Instituto de Ciências Matemáticas de São Carlos

ISSN - 0103-2577

Homoclinics and subharmonics of nonlinear two dimensional systems. Uniform boundedness of generalized inverses

H. M. RODRIGUES
J. G. RUAS FILHO

Nº 13

NOTAS DO ICMSC
Série Matemática

São Carlos
jan. / 1994

Homoclinics and Subharmonics of Nonlinear Two Dimensional Systems. Uniform Boundedness of Generalized Inverses

H. M. Rodrigues[1] and J. G. Ruas-Filho[2]

[1]ICMSC, USP, S. Carlos, SP, Brazil and Georgia Tech, Atlanta, GA 303 32-0190

[2]ICMSC, USP, S. Carlos, SP, Brazil

Abstract

In this work, using Lyapunov-Shmidt Method, we study existence and bifurcation of homoclinic and subharmonic solutions of the nonlinear equation: $\dot{x} = Ax + f(x) + F(t, x, \lambda)$, where $x \in R^2$, $F(t, x, 0) \equiv 0$ and $\lambda \in R^2$ is a small parameter. We assume that A and f are reversible with respect to a symmetric matrix S, such that $S^2 = I$ and that A is hyperbolic. A major difficulty handled in this paper was to prove the uniform boundedness of some generalized inverses. Some examples of nonlinear ordinary equations are analyzed.

Key words. homoclic solution, subharmonic solution, symmetry, reversible system,bifurcation, nonlinear equations.

AMS(MOS) subject classifications. primary 34C37, 34C25, 34C15

1 Introduction

Consider the equation:

$$\dot{x} = Ax + f(x) + F(t, x, \lambda), \tag{1.1}$$

where $x \in R^2$, $F(t, x, 0) \equiv 0$ and $\lambda \in R^2$ is a small parameter. We assume that A and f are reversible with respect to a symmetric matrix S, such that $S^2 = I$.

The object of this paper is to study existence and bifurcation of homoclinic and subharmonic solutions of equation (1.1). We find the bifurcation curves in both cases. A major difficulty that arises in this problem is to prove the uniform boundedness of the generalized inverse that appears when we use the Lyapunov-Schmidt Method.

Using this approach, this problem was considered by Chow, Hale and Mallet-Paret [3], for nonlinear second order equations. This problem is also analyzed in Chow and Hale[2]. Rodrigues and Silveira[7] studied the homoclinic case for infinite dimensional systems, with applications to a nonlinear beam equation.

The problem of the uniform boundedness of the right inverse was considered by Chan and Chow in [1], where they analyzed some examples.

In this work we consider the general case, where the unperturbed nonlinear equation $\dot{x} = Ax + f(x)$ is reversible with respect to a symmetric matrix S. A consequence of the uniform boundedness is that the bifurcation curves for any subharmonic and for the homoclinic solutions can be found altogether, that is, we can find a neighborhood of the origin in the λ-space and all the bifurcation curves in this neighborhood. The symmetric Hartman-Grobman Theorem as proved in Rodrigues[8], plays an important role in our approach.

In the last chapter of this work we analyze some applications of our results to nonlinear two dimensional system of ordinary differential equations.

2 Uniform Boundedness of the Generalized Inverses

Now we present some basic results which will be important for the proofs of our main results.

Let S be a $n \times n$ real symmetric matrix such that $S^2 = I$. Let $\mathcal{S} = \{x \in R^n : Sx = x\}$ and $\mathcal{N} = \{x \in R^n : -Sx = x\}$. Then $\mathcal{S}$ and $\mathcal{N}$ are orthogonal spaces and $R^n = \mathcal{S} \oplus \mathcal{N}$.

Definition 2.1 *Let $g : R \times R^n \rightarrow R^n$ be a Lipschitz continuous function. Consider the differential equation: $\dot{x} = g(t, x)$. Let S be a $n \times n$ real symmetric matrix such that $S^2 = I$. We say that the above system is Reversible, or has the Property E, with respect to S if $Sg(-t, Sx) = -g(t, x)$ for all t, x. In this case we also will say that $g(t, x)$ is reversible.*

Throughout this paper we assume that $S \neq \pm I$.

For the notations and basic results on reversible systems or systems with the Property E we suggest Hale[5] and Vanderbauwhede[9].

If $x(t)$ is a solution of a reversible system then $Sx(-t)$ is also a solution. If $x(t, x_0)$ denotes the solution of the above equation with initial condition x_0 at $t = 0$, then $x(t, Sx_0) = Sx(-t, x_0)$, for every $t \in R$.

If A is a $n \times n$ real matrix which is reversible with respect to S and β is an eigenvalue of A associated to the eigenvector x_0, then $-\beta$ is also an eigenvalue of A associated to the eigenvector Sx_0. In other words, the spectrum of A is symmetric with respect to the origin in the complex plane.

Consider the systems:

$$\dot{y} = Ay \tag{2.1}$$
$$\dot{x} = Ax + f(x), \tag{2.2}$$

Theorem 2.2 *(The Symmetric Hartman-Grobman Theorem) If A has no eigenvalue on the imaginary axis, $f : R^n \rightarrow R^n$ is a C^1 function such that $f(x) = o(|x|)$, as x goes to 0, and if equation (2.2) is reversible with respect to S, then there exists a local homeomorphism h that conjugates the orbits of (2.1) and (2.2) in a neighborhood of the origin, such that $h \circ S = S \circ h$. In particular, if x is in this neighborhood and $Sx = x$, then $S(hx) = hx$.*

For a proof of the above result see Rodrigues[8].

Using the symmetry assumptions we can prove the following lemma:

Lemma 2.3 *Suppose the assumptions of Theorem (2.2) satisfied. Let V^s and V^u, respectively, be the global stable and unstable manifolds of the origin with respect to system (2.2). Then the following hold:*

(i) $V^s = SV^u$

(ii) If $x_0 \in V^u \cap S$, $x_0 \neq 0$ then $x_0 \in V^s$, $x(-t, x_0) = Sx(t, x_0)$, for every $t \in R$, and $x(t, x_0)$ is homoclinic to the origin.

(iii) If $T > 0$, $x_0 \in S$, $x(T, x_0) \in S$ and $x(t, x_0) \notin S$, for every $t \in (0, T)$ then $x(t, x_0)$ has minimum period $2T$.

