nº 11

Characterization of the dual of an Orlicz Space

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CHARACTERIZATION OF THE DUAL OF AN ORLICZ SPACE

Our objective is to characterize the dual of an Orlicz space $L_A(X,\Sigma,\mu)$, with the only hypothesis that (X,Σ,μ) is a measure space with no atoms of infinite measure.

This work originates from the reading of [12] and [13] of M.M. Rao, where we found some statements which were unclear to us. In particular the characterization in [13] seems to be incomplete, possibly for some fault in the fundamental definition. Here we present another characterization.

1. Preliminaries. If ψ is a nondecreasing function from $[0,\infty[$ to $[0,\infty]$ such that $0 < \psi(t_o) < \infty$ for some $t_o \neq 0$, the function A defined on $[0,\infty[$ by the equality $A(u) = \int_0^u \psi(t)dt$ is called Young's function.

(1.1) Remark. Let A be a Young's function. Then

(i) A is nondecreasing and convex;

(ii) the right derivative A' of A exists, is nondecreasing and is finite-valued on $[0,b[$, where b is as in (iv) below;

(iii) the function $\bar{A}$ defined on $[0,\infty[$ as

$$\bar{A}(u) = \sup\{uv-A(v): v \in [0,\infty[\} ,$$

is a Young's function, called the complement of A;

(iv) $uv \leq A(u)+\bar{A}(v)$ for $u,v \in [0,\infty[$, the equality holding if, and only if, at least one of the relations $v = p(u)$ or $u = q(v)$ is satisfied by u and v, where

$$p(t) = \begin{cases} \lim\limits_{s\downarrow t} A'(s), & \text{if } t\in[0,b[, \\ \infty & \text{if } t\in[b,\infty[, \end{cases} \qquad q(t) = \begin{cases} \lim\limits_{s\downarrow t} \bar{A}'(s), & \text{if } t\in[0,\bar{b}[, \\ \infty & \text{if } t\in[\bar{b},\infty[, \end{cases}$$

$$b = \inf\{u\in[0,\infty[: A(u) = \infty\} \text{ and } \bar{b} = \inf\{u\in[0,\infty[: \bar{A}(u) = \infty\}.$$

In all that follows A, $\bar{A}$, b and $\bar{b}$ will be as in (1.1), and we will denote

$$a = \sup\{u\in[0,\infty[: A(u) = 0\} \quad , \quad \bar{a} = \sup\{u\in[0,\infty[: \bar{A}(u) = 0\}.$$

Moreover (X,Σ,μ) will be a fixed but arbitrary measure space with no atoms of infinite measure (i.e, if $E\in\Sigma$ and $\mu(E)=\infty$, then there exists $F\in\Sigma$, $F\subset E$

such that $0 < \mu(F) < \infty$). Unless otherwise stated, our functions will be from X to $\mathbb{R}$, and we shall employ the conventions that $0/0 = 1/\infty = 0.\infty = 0$ and $\inf\emptyset = \infty$.

The Orlicz space L_A is the space of all measurable functions, such that $\int_X A(k|f|)d\mu < \infty$, for some $k\in]0,\infty[$. This is a complete space with the seminorm $\|\cdot\|_A$, defined by

$$\|f\|_A = \inf\{k\in]0,\infty[: \int_X A(\frac{|f|}{k})d\mu \le 1\} .$$

(1.2) Proposition. The following assertions are true:

(i) if $E\in\Sigma$ and $\xi_E \in L_A$, then $\|\xi_E\|_A \le \bar{b}\mu(E)$;

(ii) for $E\in\Sigma$ the relation $\mu(E)<\infty$ implies that $\xi_E \in L_A$, the converse holding if $a = 0$; moreover if $\mu(E) = \infty$ and $a > 0$, then $\|\xi_E\|_A = \frac{1}{a}$;

(iii) for $f\in L_A$ one has, $\int_X A(|f|)d\mu \le 1$ if and only if $\|f\|_A \le 1$;

(iv) if $\delta\in[0,\infty[$ and $f \in L_A$, then

$$A(\delta).\mu(\{x\in X: |f(x)| \ge \delta\|f\|_A\}) \le 1 ;$$

(v) if $f\in L_A$, then $\|f\|_A \le \int_X A(|f|)d\mu + 1$.

Proof. Assertion (i) follows immediately from

$$\frac{A(u)}{u} \le \bar{b} \quad \text{for } u \in]0,\infty[,$$

and for this, observe that $\frac{A(u)}{u}$ is nondecreasing.

To prove (iv), use (1.1.i) and (iii). For the remaining, see [2].//

(1.3) Proposition. For f in L_A there is a sequence (s_n) of simple functions in L_A such that

(i) $0 \le s_n \le s_{n+1}$, for all $n\in\mathbb{N}$, if $f \ge 0$;

(ii) $|f-s_n| \le |f|$ and $|s_n| \le |f|$ for all $n\in\mathbb{N}$;

(iii) (s_n) converges to f ;

(iv) (s_n) converges in μ-measure to f ;

(v) if $k \in]0,\infty[$ is such that $A(|f|/k) \in L_1$, then

$$\lim_{n\to\infty} \int_X A(\frac{|f-s_n|}{k})d\mu = 0 .$$

Proof. Discarding a trivial case, let $\|f\|_A \neq 0$. The measurability of f guarantees the existence of a sequence (s_n) of simple functions satisfying (i), (ii) and (iii). Moreover, if $E \in \Sigma$ is such that f is bounded on E, then $(s_n \xi_E)$ converges uniformly to $f\xi_E$. From (ii) it follows that each $s_n \in L_A$.

To establish (v) it suffices to use (ii) and (iii) and apply Lebesgue's Convergence Theorem.

It remains to prove (iv), and we first consider the case in which $a > 0$. Let $B = \{x \in X: |f(x)| \geq 2a\|f\|_A\}$. Clearly $(s_n \xi_{B^c})$ converges uniformly to $f\xi_{B^c}$. By (1.2.iv) we have $\mu(B) < \infty$ and hence, by Egorov's Theorem, that $(s_n \xi_B)$ converges to $f\xi_B$ in μ-measure. So (iv) holds in this case.

If $a=0$, let $c \in]0,\infty[$ be such that $0 < A(c) < \infty$. Replacing k by $\|f\|_A$ in (v) we have $\lim_{n\to\infty} \int_X A\left(\frac{|f-s_n|}{\|f\|_A}\right) d\mu = 0$, and so there is a subsequence (s_{n_k}) such that

$$\int_X A\left(\frac{|f-s_{n_k}|}{\|f\|_A}\right) d\mu < \frac{1}{2^k} A\left(\frac{c}{2^k}\right).$$

From this we see that $\mu(\{x \in X: |f(x)-s_{n_k}(x)| > \frac{c\|f\|_A}{2^k}\}) < \frac{1}{2^k}$ for all $k \in \mathbb{N}$, and so (s_{n_k}) converges to f in μ-measure. Replace (s_n) by (s_{n_k}).//

2. Characterization of the dual space $(M_A)^*$. Denote by M_A the closed subspace of L_A spanned by the simple functions.

