model which is derived from a

nonhomogeneous Poisson process, with a hazard rate function $h(w; t) = h_0(w + t)$. The rate function of recurrence process $N(w + t_{j-1})$ is given by

$$h(w; t_{j-1}, \phi, \beta) = \frac{\phi \beta e^{-\beta(w+t_{j-1})}}{1 - \exp(-\phi e^{-\beta(w+t_{j-1})})}, \quad (21)$$

where $w, \phi, \beta > 0$. The rate function (21) has decreasing behavior. Further, Louzada et al. [12] developed a similar study based on the Poisson-exponential distribution. Both models can be unified in one where the p.d.f of the new distribution for recurrent event data where baseline rate function is EEP distribution is given by

$$f(w; t_{j-1}, \phi, \beta) = \frac{\phi \beta \exp(-\beta(w + t_{j-1})) + \phi e^{-\beta(w+t_{j-1})}}{\exp(\phi e^{-\beta t_{j-1}}) - 1}, \quad (22)$$

where $w, \beta > 0$ and $\phi \in \mathbb{R}^*$. The hazard function for $\phi > 0$ ($\phi < 0$) has decreasing (increasing) shape.

Table 1. The Bias(MSE) for the estimates of θ and λ considering different sample sizes.

Model	C	n	λ_1	β_1	α_1	CP(λ_1)	CP(β_1)	CP(α_1)
EEP	30%	50	0.2549(0.9006)	0.1708(0.4473)	————	97.90%	96.00%	————
		100	0.1064(0.4753)	0.0711(0.2316)	————	96.50%	94.80%	————
		200	0.0163(0.2392)	0.0163(0.1123)	————	95.60%	95.10%	————
		300	0.0243(0.1647)	0.0177(0.0754)	————	94.90%	94.70%	————
		400	0.0168(0.1221)	0.0143(0.0562)	————	94.70%	94.70%	————
	50%	50	0.2761(1.0738)	0.1864(0.6596)	————	97.50%	95.60%	————
		100	0.1630(0.5745)	0.1154(0.3599)	————	96.90%	95.90%	————
		200	0.0192(0.3256)	0.0174(0.1979)	————	96.80%	95.50%	————
		300	0.0056(0.2160)	0.0055(0.1279)	————	96.20%	95.40%	————
		400	0.0056(0.1684)	0.0063(0.0985)	————	95.20%	94.80%	————
EWP	30%	50	0.0265(2.2973)	-0.0057(0.2259)	0.0306(0.0227)	99.10%	97.90%	97.10%
		100	-0.3926(1.5546)	-0.1251(0.1515)	0.0478(0.0185)	97.90%	97.60%	95.90%
		200	-0.1407(0.9269)	-0.0464(0.0874)	0.0174(0.0089)	98.70%	98.60%	96.10%
		300	-0.1813(0.7587)	-0.0605(0.0760)	0.0180(0.0065)	98.20%	97.90%	96.40%
		400	-0.1584(0.5957)	-0.0536(0.0617)	0.0165(0.0050)	98.60%	98.10%	97.20%
	50%	50	0.8497(3.7558)	0.1942(0.2694)	-0.0259(0.0212)	99.80%	97.40%	96.70%
		100	-0.2244(1.5073)	-0.0641(0.1348)	0.0393(0.0182)	99.30%	99.00%	97.30%
		200	-0.0646(1.0365)	-0.0284(0.0839)	0.0174(0.0121)	98.90%	98.80%	96.10%
		300	-0.1240(0.9821)	-0.0443(0.0779)	0.0189(0.0101)	98.30%	97.90%	94.90%
		400	-0.2153(0.8449)	-0.0696(0.0775)	0.0244(0.0088)	98.00%	98.10%	95.00%
G2EP	30%	50	-0.3867(8.8114)	-0.1703(0.2602)	0.3855(1.9715)	96.00%	95.40%	95.10%
		100	-0.0695(4.8959)	-0.1201(0.1265)	0.0865(0.6004)	96.10%	96.50%	93.50%
		200	0.0706(2.3952)	-0.0865(0.0577)	-0.0558(0.2426)	95.20%	96.00%	92.90%
		300	0.1075(1.5907)	-0.0766(0.0381)	-0.1032(0.1592)	96.30%	95.30%	92.00%
		400	0.1123(1.1818)	-0.0726(0.0292)	-0.1239(0.1231)	96.50%	94.80%	91.10%
	50%	50	-0.6999(10.6788)	-0.2683(0.4341)	0.4395(2.1936)	97.60%	95.40%	95.60%
		100	-0.3131(6.5059)	-0.1973(0.2485)	0.1273(0.6688)	98.10%	96.10%	93.70%
		200	-0.0891(3.2484)	-0.1363(0.1185)	-0.0217(0.2730)	98.00%	97.20%	92.90%
		300	-0.0296(2.0539)	-0.1150(0.0748)	-0.0707(0.1774)	96.90%	96.90%	92.70%
		400	-0.0032(1.5007)	-0.1043(0.0550)	-0.0956(0.1347)	96.50%	96.10%	92.10%

