

n° 26

UNITS OF GROUP RINGS

A short survey

C. Polcino Milies

Junho 1981

## UNITS OF GROUP RINGS

A short survey

C. Polcino Milies

§ 1 - INTRODUCTION.- Historically, group rings appeared for the first time in a paper by A. Cayley [14] which is also considered by many authors as the starting point of abstract group theory (e.g. Bourbaki [5] or M. Kline [40]). They were studied later by T. Molien [45],[46] and G. Frobenius [27] and earned a definitive status, in connection with group representation theory, after the work of R. Brauer and E. Noether [12],[13],[51] (regarding the history of group rings see [32]).

In recent times, the subject gained impetus after inclusion of questions on group rings in I. Kaplansky's famous lists of problems [37],[38]. Other important facts to stimulate the area were the inclusion of sections on group rings in the books on ring theory by J. Lambeck [41] and P. Ribemboim [73] as well as the publication of the first book entirely devoted to the subject, due to D.S. Passman [55].

Since then, several survey articles have appeared, namely, those by A.E. Zaleskii and A.V. Mikhalev [83], D.S.

Passman [56],[57], K. Dennis [23] and D. Farkas [25]. Also, new books on the subject have been published in recent years: A.A. Bovdi [8], I.B.S. Passi [53], D.S. Passman [59] and S.K. Sehgal [76].

Considerable work has been done lately on the structure and group-theoretical properties of the group of units of a group ring. In fact, the survey by K. Dennis [23] covers most of the results obtained up to 1977 and two chapters in S.K. Sehgal's book [76] include a large amount of new material.

We do not intend to give a full account of what has been done since then but rather describe briefly some of the progress done in connection with several specific problems and raise a few questions. Most of the results we shall cover here are either very recent or still unpublished.

We start by introducing some notations. We shall denote by  $RG$  the group ring of a group  $G$  over a commutative ring with unity  $R$ , by  $U(RG)$  the group of units of this ring and by  $V(RG)$  the subgroup  $V(RG) = \{ \mu \in U(RG) \mid \epsilon(\mu) = 1 \}$  where  $\epsilon: RG \rightarrow R$  denotes the augmentation mapping. As usual, we shall be interested in the cases where  $R$  is either  $\mathbb{Z}$ , the ring of rational integers;  $\mathbb{Z}_p$  the ring of  $p$ -adic integers or  $R=K$  a field.

Also, for an arbitrary group  $X$ , we shall denote by  $\zeta(X)$  the center of  $X$  and by  $T(X)$  the set of all elements of finite order in  $X$ , which we shall call the torsion set of  $X$  (or the torsion subgroup, whenever this is the case).

§ 2 - "ALGEBRAIC" QUESTIONS.- We start by considering the following question, which is listed as problem 35 in [76]:

(2.1) Problem.- When is  $U(RG)^n \subset \zeta(U(RG))$ ?

A closely related question is also stated in [17]:

(2.2) Problem.- Determine those groups  $G$  such that  $U(RG)/\zeta(U(RG))$  is torsion.

It should be noted that it is relevant to consider these situations since several problems are either related to or lead directly to them (see [76]):

(2.3) If  $G$  is a finitely generated group such that  $U(RG)$  is FC then  $U(RG)/\zeta(U(RG))$  is finite.

(2.4) Let  $\text{char}(R)=p$ , a rational prime. If  $RG$  is either Lie-nilpotent or Lie  $m$ -Engel then exists a positive integer  $n$  such that  $(RG)^{p^n} \subset \zeta(RG)$ .

(2.5) For a finite group  $G$ , it is easily seen that if  $G \triangleleft U(RG)$  then there exists a positive integer  $n$  such that  $U(RG)^n \subset \zeta U(RG)$ .

In the case where  $R=\mathbb{Z}$  several statements related to this question appear in [76, proposition II.2.14]. Also [76, theorem II.2.15] which appeared originally as lemmas 2.2 and 2.3 in [77] gives necessary conditions:

(2.6) Theorem. - Let  $G$  be a group such that  $U(RG)^n \subset \zeta(U(RG))$  for some  $n$ . Then  $G^m \subset \zeta(G)$  for some  $m$ ,  $T(G)$  is either abelian or a hamiltonian 2-group. Moreover, for any abelian subgroup  $T_1$  of  $T(G)$  and  $x \in G$  we have that either:

(i)  $x$  centralizes  $T_1$

or

(ii)  $x^{-1}tx = t^{-1}$  for all  $t \in T_1$ .

As a rule, it is probably very difficult to give answers to most of the questions we shall consider here in the general case. If we do not introduce certain restrictive hypothesis, we have little information about units available and it is likely that we would rapidly get involved with conjectures that have been open for quite a long time now, like the possibility of existence of non-trivial units in  $KG$  when  $G$  is torsion-free.

If we make the additional assumption that  $G$  is an extension of  $T(G)$  by a torsion-free nilpotent group, it is easily seen that the conditions in theorem (2.6) are also sufficient. In fact, in this case we have a good description of the units [76, theorem VI.3.22]; we know that  $U(ZG) = U(ZT).G$ . Thus, given  $\mu \in U(ZG)$  we can write  $\mu = vg$  with  $v \in U(ZT)$  and  $g \in G$ . Then  $\mu^m = v^*g^m$  where  $v^* \in U(ZT)$  and  $g^m \in \zeta(G)$  and [76, theorem II.4.1] shows that in this case  $U(ZT) = \pm T$ ; hence  $\mu^{m^2} \in \zeta(U(ZT))$ .

In the case where  $R=K$  a field, problem (2.1) has been fully answered for solvable or FC groups. If  $\text{char}(K)=0$  the results are due to G.H.Cliff and S.K.Sehgal [17] who also gave partial results where  $\text{char}(K)=p>0$  discussing separately the cases where  $n=p^r$  for some  $r$  or where  $p \nmid n$ .

(2.7) Theorem.- *Let  $G$  be a solvable or FC group and  $K$  a field of characteristic 0. Then  $U(KG)^n \subset \zeta(U(KG))$  for some  $n$  if and only if  $G^m \subset \zeta(G)$  for some  $m$  and  $T(G)$  is central in  $G$ .*

For  $\text{char}(K)=p>0$  a complete answer-for solvable or FC groups-was given by S.P.Coelho [19]:

(2.8) Theorem.- *Let  $G$  be a locally finite group. Then  $U(KG)^n \subset \zeta(KG)$  for some  $n$  if and only if the following conditions hold:*

- (i)  $G^m \subset \zeta(G)$  for some  $m$ .
- (ii)  $G$  contains a normal  $p$ -abelian subgroup of finite index.
- (iii) either every  $p'$ -element in  $G$  is central or  $G$  is of bounded exponent and  $K$  is finite.