In [13] the author gives a representation for $x^* \in (M_A)^*$ as an integral, in the sense of Dunford and Schwartz [3], relatively to a measure G defined as $G(E) = x^*(\xi_E)$. Clearly this defines G(X) only if $\xi_X \in L_A$. Since this does not occur when $a = 0$ and $\mu(X) = \infty$, the measure G is not defined on Σ in this case, and so we cannot apply the theory of [3]. Thus in [5] we introduce the concept of integration relative to measure defined on ideal.

To facilitate the reading, we transcribe from [5] some definitions and one proposition. For this, let $\mathcal{A}$ be an algebra of subsets of X, and H, an ideal of $\mathcal{A}$(i.e, H is a ring, and for $E \in \mathcal{A}$ and $F \in H$ one has $E \cap F \in H$). Moreover let G be an extended real-valued finitely additive measure defined on H, and $v(G,\cdot)$ be the total variation of G, which is obtained replacing Σ by H in III.1.4 of [3].

(2.1) Definition. A function s is H-simple if there is a pairwise disjoint, finite sequence $(E_1,E_2,\ldots,E_n)$ in H and there exists $(c_1,c_2,\ldots,c_n) \in \mathbb{R}^n$ such that $s = \sum_{i=1}^{n} c_i \xi_{E_i}$. If $v(G,\{x\in X: s(x) = c_i\}) < \infty$ for $c_i \neq 0$, we shall say that s is G-integrable and for $E\in A$ we define

$$\int_E s\,dG = \sum_{i=1}^{n} c_i G(E \cap E_i).$$

(2.2) Definition. A function f is G-integrable if there is a sequence (s_n) of G-integrable simple functions such that

(i) (s_n) converges in G-measure to f, i.e. for every $\delta\in]0,\infty[$ one has

$$\lim_{n\to\infty}(\inf\{v(G,F): F\in H \text{ and } \{x\in X: |f_n(x)-f(x)| > \delta\} \subset F\}) = 0;$$

(ii) $\lim_{m,n\to\infty} \int_X |s_n - s_m|\,dv(G,\cdot) = 0.$

If f is G-integrable and $E\in A$ we define

$$\int_E f\,dG = \lim_{n\to\infty} \int_E s_n\,dG.$$

(2.3) Proposition (Vitali Convergence Theorem). Let f be a function and (f_n) a sequence of G-integrable functions, such that

(i) (f_n) converges in G-measure to f;

(ii) $\lim_{v(G,E)\to 0} \int_E |f_n|\,dv(G,\cdot) = 0$ uniformly in $n\in\mathbb{N}$;

(iii) for each $\varepsilon\in]0,\infty[$ there is a set F_ε in H with $v(G,F_\varepsilon) < \infty$ and such that

$$\int_{F_\varepsilon^c} |f_n|\,dv(G,\cdot) < \varepsilon, \quad \text{for all } n\in\mathbb{N}.$$

Then f is G-integrable and for all $E\in A$ one has

$$\int_E f\,dG = \lim_{n\to\infty} \int_E f_n\,dG.$$

Next we take up our characterization of $(M_A)^*$.

From here on we agree that $A = \Sigma$ and $\Sigma_1=\{E\in\Sigma: \mu(E) < \infty\}$; moreover, if $a = 0$ we set $H = \Sigma_1$, and if $a > 0$, $H = \Sigma$.

(2.4) Definition. We shall say that $G \in G_{\bar{A}}(X,\Sigma,\mu,H)$ if G is a real--valued, finitely additive, measure defined on H such that

(i) $G \ll \mu$, i.e. if $E \in \Sigma$ and $\mu(E) = 0$, then $G(E) = 0$;

(ii) $0 \leq v(G,E) < \infty$ for all $E \in H$;

(iii) $\alpha_G^{\bar{A}} = \inf\{k \in]0,\infty[: I_{\bar{A}}(\frac{G}{k}, X) \leq 1\} < \infty$, where

$$I_{\bar{A}}(\frac{G}{k}, E) = \sup\{ \sum_{i=1}^{n} \bar{A} (\frac{|G(E_i)|}{k\mu(E_i)})\mu(E_i) : (E_1,E_2,\ldots,E_n) \in \mathcal{D}(\Sigma_1,E)\} \text{ , for } E \in \Sigma \text{ ,}$$

and $\mathcal{D}(\Sigma_1,E)$ is the set of all pairwise disjoint finite sequences $(E_1,E_2,\ldots,E_n)$ in Σ_1 such that $\bigcup_{i=1}^{n} E_i \subset E$.

Unless otherwise stated, G will denote a real-valued, finitely additive measure, defined on H such that $G \ll \mu$.

(2.5) Remarks. (i) For $k \in]0,\infty[$, the function $I_{\bar{A}}(\frac{G}{k},\cdot)$ is a real-valued finitely additive measure defined on Σ.

(ii) If $0 < \alpha_G^{\bar{A}} < \infty$, then $I_{\bar{A}}(\frac{G}{\alpha_G^{\bar{A}}}, X) \leq 1$.

(iii) If $\alpha_G^{\bar{A}} = 0$, then $G(E) = 0$ for all $E \in \Sigma_1$.

(iv) The function $\alpha_{(\cdot)}^{\bar{A}}$ is a seminorm on the vector space $G_{\bar{A}}(X,\Sigma,\mu,H)$, and is a norm if $H = \Sigma_1$ (i.e, a=0).

(v) If $G \in G_{\bar{A}}(X,\Sigma,\mu,H)$, then by (1.2.ii) and (2.4.ii), every simple function in L_A is H-simple and G-integrable.

In (2.8) below, for a fixed $f \in L_A$ we shall find a sequence (s_n) of simple functions in L_A such that $\int_E f dG = \lim_{n\to\infty} \int_E s_n dG$ for every $E \in \Sigma$ and all $G \in G_{\bar{A}}(X,\Sigma,\mu,H)$. First, some propositions.

(2.6) Proposition. If $b = \infty$ and G is such that $\alpha_G^{\bar{A}} < \infty$, then G is μ-continuous, i.e. $\lim_{\mu(A)\to 0} G(A) = 0$.

Proof. See Lemma 6 in [12].//

(2.7) Proposition. Let $G \in G_{\bar{A}}(X,\Sigma,\mu,H)$. If $s \in L_A$ is a simple, nonnegative function, then

$$\int_X s dv(G,\cdot) \leq c\|s\|_A ,$$

where $c = 2\alpha_G^{\bar{A}}$ if $a = 0$ or $\mu(X) < \infty$, and $c = 2\alpha_G^{\bar{A}} + av(G,X)$ if $a>0$ and $\mu(X)=\infty$.

Proof. Eliminating a trivial case, suppose $\|s\|_A \neq 0$.

Let $s = \sum_{i=1}^{n} c_i \xi_{E_i}$ be as in (2.1) with $c_i \geq 0$.