5. Numerical evaluation

In this section a simulation study is presented to in order to check the efficiency of the ML estimates under random censoring by computing the bias and the mean square

Table 2. The Bias(MSE) for for the estimates of θ and λ considering different sample sizes.

Model	C	n	λ_1	β_1	α_1	CP(λ_1)	CP(β_1)	CP(α_1)
EEP	30%	50	0.1919(2.5781)	0.1175(0.1228)	—	98.10%	98.10%	—
		100	0.0581(2.0995)	0.0625(0.0520)	—	99.20%	97.70%	—
		200	-0.3527(2.4040)	0.0085(0.0410)	—	96.90%	94.20%	—
		300	-0.3098(1.2802)	-0.0094(0.0168)	—	99.10%	94.60%	—
		400	-0.0905(1.4225)	0.0220(0.0263)	—	97.30%	97.30%	—
	50%	50	0.2003(2.6696)	0.1170(0.1074)	—	99.90%	97.80%	—
		100	0.1508(2.6916)	0.0827(0.0718)	—	99.80%	99.10%	—
		200	0.1447(2.9435)	0.0908(0.0892)	—	98.00%	97.00%	—
		300	-0.4525(1.9526)	-0.0168(0.0207)	—	96.60%	91.90%	—
		400	-0.2489(1.5965)	0.0053(0.0258)	—	96.10%	94.30%	—
EWP	30%	50	1.3965(6.4446)	0.3083(0.2869)	-0.0231(0.0654)	92.00%	96.20%	95.80%
		100	0.8055(2.5338)	0.1431(0.0854)	0.0093(0.0284)	95.50%	98.50%	96.60%
		200	0.4309(1.1024)	0.0653(0.0249)	0.0112(0.0133)	97.80%	99.40%	96.10%
		300	0.1442(0.6853)	0.0252(0.0112)	0.0094(0.0092)	97.70%	99.60%	95.20%
		400	0.1388(0.6151)	0.0236(0.0101)	0.0025(0.0066)	97.50%	99.40%	95.00%
	50%	50	2.8575(11.6493)	0.6097(0.6121)	-0.0465(0.1151)	90.10%	95.40%	96.00%
		100	1.6103(5.9828)	0.3198(0.2502)	-0.0131(0.0499)	92.10%	97.30%	95.60%
		200	1.0107(3.1389)	0.1830(0.1148)	0.0089(0.0209)	93.10%	97.80%	96.30%
		300	0.5147(1.5409)	0.0867(0.0422)	0.0092(0.0138)	96.50%	98.40%	94.90%
		400	0.2043(0.8367)	0.0370(0.0175)	0.0086(0.0102)	98.00%	98.40%	94.90%
G2EP	30%	50	0.4124(1.2689)	0.0569(0.0148)	0.0200(0.0349)	98.00%	96.00%	95.90%
		100	0.2857(0.6519)	0.0327(0.0063)	-0.0039(0.0155)	97.50%	96.40%	94.90%
		200	0.1981(0.3934)	0.0229(0.0036)	-0.0058(0.0105)	96.90%	97.00%	95.30%
		300	0.1786(0.3091)	0.0190(0.0026)	-0.0108(0.0081)	96.10%	96.90%	94.50%
		400	0.0712(0.2904)	0.0098(0.0023)	-0.0144(0.0062)	95.50%	95.60%	91.10%
	50%	50	0.9410(2.4972)	0.1239(0.0400)	0.0123(0.0410)	97.60%	95.10%	95.40%
		100	0.5445(1.2879)	0.0667(0.0159)	-0.0012(0.0195)	96.30%	96.70%	95.30%
		200	0.4949(1.0424)	0.0552(0.0116)	-0.0158(0.0136)	94.30%	95.40%	94.70%
		300	0.4626(0.8084)	0.0503(0.0088)	-0.0162(0.0107)	93.10%	95.40%	93.40%
		400	0.4259(0.5968)	0.0408(0.0057)	-0.0181(0.0077)	91.60%	95.30%	90.50%