Lemma 2.4 *Suppose $n = 2$ and that the eigenvalues of A are β and $-\beta$, where $\beta > 0$. Let the system (2.2) be reversible with respect to S, where $f : R^2 \to R^2$ is a C^1 function such that $f(x) = o(|x|)$, as x goes to 0. If there exists $x_\infty \in V^u \cap S$, $x_\infty \neq 0$, then there are positive constants, δ and T_0, such that for each $T > T_0$, there exists $x_T \in S$, $|x_\infty - x_T| < \delta$, $x_T \to x_\infty$, as $T \to \infty$, such that the solution $x(t, x_T)$, of (2.2), is periodic and has minimum period $2T$.*

Proof. Since $x(-t, x_\infty) = Sx(t, x_\infty)$, for every $t \in R$, the set $C = \{x(t, x_\infty) : t \in R\} \cup \{0\}$ is a Jordan curve, transversal to S at x_∞. We can define an order on S, in such a way that $0 < x_\infty$. Let $\varepsilon > 0$, be such that the phase portrait of (2.2) in a neighborhood of the origin is given as in Fig.(2.1). Let $a = a_\varepsilon > 0$ be such that $|x(t, x_\infty)| < \varepsilon/2$, for every $t \geq a$. From the continuity with respect to the initial data, it follows that there exists $\delta > 0$, such that if $y \in S$, $y < x_\infty$ and $|y - x_\infty| < \delta$ then $|x(a, y) - x(a, x_\infty)| < \varepsilon/2$. Therefore there is $T = T(y) \geq a$ such that $x(T, y) \in S$ and so $x(t, y)$ is $2T$-periodic. Moreover $x(T, y)$ is in the interior of the Jordan curve C. For a fixed $\bar{y} \in S$, $\bar{y} < x_\infty$, $|\bar{y} - x_\infty| < \delta$, let T_0 be such that $x(T_0, \bar{y}) \in S$. For each x, $\bar{y} < x < x_\infty$, there exists a least $T > T_0$, such that $x(T, x) \in S$, and so $x(t, x)$ is $2T$-periodic. We denote $x_T = x$. This completes the proof. ∎

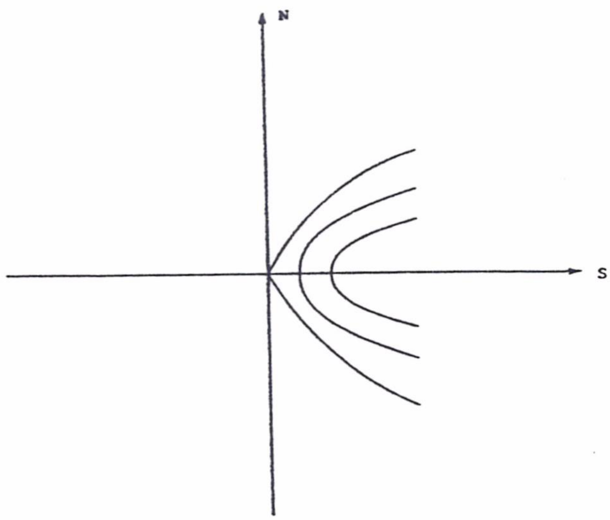

Fig. 2.1

Remark 2.5 *Fiedler and Vanderbauwhede [4] proved a similar result for dimension greater than two. We present our proof in the two dimensional case because it is very simple.*

Under the conditions of Lemma (2.4), for each T, $T > T_0 > 0$, let $p^T(t) = x(t, x_T)$, where x_T is as in the proof of that lemma. Then $Sp^T(-t) = p^T(t)$ and $S\dot{p}^T(t) = -\dot{p}^T(t)$, for every $t \in R$ and if T is finite then $p^T(t)$ is $2T$-periodic.

Let S, A and f as in Lemma (2.4). Consider the system:

$$\dot{x} = Ax + f(x) + F(t, x, \lambda) \tag{2.3}$$

where F is a smooth function 1-periodic in t, such that $F(t, x, 0) \equiv 0$ and $\lambda \in R^2$.

Our purpose is to study the existence and bifurcation of m-periodic and bounded solutions of (2.3).

Without loss of generality we can assume that $\beta = 1$.

Now we consider the following basis for R^2, $\{e_1, e_2\}$, where $Se_1 = e_1$, $Se_2 = -e_2$, and $\det(e_1, e_2) = 1$. Let $x = x_1e_1 + x_2e_2$.

In the new basis the systems (2.1), (2.2) and (2.3) will be similar to the original ones, but (2.1), (2.2) will be reversible with respect to the following matrix $S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Furthermore, in the new basis, $A = \begin{pmatrix} 0 & b \\ 1/b & 0 \end{pmatrix}$, where $b > 0$ and $f = \mathrm{col}(f^1, f^2)$, where $f^1(x_1, -x_2) = -f^1(x_1, x_2)$, $f^2(x_1, -x_2) = f^2(x_1, x_2)$, for any $x_1, x_2 \in R$.

The new expressions for the solutions $p^T(t), \dot{p}^T(t)$ will have the correspondings properties of the old ones. For example, $\dot{p}_1^T(-t) = -\dot{p}_1^T(t)$, $\dot{p}_2^T(-t) = \dot{p}_2^T(t), p_1^T(-t) = p_1^T(t)$ and $p_2^T(-t) = -p_2^T(t)$, for every t in R. Therefore, $\dot{p}_1^T(0) = 0$.

Consider the system:

$$\dot{z} = Az + f_x(p^T(t))z \tag{2.4}$$

Lemma 2.6 *Under the assumptions of Lemma (2.4) the following hold:*

(i) $\mathrm{tr}(A) = 0$ *and* $\int_{-T}^{T} \mathrm{tr}(f_x(p^T(t))dt = 0$.

(ii) There exists a unique solution $q^T(t)$ *of (2.4), such that* $Sq^T(0) = q^T(0)$ *and* $\det Y_T(0) = 1$, *where* $Y_T(t) = (q^T(t), \dot{p}^T(t))$.

Proof. It is obvious that $\mathrm{tr}(A) = 0$. From the reversibility of f it follows that $f_x(Sx) = -Sf_x(x)S$ and so, $\int_{-T}^{0} \mathrm{tr}(f_x(p^T(t)))dt = \int_0^T \mathrm{tr}(f_x(p^T(-t)))dt =$
$\int_0^T \mathrm{tr}(f_x(Sp^T(t)))dt = -\int_0^T \mathrm{tr}(Sf_x(p^T(t))S)dt = -\int_0^T \mathrm{tr}(f_x(p^T(t)))dt$

The second part can be proved as follows. Let us suppose that there are two solutions, q and $\bar{q}$. Then $\det(q(0) - \bar{q}(0), \dot{p}^T(0)) = 0$ and so $q(t) - \bar{q}(t)$ and $\dot{p}^T(t)$ are linearly dependent solutions of (2.4). Therefore there exists c such that $q(t) - \bar{q}(t) = c\dot{p}^T(t)$. If we let $t = 0$ and multiply by S the last equality, we obtain $q(0) - \bar{q}(0) = -c\dot{p}^T(0)$. The two last equalities imply that $c = 0$ and so $q(t) = \bar{q}(t)$, for every $t \in R$. ∎

The symmetry assumptions imply that $q_1^T(-t) = q_1^T(t)$ and $q_2^T(-t) = -q_2^T(t)$, for every t in R. Therefore $q_2^T(0) = 0$.

From now on we are going to assume that the space of the $2T$-periodic solutions of (2.4) has dimension 1. This implies that $\dot{p}^T(t) = \text{col}(\dot{p}_1^T(t), \dot{p}_2^T(t))$ is a basis of that space.

It follows from Lemma (2.5) that

$$e^{-\int_0^t \text{tr}(f_x(p^T(s)))ds}(\dot{p}_2^T(t), -\dot{p}_1^T(t))$$

is a basis of the space of the $2T$-periodic solutions of the adjoint equation of (2.4).