First we observe that if $(F_1, F_2, \ldots, F_\ell)$ is a pairwise disjoint finite sequence in Σ_1 and $\alpha_G^{\bar{A}} \neq 0$, then by (1.1.iv) we have

$$\sum_{j=1}^{\ell} c_i |G(F_j)| \leq \alpha_G^{\bar{A}} \|s\|_A [\sum_{j=1}^{\ell} (\bar{A}(\frac{|G(F_j)|}{\alpha_G^{\bar{A}} \mu(F_j)}) + A(\frac{c_i}{\|s\|_A}))\mu(F_j)]. \quad (1)$$

Let $J = \{i \in \mathbb{N}: 1 \leq i \leq n \text{ and } E_i \in \Sigma_1\}$. Using (1.2.iii), (2.5.i), (2.5.ii), (1) and observing (2.5.iii) it follows that

$$\sum_{i \in J} c_i v(G, E_i) \leq 2\alpha_G^{\bar{A}} \|s\|_A. \quad (2)$$

By (1.2.ii) we obtain

$$\sum_{i \notin J} c_i v(G, E_i) = a \sum_{i \notin J} c_i \|\xi_{E_i}\|_A v(G, E_i) \leq a \|s\|_A v(G, X). \quad (3)$$

From (2), (3) and (1.2.ii) we obtain the desired result.//

(2.8) Theorem. For f in L_A there is a sequence (s_n) of simple functions in L_A satisfying (1.3.i), (1.3.ii), (1.3.iii) and such that for all $G \in G_{\bar{A}}(X, \Sigma, \mu, H)$ one has

(i) (s_n) converges in G-measure to f;

(ii) f is G-integrable and $\int_E f dG = \lim_{n \to \infty} \int_E s_n dG$, for $E \in \Sigma$.

As a consequence, the following hold:

(iii) $|\int_X f dG| \leq c\|f\|_A$, with c as in (2.7);

(iv) $\int_X f d(\gamma G_1 + G_2) = \gamma \int_X f dG_1 + \int_X f dG_2$, for $\gamma \in \mathbb{R}$ and $G_1, G_2 \in G_{\bar{A}}(X, \Sigma, \mu, H)$.

Proof. To establish (i) and (ii), it suffices to consider $f \geq 0$.

If $b = \infty$, let (s_n) be as in (1.3). From (2.6) we obtain that (s_n) converges in G-measure to f.

If $b < \infty$, we have that $|f| \leq 2b\|f\|_A$ μ-a.e. (1.2.iv) and, since f is measurable, we can take a sequence (s_n) of simple functions satisfying (1.3.i), (1.3.iii) and converging uniformly to f except on a set $E \in \Sigma$ with $\mu(E) = 0$. Clearly $s_n \in L_A$ for $n \in \mathbb{N}$, $v(G, \cdot) \ll \mu$ and (s_n) converges in G-measure to f.

Now we will prove that the sequence (s_n) above also satisfies (2.3.ii) and (2.3.iii), and thus obtain (ii).

For all $n \in \mathbb{N}$, let $y_n = \int_X s_n dv(G,\cdot)$. Since (y_n) is bounded (2.7), and nondecreasing, given $\varepsilon \in]0,\infty[$ there is an $n_o \in \mathbb{N}$ such that

$$\int_E s_n dv(G,\cdot) = \int_X s_n dv(G,\cdot) - \int_{E^c} s_n dv(G,\cdot) < \frac{\varepsilon}{2} + \int_E s_{n_o} dv(G,\cdot) ,$$

for all $n \geq n_o$ and $E \in \Sigma$.

If $M = 1 + \max\{|s_{n_o}(x)|: x \in X\}$, $\delta = \varepsilon/(2M)$ and $F_\varepsilon = \{x \in X: s_{n_o}(x) \neq 0\}$, we have that $F_\varepsilon \in H$, and (s_n) satisfies (2.3.ii) and (2.3.iii).

For assertion (iii), use (ii), (2.7) and note that $|\int_X s_n dG| \leq \int_X |s_n| dv(G,\cdot)$, for $n \in \mathbb{N}$.

To establish (iv) it suffices to use (ii).//

Our next objective is to define a norm on $G_{\bar{A}}(X,\Sigma,\mu,H)$ and to prove that this space is isometrically isomorphic to $(M_A)^*$.

(2.9) <u>Proposition</u>. The function defined by

$$\|G\|_{\bar{A}} = \sup\{|\int_X f dG|: f \in L_A \text{ and } \|f\|_A \leq 1\} ,$$

is a norm on $G_{\bar{A}}(X,\Sigma,\mu,H)$. Moreover, one has

$$\|G\|_{\bar{A}} = \sup\{|\int_X f dG|: f \in M_A \text{ and } \|f\|_A \leq 1\}.$$

Proof. The first assertion follows trivially from (2.8.iii) and (2.8.iv).

For the second, observe that simple functions in L_A belong to M_A and use (2.8.ii).//

(2.10) <u>Proposition</u>. If $G \in G_{\bar{A}}(X,\Sigma,\mu,H)$, then the function x^* defined on M_A $[L_A]$ by $x^*(f) = \int_X f dG$, belongs to $(M_A)^*$ $[(L_A)^*]$ and $\|x^*\| = \|G\|_{\bar{A}}$.

(2.11) <u>Proposition</u>. Let $x^* \in (M_A)^*$ and let G be the function defined on H by $G(E) = x^*(\xi_E)$. Then $G \in G_{\bar{A}}(X,\Sigma,\mu,H)$ and the following holds:

(i) $x^*(f) = \int_X f dG$ for all $f \in M_A$, and $\|x^*\| = \|G\|_{\bar{A}}$.

Proof. We may syppose $\|x^*\| \neq 0$. It is clear that G is a real-valued, finitely additive measure, and $G \ll \mu$.

To prove that $v(G,E) < \infty$ for all $E \in H$, it is enough to observe that, if $E \in H$ and $(E_1,E_2,\ldots,E_n)$ is a pairwise disjoint finite sequence in H with

$\bigsqcup_{i=1}^{n} E_i \subset E$ then taking $J = \{i \in \mathbb{N}: 1 \le i \le n$ and $x^*(\xi_{E_i}) \ge 0\}$, we have

$$\sum_{i=1}^{n} |G(E_i)| = x^*(\xi_{\bigsqcup_{i \in J} E_i}) - x^*(\xi_{\bigsqcup_{i \notin J} E_i}) \le 2 \|x^*\| \|\xi_E\|_A .$$

Next we will prove that $\alpha_G^{\bar{A}} \le \|x^*\| < \infty$.