errors (MSE), given by

$$\text{Bias}_i \frac{1}{N} \sum_{j=1}^N (\hat{\Theta}_{i,j} - \Theta_i) \quad \text{and} \quad \text{MSE}_i = \sum_{j=1}^N \frac{(\hat{\Theta}_{i,j} - \Theta_i)^2}{N}, \quad i = 1, \dots, k,$$

where $\Theta = (\theta, \lambda)$ are the parameters related to $G(t; \Theta)$ and $N = 200,000$ is the number of estimates obtained through the ML estimators. Under this approach, the best estimators should provide both Bias and MSE closer to zero. In addition, the 95% coverage probability ($CP_{95\%}$) of the confidence intervals are also evaluated in which for a large N under 95% confidence level, the frequencies of intervals that covered the true values of Θ should be closer to 95%.

The simulation study was carry out using the software R and the sample sizes were $n = (50, 100, 200, 400)$. The distributions used in the simulation study are the EP, EW and the G2EP distribution. The chosen values to perform this study were $(\lambda_1, \beta_1) = (2, 3)$ and $(\lambda_2, \beta_2) = (-5, 0.5)$ for the EP distribution, $(\lambda_1, \beta_1, \alpha_1) = (3, 2, 0.5)$ and $(\lambda_2, \beta_2, \alpha_2) = (-3, 0.3, 1.5)$ for the EW distribution and $(\lambda_1, \beta_1, \alpha_1) = (2, 2, 1.5)$ and $(\lambda_2, \beta_2, \alpha_2) = (-1, 0.2, 0.8)$ for the G2EP distribution. However, the following results were similar for other choices of Θ . The samples were generated using random sampling with respectively 30% and 50% of censoring. Tables 1 and 2 present the Bias, MSEs and $CP_{95\%}$ for the obtained estimates.

From these results, we observed that both Bias and MSE tend to zero as there is an increase of n , i.e., the ML estimators are asymptotically unbiased for the parameters.

Moreover, the coverage probability tends to the nominal level as n increase. Other estimation procedures can be considered for these models. For instance, Rodrigues et al. [21] compared ten different estimation methods for the parameters of PE distribution under complete data and concluded that a minimum distance estimator provided better results than the ML estimators, a similar study can be conducted in the presence of censored data and for the other models. Additionally, it is important to point out that a simulation study was not presented for EGEVP, since its ML estimators showed to be non-identifiable, leading to different roots depending on the data set, in this case the conditions for the asymptotic properties were not fulfill and confidence intervals could not be constructed. Further research are need considering other estimation procedures for this particular model.

6. Application

In this section we considered a data set related to failure time of devices of an airline company collected in the period of 2014-2015. The study of its failure can prevent customer dissatisfaction and customer attrition which avoid company loss. Table 1 presents the data related to failure time of (in days) of 131 devices in an aircraft (+ indicates the presence of censorship).

Table 3. Data set related to the failure time of 131 devices in an aircraft.

36	15	4	9	20	2	127+	97	28	22	329+	158	43
45	21	24	9	84+	237	56+	18	2	1	2	9	4
1	2	1	19	20	3	1	2	1	6	10	7	33
16	2	17	10	8	30+	25	13	36	7	2	2	93
44	3	3	12	11	1	15	16	2	18	10	18	76
16	92	3	28	53+	29	46	11	94	95	1	33	40
22	12	15	46	20	53+	74	126	27	14	22	79	15
8	68	81+	51	7	2	20	24	11	16	3	42	2
10	52	5	46	5	37	14	40	95+	24	10	3	20
167+	44	8	1	18	28	17	11	10	16	79	20	55
115+												

Different discrimination criterion methods based on log-likelihood function evaluated at the ML estimates were also considered. The discrimination criterion methods are respectively: Akaike information criterion (AIC) computed through $AIC = -2l(\hat{\Theta}; \mathbf{t}) + 2k$ and the corrected Akaike information criterion $AICC = AIC + \frac{2k(k+1)}{(n-k-1)}$, where k is the number of parameters to be fitted and $\hat{\Theta}$ is the estimates of Θ . The best model is the one which provides the minimum values of those criteria. Table 4 presents the results of AIC, AICc criteria, for different probability distributions.