Let $\mathcal{P}_T$ be the space of $2T$-periodic functions from R to R^2 if T is finite and $\mathcal{P}_\infty$ be the space of bounded functions from R to R^2 with the usual sup norm.

Consider the projections: $P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $P_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $Y_T(t)P_1$ and $P_2Y_T^{-1}(t)$ are $2T$-periodic if T is finite and are bounded if $T = \infty$.

Let M_T and Q_T be projections defined as follows: for each function $g \in \mathcal{P}_T$, let $(M_Tg)(t) = g_2(0)\dot{p}^T(t)/\dot{p}_2^T(0)$ and

$$(Q_Tg)(t) = (1/d_T)\int_{-T}^{T} \psi_T(s)g(s)ds\ \psi_T'(t),$$

where $\psi_T(t) = e^{-\int_0^t \text{tr}(f_x(p^T(s)))ds}(\dot{p}_2^T(t), -\dot{p}_1^T(t))$, and $d_T = \int_{-T}^{T} \psi_T(t)\psi_T'(t)dt$, where $'$ indicates the transpose of the matrix.

Consider now the nonhomogeneous system:

$$\dot{z} = Az + f_x(p^T(t))z + g(t) \qquad (2.5)$$

Theorem 2.7 (The Fredholm Alternative and The Uniform Boundedness) *Let f be a C^3 function. Under the above assumptions, if $g \in \mathcal{P}_T$, then system (2.5) has a solution in $\mathcal{P}_T$ if and only if*

$$\int_{-T}^{T} P_2Y_T^{-1}(s)g(s)ds = 0 \quad (\text{or } Q_Tg = 0).$$

If this last condition is satisfied then system (2.5) has a unique solution $(\mathcal{K}_Tg)(t)$ in $\mathcal{P}_T$, such that $M_T(\mathcal{K}_Tg) = 0$. Moreover there exist positive constants, T_0 and K, such that, for any $T \geq T_0$, $\|\mathcal{K}_Tg\| \leq K\|g\|$, for any function g in $\mathcal{P}_T$, where $\|\cdot\|$ indicates the sup norm.

Proof.

We will consider only the case $T < \infty$, because the case $T = \infty$ is simpler. If $Q_Tg = 0$, from the variation of constants formula it follows that the $2T$-periodic solution of (2.5) is given by:

$$\begin{aligned} z(t) &= Y_T(t)Y_T^{-1}(0)z(0) + \int_0^t Y_T(t)Y_T^{-1}(s)g(s)ds \\ &= Y_T(t)Y_T^{-1}(0)z(0) + \int_0^t Y_T(t)P_1Y_T^{-1}(s)g(s)ds + \int_0^t Y_T(t)P_2Y_T^{-1}(s)g(s)ds \\ &= Y_T(t)[Y_T^{-1}(0)z(0) + \int_0^T P_2Y_T^{-1}(s)g(s)ds] + \int_0^t Y_T(t)P_1Y_T^{-1}(s)g(s)ds + \end{aligned}$$

$$
\begin{aligned}
&\int_T^t Y_T(t) P_2 Y_T^{-1}(s) g(s) ds \\
&= Y_T(t) w + \int_0^t Y_T(t) P_1 Y_T^{-1}(s) g(s) ds + \int_T^t Y_T(t) P_2 Y_T^{-1}(s) g(s) ds
\end{aligned}
$$

where $w = Y_T^{-1}(0) z(0) + \int_0^T P_2 Y_T^{-1}(s) g(s) ds$.

The condition $Mz = 0$ implies that $P_1 w = 0$. Let

$$
(\mathcal{K}_T g)(t) = Y_T(t) w + \int_0^t Y_T(t) P_1 Y_T^{-1}(s) g(s) ds + \int_T^t Y_T(t) P_2 Y_T^{-1}(s) g(s) ds
$$

where w will be determined below.

The periodicity condition implies that w must satisfy:

$$
[Y_T(T) - Y_T(-T)] w = -\int_{-T}^T Y_T(T) P_1 Y_T^{-1}(s) g(s) ds
$$

and then

$$
[I - Y_T^{-1}(T) Y_T(-T)] w = -\int_{-T}^T P_1 Y_T^{-1}(s) g(s) ds,
$$

and

$$
[P_1 Y_T^{-1}(T) Y_T(-T)] w = \int_{-T}^T P_1 Y_T^{-1}(s) g(s) ds.
$$

From this expression we can find w, such that $P_1 w = 0$. In fact, $w = col(\rho, 0)$, where

$$
\rho = \frac{e^{\int_0^T tr f_x(p^T(s)) ds} \nu}{2 q_2(-T) q_1(T)}
$$

and ν is the second component of $\int_{-T}^T P_1 Y_T^{-1}(s) g(s) ds$.

Therefore, $[Y_T(T) P_1 Y_T^{-1}(T) Y_T(-T)] w = \int_{-T}^T Y_T(T) P_1 Y_T^{-1}(s) g(s) ds$.

Since $Y_T(-T) P_2 = S Y_T(T) P_2$, we have:

$$
Y_T(T) P_1 Y_T^{-1}(T) S Y_T(T) w = \int_{-T}^T Y_T(T) P_1 Y_T^{-1}(s) g(s) ds \tag{2.6}
$$

But,

$$
-Y_T(T) P_1 Y_T^{-1}(T) S = \begin{pmatrix} 0 & 0 \\ \dot{p}_2(T) q_2(T) e^{-\int_0^T tr(f_x p(s)) ds} & 1 \end{pmatrix}
$$

Let $B(T) = \begin{pmatrix} 0 & q_1^T(T)/2q_2^T(T) \\ 0 & 1/2 \end{pmatrix}$.

Since $P_1 w = 0$,one can show that $-B(T) Y_T(T) P_1 Y_T^{-1}(T) S Y_T(T) w = Y_T(T) w$. Therefore, from equation (2.6) it follows that,

$$
Y_T(T) w = -B(T) \int_{-T}^T Y_T(T) P_1 Y_T^{-1}(s) g(s) ds. \tag{2.7}
$$

Our next purpose is to prove that there exist positive constants T_0 and K, such that $|B(T)| \le K$,

$$
\int_0^t |Y_T(t) P_1 Y_T^{-1}(s)| ds + \int_t^T |Y_T(t) P_2 Y_T^{-1}(s)| ds \le K,
$$

for every $t \in [0,T]$ and $|Y_T(t)P_2Y_T^{-1}(s)| \leq K$, for $0 \leq t \leq s \leq T$ and for every $T \geq T_0$. With a similar procedure, using the symmetry hypotheses we can prove the analog estimates for $t \in [-T,0]$. Since $|Y_T(t)w| = |Y_T(t)P_2w| \leq |Y_T(t)P_2Y_T^{-1}(T)||Y_T(T)w|$ and $Y_T(-t)P_2 = SY_T(t)P_2$, the proof of the uniform boundedness will follow from the above estimates.