Let $\bar{c} \in]0,1[$ and we take $(E_1, E_2, \ldots, E_n)$ an arbitrary pairwise disjoint finite sequence in Σ_1. If $u_i = \bar{c}|G(E_i)|/(\|x^*\|\mu(E_i))$ belongs to $[0,\bar{b}[$ (1.2.i),

$c_i = \lim_{t \downarrow u_i} \bar{A}'(t)$ for $i \in \{1,2,\ldots,n\}$ and $f = \sum_{i=1}^{n} c_i \operatorname{sgn}(G(E_i))\xi_{E_i}$, then $f \in L_A$

and by (1.1.iv) and (1.2.v) we have

$$\|f\|_A \ge \frac{\bar{c}|x^*(f)|}{\|x^*\|} = \sum_{i=1}^{n} [A(c_i) + \bar{A}(\frac{\bar{c}|G(E_i)|}{\|x^*\|\mu(E_i)})]\mu(E_i)$$

$$\ge \|f\|_A - 1 + \sum_{i=1}^{n} \bar{A}(\frac{\bar{c}|G(E_i)|}{\|x^*\|\mu(E_i)})\mu(E_i).$$

From this relation it is easy to see that $\alpha_G^{\bar{A}} \le \|x^*\|/\bar{c}$, and since $\bar{c} \in]0,1[$ is arbitrary, we conclude that $\alpha_G^{\bar{A}} \le \|x^*\|$. Thus $G \in G_{\bar{A}}(X,\Sigma,\mu,H)$.

To obtain (i) it is enough to observe that (2.10) is true and that $x^*(s) = \int_X s\,dG$, where s is a simple function in L_A.//

From (2.10), (2.11) and (2.8.iv) we obtain in (2.12) a characterization of $(M_A)^*$; it will follow that $G_{\bar{A}}(X,\Sigma,\mu,H)$ is a Banach space.

(2.12) Theorem. There exists an isometric isomorphism of $(M_A)^*$ onto $G_{\bar{A}}(X,\Sigma,\mu,H)$, given by the mapping $x^* \longmapsto G$ where $G(E) = x^*(\xi_E)$ for $E \in H$, and the following holds:

(i) $x^*(f) = \int_X f\,dG$ for all $f \in M_A$ and $\|x^*\| = \|G\|_{\bar{A}}$.

(2.13) Comment. In [13], given $x^* \in (M_A)^*$ the author associates a measure G defined on Σ, satisfying some conditions. To obtain (2.12) it was necessary to replace the domain of G by H, and condition (i) of Definition 3 in [13] by (2.4.ii).

3. Characterization of the dual space $(N_A)^*$. Let $N_A = L_A/M_A$, write $\bar{f} = f + M_A$ for any $f \in L_A$, and denote by $d(\cdot)$ the quotient norm on N_A. One can easily show that

$d(\bar{f}) = \inf \{\|f+s\|_A : s$ is a simple function in $L_A\}$.

(3.1) Definition. Let $\bar{f},\bar{g} \in N_A$. We define $\bar{f} \leq \bar{g}$ if there exist $f_1 \in \bar{f}$ and $g_1 \in \bar{g}$ such that $f_1 \leq g_1$.

In [13] it is proved that the above relation is a partial ordering on N_A and that $(N_A)^*$ is a vector lattice. Two results which are used in the proofs, and are not obvious to us, are established in (3.4) and (3.5). First, two propositions.

(3.2) Proposition. Let $a > 0$ or $\mu(X) < \infty$. If f is a measurable function, bounded μ-a.e, then $f \in M_A$.

Proof. We may suppose f to be bounded. Since there exist $\bar{x} \in]0,\infty[$ such that $A(\bar{x}) \leq 1/(\mu(X)+1)$, and a sequence (s_n) of simple functions, converging uniformly to f, it is easy to verify that $\lim_{n\to\infty}\|f-s_n\|_A = 0$. From $a > 0$ or $\mu(X) < \infty$ it follows that $f, s_n \in L_A$ for all $n \in \mathbb{N}$. Thus $f \in M_A$.//

(3.3) Proposition. Let $J_A = \{f \in L_A : \int_X A(k|f|)d\mu < \infty$ for all $k \in]0,\infty[\}$. Then J_A is a closed subspace of M_A and the following assertions are true:

(i) $J_A \subset M_A \subset L_A$;

(ii) if $a=0$ and $b=\infty$, or if $\mu(X) < \infty$ and $b=\infty$, then $J_A = M_A$;

(iii) if $a > 0$ and $b < \infty$, or if $\mu(X) < \infty$ and $b < \infty$, then $L_A = M_A$.

Proof. From (1.3.v) it follows easily that $J_A \subset M_A$. The remaining assertions are proved in Lemma 2 in [13].//

(3.4) Proposition. If $g \in M_A$ and f is a measurable function such that $|f| \leq |g|$ μ-a.e, then $f \in M_A$.

Proof. We may suppose $|f| \leq |g|$. Let (s_n) be a sequence of simple functions in L_A such that $\lim_{n\to\infty} \|g-s_n\|_A = 0$.

First we observe that for $k \in]0,\infty[$, if $n \in \mathbb{N}$ is such that $2\|g-s_n\|_A \leq k$, then by (1.1.i) and (1.2.iii), for all $H \in \Sigma$ we have

$$\int_H A\left(\frac{|f|}{k}\right)d\mu \leq \int_H A\left(\frac{|g|}{k}\right)d\mu \leq \frac{1}{2} + \frac{1}{2}\int_H A\left(\frac{2|s_n|}{k}\right)d\mu. \qquad (1)$$

Suppose $b = \infty$ and $a > 0$. From $\lim_{n\to\infty} \|g-s_n\|_A = 0$ there exists an $n_0 \in \mathbb{N}$

such that $\|g-s_{n_o}\|_A \le 1$.

Let $B = \{x\in X: |g(x)-s_{n_o}(x)| \ge 2a\}$. Then $\mu(B)<\infty$ (1.2.iv), and the inequalities $|f| \le |g| \le |g-s_{n_o}|+|s_{n_o}|$, together with (3.2), imply that $f\xi_{B^c} \in M_A$. Moreover it follows from (1) that $f\xi_B \in J_A \subset M_A$. So $f \in M_A$ in this case.

Let $b<\infty$, $a = 0$ and $E_n = \{x \in X: s_n(x) \ne 0\}$ for $n\in\mathbb{N}$. Then $|f|\le 2b\|f\|_A$ μ-a.e. (1.2.iv), $\mu(E_n)<\infty$ (1.2.ii) and $f\xi_{E_n} \in M_A$(3.2), for all $n \in \mathbb{N}$. Using (1) we obtain that $\lim_{n\to\infty}\|f-f\xi_{E_n}\|_A = 0$, and since M_A is closed in L_A, we conclude that $f \in M_A$ also in this case.

The remaining cases are easy (3.3).//

(3.5) Proposition. If $f,g \in L_A$ and $|f| \le |g|$ μ-a.e, then $d(\hat{f}) \le d(\hat{g})$.

Proof. We may suppose $|f| \le |g|$ and, by (3.4), that $g \notin M_A$.

It suffices to prove that for all $\varepsilon \in]0,\infty[$ we have

$$d(\hat{f}) \le d(\hat{g}) + \varepsilon. \qquad (1)$$

For a fixed $\varepsilon \in]0,\infty[$, let $s\in L_A$ be a simple function such that $0 < d(\hat{g}) \le \|g+s\|_A < d(\hat{g}) + \varepsilon$ and let $H_1 = \{x\in X: s(x) \ne 0\}$.

As $g \notin M_A$ we have $b<\infty$ and $a = 0$, or $b = \infty$ (3.3.iii).