(2.9) Theorem.- *Let  $G$  be a solvable or FC group which is not torsion. Then,  $U(KG)^n \subset \zeta(U(KG))$  for some  $n$  if and only if either  $KG$  is  $m$ -Engel for some  $m$  or the following conditions hold:*

- (i)  $G^S \subset \zeta(G)$  for some  $S$ .

(ii)  $A$ , the set of  $p'$ -elements in  $T(G)$  is an abelian subgroup of  $G$  and if  $A$  is not central, then it is of bounded exponent and for all  $a \in A$  and all  $x \in G$  there exists an integer  $r$  such that  $xtx^{-1} = tP^r$ . Furthermore, in the second case,  $K$  must be finite and  $[K:GF(p)] \mid r$  for all possible values  $r$ .

(iii)  $P$ , the set of all  $p$ -elements in  $T(G)$  is a subgroup of bounded exponent, centralizing  $A$  and, if  $P$  is not finite, then  $G$  contains a normal  $p$ -abelian subgroup of finite index.

We recall that those groups  $G$  such that  $KG$  is Lie  $m$ -Engel, where  $\text{char}(K) = p > 0$ , are completely described in [76, theorem V.6.1].

Many of the conditions in the theorem above might look familiar for those who have worked with group rings with polynomial identities. This is so because the techniques in [19] consist mainly in showing that, in several situations, if  $U(KG)^n \subset \zeta U(KG)$  for some  $n$ , then  $KG$  satisfies a polynomial identity. Also, we have been told by S.P. Coelho that these techniques had led to some progress in regard to problem 31 of [76]:

(2.10) Problem.- Characterize groups  $G$  such that  $\Delta(G)$  is nil.

Finally, problem (2.2) is solved in [17] for solvable or FC groups:

(2.11) Theorem.- Let  $G$  be a solvable or FC group and  $K$  a field of characteristic 0. Then  $U(KG)/\zeta(U(KG))$  is torsion if and only if  $G/\zeta(G)$  is torsion and  $T(G)$  is central.

(2.12) Theorem.- Let  $G$  be a solvable or FC group and  $K$  a field of characteristic  $p > 0$ . Then  $U(KG)/\zeta U(KG)$  is torsion if and only if  $G/\zeta(G)$  is torsion and one of the following conditions holds:

- (i)  $U(KG)$  is torsion.
- (ii) Every  $p'$ -element is central.
- (iii)  $K$  is algebraic over  $GF(p)$  and every idempotent in  $KG$  is central

It should be noted that the theorems in [17] are stated for solvable groups, but the proofs of those results which we mention here are also valid in the FC case.

We remark that question (2.2) is still opened for integral group rings and that no work has yet been done in the case where  $R$  is a ring of  $p$ -adic integers.

It is shown in [75, theorem II.2.14] that if  $G$  is a finite group such that  $G \triangleleft U(RG)$  then  $U(RG)^\cap \zeta(U(RG))$

for some  $n$ . Hence it seems reasonable, in this context, to consider the following:

(2.13) Problem.- Determine all groups  $G$  such that  $G \triangleleft U(RG)$ .

This appears, in a slightly more general form, as problem 8 of [23].

(2.14) Problem.- Which subgroups  $H$  of  $G$  are normal subgroups of  $V(RG)$ ?

Problem (2.14) has been considered by K.R.Pearson [61],[62], A.A. Bovdi [6],[7] and A.A. Bovdi and I.I. Khripta [9] in the case where  $G$  is a finite group. These results were mentioned in [23].

An answer to problem (2.13) was recently given by G.H.Cliff and S.K.Sehgal [16]:

(2.15) Theorem.-  $G$  is normal in  $U(\mathbb{Z}G)$  if and only if the following conditions are satisfied:

(i)  $T(G)$  is either abelian or a hamiltonian 2-group.

(ii)  $U(\mathbb{Z}G) = U(\mathbb{Z}T).G$

(iii) for any abelian subgroup  $T_1$  of  $T(G)$  and  $x \in G$  either  $x$  centralizes  $T_1$  or  $x^{-1}tx = t^{-1}$  for all  $t \in T_1$ .

(2.16) Theorem.- Let  $G$  be torsion free and  $R$  an integral domain. Then  $G$  is normal in  $U(RG)$  if and only if every unit

of  $RG$  is trivial.

(2.17) Theorem.- Let  $K$  be a field of characteristic  $p \geq 0$ . Then a nonabelian group  $G$  with torsion is normal in  $U(KG)$  if and only if one of the following conditions holds:

(i)  $K = GF(2)$  and  $G = S_3$ , the symmetric group on three letters.

(ii)  $p = 2$  and  $|G'| = |T(G)| = 2$

(iii)  $T(G)$  is a subgroup of  $\mathbb{Z}(q^\infty)$  for some prime  $q \neq p$ ,  $G'$  is of order  $q$  and:

(a) Whenever  $T(G)$  has an element of order  $q^n$ , the  $q^n$ -th cyclotomic polynomial  $\Phi_{q^n}$  is irreducible over  $K$ ,

(b) either  $T(G)$  is central in  $G$  or  $K = GF(2)$ ,  $|T(G)| = 3$  or  $5$  and if  $x \in G$  does not centralize  $T$  then  $t^x = t^{-1}$  for all  $t \in T$ .

In the same line as (2.14) one might ask:

(2.18) Problem.- Which subgroups  $H$  of  $G$  are subnormal in  $V(RG)$ ?

As far as we know, this question has been considered only by K.R.Pearson and T.E.Taylor [63], who answered it for subgroups of finite groups  $G$  over rings  $R$  such that  $\text{char}(R) \neq 0$ .

Closely related is the following:

(2.19) Problem.- Determine the normalizers (centralizers) of subgroups  $H$  of  $G$  in  $V(RG)$ .

D.B.Coleman has shown in [20] that if  $G$  is a finite  $p$ -group and  $\text{char}(K)=p>0$  then the normalizer of  $G$  in  $V(KG)$  is  $N_V(G)=G.\zeta(V(KG))$ . A reformulation of his argument actually gives a relation between normalizers and centralizers of certain groups:

(2.20) Proposition.- Let  $H$  be a finite  $p$ -subgroup of a group  $G$  and let  $K$  be a field of characteristic  $p$ . Then  $N_V(H)=G.Z_V(H)$ , where  $Z_V(H)$  denotes the centralizer of  $H$  in  $V(KG)$ .