Since it is not our goal to find optimal estimates, in each step below we will be changing the constants K and T_0, by larger ones, but we will maintain the same symbols.

Let $\varepsilon_0 > 0$ be such that the Symmetric Hartman-Grobman Theorem holds in $B_{\varepsilon_0} = \{x \in R^2 : |x| < \varepsilon_0\}$. Let h be the homeomorphism that conjugates the orbits of the non linear system (2.2) and of the linear one. From Hartman [5], it follows we can assume that h is C^1.

Let $\mu_0 < \varepsilon_0$. Let $a_0 = a_0(\mu_0)$ be such that $|p_\infty(t)| < \mu_0/4$, for $t \geq a_0$. Since $p_T(a_0) \to p_\infty(a_0)$, as $T \to \infty$, there exists $T_0 = T_0(\mu_0)$, such that if $T \geq T_0$ then $|p_\infty(a_0) - p_T(a_0)| < \mu_0/4$. Therefore for $T \geq T_0$ we have $|p_T(a_0)| < \mu_0/2$ and then $|p_T(t)| < \mu_0/2$, for every $t \in [a_0, T]$.

Since $Sp^T(T) = p^T(T)$, from The Symmetric Hartman-Grobman Theorem it follows that $Sh^{-1}(p^T(T)) = h^{-1}p^T(T)$, that is $Sh^{-1}(p^T(T)) = \text{col}(\alpha, 0)$, for some $\alpha > 0$. Let $u(t) = h^{-1}(p^T(t))$, for every $t \in [a_0, T]$. Then $u(t) = \alpha(\cosh(T-t), -\sinh(T-t))$, for every $t \in [a_0, T]$.

Using the fact that h is a perturbation of the identity and the fact that h preserves the symmetry, we can prove that for μ_0 sufficiently small we have:

$$|u_1(t)| \leq (3/2)|p_1^T(t)|, \quad |u_2(t)| \leq (3/2)|p_2^T(t)|, \tag{2.8}$$

for every $t \in [a_0, T]$. Similarly, we can prove that

$$|p_1^T(t)| \leq (3/2)|u_1(t)|, \quad |p_2^T(t)| \leq (3/2)|u_2(t)|, \tag{2.9}$$

for every $t \in [a_0, T]$.

To simplify the calculation, given two functions $w(t), v(t)$, we will say that $w(t) \simeq_u v(t)$ if there exist positive constants K, T_0, such that $(1/K)v(t) \leq w(t) \leq Kv(t)$, for every $t \in [a_0, T]$, for every $T \geq T_0$. Then we have $|p_1^T(t)| \simeq_u |u_1(t)|$ and $|p_2^T(t)| \simeq_u |u_2(t)|$.

If we let $f = \text{col}(f^1, f^2)$, from the symmetric properties of f it follows that there exist C^∞ functions $F(x_1, x_2)$, $G_1(x_1)$, $G_2(x_1, x_2)$, such that $f^1(x_1, x_2) = x_2F(x_1, x_2)$, $f^2(x_1, x_2) = x_1^2G_1(x_1) + x_2^2G_2(x_1, x_2)$, where $F(x_1, x_2) = O(|x_1| + |x_2|^2)$ and $G_1(x_1)$, $G_2(x_1, x_2)$ are $O(1)$.

Using (2.8) and (2.9) we obtain,

$$|p_2^T(t)/p_1^T(t)| \leq K|u_2(t)/u_1(t)| \leq K|\tanh(T-t)| \leq K,$$

for every $t \in [a_0, T]$ and for every $T \geq T_0$.

Under the above assumptions, system (2.2) can be written as:

$$\begin{aligned} \dot{p}_1 &= bp_2 + p_2F(p_1, p_2) \\ \dot{p}_2 &= (1/b)p_1 + p_1^2G_1(p_1) + p_2^2G_2(p_1, p_2) \end{aligned} \tag{2.10}$$

Therefore, $(1/K)|p_2^T(t)| \leq |p_1^T(t)| \leq K|p_2^T(t)|$, $(1/K)|p_1^T(t)| \leq |p_2^T(t)| \leq K|p_1^T(t)|$, and from (2.8) and (2.9), we obtain:

$$\begin{aligned}(1/K)|u_2(t)| &\leq |\dot{p}_1^T(t)| \leq K|u_2(t)|,\\ (1/K)|u_1(t)| &\leq |\dot{p}_2^T(t)| \leq K|u_1(t)|\end{aligned}$$

and so,

$$\begin{aligned}(1/K)\alpha\sinh(T-t) &\leq |\dot{p}_1^T(t)| \leq K\alpha\sinh(T-t),\\ (1/K)\alpha\cosh(T-t) &\leq |\dot{p}_2^T(t)| \leq K\alpha\cosh(T-t)\end{aligned} \tag{2.11}$$

for every $t \in [a_0, T]$ and for every $T \geq T_0$, that is, $|\dot{p}_1^T(t)| \simeq_u |u_2(t)|$ and $|\dot{p}_2^T(t)| \simeq_u |u_1(t)|$.

Our next purpose is to find a solution $Q(t) = \mathrm{col}(Q_1(t), Q_2(t))$, of (2.4), such that $Q_1(a_0)) = 0$ and $\det(Q(0), \dot{p}^T(0)) = 1$. Then if we let $f = \mathrm{col}(f^1, f^2)$, we obtain,

$$Q_2(t) = [1/(b + f^1_{x_2}(p^T(t))]\dot{Q}_1(t) - [1/(b + f^1_{x_2}(p^T(t))]f^1_{x_1}(p(t))Q_1(t)$$

and $Q_1(t)\dot{p}_2^T(t) - Q_2(t)\dot{p}_1^T(t) = e^{\int_0^t \mathrm{tr}(f_x(p^T(s)))ds}$. Thus

$$Q_1(t)\dot{p}_2^T(t) - \{[1/(b + f^1_{x_2}(p^T(t))]\dot{Q}_1(t) - [1/(b + f^1_{x_2}(p^T(t))Q_1(t)]f^1_{x_1}(p^T(t))\}\dot{p}_1^T(t) = e^{\int_0^t \mathrm{tr}(f_x(p^T(s)))ds}$$

and we obtain the equation,

$$\dot{Q}_1(t) = [\frac{(b + f^1_{x_2}(p^T(t))\dot{p}_2^T(t)}{\dot{p}_1^T(t)} + f^1_{x_1}(p^T(t))]Q_1(t) - \frac{(b + f^1_{x_2}(p^T(t))e^{\int_0^t \mathrm{tr}(f_x(p^T(s)))ds}}{\dot{p}_1^T(t)} \tag{2.12}$$

with the initial condition $Q_1(a_0)) = 0$.

If we let

$$w(t) = w_T(t) = \frac{(b + f^1_{x_2}(p^T(t))\dot{p}_2^T(t)}{\dot{p}_1^T(t)} + f^1_{x_1}(p^T(t)),$$

$$W(t) = W_T(t) = -\frac{(b + f^1_{x_2}(p^T(t))e^{\int_0^t \mathrm{tr}(f_x(p^T(s)))ds}}{\dot{p}_1^T(t)},$$

we obtain the equation,

$$\dot{Q}_1(t) = w(t)Q_1(t) + W(t) \tag{2.13}$$

Therefore $Q_1(t) = \int_{a_0}^t e^{\int_s^t w(\tau)d\tau}W(s)ds$.