If $b<\infty$ and $a=0$, then $\mu(H_1)<\infty$ (1.2.ii) and $|f|\le 2b\|f\|_A$ μ-a.e. (1.2.iv). Since $f\xi_{H_1} \in M_A$ (3.2) and

$$\|f\xi_{H_1^c}\|_A \le \|g\xi_{H_1^c}\|_A \le \|(g+s)\xi_{H_1^c}\|_A < d(\hat{g}) + \varepsilon,$$

we conclude (1).

For the remaining cases, we first note that if $H\in\Sigma$ and $A(|s|/((\alpha-1)\|g+s\|_A))\xi_H \in L_1$ for all $\alpha \in]1,\infty[$, then

$$d((f\xi_H)^\wedge) \le d(\hat{g}) + \varepsilon. \qquad (2)$$

In fact, by (1.1.i) and (1.2.iii) it is true that

$$\int_H A\left(\frac{|f|}{\alpha\|g+s\|_A}\right)d\mu \le \int_H A\left(\frac{|g|}{\alpha\|g+s\|_A}\right)d\mu$$

$$\le \frac{1}{\alpha} + \frac{\alpha-1}{\alpha}\int_H A\left(\frac{|s|}{(\alpha-1)\|g+s\|_A}\right)d\mu,$$

and hence $A(|f| / (\alpha\|g+s\|_A))\xi_H \in L_1$. From (1.3) there exists a simple function $s_o \in L_A$ such that $\|(f-s_o)\xi_H\|_A \le \alpha\|g+s\|_A$. Thus $d((f\xi_H)^{\sim}) \le \|(f-s_o)\xi_H\|_A \le \alpha\|g+s\|_A < \alpha(d(\tilde{g})+\varepsilon)$ and, since $\alpha \in]1,\infty[$ is arbitrary, (2) holds.

If $b = \infty$ and $a = 0$, then $\mu(H_1) < \infty$ (1.2.ii) and we obtain (1) replacing H by X in (2).

If $b = \infty$ and $a > 0$, let $H_2 = \{x \in X: |g(x)+s(x)| \ge 2a\|g+s\|_A\}$. Then (1.2.iv) tells us that $\mu(H_2) < \infty$ and we may replace H by H_2 in (2). Moreover, it follows from $|f| \le |g| \le |g+s|+|s|$ that $f\xi_{H_2^c}$ is bounded, and thus $f\xi_{H_2^c} \in M_A$ (3.2). Therefore (1) holds.//

Let E be a pseudonormed vector lattice, $x \in E$ and $z^* \in E^*$. Then we will set as usual $x_+ = x \vee 0$, $x_- = (-x)\vee 0$ and $|x| = x_+ + x_-$. Also, we will write that $z^* \ge 0$ if $z^*(y) \ge 0$ whenever $y \ge 0$.

(3.6) Proposition. The following assertions are true:

(i) N_A is a vector lattice, and if $\tilde{f},\tilde{g} \in N_A$, then $\tilde{f}\vee\tilde{g} = (\max\{f,g\})^{\sim}$, $\tilde{f}\wedge\tilde{g} = (\min\{f,g\})^{\sim}$ and $(|f|)^{\sim} = |\tilde{f}|$;

(ii) N_A and $(N_A)^*$ are Banach lattices;

(iii) if $x^* \in (N_A)^*$ and $x^* \ge 0$, then

$$\|x^*\| = \sup\{x^*(\tilde{f}): f \ge 0 \text{ and } \int_X A(f)d\mu < \infty\} ;$$

(iv) if $x^* \in (N_A)^*$, then $\|x^*\| = \|(x^*)_+\|+\|(x^*)_-\|$;

(v) if $x^*,y^* \in (N_A)^*$ and $x^*,y^* \ge 0$, then

$$\|x^*+y^*\| = \|x^*\| + \|y^*\| ;$$

(vi) if $\tilde{f},\tilde{g} \in N_A$ and $\tilde{f},\tilde{g} \ge \tilde{0}$, then $d(\tilde{f}\vee\tilde{g}) = \max\{d(\tilde{f}),d(\tilde{g})\}$.

Proof. Except for (iii), these assertions are proved in [13]. Assertion (iii) follows from Lemma 6 in [13], observing that $|\tilde{f}| = (|f|)^{\sim}$ for $f \in L_A$.//

From (3.6.ii) and (3.6.v), and from (3.6.ii) and (3.6.vi) we have, respectively, that $(N_A)^*$ is an L-space and that N_A is an M-space.

From here on ν will denote a real-valued finitely additive, bounded measure defined on Σ , $v(\nu,\cdot)$ the total variation of ν, $\nu_1 = (\nu+v(\nu,\cdot))/2$ and $\nu_2 = (v(\nu,\cdot) - \nu)/2$. Also we will denote by P the set of all the pairwise disjoint finite sequences $(E_1,E_2,\ldots,E_n)$ in Σ, such that $\bigsqcup_{i=1}^{n} E_i = X$.

(3.7) Definition. Let $f \in L_A$. If f is nonnegative, we define

$$\int_X f d\nu_j = \inf\{\sum_{i=1}^{n} d((f\xi_{E_i})^{\sim})\nu_j(E_i): (E_1,E_2,\ldots,E_n) \in P\},$$

for $j \in \{1,2\}$.

In the general case we define

$$\int_X f d\nu = (\int_X f_+ d\nu_1 - \int_X f_+ d\nu_2) - (\int_X f_- d\nu_1 - \int_X f_- d\nu_2).$$

(3.8) Remarks. (i) From Lemma 10 in [13] and (3.5), one has that

$$\int_X (\gamma f+g)d\nu = \gamma\int_X f d\nu + \int_X g d\nu \text{ , for } \gamma\in\mathbb{R} \text{ and } f,g \in L_A \text{ ,}$$

and

$$|\int_X f d\nu| \le \int_X |f| d\nu_1 + \int_X |f| d\nu_2 \le d(|f|^{\sim})(\nu_1+\nu_2)(X) = d(\tilde{f})v(\nu,X), \text{ for } f\in L_A.$$

(ii) If $x^*(\tilde{f}) = \int_X f d\nu$, for $\tilde{f}\in N_A$, then $x^* \in (N_A)^*$ and $\|x^*\| \le v(\nu,X)$.

(iii) If ν is nonnegative, $E \in \Sigma$ and ν_E is defined on Σ by $\nu_E(F) = \nu(E\cap F)$, then $\int_X f\xi_E d\nu = \int_X f d\nu_E$ for all $0 \le f \in L_A$.

(iv) If ν and ϑ are real-valued finitely additive, bounded measures defined on Σ, $f \in L_A$ and $\gamma \in \mathbb{R}$, then

$$\int_X f d(\gamma\nu+\vartheta) = \gamma\int_X f d\nu + \int_X f d\vartheta .$$

(v) If $\tilde{f}\in N_A$ and $\tilde{f} \ge \tilde{0}$, there exists a nonnegative representative of $\tilde{f}$.

Next we take up our characterization of $(N_A)^*$.