Table 4. Results of AIC, AICc criteria for different probability distributions considering the data set related to the failure time of 131 of devices in an aircraft.

Test	EP	EW	GE2P	EGEV
AIC	1084.38	1084.04	1085.73	1085.75
AICc	1084.48	1084.22	1086.05	1085.94

Figure 5 presents the survival function adjusted by different distributions and the Kaplan-Meier estimator.

Comparing the empirical survival function with the adjusted models we observed a goodness of the fit EW distribution. Additionally, from the results obtained by the AIC, AICc the EW returned the minimum value, i.e., among the proposed models the EW distribution fits better the data related to the failure time of aircraft devices. Table 5 displays the ML estimates, standard-error (SE) and 95% confidence intervals for λ , β and α .

Under this approach the data set allow us to discovery if the activation mechanism comes from the minimum or maximum. Since the ML estimates of λ returned negative value we concluded the activation mechanism comes from the minimum of Weibull distributions, i.e., if T_i , $i = 1, \dots, 131$ is our data set than T_i represents the lifetime of the minimum of $X_{i,j}$, $j = 1, \dots, N$ where $X_{i,j}$ follows a Weibull distribution and N is random and not observable.

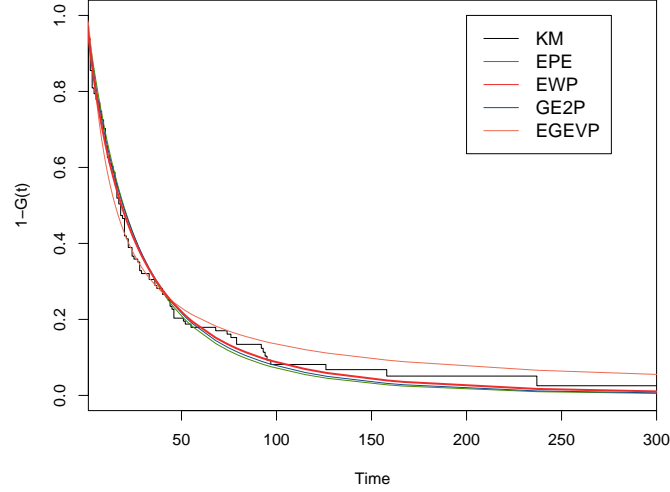


Figure 5. Survival function adjusted by different distributions and the Kaplan-Meier estimator considering data set related to the failure time of 131 devices.

Table 5. ML estimates (MLE), SE (standard errors) and 95% confidence intervals for λ , β and α

Θ	MLE	SE	$CI_{95\%}(\Theta)$
λ	-3.68674	3.09309	(-7.13376; -0.23971)
β	0.01463	0.00004	(0.00189; 0.02737)
α	0.89760	0.00466	(0.76379; 1.03141)

7. Concluding remarks

In this paper we proposed a new approach to generate flexible parametric families of distributions for modeling survival data. These models arise on CCR scenario, where the latent variables have a ZTP distribution. We observed that if $\lambda < 0$ ($\lambda > 0$) then random variable represents the lifetime of the minimum (maximum) among all elements in risk. Therefore, the extra shape parameter has an important physical interpretation in CCR modeling.

Moreover, we also proved that depending on the behavior of the baseline hazard function, some of the resulting models will be able to fit data with zero value (instantaneous failures). The parameter estimators are also discussed considering the ML estimation in the presence of random censoring. Furthermore, our proposed methodology is applied in common distributions such as Exponential, Weibull, among other. Many other distributions are also cited, for those our results are valid and may be applied further with success. Finally our proposed methodology is used to describe an real data set related to failure time of devices in an aircraft.

There are a large number of possible extensions of this current work. The presence of covariates as well as long-term survivals are very common in practice [19]. Our approach should be investigated in these contexts. Another possible approach is to consider bivariate versions using the idea presented by Marshall and Olkin [17].

Acknowledgment

We are grateful to two anonymous referees and the associate editor for very useful comments and suggestions, which greatly improved this paper. Pedro L. Ramos acknowledges support from to the São Paulo State Research Foundation (FAPESP Proc. 2017/25971-0).

Disclosure statement

No potential conflict of interest was reported by the author(s).