From the above estimates it follows that,

$$\int_s^t b\frac{\dot{p}_2^T(\tau)}{-\dot{p}_1^T(\tau)}d\tau - K \leq -\int_s^t w(\tau)d\tau \leq \int_s^t b\frac{\dot{p}_2^T(\tau)}{-\dot{p}_1^T(\tau)}d\tau + K$$

Using (2.10), we obtain,

$$\int_s^t \frac{\dot{p}_2^T(\tau)}{-p_2^T(\tau)}d\tau - K \leq -\int_s^t w(\tau)d\tau \leq \int_s^t \frac{\dot{p}_2^T(\tau)}{-p_2^T(\tau)}d\tau + K$$

Then,

$$\int_s^t \frac{\dot p_2^T(\tau)}{p_2^T(\tau)}d\tau - K \le \int_s^t w(\tau)d\tau \le \int_s^t \frac{\dot p_2^T(\tau)}{p_2^T(\tau)}d\tau + K$$

and so,

$$(1/K)e^{\int_s^t \frac{\dot p_2^T(\tau)}{p_2^T(\tau)}d\tau} \le e^{\int_s^t w(\tau)d\tau} \le Ke^{\int_s^t \frac{\dot p_2^T(\tau)}{p_2^T(\tau)}d\tau}$$

Thus

$$(1/K)\frac{p_2^T(t)}{p_2^T(s)} \le e^{\int_s^t w(\tau)d\tau} \le K\frac{p_2^T(t)}{p_2^T(s)}$$

and so,

$$(1/K)|p_2^T(t)|\int_{a_0}^t \frac{1}{[p_2^T(s)]^2}ds \le \int_{a_0}^t e^{\int_s^t w(\tau)d\tau}W(s)ds \le K|p_2^T(t)|\int_{a_0}^t \frac{1}{[p_2^T(s)]^2}ds$$

Since $|p_2^T(t)| \simeq_u |u_2(t)|$, it follows that,

$$Q_1(t) \simeq_u \frac{\sinh(t-a_0)}{\alpha\sinh(T-a_0)} \tag{2.14}$$

Now we are going to estimate $q_1^T(t)$ and $q_2^T(t)$, where $q^T(t)$ is given by Lemma 2.5. An easy calculation shows that $q_1^T(t)$ and $Q_1(t)$ are solutions of (2.13). Therefore,

$$q_1^T(t) = Q_1(t) + e^{\int_{a_0}^t w(s)ds}q_1^T(a_0),$$

for every $t \in R$.

We can assume that $q_1^T(a_0) > 0$, for every $T \ge T_0$. In fact. Let us consider now the case $T=\infty$. In this case, there exists a solution, $\bar q(t)$, of (2.4), such that $|\bar q(t) - e^t\mathrm{col}(b,1)| = o(e^t)$, as $t\to\infty$. Then $\bar q_i(t)\to\infty$, as $t\to\infty$, and we can assume that $\bar q_i(t) > 0$, for $t \ge a_0$, for $i=1,2$. Therefore there exist constants c, d such that $q^\infty(t) = c\bar q(t) + d\dot p^\infty(t)$, for every $t \in R$. From the condition $\det(q^\infty(t), \dot p^\infty(t)) = e^{\int_0^t \mathrm{tr} f_x(p^\infty(s))ds}$, we obtain, $c\det(\bar q(t), \dot p^\infty(t)) = e^{\int_0^t \mathrm{tr}(f_x(p^\infty(s))ds}$. Therefore $c>0$ and $q_1^\infty(t)\to\infty$, as $t\to\infty$ and we can assume $q_1^\infty(a_0) > 0$. Using the continuity with respect to the initial data, we can prove that there exists $T_0 > 0$ such that $q_1^T(a_0) > 0$ for every $T \ge T_0$. Then we have $Q_1(t) \le q_1^T(t) = Q_1(t) + e^{\int_{a_0}^t w(s)ds}q_1^T(a_0)$. Using the above estimates we obtain,

$$\frac{\sinh(t-a_0)}{K\alpha\sinh(T-a_0)} \le q_1^T(t) \le K\Big[\frac{\sinh(t-a_0)}{\alpha\sinh(T-a_0)} + \frac{|p_2^T(t)q_1^T(a_0)|}{|p_2^T(a_0)|}\Big]$$

Since $q_1^T(a_0) \to q_1^\infty(a_0)$, as $T\to\infty$, we have for every $t\in[a_0,T]$ and for every $T \ge T_0$,

$$\frac{\sinh(t-a_0)}{K\alpha\sinh(T-a_0)} \le q_1^T(t) \le K\frac{\sinh(t-a_0)+1}{\alpha\sinh(T-a_0)} \tag{2.15}$$

for every $t \in [a_0, T]$, for every $T \geq T_0$. Therefore,

$$\frac{\sinh(t-a_0)}{K\alpha\sinh(T-a_0)} \leq q_1^T(t) \leq K\frac{\sinh(t-a_0)}{\alpha\sinh(T-a_0)} \tag{2.16}$$

for every $t \in [a_0+1, T]$, for every $T \geq T_0$.

Since $q_2^T(t)$, satisfies

$$\dot{q}_2^T(t) = [(1/b) + f_{x_1}^2(p^T(t))]q_1^T(t) + f_{x_2}^2(p^T(t))q_2^T(t),$$

we have, $q_2^T(t) = e^{\int_{a_0}^t f_{x_2}^2(p^T(s))ds} q_2^T(a_0) + \int_{a_0}^t e^{\int_s^t f_{x_2}^2(p^T(\tau))d\tau}[(1/b) + f_{x_1}^2(p^T(s))]q_1^T(s)ds$. As before, we can assume that $q_2^T(a_0) > 0$ for every $T \geq T_0$. Then,

$$(1/K)\int_{a_0}^t q_1^T(s)ds \leq q_2^T(t) \leq K[q_2^T(a_0) + \int_{a_0}^t q_1^T(s)ds]$$

Using (2.16), we obtain,

$$\frac{\cosh(t-a_0)-1}{K\alpha\sinh(T-a_0)} \leq q_2^T(t) \leq K\frac{\cosh(t-a_0)}{\alpha\sinh(T-a_0)}$$

for every $t \in [a_0, T]$ and for every $T \geq T_0$ and then,

$$\frac{\cosh(t-a_0)}{K\alpha\sinh(T-a_0)} \leq q_2^T(t) \leq K\frac{\cosh(t-a_0)}{\alpha\sinh(T-a_0)} \tag{2.17}$$

for every $t \in [a_0+1, T]$ and for every $T \geq T_0$

Consider now the matrices:

$$Y_T(t)P_1Y_T^{-1}(s) = e^{-\int_0^s \operatorname{tr} f_x(p^T(\tau))d\tau}\begin{pmatrix} -\dot{p}^T{}_1(t)q_2^T(s) & \dot{p}_1^T(t)q_1^T(s) \\ -\dot{p}_2^T(t)q_2^T(s) & \dot{p}_2^T(t)q_1^T(s) \end{pmatrix}$$

$$Y_T(t)P_2Y_T^{-1}(s) = e^{-\int_0^s \operatorname{tr} f_x(p^T(\tau))d\tau}\begin{pmatrix} \dot{p}_2^T(s)q_1^T(t) & -\dot{p}_1^T(s)q_1^T(t) \\ \dot{p}_2^T(s)q_2^T(t) & -\dot{p}_1^T(s)q_2^T(t) \end{pmatrix}$$