(3.9) Definition. We shall say that $\nu\in V_{\bar{A}}(X,\Sigma,\mu)$ if $\nu \ll \mu$ and there exists $\tilde{f} \in N_A$, $\tilde{f} \ge \tilde{0}$ with $d(\tilde{f}) \le 1$, for which the following holds:

(i) if $E \in \Sigma$ and $v(\nu,E) \ne 0$, then $d((f\xi_E)^{\sim}) = 1$.

It is easy to see, using (3.6.vi), that $V_{\bar{A}}(X,\Sigma,\mu)$ is a vector space.

(3.10) Comment. In [13] (page 573), in place of $V_{\bar{A}}(X,\Sigma,\mu)$, the author uses $B_{\bar{A}}(\mu)$, the space of all $\nu \ll \mu$ such that there exists $f \in L_A$ with $\int_X A(|f|)d\mu < \infty$, $f \notin M_A$ and the following holds:

(i) the support of ν lies in the support of f ,

where "support of ν" is defined in [13] (page 571) as "the sets E for which $v(\nu,E) > 0$".

The author claims that if $\nu \in B_{\bar{A}}(\mu)$ is nonnegative and $x^*(\hat{f}) = \int_X f d\nu$ for all $\hat{f} \in N_A$, then $\|x^*\| = \nu(X)$. To prove this, he considers $\pi = (E_1, E_2, \ldots, E_n) \in P$ with $\nu(E_i) > 0$ and defines $f = f_1 + f_2 + \ldots + f_n$ where $\int_X A(|f_i|)d\mu < \infty$, $d(\hat{f}_i) = 1$ and support of f_i is E_i, for $i \in \{1,2,\ldots,n\}$. Then he writes: "Thus $\sum_{i=1}^{n} d((f\xi_{E_i})^\frown)\nu(E_i) = \nu(X)$ and refining the partition π on the left yields $x^*(\hat{f}) = \nu(X)$". This last assertion is not clear to us, for f depends on the partition π. However we observe that if there is an $\hat{f}$ satisfying (3.9.i), $\hat{f} \geq \hat{0}$, then it is easy to see that $x^*(\hat{f}) = \nu(X)$. Thus we have replaced, in our characterization, (3.10.i) by (3.9.i).

(3.11) <u>Notation</u>. If $E \in \Sigma$ and $x^* \in (N_A)^*$, we will denote by x^*_E the function defined on N_A by $x^*_E(\hat{f}) = x^*((f\xi_E)^\frown)$.

(3.12) <u>Proposition</u>. Let $0 \leq x^* \in (N_A)^*$. Then there exists $g \in L_A$, $g \geq 0$, with $d(\hat{g}) \leq 1$, such that

(i) $\|x^*_E\| = x^*((g\xi_E)^\frown)$ for all $E \in \Sigma$;

(ii) if $E \in \Sigma$ and $\|x^*_E\| \neq 0$, then $d((g\xi_E)^\frown) = 1$.

Proof. By (3.6.iii), for $n \in \mathbb{N}$ there exists a nonnegative $f_n \in L_A$ such that $\int_X A(f_n)d\mu < \infty$ and $x^*(\hat{f}_n) > \|x^*\| - \frac{1}{n}$. Moreover from (1.3) we know that there is a nonnegative $h_n \in L_A$ such that $\int_X A(h_n)d\mu < \frac{1}{2^n}$ and $\hat{h}_n = \hat{f}_n$.

Let $h = \lim_{n\to\infty}(\max\{h_1, \ldots, h_n\})$ and observe that if $E = \{x \in X: h(x) = \infty\}$, then $\mu(E) = 0$. In fact, if $\mu(E) > 0$, there exists $F \in \Sigma_1$, $F \subset E$ such that $\mu(F) > 0$. By (1.1.iv) and (1.2.iii) we have

$$\infty \cdot \mu(F) = \int_X \xi_F h d\mu \leq \|\xi_F\|_{\bar{A}} \lim_{k\to\infty}\left[\int_X \bar{A}\left(\frac{\xi_F}{\|\xi_F\|_{\bar{A}}}\right)d\mu + \sum_{n=1}^{k} \int_X A(h_n)d\mu\right] < \infty,$$

and so $\mu(F) = 0$, which is a contradiction.

Let $g = h\xi_{E^c}$. Then $\int_X A(g)d\mu \leq \sum_{n=1}^{\infty} \int_X A(h_n)d\mu \leq 1$, and since

$x^*(\hat{g}) \geq x^*(\hat{h}_n) = x^*(\hat{f}_n) > \|x^*\| - \frac{1}{n}$, for all $n \in \mathbb{N}$, we conclude that $\|x^*\| = x^*(\hat{g})$. So for $E \in \Sigma$, using (3.6.v), we obtain

$$0 \leq \|x^*_E\| - x^*((g\xi_E)^\frown) = x^*((g\xi_{E^c})^\frown) - \|x^*_{E^c}\| \leq 0 ,$$

and thus

$$\|x^*_E\| = x^*((g\xi_E)^\frown) \leq \|x^*_E\| \, d((g\xi_E)^\frown) \leq \|x^*_E\| . //$$

(3.13) Theorem. Let $0 \le x^* \in (N_A)^*$. Then there exists a unique $\nu \in V_{\bar{A}}(X,\Sigma,\mu)$, nonnegative, defined by $\nu(E) = \|x^*_E\|$, such that

(i) $x^*(\hat{f}) = \int_X f d\nu$ for all $\hat{f} \in N_A$, and $\|x^*\| = \nu(X)$.

Proof. Let $\nu(E) = \|x^*_E\|$ for $E \in \Sigma$. Then $\nu \in V_{\bar{A}}(X,\Sigma,\mu)$ ((3.6.v) and (3.12)). To show that (i) holds define $z^*(\hat{f}) = \int_X f d\nu$ for $\hat{f} \in N_A$; we shall prove that $\|z^*-x^*\| = 0$.

For $\hat{0} \le \hat{f} \in N_A$ and $\pi = (E_1,E_2,\ldots,E_n) \in P$ we have

$$\sum_{i=1}^{n} d((f\xi_{E_i})^{\frown})\nu(E_i) = \sum_{i=1}^{n} d((f\xi_{E_i})^{\frown}) \|x^*_{E_i}\| \ge \sum_{i=1}^{n} x^*((f\xi_{E_i})^{\frown}) = x^*(\hat{f}),$$

and from this it is clear that $z^* \ge x^*$. Thus $\|z^*\| = \|x^*\|$, for $\|z^*\| \le \nu(X) = \|x^*\|$ (3.8.ii), and also $\|(z^*-x^*)+x^*\| = \|z^*-x^*\|+\|x^*\|$ (3.6.v). Hence $\|z^*-x^*\| = 0$.

To prove the uniqueness of ν suppose that $\tilde{\nu} \in V_{\bar{A}}(X,\Sigma,\mu)$, nonnegative, and $x^*(\hat{f}) = \int_X f d\tilde{\nu}$ for all $\hat{f} \in N_A$. Then by (3.8.iii) and (3.8.i) we have that

$$x^*_E(\hat{f}) = x^*((f\xi_E)^{\frown}) = \int_X f\xi_E d\tilde{\nu} \le d(\hat{f})\tilde{\nu}(E) ,$$

for $\hat{f} \ge \hat{0}$ and $E \in \Sigma$. So $\tilde{\nu} - \nu$ is nonnegative, belongs to $V_{\bar{A}}(X,\Sigma,\mu)$ and $\int_X f d(\tilde{\nu}-\nu) = 0$ for all $f \in L_A$ (3.8.iv).