§ 3 - GROUP-THEORETICAL PROPERTIES.- First, we turn our attention to problem 37 in [76]:

(3.1) Problem.- When is  $U(RG)$  an FC group?

An answer to the question in the case where  $R=\mathbb{Z}$  was given by S.K.Sehgal and H.J.Zassenhaus [77] and is included in [76] as theorem VI.5.3. The same paper gave a solution when  $R=K$ , a field of characteristic 0. Subsequently, the case where  $K$  is an infinite field with  $\text{char}(K)=p\neq 2$  was studied by C.Polcino Milies [67],[68] and a final solution was obtained by G.H.Cliff and S.K.Sehgal [15] where a complete answer is stated as follows:

(3.2) Theorem. - If  $K$  is a finite field of characteristic  $p$  and  $G$  a group which has no  $p$ -elements, then  $U(KG)$  is an FC group if and only if one of the following conditions holds:

(i)  $G$  is finite.

(ii)  $G$  is abelian.

(iii)  $G$  is an infinite non-abelian FC group with finite abelian torsion and such that every idempotent in  $K.T(G)$  is central in  $KG$ .

(iv)  $G$  is a non-abelian FC group with central torsion subgroup  $T(G) = \mathbb{Z}(q^\infty) \times B$ ,  $|B| < \infty$ ,  $q \neq p$ , such that  $G' \subset \mathbb{Z}(q^\infty)$ .

(3.3) Theorem. - If  $K$  is a field of characteristic  $p > 0$  and  $G$  contains an element of order  $p$ , then  $U(KG)$  is FC if and only if one of the following conditions holds:

(i)  $KG$  is finite.

(ii)  $G$  is abelian.

(iii)  $p=2$ ,  $|G'|=2$  and  $T(G) = G' \times A$  where  $A$  is finite, central, of odd order.

(3.4) Theorem. - If  $K$  is an infinite field of characteristic  $p \geq 0$  and  $G$  contains no  $p$ -elements (when  $p > 0$ ), then  $U(KG)$  is FC if and only if one of the following conditions holds:

(i)  $G$  is abelian

(ii)  $G$  is a non-abelian FC group with finite central torsion.

(iii)  $G$  is a non-abelian FC group with  $T(G)$  central of the form  $T(G) = \mathbb{Z}(q^\infty) \times B$ ,  $|B| < \infty$ ,  $G' \subset \mathbb{Z}(q^\infty)$ ,  $q \neq p$ .

Moreover, if  $k_\infty$  is the field obtained by adjoining all  $q^i$ -th roots of unity to the prime subfield  $k$  of  $K$  then  $[k_\infty : k] < \infty$ .

The case where  $R = \mathbb{Z}_{(p)}$ , a localization of  $\mathbb{Z}$  at a prime ideal  $(p)$  was studied by H. Merklen and C. Polcino Milies [43] and extended to other coefficient rings in an unpublished note by H. Merklen. We give a sketch of his arguments below.

(3.5) Theorem. - Let  $R$  be an integrally closed domain of characteristic 0, with quotient field  $K$  such that  $R \cap \mathbb{Q} = \mathbb{Z}_{(p)}$  for some rational prime  $p$ . Then  $U(RG)$  is FC if and only if one of the following conditions holds:

(i)  $G$  is abelian

(ii)  $G$  is an FC group such that  $T(G)$  is central; the subgroup of  $p'$ -elements in  $T(G)$  is either finite or of the form  $T' = \mathbb{Z}(q^\infty) \times H$ ,  $q \neq p$ ,  $G' \subset \mathbb{Z}(q^\infty)$ ,  $H$  is finite and there exists a natural number  $k$  such that  $K$  does not contain primitive roots of unity of order  $q^k$ .

The proof is quite similar to the one given in [43]. To show the necessity of the conditions, by using the arguments in [43], one needs only to show that (ii) holds. If not,  $K$  contains roots of unity of order  $q^m$  for all  $m > 0$ . It is then possible to determine, by induction, two sequences:

$$\xi_m \in R, \text{ such that } \xi_0 = 1 \text{ and } \xi_{m+1}^q = \xi_m.$$

$$z_m \in \mathbb{Z}(q^\infty), \text{ such that } z_0 = 1 \text{ and } z_{m+1}^q = z_m.$$

For  $\xi$  of order  $q^h \leq q^m$  consider the idempotent:

$$e(\xi, z_m) = \frac{1}{q^m} \sum_{0 \leq j \leq q^m - 1} \xi^j z_m^j \in RZ(q^\infty).$$

Following [43], it can be seen that  $z_1$  is a commutator of the form  $z_1 = (x, y)$  and considering the units  $\mu_m = e(\xi_m, z_m)x + (1 - e(\xi_m, z_m))$ , an easy computation shows that  $(\mu_m, y) = (\mu_m, , y)$  if and only if  $e(\xi_m, z_m) = e(\xi_m, , z_m)$  and hence all conjugates of the form  $\mu_m y \mu_m^{-1}$  are different from one another, a contradiction.

The proof of the sufficiency also follows as in [43] with some extra computations.

We would like to mention a closely related problem. Given a group  $G$ , its FC-subgroup  $\Phi(G)$  is defined as follows:

$$\Phi(G) = \{g \in G \mid g \text{ has a finite number of conjugates in } G\}$$

Clearly,  $G$  is FC if and only if  $G = \Phi(G)$  and it is easy to see that  $U(RG)$  is FC if and only if  $G \subset \Phi(U(RG))$ .

Now, we consider the group:

$$S_R(G) = \{g \in G \mid g \text{ has a finite number of conjugates in } U(RG)\} = G \cap \Phi(U(RG)).$$

which has been called the R-supercenter of  $G$ .

This subgroup of  $U(RG)$  might be of relevance to the isomorphism conjecture for integral group rings of finite groups since it will follow from theorem (3.8) that

in the finite case,  $S_{\mathbb{Z}}(G)$  is precisely the intersection of all the group bases of  $\mathbb{Z}G$ .

This group was first studied, for integral group rings, by S.K. Sehgal and H.J. Zassenhaus [79] who gave a description in the case where  $G$  is either torsion, nilpotent or FC and obtained results on the torsion subgroup  $T S_{\mathbb{Z}}(G)$  for arbitrary groups  $G$ . We shall not cover these results here since the statements are rather long and technical.