Using (2.11), (2.16) and (2.17), we obtain for $a_0+1 \leq t \leq s \leq T$,

$$\begin{aligned}
|\dot{p}_2^T(s)q_1^T(t)| &+ |\dot{p}_1^T(s)q_1^T(t)| + |\dot{p}_2^T(s)q_2^T(t)| + |\dot{p}_1^T(s)q_2^T(t)| \leq \\
&\leq \frac{K\alpha}{\alpha\sinh(T-a_0)}[\cosh(T-s)\sinh(t-a_0) + \sinh(T-s)\cosh(t-a_0)] \\
&\quad + \frac{K\alpha}{\alpha\sinh(T-a_0)}[\cosh(T-s)\cosh(t-a_0) + \sinh(T-s)\sinh(t-a_0)] \\
&\leq \frac{K}{\sinh(T-a_0)}[\sinh(T-s+t-a_0) + \cosh(T-s+t-a_0)]
\end{aligned}$$

Therefore,

$$|Y_T(t)P_2Y_T^{-1}(s)| \leq \frac{K}{\sinh(T-a_0)}[\sinh(T-a_0-(s-t)) + \cosh(T-a_0-(s-t))] \tag{2.18}$$

for $a_0+1 \leq t \leq s \leq T$.

Then

$$|Y_T(t)P_2Y_T^{-1}(s)| \leq \frac{K}{\sinh(T-a_0)}[\sinh(T-a_0)+\cosh(T-a_0)]$$

and so,

$$|Y_T(t)P_2Y_T^{-1}(s)| \leq K[1+\frac{\cosh(T-a_0)}{\sinh(T-a_0)}]$$

which implies,

$$|Y_T(t)P_2Y_T^{-1}(s)| \leq K, \text{ for } a_0+1 \leq s \leq t \leq T.$$

From the continuity with respect to the initial data it follows that,

$$|Y_T(t)P_2Y_T^{-1}(s)| \leq K, \text{ for } 0 \leq s \leq t \leq T.$$

Let us prove first that $\int_t^T |Y_T(t)P_2Y_T^{-1}(s)|ds \leq K$, for $t \in [0,T]$. Consider first $t \in [a_0+1, T]$.

From (2.18) it follows that,

$$\int_t^T |Y_T(t)P_2Y_T^{-1}(s)|ds \leq K\int_t^T \frac{K}{\sinh(T-a_0)}[\sinh(T-s+t-a_0)+\cosh(T-s+t-a_0)]ds$$

and then

$$\int_t^T |Y_T(t)P_2Y_T^{-1}(s)|ds \leq K\frac{[\sinh(T-a_0)+\cosh(T-a_0)]}{\sinh(T-a_0)}.$$

Therefore,

$$\int_t^T |Y_T(t)P_2Y_T^{-1}(s)|ds \leq K, \text{ for } t \in [a_0+1,T].$$

For $t \in [0, a_0+1]$, we have,

$$\int_t^T |Y_T(t)P_2Y_T^{-1}(s)|ds \leq \int_t^{a_0+1} |Y_T(t)P_2Y_T^{-1}(s)|ds + \int_{a_0+1}^T |Y_T(t)P_2Y_T^{-1}(s)|ds \leq$$

$$\int_t^{a_0+1} |Y_T(t)P_2Y_T^{-1}(s)|ds + |Y_T(t)Y_T^{-1}(a_0+1)|\int_{a_0+1}^T |Y_T(a_0+1)P_2Y_T^{-1}(s)|ds$$

¿From the continuity with respect to the initial data and from the above inequality proved for $t \in [a_0+1,T]$, we obtain,

$$\int_t^T |Y_T(t)P_2Y_T^{-1}(s)|ds \leq K, \text{ for } t \in [0,T],$$

for every $T \geq T_0$.

Now we are going to estimate $\int_0^t |Y_T(t)P_1Y_T^{-1}(s)|ds$. As before, consider first the case $t \in [a_0+1,T]$.

$$\int_0^t |Y_T(t)P_1Y_T^{-1}(s)|ds \leq \int_0^{a_0+1} |Y_T(t)P_1Y_T^{-1}(s)|ds + \int_{a_0+1}^t |Y_T(t)P_1Y_T^{-1}(s)|ds$$

As before, from the continuity with respect to the initial data, it follows that the first integral is uniformly bounded. It remains to estimate the last integral.

$$\int_{a_0+1}^{t} |Y_T(t)P_1Y_T^{-1}(s)|ds \leq$$

$$\leq K\int_{a_0+1}^{t} [|\dot{p}_1^T(t)q_2^T(s)| + |\dot{p}_1^T(t)q_1^T(s)| + |\dot{p}_2^T(t)q_2^T(s)| + |\dot{p}_2^T(t)q_1^T(s)|]ds$$

$$\leq \frac{K}{\sinh(T-a_0)}\int_{a_0+1}^{t} [\sinh(T-t)\cosh(s-a_0) + \sinh(T-t)\sinh(s-a_0)$$
$$+ \cosh(T-t)\cosh(s-a_0) + \cosh(T-t)\sinh(s-a_0)]ds$$

$$\leq \frac{K}{\sinh(T-a_0)}\int_{a_0+1}^{t} [\sinh(T-a_0-(t-s)) + \cosh(T-a_0-(t-s))]ds$$

$$\leq \frac{K}{\sinh(T-a_0)}[\cosh(T-a_0) - \cosh(T-t-1) + \sinh(T-a_0) - \sinh(T-t-1)]$$

$$\leq \frac{K}{\sinh(T-a_0)}[\cosh(T-a_0) + \sinh(T-a_0)]$$

Therefore, $\int_{a_0+1}^{t} |Y_T(t)P_1Y_T^{-1}(s)|ds \leq K$, for every $t \in [a_0+1, T]$.

From the continuity with respect to the initial data it follows that, $\int_0^t |Y_T(t)P_1Y_T^{-1}(s)|ds \leq K$, for every $t \in [0, a_0+1]$.

Let us prove now that $|B(T)| \leq K$, where,

$$B(T) = \begin{pmatrix} 0 & q_1^T(T)/2q_2^T(T) \\ 0 & 1/2 \end{pmatrix}.$$

From (2.16) and (2.17) it follows that,

$$\frac{q_1^T(T)}{q_2^T(T)} \leq K\frac{\alpha\sinh(T-a_0)}{\alpha\cosh(T-a_0)}.$$

Therefore, $|B(T)| \leq K$, for every $T \geq T_0$.