Since $\tilde{\nu}-\nu \in V_{\bar{A}}(X,\Sigma,\mu)$, there exists $\hat{0} \le \hat{g} \in N_A$ such that $d((g\xi_E)^{\frown}) = 1$ for $E \in \Sigma$ with $(\tilde{\nu}-\nu)(E) \ne 0$. Thus $\int_X g d(\tilde{\nu}-\nu) = (\tilde{\nu}-\nu)(X)$, and hence $(\tilde{\nu}-\nu)(X) = 0$.//

(3.14) Theorem. There exists an isometric isomorphism of $(N_A)^*$ onto $V_{\bar{A}}(X,\Sigma,\mu)$, given by the mapping $x^* \longmapsto \nu$, such that the following holds:

(i) $x^*(\hat{f}) = \int_X f d\nu$ for all $\hat{f} \in N_A$, and $\|x^*\| = v(\nu,X)$.

Proof. If $x^* \in (N_A)^*$ we will write $y^* = (x^*)_+$, $z^* = (x^*)_-$ and define, for all $E \in \Sigma$, $\nu_3(E) = \|y^*_E\|$ and $\nu_4(E) = \|z^*_E\|$. Let $\nu = \nu_3 - \nu_4$.

To prove (i), use (3.13) and (3.8.iv) to write

$$x^*(\hat{f}) = \int_X f d\nu \quad \text{for all } \hat{f} \in N_A,$$

and, (3.6.iv) and (3.8.ii), to obtain

$$\nu_3(X)+\nu_4(X) = \|y^*\| + \|z^*\| = \|x^*\| \le v(\nu,X) \le \nu_3(X) + \nu_4(X).$$

To show that the mapping $x^* \longmapsto \nu$ is onto, consider $\tilde{\nu} \in V_{\bar{A}}(X,\Sigma,\mu)$ and define $x^*(\hat{f}) = \int_X f d\tilde{\nu}$ for $\hat{f} \in N_A$. Then, writing $\tilde{\nu}_1 = (\tilde{\nu}+v(\tilde{\nu},\cdot))/2$ and $\tilde{\nu}_2 = (v(\tilde{\nu},\cdot)-\tilde{\nu})/2$, we have that

$$\int_X f d\tilde{\nu}_1 - \int_X f d\tilde{\nu}_2 = \int_X f d\tilde{\nu} = x^*(\tilde{f}) = \int_X f d\nu = \int_X f d\nu_3 - \int_X f d\nu_4 ,$$

for all $\tilde{f} \in N_A$, and thus

$$\int_X f d(\tilde{\nu}_1+\nu_4) = \int_X f d(\nu_3+\tilde{\nu}_2) \quad \text{for all } f \in L_A .$$

Since $\tilde{\nu} \in V_{\bar{A}}(X,\Sigma,\mu)$, it is easy to verify that $\tilde{\nu}_1,\tilde{\nu}_2 \in V_{\bar{A}}(X,\Sigma,\mu)$, and thus $\tilde{\nu}_1+\nu_4 = \nu_3+\tilde{\nu}_2$ (3.13). Hence $\nu = \tilde{\nu}$.

Finally observing that $x^* \longmapsto \nu$ has a linear inverse, we conclude that the mapping is linear.//

It is immediate, by (3.14), that $V_{\bar{A}}(X,\Sigma,\mu)$ is a Banach space.

4. Characterization of the dual space $(L_A)^*$. As in 2, set $H = \Sigma$ if $a > 0$, and $H = \Sigma_1$ if $a = 0$.

(4.1) Theorem. Let $x^* \in (L_A)^*$. Then there exist a unique $G \in G_{\bar{A}}(X,\Sigma,\mu,H)$, defined by $G(E) = x^*(\xi_E)$ for $E \in H$, and a unique $z^* \in (M_A)^\perp$ such that

(i) $x^*(f) = \int_X f dG + z^*(f)$, for all $f \in L_A$.

Proof. The same as in Proposition 2 of [13], noting that (2.10), (2.11) and (2.12) hold.//

(4.2) Proposition. Let $x^* \in (L_A)^*$, G and z^* as in (4.1), and $y^*(f) = \int_X f dG$ for all $f \in L_A$. The following assertions are true:

(i) if $x^* \geq 0$, then $y^* \geq 0$ and $z^* \geq 0$;

(ii) $\| |y^*| + |z^*| \| = \|y^*\| + \|z^*\|$;

(iii) $|x^*| = |y^*| + |z^*|$ and $|y^*| \wedge |z^*| = 0$.

Proof. If $x^* \geq 0$, it is clear that $y^* \geq 0$. Thus for (i) it suffices to prove that $x^* \geq y^*$. For this let $0 \leq f \in L_A$ and take (s_n) as in (2.8). Since $z^* \in (M_A)^\perp$ and $x^* \geq 0$ we have

$$y^*(f) = \lim_{n\to\infty} \int_X s_n dG = \lim_{n\to\infty} x^*(s_n) \leq x^*(f).$$

For (ii), let $\varepsilon \in]0,\infty[$. Observing that if $w^* \in (L_A)^*$, then $\|w^*\| = \| |w^*| \|$ (page 239 in [7]), there exist $f,g \in L_A$, nonnegative, with $\|f\|_A \leq 1$, $\|g\|_A \leq 1$ and such that

$$\|y^*\| < |y^*|(f) + \frac{\varepsilon}{2} \quad \text{and} \quad \|z^*\| < |z^*|(g) + \frac{\varepsilon}{2} .$$

From (1.3), for $k \in \mathbb{N}$ fixed there exists $s \in M_A$ with $0 \le s \le g$ and such that

$$\int_X A(g-s)d\mu < (1 + \frac{1}{k}) - \int_X A(f)d\mu;$$

Let $h = \max\{f,(g-s)\}$. Then $h \in L_A$ and

$$\int_X A(\frac{kh}{k+1})d\mu \le \frac{k}{k+1} \int_X A(h)d\mu \le \frac{k}{k+1}[\int_X A(f)d\mu + \int_X A(g-s)d\mu] \le 1 .$$

Thus $\|h\|_A \le (k+1)/k$.

Hence we have that

$$\begin{aligned} \|y^*\| + \|z^*\| &< |y^*|(f) + |z^*|(g) + \varepsilon = |y^*|(f) + |z^*|(g-s) + \varepsilon \\ &\le |y^*|(h) + |z^*|(h) + \varepsilon = (|y^*| + |z^*|)(h) + \varepsilon \\ &\le \| |y^*| + |z^*| \| \|h\|_A + \varepsilon \le \| |y^*| + |z^*| \| ((k+1)/k) + \varepsilon . \end{aligned}$$

Since ε and k are arbitrary we conclude that (ii) holds.