In the case where  $R=K$  is an infinite field, partial results have been given by C. Polcino Milies and S.K. Sehgal [71]. It is shown there that, for any group  $G$ ,  $T S_K(G)$  is central in  $T(G)$ . As a consequence it follows:

(3.6) Theorem. - *Let  $G$  be a torsion group and  $K$  an infinite field. Then  $S_K(G) = \zeta(G)$ .*

Also, it is shown:

(3.7) Theorem. - *Let  $K$  be an infinite field with  $\text{char}(K) = p > 0$  and  $G$  a group which contains a normal  $p$ -group. Then either  $S_K(G)$  is central or  $p=2$  and  $T S_K(G) = T(\zeta(G)) = \langle t \rangle \times A$ , where  $t$  is of order 2 and  $A$  is a finite group of odd order. Furthermore, for the commutator subgroup  $(G, S_K(G))$  we have that  $(G, S_K(G)) = \langle t \rangle$ .*

It seems more difficult to describe  $S_K(G)$  and  $T S_K(G)$  if  $G$  contains no  $p$ -element. In [71] some examples are given to illustrate the situations that may occur.

Other coefficient rings were studied by H. Merklein [42]. In the following theorems, we shall denote by  $B$  the intersection of all the group bases of the group ring  $RG$ .

(3.8) Theorem. - Let  $G$  be a torsion group.

(i) If  $R$  is an integral domain with  $\text{char}(R)=0$  and such that  $\{0 \neq (g) \mid g \in G\} \cap U(R) = \{1\}$ , then  $S_R(G) = B$ .

(ii) If  $R$  is a commutative ring containing a subring  $R'$  which is a domain of characteristic 0 with non-zero Jacobson radical and  $\{0 \neq (g) \mid g \in G\} \cap U(R') = \{1\}$ , then  $S_R(G) = \zeta(G)$ .

(iii) If  $R=R'$  then also  $T \nsubseteq (U(RG)) = S_R(G)$ .

(3.9) Theorem. - Let  $R$  be an integral domain such that no rational prime is a unit in  $R$ . Then  $TS_R(G) = T(B)$ .

As a whole, the groups  $T_R(G)$  are not well-known yet; particularly when  $R=K$ , a field of characteristic  $p>0$ , and  $G$  has no  $p$ -elements, not even for special classes of groups  $G$ . Also, no information about  $S_K(G)$ , when  $K$  is finite, has been given. Hence, we can still consider.

(3.10) Problem. - Determine  $S_R(G)$ .

If  $G$  is a finite group, it is easy to see that several group-theoretical properties are equivalent for  $U(\mathbb{Z}G)$ : FC, solvability, nilpotence and the fact that the torsion units are trivial or that they form a subgroup. Any of these holds if and only if  $G$  is either abelian or a Hamiltonian 2-group (see [76] and [52]).

It has been announced recently [29] that these equivalences are also valid if the coefficient ring is the ring of algebraic integers of a totally real algebraic number field.

B. Hartley and P. F. Pikel [31] have shown that if  $G$  is not an abelian group or a Hamiltonian 2-group, then  $U(\mathbb{Z}G)$  contains a free subgroup of rank 2. Also, they made the following conjecture, which we state as a separate problem:

(3.11) Problem.- Show that if  $G$  is a group such that  $U(\mathbb{Z}G)$  does not contain a free subgroup of rank 2 then every subgroup of  $G$  is normal in  $G$  and  $T(G)$  is either abelian or a Hamiltonian 2-group.

They have shown that this is so if  $G$  is assumed to be a solvable-by-finite group.

Nilpotent or FC groups such that the torsion units form a subgroup have been studied by C. Polcino Milies in [69] and [70].

The nilpotence of the group of units was studied by J. M. Bateman and D. B. Coleman [3], P. B. Bhattacharya and S. K. Jain [4], I. I. Khripta [39], J. Fisher, M. M. Parmenter and S. K. Sehgal [26], K. Motose and H. Tominaga [48], C. Polcino Milies [65], [66] and S. K. Sehgal and H. J. Zassenhaus [78]. Since most of these results are included in [76], we shall not discuss them here.

The residual nilpotence of  $U(\mathbb{Z}G)$  was studied recently by I. Musson and A. Weiss [50] who proved, for finite groups  $G$ , that  $U(\mathbb{Z}G)$  is residually nilpotent if and only if  $G$  is a nilpotent  $p$ -abelian group. The authors also consider the question when  $G$  is finitely generated nilpotent or finitely generated FC, but in this case the answer is rather technical and some particular cases are not covered.

The study of those groups  $G$  such that  $U(RG)$  is solvable was initiated independently by K. Motose and H. Tominaga [49] and J. M. Bateman [2], who considered mainly the case where  $R=K$ , a field, and  $G$  a finite group. Some oversights of [2] were corrected by K. Motose and Y. Ninomiya [47] and an alternative characterization was given by A. A. Bovdi and I. I. Khripta [10]. A nice exposition of these results was latter given by D. S. Passman [58]:

[3.12] Theorem. - *Let  $K$  be a field and let  $G$  be a finite group. Then  $U(KG)$  is solvable if and only if one of the following occurs.*

- (i)  $\text{Char } K=0$  and  $G$  is abelian.
- (ii)  $\text{Char } K=p>0$  and  $G/\Phi_p(G)$  is abelian.
- (iii)  $K=GF(2)$  and if we set  $\bar{G}=G/\Phi_2(G)$ , then  $\bar{G}=\bar{A}x_{\bar{G}}(\bar{x})$  is the semidirect product of the elementary abelian 3-group  $\bar{A}$  by the group  $(\bar{x})$  of order 2, where  $\bar{x}$  acts on  $\bar{A}$  by inverting its elements.

(iv)  $K = GF(3)$  and  $\bar{G} = G/\Phi_3(G)$  is a 2-group having an abelian subgroup  $\bar{A}$  of index 2 so that  $\bar{G} = (\bar{A}, \bar{x})$ . Furthermore, either (a).  $\bar{A}$  is elementary abelian, or (b).  $\bar{A}$  has period  $\leq 8$  and conjugation by  $\bar{x}$  maps each element of  $\bar{A}$  to its cube, or (c)  $[\bar{G} : \zeta(\bar{G})] = 4$  and  $\zeta(\bar{G})$  is elementary abelian.