This completes the proof of our theorem. ∎

3 Applications

Consider the equations:

$$\dot{y} = Ay + f(y) \tag{3.1}$$
$$\dot{x} = Ax + f(x) + F(t, x, \lambda) \tag{3.2}$$

where A and f are reversible with respect to $S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, f is a C^3 function $F(t, x, \lambda) = \lambda_1 Bx + \lambda_2 h(t) + R(t, x, \lambda)$, where $R(t, x, \lambda) = O(|\lambda|^2)$ is C^3, B is a constant 2×2 matrix and h is continuous in R. Let $F(t, x, \lambda)$ be a 1-periodic functions of t.

We suppose that the eigenvalues of A are real and different from zero. In fact, without loss of generality we can assume that $A = \begin{pmatrix} 0 & b \\ 1/b & 0 \end{pmatrix}$, where b is a positive number.

We assume that there exists $x_\infty \in V^u \cap S$, $x_\infty \neq 0$. Therefore there is a homoclinic solution to the origin, $p^\infty(t)$, of equation (3.1). Let $p^T(t)$ be the $2T$-periodic solution, given by Lemma 2.4, for T sufficiently large.

¿From now on we are going to use a notation a little different from the one used in Chapter 2. Let us indicate by $p^m(t)$, the m-periodic solution of (3.1), where m is a positive integer, such that $Sp_m(0) = p_m(0)$.

Following the procedure of Chow, Hale and Mallet-Paret[3], we seek solution $x(t)$ of (3.2) close to $p^m(t+\alpha)$, where $1 \leq m \leq \infty$. This suggests to consider $x(t-\alpha) = p^m(t) + z(t)$, where $z_2(0) = 0$.

Then we obtain the following equation:

$$\dot{z} = Az + f_x(p^m(t))z + G(t,z) + F(t-\alpha, p^m(t) + z, \lambda) \tag{3.3}$$

where,

$$G(t,z) = \int_0^1 [f_x(p^m(t) + \theta z) - f_x(p^m(t))] d\theta z = O_u(|z|^2),$$

and $O_u(|z|^2)$ means that there exist positive constants K, m_0 and δ, such that $|O_u(|z|^2)| \leq K|z|^2$, for every z satisfying $|z| \leq \delta$, for any $m \in [m_0, \infty]$.

Let $\mathcal{P}_m$ be the space of m-periodic functions from R to R^2, if m is an integer and $\mathcal{P}_\infty$ be the space of bounded functions from R to R^2, with the sup norm.

Let M_m and Q_m be projections defined as follows: for each m-periodic function $h(t)$, let $(M_m h)(t) = h_2(0)\dot{p}^m(t)/\dot{p}_2^m(0)$ and

$$(Q_m h)(t) = (1/d_m) \int_{-m/2}^{m/2} \psi_m(s) h(s) ds \, \psi_m'(t),$$

where,

$$\psi_m(t) = e^{-\int_0^t tr(f_x(p^m(s)))ds} (\dot{p}_2^m(t), -\dot{p}_1^m(t)),$$

and $d_m = \int_{-m/2}^{m/2} \psi_m(t) \psi\prime_m(t) dt$.

Using the Lyapunov-Schmidt Method we see that $z(t)$ is a solution of equation (3.3), in $\mathcal{P}_m$, such that $M_m z = 0$, if and only if, $z(t)$ is a solution of the following system of equations:

$$z = \mathcal{K}_m (I - Q_m)(G(\cdot, z) + F(\cdot - \alpha, p^m(\cdot) + z, \lambda)) \tag{3.4}$$

$$Q_m (G(\cdot, z) + F(\cdot - \alpha, p^m(\cdot) + z, \lambda)) = 0 \tag{3.5}$$

where $\mathcal{K}_m$ is given in the proof of Theorem (2.6).

From Theorem 2.6 and from the implicit function theorem it follows that there exists positive constants m_0, σ and δ, such that equation (3.4) has a unique solution $\bar{z}(t) = \bar{z}(\lambda, \alpha)(t)$, such that $|\bar{z}| \leq \delta$, for every λ, α, $|\lambda| \leq \sigma$ and for every $m \in (m_0, \infty]$. Moreover $|\bar{z}(\lambda, \alpha)| = O_u(|\lambda|)$.

If we substitute $\bar{z}$ into equation (3.5) we obtain,

$$Q_m (G(\cdot, \bar{z}) + F(\cdot - \alpha, p^m(\cdot) + \bar{z}, \lambda)) = 0 \tag{3.6}$$

Using the expression of F, given in the beginning of this chapter, in equation (3.6), we obtain,

$$\frac{\lambda_1}{d_m}\int_{-m/2}^{m/2}\psi_m(t)Bp^m(t)dt + \frac{\lambda_2}{d_m}\int_{-m/2}^{m/2}\psi_m(t)h(t-\alpha)dt + O_u(|\lambda|^2) = 0 \qquad (3.7)$$

Lemma 3.1 *Let $V : R^4 \to R$, be a smooth function such that $|V(x,y)| \leq K(|x|+|y|)$, for every $x, y \in R^2$, where K is a positive constant. Then $\int_{-m/2}^{m/2} V(p^m(t), \dot{p}^m(t))dt \to \int_{-\infty}^{\infty} V(p^\infty(t), \dot{p}^\infty(t))dt$, as $m \to \infty$, if the last integral is finite.*

Proof: The symmetry assumptions imply that we only have to consider the integrals on R_+.

Let a_0 as in Theorem 2.6. Then

$$|\int_0^\infty V(p^\infty(t), \dot{p}^\infty(t))dt - \int_0^{m/2} V(p^m(t), \dot{p}^m(t))dt| \leq$$
$$|\int_0^{a_0} V(p^\infty(t), \dot{p}^\infty(t)) - V(p^m(t), \dot{p}^m(t))dt| + |\int_{a_0}^{m/2} V(p^m(t), \dot{p}^m(t))dt|$$
$$+ |\int_{a_0}^{\infty} V(p^\infty(t), \dot{p}^\infty(t))dt|.$$

From the continuity with respect to the initial data it follows that the first integral goes to zero as $m \to \infty$.