References

- [1] K. Adamidis and S. Loukas, *A lifetime distribution with decreasing failure rate*, Statistics & Probability Letters 39 (1998), pp. 35–42.
- [2] K. Adamidis, T. Dimitrakopoulou, and S. Loukas, *On an extension of the exponential-geometric distribution*, Statistics & probability letters 73 (2005), pp. 259–269.
- [3] J. Balka, A.F. Desmond, and P.D. McNicholas, *Review and implementation of cure models based on first hitting times for wiener processes*, Lifetime data analysis 15 (2009), pp. 147–176.
- [4] J. Balka, A.F. Desmond, and P.D. McNicholas, *Bayesian and likelihood inference for cure rates based on defective inverse gaussian regression models*, Journal of Applied Statistics 38 (2011), pp. 127–144.
- [5] W. Barreto-Souza and F. Cribari-Neto, *A generalization of the exponential-poisson distribution*, Statistics & Probability Letters 79 (2009), pp. 2493–2500.
- [6] W. Barreto-Souza, A.J. Lemonte, and G.M. Cordeiro, *General results for the marshall and olkin’s family of distributions*, Anais da Academia Brasileira de Ciências 85 (2013), pp. 3–21.
- [7] V.G. Cancho, F. Louzada-Neto, and G.D. Barriga, *The poisson-exponential lifetime distribution*, Computational Statistics & Data Analysis 55 (2011), pp. 677–686.
- [8] F. Hemmati, E. Khorram, and S. Rezakhah, *A new three-parameter ageing distribution*, Journal of statistical planning and inference 141 (2011), pp. 2266–2275.
- [9] A.F. Jenkinson, *The frequency distribution of the annual maximum (or minimum) values of meteorological elements*, Quarterly Journal of the Royal Meteorological Society 81 (1955), pp. 158–171.
- [10] C. Kuş, *A new lifetime distribution*, Computational Statistics & Data Analysis 51 (2007), pp. 4497–4509.
- [11] E.L. Lehmann, *The power of rank tests*, The Annals of Mathematical Statistics (1953), pp. 23–43.
- [12] F. Louzada, M.A. Macera, and V.G. Cancho, *The poisson-exponential model for recurrent event data: an application to bowel motility data*, Journal of Applied Statistics 42 (2015), pp. 2353–2366.
- [13] F. Louzada, M. Roman, and V.G. Cancho, *The complementary exponential geometric distribution: Model, properties, and a comparison with its counterpart*, Computational Statistics & Data Analysis 55 (2011), pp. 2516–2524.
- [14] F. Louzada-Neto, *Polyhazard models for lifetime data*, Biometrics 55 (1999), pp. 1281–1285.
- [15] W. Lu and D. Shi, *A new compounding life distribution: the weibull-poisson distribution*, Journal of Applied Statistics 39 (2012), pp. 21–38.
- [16] M.A. Macera, F. Louzada, V.G. Cancho, and C.J. Fontes, *The exponential-poisson model*

- for recurrent event data: An application to a set of data on malaria in brazil, *Biometrical Journal* 57 (2015), pp. 201–214.
- [17] A.W. Marshall and I. Olkin, *A new method for adding a parameter to a family of distributions with application to the exponential and weibull families*, *Biometrika* 84 (1997), pp. 641–652.
 - [18] H.S. Migon, D. Gamerman, and F. Louzada, *Statistical inference: an integrated approach*, CRC press, 2014.
 - [19] G.C. Perdoná and F. Louzada-Neto, *A general hazard model for lifetime data in the presence of cure rate*, *Journal of Applied Statistics* 38 (2011), pp. 1395–1405.
 - [20] R. Rocha, S. Nadarajah, V. Tomazella, and F. Louzada, *A new class of defective models based on the marshall–olkin family of distributions for cure rate modeling*, *Computational Statistics & Data Analysis* 107 (2017), pp. 48–63.
 - [21] G.C. Rodrigues, F. Louzada, and P.L. Ramos, *Poisson–exponential distribution: different methods of estimation*, *Journal of Applied Statistics* (2016), pp. 1–17.
 - [22] M.H. Tahir and G.M. Cordeiro, *Compounding of distributions: a survey and new generalized classes*, *Journal of Statistical Distributions and Applications* 3 (2016), pp. 1–35.
 - [23] G. Whitmore, *An inverse gaussian model for labour turnover*, *Journal of the Royal Statistical Society. Series A (General)* (1979), pp. 468–478.
 - [24] X. Zhao and X. Zhou, *Modeling gap times between recurrent events by marginal rate function*, *Computational Statistics & Data Analysis* 56 (2012), pp. 370–383.