A first step towards the study of the solvability of the group of units of not necessarily finite groups over fields was given by S.K. Sehgal [75] who proved that if  $K$  is a field of characteristic  $p \geq 0, p \neq 2, 3$ , and  $G$  is either a nilpotent or an FC group which contains no elements of order  $p$  then  $U(KG)$  is solvable if and only if every idempotent of  $KG$  is central and  $T(G)$  is abelian.

Torsion groups were studied by A.A. Bovdi and I.I. Khripta [11] who showed that if  $\text{char}(K) = p \geq 0, p \neq 2, 3$ , then  $U(KG)$  is solvable if and only if  $G'$  is a finite  $p$ -group. This readily implies that, for torsion groups  $G$ ,  $U(KG)$  is solvable if and only if  $KG$  is Lie-solvable [76, theorem V.4.6].

These statements are not equivalent in general, not even if  $p > 0$  and  $G$  contains a  $p$ -element, as is shown by the following example, due J.Z. Gonçalves [30]. Set

$$G = \langle a, b, c \mid a^{pq} = 1, [a, b] = 1, [a, c] = 1, [c, b] = a^p \rangle$$

and let  $K$  be a field of characteristic  $p$ . Set  $\pi: KG \rightarrow KG / \langle a^q \rangle$  the natural projection.

Since  $G/\langle a^q \rangle$  contains no  $p$ -elements, [75, proposition 4.5] shows that  $U(KG/\langle a^q \rangle)$  is solvable. Since  $\text{Ker}(\pi) = 1 + \Delta_K(G; \langle a^q \rangle)$  (where  $\Delta_K(G; \langle a^q \rangle)$  denotes the ideal of  $G$  generated by the set  $\{x-1 \mid x \in \langle a^q \rangle\}$ ) is nilpotent, it follows that  $U(KG)$  is solvable. On the other hand  $G/\langle a^q \rangle$  is not a  $p$ -group.

The known results in the case where  $R = \mathbb{Z}$  are due to S.K. Sehgal and are given in [76, theorem VI.4.8]. They were extended to  $p$ -adic group rings by J.Z. Gonçalves [30]:

(3.13) Theorem. - Let  $R = \mathbb{Z}_{(p)}$  and assume that  $U(RG)$  is solvable. Then  $T(G)$  is an abelian group such that every subgroup of  $T(G)$  is normal in  $G$ .

Conversely, if  $T(G)$  is as above,  $T(G)$  is a  $p$ -group and  $G/T(G)$  is nilpotent, then  $U(\mathbb{Q}_p G)$  is solvable, where  $\mathbb{Q}_p$  denotes the  $p$ -adic completion of the field of rational numbers.

Also, the following appears in [30]:

(3.14) Theorem. - Let  $G$  be a group such that  $T(G)$  is a subgroup and  $G/T(G)$  is nilpotent and let  $R$  be the ring of integers of a totally real algebraic number field. Then the following conditions are equivalent:

- (i)  $U(RG)$  is solvable
- (ii)  $T(G)$  is either abelian or a Hamiltonian 2-group such that every subgroup of  $T$  is normal in  $G$ .

(iii)  $T(U(RG))$  is a subgroup of  $U(RG)$ .

(3.15) Theorem.- If  $G$  is a torsion group and  $R$  is a ring of algebraic integers which contains a complex root of unity, then  $U(RG)$  is solvable if and only if  $G$  is abelian.

Theorems (3.14) and (3.15) above show that the structure of the ring of algebraic integers does have an influence on the unit groups. It might be interesting to complete the study of solvability, nilpotence and the FC property in this case. Also, one might study problem (3.11) in this context.

§ 4 - EXPLICIT COMPUTATIONS AND NORMAL COMPLEMENTS.-  $G$ .

Higman, in his famous paper [33] determined those groups  $G$  such that every unit in  $\mathbb{Z}G$  is trivial, i.e., such that  $U(\mathbb{Z}G) = \pm G$ . As a rule, it would be useful to know explicit examples of groups of units of group rings; in other words, it is natural to consider the following

(4.1) Problem.- Describe the group of units of the (integral) group ring of a given group (or family of groups).

In each particular case, once a description has been obtained, several other questions may be considered:

(4.2) Problem.- Decide if every unit of finite order in  $V(\mathbb{Z}G)$  is conjugate to a trivial unit.

or, alternatively:

How many conjugacy classes are there in  $V(\mathbb{Z}G)$  of subgroups of  $V(\mathbb{Z}G)$  conjugate to  $G$ ?

According to [34] this question was raised by H.J.Zassenhaus. It should be noted that it is closely related to problem 21 in [76] which, in turn, is relevant to the isomorphism conjecture and the study of automorphisms of group rings.

(4.3) Problem.- Let  $G$  be finite. Given an automorphism  $\gamma: \mathbb{Z}G \rightarrow \mathbb{Z}G$ , does there exist an automorphism  $\lambda: G \rightarrow G$  and a unit  $\alpha \in U(QG)$  such that  $\gamma(g) = \pm \alpha g^\lambda \alpha^{-1}$ , for all  $g \in G$ ?

Still another interesting question is the following:

(4.4) Problem.- Does  $G$  have a normal complement in  $V(RG)$ ; i.e., does there exist a normal subgroup  $F$  of  $V(RG)$  such that  $V(RG) = F.G$ ?

Clearly, if  $G$  has a torsion-free normal complement in  $V(\mathbb{Z}G)$  then  $G$  is determined by its integral group ring.

This question seems to have been considered for the first time by D.L.Johnson [35] (notice that this paper was only published in 1978, but was received by the editors in 1973), who considered the case where  $R=GF(p)$  and showed that finite abelian  $p$ -groups and the  $p$ -Sylow subgroup of  $GL(n,p)$  have normal complements.

The problem was also studied by K.Dennis [22] and the results included on his survey [23]. It is to these two papers that subsequent authors refer when considering the problem.

We return to specific examples. The group of units of  $\mathbb{Z}S_3$  and  $\mathbb{Z}D_4$  were studied by I.Hughes and K. R. Pearson [34] and by C. Polcino Milies [64] respectively, who gave, in each case, an answer to problems (4.2) and (4.3).

The group of units of  $\mathbb{Z}A_4$  was described by P.J. Allen and C.Hobby [1], who showed that all elements of order 2 in  $V(\mathbb{Z}A_4)$  are conjugate in this group, and by K. Sekiguchi [81] who also studied  $\mathbb{Z}S_4$  and showed that there are 4 conjugacy classes in  $V(\mathbb{Z}A_4)$  of subgroups of  $V(\mathbb{Z}A_4)$  isomorphic to  $A_4$  and that there are 16 conjugacy classes in  $V(\mathbb{Z}S_4)$  of subgroups of  $V(\mathbb{Z}S_4)$  isomorphic to  $S_4$  (these results were also announced in [80]). Both authors showed the existence of torsion-free normal complements.