From our assumptions and from the results of Chapter 2 it follows that

$$|\int_{a_0}^{m/2} V(p^m(t), \dot{p}^m(t))dt| \leq$$
$$K|\int_{a_0}^{m/2} [|p^m(t)| + |\dot{p}^m(t))|]dt \leq K_1 \int_{a_0}^{m/2} |\dot{p}^m(t))|dt \leq K_2|p^m(a_0)|.$$

Therefore, given $\varepsilon > 0$, there exists $m_0 > 0$, such that for any $m \geq m_0$,

$$|\int_0^\infty V(p^\infty(t), \dot{p}^\infty(t))dt - \int_0^{m/2} V(p^m(t), \dot{p}^m(t))dt| \leq \varepsilon$$

■

Let us assume that $\int_{-\infty}^{\infty} \psi_\infty(t)Bp^\infty(t)dt \neq 0$. From Lemma 3.1 it follows that $\int_{-m/2}^{m/2} \psi_m(t)Bp^m(t)dt \neq 0$, for every $m \geq m_0$, for a sufficiently large m_0 and the bifurcation equation (3.7) is then equivalent to:

$$\lambda_1 = \lambda_2 H_m(\alpha) + O_u(|\lambda|^2) \qquad (3.8)$$

where

$$H_m(\alpha) = -\frac{\int_{-m/2}^{m/2} \psi_m(t)h(t-\alpha)dt}{\int_{-m/2}^{m/2} \psi_m(t)Bp^m(t)dt}$$

If in the interval $[0,1]$, $H_\infty(\alpha)$ has a unique maximum (resp. minimum) point α_0^∞ (resp. α_1^∞), and $\frac{d^2 H_\infty(\alpha_0^\infty)}{d\alpha^2} < 0$, (resp. $\frac{d^2 H_\infty(\alpha_1^\infty)}{d\alpha^2} > 0$), then there exists $m_0 > 0$, such that, for any $m \geq m_0$, there is α_0^m (respect. α_1^m) in such a way that $H_m(\alpha_0^m)$ (respect. $H_m(\alpha_1^m)$) is the maximum value (respect. minimum value) of H on $[0,1]$.

Using the ideas of Chow, Hale and Mallet-Paret[3], we can prove the following theorem,

Theorem 3.2 *Suppose A and f are reversible with respect to $S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, f is a C^3 function, $F(t,x,\lambda) = \lambda_1 Bx + \lambda_2 h(t) + R(t,x,\lambda)$, where $R(t,x,\lambda) = O(|\lambda|^2)$ (uniformly for $t \in R$ and x in bounded sets) is C^3, B is a constant 2×2 matrix and h is continuous in R. Suppose that $F(t,x,\lambda)$ is a 1-periodic function of t and that the eigenvalues of A are real and different from 0. We assume that there exists $x_\infty \in V^u \cap S$, $x_\infty \neq 0$, that the space of m-periodic solution of (2.4) is one dimensional, that $\int_{-\infty}^{\infty} \psi_\infty(t) B p^\infty(t) dt \neq 0$ and that in the interval $[0,1]$, $H_\infty(\alpha)$ has a unique maximum (respect. minimum) point α_0^∞ (resp. α_1^∞), and $\frac{d^2 H_\infty(\alpha_0^\infty)}{d\alpha^2} < 0$ (respect. $\frac{d^2 H_\infty(\alpha_1^\infty)}{d\alpha^2} > 0$).*

Under the above assumptions, there exist positive numbers m_0 and δ such that in the neighborhood of the origin $U = \{\lambda : |\lambda| < \delta\}$, there exist bifurcation curves C_m^{max} (respect. C_m^{min}), of subharmonic solutions, if $m < \infty$ and of homoclinic solutions if $m = \infty$, which are tangent to the straight lines $\lambda_2 = \lambda_1 H_m(\alpha_0^m)$ (respect. $\lambda_2 = \lambda_1 H_m(\alpha_1^m)$). Moreover $C_m^{max} \to C_\infty^{max}$ (respect. $C_m^{max} \to C_\infty^{max}$), as $m \to \infty$.

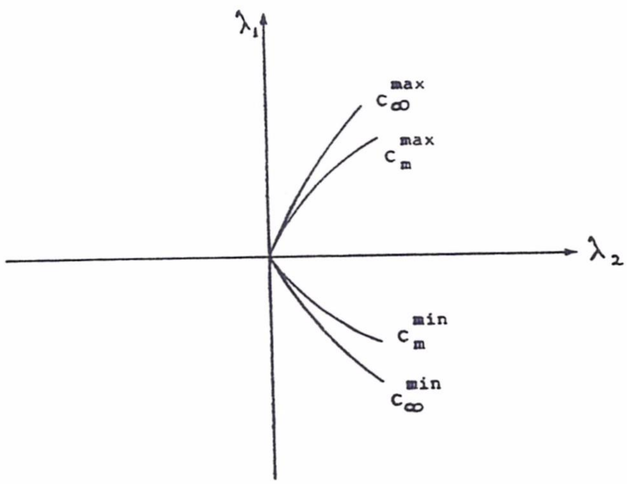

Fig. 3.1

Proof: We just have to analyze the bifurcation equation (3.8), using the a priori estimate: $|\lambda_1/\lambda_2| \leq C$, for (λ_1, λ_2) in a neighborhood of the origin, where C is a constant. ■

Fig. 3.1 shows a possible picture of the bifurcation curves.

Now we present some examples where our assumptions are satisfied. An interesting case is the one that comes from the second order equation:

$$\ddot{x} + g(x) = -\lambda_1 \dot{x} + \lambda_2 h(t)$$

that is, we consider the system:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 + g(x_1) - \lambda_1 \dot{x} + \lambda_2 h(t) \end{aligned}$$

where h is a 1-periodic continuous function, with suitable conditions on $g(x_1)$. We can take por example $g(x_1) = -(x_1)^2$, or $g(x_1) = -(x_1)^3$. In this case we take $B = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$.

We can also consider a perturbation of:

$$\begin{aligned} \dot{x}_1 &= x_2 - 1.88(x_2)^3 + (x_2)^5 \\ \dot{x}_2 &= x_1 - (x_1)^2. \end{aligned}$$

Acknowledgements

- This research was partially supported by Fundação de Amparo à Pesquisa do Estado de São Paulo, SP, Brazil.
- We thanks to Prof. Jack K. Hale for his valuable comments.

References

[1] W. C. Chan and S. N. Chow, *Uniform Boundedness and Generalized Inverses in Lyapunov-Schmidt Method for Subharmonics*, Lecture Notes 1248, Springer Verlag, (1987), 28-39.

[2] S. N. Chow and J. K. Hale, *Methods of Bifurcation Theory*, Springer-Verlag (1982).

[3] S. N. Chow, J. K. Hale and Mallet-Paret, *An example of bifurcation to homoclinic orbits*, Journal of Differential Equations, **37** (1980), 351-353.

[4] B.Fiedler and A. Vanderbauwhede, *Homoclinic period blow-up in reversible and conservative systems*, Z angew Math Phys **43** (1992), 292-318.

[5] J. K. Hale, *Ordinary Differential Equations*, Krieger, (1980)

[6] P. Hartman, *On local Homeomorphisms of Euclidean Spaces*, Bol. Soc. Mat. Mexicana, **5**, 1960, 154-188.

[7] H. M. Rodrigues and M. Silveira,*Properties of Bounded Solutions of Linear and Nonlinear Evolution Equations*, Journal of Differential Equations, **70** (1987), 403-440.

[8] H. M. Rodrigues, *The symmetric Hartman-Grobman Theorem*, Notes of CDSNS, CDSNS93-112 , Center for Dynamical Systems and Nonlinear Studies, Georgia Tech, (1993).

[9] A. Vanderbauwhede, *Local Bifurcation and Symmetry*, Pitman, (1982)

N O T A S D O I C M S C

SERIE: Matematica