Assertion (iii) may be found in [11] (page 40).//

(4.3) Proposition. There is an isometric isomorphism of $(M_A)^{\perp}$ onto $(N_A)^*$ given by the mapping j: $z^* \mapsto x^*$, where x^* is defined by $x^*(\bar{f}) = z^*(f)$.

In the next theorem we present our characterization of $(L_A)^*$.

(4.4) Theorem. There is an isometric isomorphism of $(L_A)^*$ onto the Banach space $G_{\bar{A}}(X,\Sigma,\mu,H) \times V_{\bar{A}}(X,\Sigma,\mu)$ given by the mapping $x^* \mapsto (G,\nu)$ such that the following hold:

(i) $x^*(f) = \int_X f dG + \int_X f d\nu$ for all $f \in L_A$;

(ii) $\|x^*\| = \|G\|_{\bar{A}} + v(\nu,X)$,

the first integral being as in (2.2) and the second, as in (3.7).

Proof. It is a consequence of (4.1), (4.3), (3.14) and (4.2), observing that $\|x^*\| = \| |x^*| \|$ for $x^* \in (L_A)^*$ [7].//

5. Concluding remarks. (I) The characterization of $(L_A)^*$ obtained by Andō in [1] goes through the spaces J_A and L_A/J_A with the hypotheses $\mu(X) < \infty$ and $b = \infty$. From (3.3) we know that under these hypotheses the spaces J_A and M_A coincide.

(II) Using (2.6), (2.8) and Radon-Nikodým's Theorem, one can prove that if $\mu(X) < \infty$ and $b = \infty$, given $G \in G_{\bar{A}}(X,\Sigma,\mu,\Sigma_1)$, there is $g \in L_{\bar{A}}$ such that

(i) $\int_X f dG = \int_X fg d\mu$, for all $f \in L_A$,

and

(ii) $\|G\|_{\bar{A}} = \sup\{\int_X |gh| d\mu : h \in L_A \text{ and } \|h\|_A \leq 1\}$.

Hence, in this case, Theorem (4.4) above tells us that given $x^* \in (L_A)^*$, there exist $g \in L_{\bar{A}}$ and $\nu \in V_{\bar{A}}(X,\Sigma,\mu)$ such that

(iii) $x^*(f) = \int_X fg d\mu + \int_X f d\nu$, for all $f \in L_A$,

and

(iv) $\|x^*\| = \sup\{\int_X |gh| d\mu : h \in L_A \text{ and } \|h\|_A \leq 1\} + v(\nu, X)$.

Thus the result obtained by Andō in [1], and Theorem (4.4) coincide when $\mu(X) < \infty$ and $b = \infty$.

(III) In [4] we also prove that equalities (iii) and (iv), above, still hold when (X,Σ,μ) is a σ-finite measure space and $J_A = M_A$, or when $\bar{a} = 0$ and $J_A = M_A$.

(IV) If $J_A \neq \{0\}$ and $J_A \neq M_A$, there exists $G \in G_{\bar{A}}(X,\Sigma,\mu,\Sigma)$ such that for all $g \in L_{\bar{A}}$ the following assertion is false:

$$\int_X f dG = \int_X fg d\mu \quad , \text{ for all } f \in M_A.$$

This will be proved below.

From (IV) it follows that equality (iii) is not true if $J_A \neq \{0\}$ and $J_A \neq M_A$.

Proof of assertion (IV). By (3.3) we know that $\mu(X) = \infty$, $b = \infty$ and $a > 0$. Therefore $\xi_X \in M_A$ and $\xi_X \notin J_A$; also $\bar{a} = 0$.

Using Hahn Banach's Theorem and (2.12) there exist $x^* \in (M_A)^*$ and $G \in G_{\bar{A}}(X,\Sigma,\mu,\Sigma)$ such that

$$x^*(\xi_X) \neq 0 \quad , \quad x^*(f) = 0 \quad \text{for all } f \in J_A ,$$

and

$$x^*(f) = \int_X f dG \text{ , for all } f \in M_A.$$

Suppose there is a $g \in L_{\bar{A}}$ such that

$$\int_X f dG = \int_X fg \, d\mu \text{ , for all } f \in M_A.$$

Let $\gamma \in]0,\infty[$ and $F = \{x \in X: |g(x)| > \gamma\}$ be such that $\mu(F) > 0$ (this F exists because $x^*(\xi_X) \neq 0$).

As $\bar{a} = 0$, and

$$\bar{A}\left(\frac{\gamma}{\|g\|_{\bar{A}}}\right)\mu(F) \leq \int_X \bar{A}\left(\frac{|g|}{\|g\|_{\bar{A}}}\right)d\mu \leq 1 ,$$

we conclude that $\mu(F) < \infty$, and so $\xi_F \operatorname{sgn} g \in J_A$.

Thus

$$0 = x^*(\xi_F \operatorname{sgn} g) = \int_X (\xi_F \operatorname{sgn} g) g \, d\mu = \int_X \xi_F |g| d\mu \geq \gamma\mu(F) > 0 ,$$

which is impossible.//

This work is part of our Master's Dissertation written under the guidance of Dr. Iracema Martin Bund.

REFERENCES

[1] T. Andô, *Linear functionals on Orlicz spaces*, Nieuw Arch. Wisk., 8(3) (1960), p. 1-16.

[2] I.M, Bund, *Birnbaum - Orlicz spaces*, IME-USP, São Paulo, 1978 (Notas do Instituto de Matemática e Estatística da Universidade de São Paulo. Série Matemática, 4).

[3] N.S. Dunford and J. T. Schwartz, *Linear operators, part I: General theory*, Interscience, New York, 1967.

[4] R. Fernandez, *Caracterização do dual de um espaço de Orlicz*, Dissertação (Mestrado), Instituto de Matemática e Estatística da Universidade de São Paulo, 1986.

[5] R. Fernandez, *Integração em relação a medidas definidas em ideais*, Trabalhos apresentados, 23º Seminário Brasileiro de Análise, Campinas, 1986, p. 199-214.

[6] H. Hudzik, *Orlicz spaces of essentially bounded functions and Banach-Orlicz algebras*, Arch. Math. 44 (1985), p.535-538.

[7] J.L. Kelley and I. Namioka, *Linear topological spaces*, Springer, New York, 1963 (Graduate Texts in Mathematics, 36).

[8] M.A. Krasnosel'skii and Ya. B. Rutickii, *Convex functions and Orlicz spaces*, (translation) P. Noordhoff Ltd., Groningen, 1961.

[9] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Mathematics, 1034, Springer, New York, 1983.

[10] W. Orlicz, *On integral representability of linear functionals over the space of φ-integrable functions*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., 8 (1960), p. 567-569.

[11] A. L. Peressini, *Ordered topological vector spaces*, Harper & Row, New York, 1967.

[12] M. M. Rao, *Linear functionals on Orlicz spaces*, Nieuw Arch. Wisk., 12(3) (1964), p. 77-98.

[13] M. M. Rao, *Linear functionals on Orlicz spaces: general theory*, Pacific J. Math., 25 (3) (1968), p. 553-584.

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