The units of  $ZG$  for certain families of groups  $G$  have been studied recently.

D.S.Passman and P.F.Smith [60] gave an interesting characterization of the units  $ZD_p$ ,  $p$  an odd rational prime, and used it to show that if  $G$  contains an abelian subgroup of index 2, then  $G$  has a torsion-free normal complement in  $V(ZG)$ .

Also, T. Miyata [44] showed that  $D_n$  admits a torsion free normal complement in  $V(ZD_n)$  and proved that there exist  $\phi(n)/2$  conjugate classes in  $V(ZD_n)$  of subgroups of  $V(ZD_n)$  isomorphic to  $D_n$  if the order of the locally free class group  $C(ZD_n)$  of  $ZD_n$  is odd. As usual,  $\phi$  denotes Euler's Totient function. Also, it is announced in [80] that problem (4.2) was solved for  $D_n$ , where  $n$  is an arbitrary positive integer by S.Endo, T. Miyata and K. Sekiguchi [24]. See also K. Sekiguchi [82].

G.H. Cliff, S.K.Sehgal and A.R.Weiss [18] have shown that if  $G$  is metabelian, i.e., contains a normal subgroup  $A$  such that both  $A$  and  $G/A$  are abelian, and if  $G/A$  is of odd order or of exponent dividing 4 or 6 then  $G$  admits a torsion-free normal complement in  $V(ZG)$ . They actually give a description of a family of normal complements in several cases separately. It should be mentioned that K.Sekiguchi [81] also had a proof of the existence of a normal complement in the case where  $A=G'$ .

Some work has been done regarding the units of integral group rings of  $p$ -groups.

F.R.De Meyer and T.J.Ford [21] have considered group rings of cyclic groups of order  $p^n$  over a ring with no non trivial idempotent such that  $p.1 \in U(R)$ . In this context they prove that  $G$  has a normal complement in  $V(RG)$  if and only if  $\xi_{p^{n+1}} \notin U(R(\xi_{p^n}))$  where  $\xi_{p^i}$  denotes a primitive  $p^i$ -th root of unity. This is also equivalent to the fact that the  $p$ -Sylow subgroup of  $U(R(\xi_{p^n}))$  (and of  $U(RG)$ ) has order  $p^n$ . Also A.Jones [36] made some observations on the explicit computation of units of integral group rings of cyclic groups of order  $p^n$ .

Recently, J. Ritter and S.K.Sehgal [74] gave a characterization of the integral group rings of the two non-abelian groups of order  $p^3$ . In the special case  $p=3$  they were able to describe the units of the group ring as a group of  $3 \times 3$  matrices over  $\mathbb{Z}[w]$ , where  $w^3=1$ .

It might be worth mentioning that many of these papers, (eg. [36],[44],[74],[80],[81]) use an exact sequence derived from a "pull back" diagram as in I.Reiner and S.Ullom [72].

We conclude recalling problem 16 of [23]:

(4.5) Problem.- Let  $G$  be a finite group.  
Find generators and relations for  $U(\mathbb{Z}G)$ .

K. Dennis himself solved the problem in [23] for  $V(ZS_3)$  and  $V(ZD_4)$ . However, no further attempts seen to have been done in the new examples which are now available.

R E F E R E N C E S

1. P. J. Allen and C. Hobby, A characterization of units in  $\mathbb{Z}[A_4]$ , *J. of Algebra*, 66 (1980) 534-543.
2. J. M. Bateman, On the solvability of unit groups of group algebras, *Trans. Amer. Math. Soc.*, 157 (1971) 73-86.
3. J. M. Bateman and D. B. Coleman, Group Algebras with nilpotent unit groups, *Proc. Amer. Math. Soc.*, 19 (1968) 448-449.
4. P. B. Bhattacharya and S. K. Jain, A note on the adjoint group of a ring, *Archiv der Math.*, 21 (1970) 366-368.
5. N. Bourbaki, *Elements d'Histoire des Mathematiques*, Hermann, Paris, 1960.
6. A. A. Bovdi, Periodic normal divisors of the multiplicative group of a group ring, *Sibirsk. Mat. Zh.*, 9 (1968) 495-498.
7. A. A. Bovdi, Periodic normal divisors of the multiplicative group of a group ring II, *Sibirsk. Mat. Zh.*, 11 (1970) 492-511.
8. A. A. Bovdi, *Group Rings* (in Russian), Uzgorod, 1974.
9. A. A. Bovdi and I. I. Khripta, Normal subgroups of a multiplicative group of a ring, *Mat. U.S.S.R. Sb.*, 16 (1972) 349-362.
10. A. A. Bovdi and I. I. Khripta, Finite dimensional group algebras having solvable unit groups, in *Trans. Science Conference Uzgorod State University*, 1974, p.p. 227-233.
11. A. A. Bovdi and I. I. Khripta, Group algebras of periodic groups with solvable multiplicative groups, *Math. Notes. Acad. Sc. USSR*, 22, 3 (1977) 725-731.

12. R. Brauer, Über Systeme hypercomplexen Zahlen, *Math. Z.*, 30 (1929) 79-160.
13. R. Brauer and E. Noether, Über minimale Zarfällungskörper irreducibler Darstellungen, *Sitzber. Press. Akad. Wiss.*, (1927) 221-228.
14. A. Cayley, On the theory of groups as depending on the symbolic equation  $\theta^n=1$ , *Phil. Magazine*, 7 (1854) 40-47.
15. G. H. Cliff and S. K. Sehgal, Group rings whose units form an FC-group, *Math Z.*, 161 (1978) 163-168.
16. G. H. Cliff and S. K. Sehgal, Groups which are normal in the unit groups of their group rings, *Archiv der Math.*, 33, 6 (1979) 529-537.
17. G. H. Cliff and S. K. Sehgal, Group rings with units torsion over their center, *Manuscripta Math.*, 33 (1980) 145-158.
18. G. H. Cliff, S. K. Sehgal and A. R. Weiss, Units of integral group rings of metabelian groups, (to appear).
19. S. P. Coelho, Group rings with units of bounded exponent over their centers, (to appear).
20. D. B. Coleman, On the modular group ring of a p-group, *Proc. Amer. Math. Soc.*, 15, 4 (1964) 511-514.
21. F. R. De Meyer and T. J. Todd, On units of group rings, *J. of Pure and Appl. Algebra*, 16 (1980) 245-248.
22. K. Dennis, Units of group rings, *J. Algebra*, 43 (1976) 655-664.
23. K. Dennis, The structure of unit group of group rings, *Lecture note in Pure and Appl. Math.* 26, M. Dekker, New York, 1977.
24. S. Endo, T. Miyata and K. Sekiguchi, Picard groups and automorphism groups of integral group rings of metacyclic groups, (to appear).
25. D. Farkas, Group Rings: an annotated questionnaire, *Comm. in Algebra*, 8, 6 (1980) 585-602.

26. J. Fisher, M. M. Parmenter and S. K. Sehgal, Group rings with solvable  $n$ -Engel unit groups, *Proc. Amer. Math. Soc.*, 59 (1976) 195-200.
27. G. Frobenius, Theorie der hyperkomplexe Grössen, parts I and II, *Sitz. Königl. Preuss. Akad. Wiss. Berlin*, (1903) 504-537 and (1903) 634-645.
28. S. Galovich, I. Reiner and S. Ullom, Class groups for integral representations of metacyclic groups, *Mathematika*, 19 (1972) 105-111.
29. J. Z. Gonçalves, Group rings over totally real fields with solvable unit groups, *Proc. of the 13<sup>o</sup> Colóquio Brasileiro de Matemática* (to appear).
30. J. Z. Gonçalves, Group rings with solvable unit groups, (to appear).
31. B. Hartley and P. F. Pickel, Free subgroups in the unit groups of integral group rings, *Canad. J. of Math.*, 32, 6 (1980) 1342-1352.
32. T. Hawkins, Hypercomplex numbers, Lie groups and the creation of Group Representation Theory, *Arch. Hist. Exact. Sci.*, 8 (1972) 243-287.
33. G. Higman, The units of group rings, *Proc. London Math. Soc.*, 2, 46 (1940) 231-248.
34. I. Hughes and K. R. Pearson, The group of units of the integral group ring  $\mathbb{Z}S_3$ , *Canad. Math. Bull.*, 15 (1972) 529-534.
35. D. L. Johnson, The modular group ring of a finite  $p$ -group, *Proc. Amer. Math. Soc.*, 68, 1 (1978) 19-22.
36. A. Jones, Units of  $\mathbb{Z}C_p$ , (to appear).
37. I. Kaplansky, Problems in the theory of rings, NAS - NRC Publ. 502, Washington, 1957, pp 1-3.
38. I. Kaplansky, "Problems in the theory of rings" revisited, *Amer. Math. Monthly*, 77 (1970) 445-454.

39. I. I. Khripta, The nilpotence of the multiplicative group of a group ring, *Math. Notes*, 11 (1972) 119-124.
40. M. Kline, *Mathematical Thought from Ancient to Modern Times*, Oxford Univ. Press, New York, 1972.
41. J. Lambeck, *Lectures on Rings and Modules*, Blaidell, Toronto, 1966.
42. H. Merklen, On the supercenter of a group over domains of characteristic 0, *Proc. Amer. Math. Soc.* (to appear).
43. H. Merklen and C. Polcino Milies, Group rings over  $\mathbb{Z}_{(p)}$  with FC unit groups, *Canad. J. of Math.*, 32, 5 (1980) 1266-1269.
44. T. Miyata, On the units of integral group rings of a dihedral group, *J. Math. Soc. Japan*, 32, 4 (1980) 703-708.
45. T. Molien, Eine Bemerkung zur Theorie der homogenen Substitutionsgruppen, *S'ber. Naturforscher-Ges Univ. Jurjeff (Dorpat)* 11 (1897) 259-274.
46. T. Molien, Uber die Anzahl der Variablen einer irreductibelen Substituonsgruppe, *S'ber Naturforscher-Ges Univ. Jurjeff (Dorpat)* 11 (1897) 277-288.
47. K. Motose and Y. Ninomiya, On the solvability of the unit groups of group rings, *Math. J. Okayama Univ.*, 15 (1972) 209-214.
48. K. Motose and H. Tominaga, Group rings with nilpotent unit groups, *Math. J. Okayama Univ.*, 14 (1969) 43-46.
49. K. Motose and H. Tominaga, Group rings with solvable unit groups, *Math. J. Okayama Univ.*, 15 (1971) 37-40.
50. I. Musson and A. Weiss, Integral group rings with residually nilpotent unit groups (to appear).
51. E. Noether, Hypercomplexe Grössen und Darstellungstheorie, *Math. Z.*, 30 (1929) 641-692.

52. M. M. Parmenter and C. Polcino Milies, Group rings whose units form a nilpotent or FC group, *Proc. Amer. Math. Soc.*, 68, 2 (1978) 247-248.
53. I. B. S. Passi, *Group rings and their augmentation ideals* Lecture notes in mathematics N° 715, Springer-Verlag, Berlin, 1979.
54. I. B. S. Passi, D. S. Passman and S. K. Sehgal, Lie solvable group rings, *Canad. J. of Math.*, 25 (1973) 748-757.
55. D. S. Passman, *Infinite Group Rings*, M. Dekker, New York, 1971.
56. D. S. Passman, Advances in Group Rings, *Israel J. of Math.*, 19 (1974) 67-107.
57. D. S. Passman, What is a group ring?, *Amer. Math. Monthly*, 83, 3 (1976) 173-185.
58. D. S. Passman, Observations in group rings, *Comm in Algebra*, 5 (1977) 1119-1162.
59. D. S. Passman, *The Algebraic Structure of Group Rings*, Interscience, New York, 1977.
60. D. S. Passman and P. F. Smith, Units in integral group rings, (to appear).
61. K. R. Pearson, On the units of a modular group ring, *Bull. Austral. Math. Soc.*, 7 (1972) 169-182.
62. K. R. Pearson, On the units of a modular group ring II, *Bull. Austral. Math. Soc.*, 8 (1973) 435-442.
63. K. R. Pearson and D. E. Taylor, Groups subnormal in the units of their modular group rings, *Proc. London Math. Soc.*, 332 (1976) 313-328.
64. C. Polcino Milies, The units of the integral group ring  $\mathbb{Z} D_4$ , *Bol. Soc. Brasileira de Mat.*, 4 (1973) 85-92.
65. C. Polcino Milies, Integral group rings with nilpotent unit groups, *Canad. J. of Math.*, 28, 5 (1976) 954-960.

66. C. Polcino Milies,  $p$ -adic group rings with nilpotent unit groups, *J. Pure and Appl. Algebra*, 12 (1978) 380-384.
67. C. Polcino Milies, Group rings whose units form an FC group, *Archiv. der Math.*, 38 (1978) 380-384.
68. C. Polcino Milies, Group rings whose units form an FC group: Corrigendum, *Archiv. der Math.*, 31, 5 (1978) 528.
69. C. Polcino Milies, Group rings whose torsion units form a sub<sub>u</sub>group I, *Proc. Amer. Math. Soc.*, 81, 2 (1981) 172-174.
70. C. Polcino Milies, Group rings whose torsion units form a sub<sub>u</sub>group II, *Comm. in Algebra* (to appear).
71. C. Polcino Milies and S. K. Sehgal, FC-elements in group rings, *Comm. in Algebra* (to appear).
72. I. Reiner and S. Ullom, A Mayer-Victoris sequence for class groups, *J. of Algebra*, 31 (1974) 305-342.
73. P. Ribenboim, *Rings and Modules*, Interscience, New York, 1967.
74. J. Ritter and S. K. Sehgal, Integral group rings of some  $p$ -groups, (to appear).
75. S. K. Sehgal, Nilpotent elements in group rings, *Manuscripta Math.*, 15 (1975) 65-80.
76. S. K. Sehgal, *Topics in Group Rings*, M. Dekker, New York, 1978.
77. S. K. Sehgal and H. J. Zassenhaus, Group rings whose units form an FC group, *Math Z.*, 153 (1977) 29-35.
78. S. K. Sehgal and H. J. Zassenhaus, Integral group rings with nilpotent unit groups, *Comm. in Algebra*, 5 (1977) 101-111.
79. S. K. Sehgal and H. J. Zassenhaus, On the supercenter of a group and its ring theoretic generalization, *Proc. Conference on orders and group rings*, Springer-Verlag (to appear).
80. K. Sekiguchi, Units in integral group rings, *Proc. 12<sup>th</sup> Symp. on Ring Theory*, Sapporo, 1979, pp. 39-50.

81. K. Sekiguchi, On the units of integral group rings, *Tokyo J. of Math.*, 3, 1 (1980) 149-162.
82. K. Sekiguchi, The group of units of the integral group ring of a metacyclic group, *Osaka J. of Math.* (to appear).
83. A. E. Zalesskii and A. V. Mikhalev, Group Rings, *J. Soviet Math.*, 4 (1975) 1-78.

TRABALHOS

DO

DEPARTAMENTO DE MATEMATICA

TITULOS PUBLICADOS

- 8001 - PLETCH, A. Local freeness of profinite groups. São Paulo, IME-USP, 1980. 10p.
- 8002 - PLETCH, A. Strong completeness in profinite groups. São Paulo, IME-USP, 1980. 8p.
- 8003 - CARNIELLI, W.A. & ALCANTARA, L.P. de. Transfinite induction on ordinal configurations. São Paulo, IME-USP, 1980. 22p.
- 8004 - JONES RODRIGUES, A.R. Integral representations of cyclic p-groups. São Paulo, IME-USP, 1980. 13p.
- 8005 - CORRADA, M. & ALCANTARA, L.P. de. Notes on many-sorted systems. São Paulo, IME-USP, 1980. /25/p.
- 8006 - POLCINO MILIES, F.C. & SEHGAL, S.K. FC-elements in a group ring. São Paulo, IME-USP, 1980. /10/p.
- 8007 - CHEN, C.C. On the Ricci condition and minimal surfaces with constantly curved Gauss map. São Paulo, IME-USP, 1980. 10p.
- 8008 - CHEN, C.C. Total curvature and topological structure of complete minimal surfaces. São Paulo, IME-USP, 1980. 21p.
- 8009 - CHEN, C.C. On the image of the generalized Gauss map of a complete minimal surface in  $R^4$ . São Paulo, IME-USP, 1980. 8p.
- 8110 - JONES RODRIGUES, A.R. Units of  $ZCp^n$ . São Paulo, IME-USP, 1981. 7p.
- 8111 - KOTAS, J. & COSTA, N.C.A. da. Problems of modal and discussive logics. São Paulo, IME-USP, 1981. 35p.

- 8112 - BRITO, F.B. & GONÇALVES, D.I. Algebras não associativas, sistemas diferenciais polinomiais homogêneos e classes características. São Paulo, IME-USP, 1981. 7p.
- 8113 - POICINO MILLES, F.C. Group rings whose torsion units form a subgroup II. São Paulo, IME-USP, 1981. 1v. (não paginado)
- 8114 - CHEN, C.C. An elementary proof of Calabi's theorems on holomorphic curves. São Paulo, IME-USP, 1981. 5p.
- 8115 - COSTA, N.C.A. da & ALVES, E.H. Relations between paraconsistent logic and many-valued logic. São Paulo, IME-USP, 1981. 8p.
- 8116 - CASTILHA, M.S.A.C. On Przymusiński's theorem. São Paulo, IME-USP, 1981. 6p.
- 8117 - CHEN, C.C. & GOFS, C.C. Degenerate minimal surfaces in  $P^4$ . São Paulo, IME-USP, 1981. 21p.
- 8118 - CASTILHA, M.S.A.C. Imagens inversas de algumas aplicações fechadas. São Paulo, IME-USP, 1981. 11p.
- 8119 - ARAGÓN VALEJO, A.J. & EVIL FILHO, P. An infinite dimensional version of Hartogs' extension theorem. São Paulo, IME-USP, 1981. 9p.
- 8120 - CONCEIÇÃO, J.Z. Groups rings with solvable unit groups. São Paulo, IME-USP, 1981. 15p.
- 8121 - CARNIELLI, W.A. & ALCANTARA, L.P. de. Paraconsistent algebras. São Paulo, IME-USP, 1981. 16p.
- 8122 - GONÇALVES, D.L. Nilpotent actions. São Paulo, IME-USP, 1981. 10p.
- 8123 - COELHO, S.P. Group rings with units of bounded exponent over the center. São Paulo, IME-USP, 1981. 25p.

8124 - PARMENTER, M.M. & POLCINO MILIES, F.C. A note on isomorphic group rings. São Paulo, IME-USP, 1981. 4p.

8125 - MERKLEN, H.A. Hereditary algebras with maximum spectra are of finite type. São Paulo, IME-USP, 1981. 10p.