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Periodic positive solutions of superlinear delay equations via topological degree

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We extend to delay equations recent results obtained by G. Feltrin and F. Zanolin for second-order ordinary equations with a superlinear term. We prove the existence of positive periodic solutions for nonlinear delay equations $-u''(t) = a(t)g(u(t), u(t-\tau))$. We assume superlinear growth for g and sign alternance for a. The approach is topological and based on Mawhin's coincidence degree.

This article is part of the theme issue 'Topological degree and fixed point theories in differential and difference equations'.

1. Introduction

Let $g:[0,+\infty)\to[0,+\infty)$ be a smooth function satisfying $g(0)=0$ and $g(s)>0$ for $s>0$ and let $a\in L^1(0,T)$. The starting point of this paper is an existence result by Gaudenzi *et al.* [1].

Theorem 1.1. *Assume that* $\{t\in[0,T]:a(t)>0\}=\bigcup_{i=1}^k J_i$, *where the sets* J_i *are pairwise disjoint intervals, and suppose that*

$$\limsup_{s\to 0^+}\frac{g(s)}{s}<\lambda_1 \text{ and } \liminf_{s\to+\infty}\frac{g(s)}{s}>\max_{i=1,\dots,k}\lambda_1^i,$$

where λ_1 *is the first eigenvalue of the eigenvalue problem*

$$\phi''+\lambda a^+(t)\phi=0,\quad \phi(0)=\phi(T)=0, \tag{1.1}$$

and λ_1^i *is the first eigenvalue of the eigenvalue problem*

$$\phi''+\lambda a^+(t)\phi=0,\quad \phi|_{\partial J_i}=0. \tag{1.2}$$

Then, there is at least one solution to

$$u'' + a(t)g(u) = 0, \quad u(0) = u(T) = 0, \tag{1.3}$$

which is positive on $(0, T)$.

Clearly, the hypotheses are satisfied if g verifies

$$\lim_{s\to 0^+} \frac{g(s)}{s} = 0 \text{ and } \lim_{s\to +\infty} \frac{g(s)}{s} = +\infty,$$

which is referred to as a *superlinear behaviour* at $+\infty$ as well as 0. The subject of superlinear equations is of central importance in the theory of ordinary differential equations and a really huge literature has been produced in the last decades.

Theorem 1.1 has been extended in [2] by Feltrin and Zanolin to Neumann and periodic boundary conditions, where the central tool is the general existence theorem:

Theorem 1.2. *Assume* $f : [0, T] \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ *is an* L^p*-Carathéodory function, for some* $1 \le p \le \infty$ *satisfying the following conditions:*

(*i*) $f(t, 0, \xi) = 0$ *for a.e.* $t \in [0, T]$ *and all* $\xi \in \mathbb{R}$,
(*ii*) *There exist a non-negative function* $k \in L^1([0, T])$ *and a constant* $\rho > 0$ *such that*

$$f(t, s, \xi) \le k(t)(|s| + |\xi|),$$

for a.e. $t \in [0, T]$ *and all* $0 \le s \le \rho$ *and* $|\xi| \le \rho$.
(*iii*) *For each* $\eta > 0$, *there exists a continuous function*

$$\phi = \phi_\eta : \mathbb{R}^+ \to \mathbb{R}^+, \text{ with } \int_1^\infty \frac{\xi^{(p-1)/p}}{\phi(\xi)} \mathrm{d}\xi = \infty,$$

and a function $\psi = \psi_\eta \in L^p([0, T], \mathbb{R}^+)$ *such that*

$$f(t, s, \xi) \le \psi(t)\phi(|\xi|) \text{ for a.e. } t \in [0, T], s \in [0, \eta] \text{ and } \xi \in \mathbb{R}.$$

In addition, suppose that there exist two constants $r, R > 0$ *with* $r \ne R$, *such that the following hypotheses hold.*

(H_r) *The average condition*

$$\int_0^T f(t, r, 0) < 0$$

is satisfied. Moreover, any solution u *of the boundary value problem*

$$\begin{cases} u'' + \vartheta f(t, u, u') = 0 \\ \text{periodic or Neumann} \end{cases}$$

for $0 < \vartheta \le 1$, *such that* $u(t) > 0$ *on* $[0, T]$, *satisfies* $\|u\|_\infty \ne r$.
(H_R) *There exist a non-negative function* $v \in L^p([0, T])$ *with* $v \not\equiv 0$ *and a constant* α_0, *such that every solution of the boundary value problem*

$$\left.\begin{aligned} &u'' + f(t, u, u') + \alpha v(t) = 0 \\ &\text{periodic or Neumann, respectively} \end{aligned}\right\} \tag{1.4}$$

for $\alpha \in [0, \alpha_0]$ *satisfies* $\|u\|_\infty \ne R$. *Moreover, there are no solutions* $u(x)$ *of (1.4) for* $\alpha = \alpha_0$ *with* $0 \le u(t) \le R$, *for all* $t \in [0, T]$.

Then, the problem

$$\begin{cases} u'' + f(t,u,u') = 0 \\ \text{periodic or Neumann, respectively} \end{cases}$$

has at least one positive solution u with

$$\min\{r,R\} < \max_{t\in[0,T]} u(t) < \max\{r,R\}.$$

Both works make use of the theory of coincidence degree developed by Mawhin [3,4] and based, as known, on the Leray–Schauder degree. It should be noted, however, that the proof of theorem 1.1 and its extension in [2] depend on *a priori* estimates that have little relation with equation (1.3) itself, but more with the eigenfunctions of the eigenvalue problems (1.1) and (1.2). In view of this remark, we propose to use similar comparison results to tackle a delay differential equation, with a positive constant time lag τ, of the form

$$-u''(t) = f(t, u(t), u(t-\tau), u'(t)), \tag{1.5}$$

in which $f : \mathbb{R} \times [0,+\infty) \times [0,+\infty) \times \mathbb{R} \to \mathbb{R}$ is T-periodic with respect to the first variable for a fixed positive T. We also assume that f is L^1-Carathéodory, that is,

(i) for almost every $t \in \mathbb{R}$, $f(t,\cdot,\cdot,\cdot)$ is continuous;
(ii) for any $x, y \in [0,+\infty)$ and any $z \in \mathbb{R}$, $f(\cdot,x,y,z)$ is measurable;
(iii) for any $\rho > 0$ there exists $g \in L^1([0,T],\mathbb{R})$ such that, for a.e. $t \in [0,T]$, every $x, y \in [0,\rho]$ and every $z \in [-\rho,\rho]$, we have

$$|f(t,x,y,z)| \le g(t).$$

We stress that the periodicity of f means that the condition

$$f(t,x,y) = f(t+T,x,y) \text{ for all } x,y \in [0,+\infty) \text{ and all } z \in \mathbb{R}$$

is satisfied for a.e. $t \in \mathbb{R}$. We define a *T-periodic solution* of (1.5) as a C^1 function $u : \mathbb{R} \to \mathbb{R}$, which is T-periodic, whose derivative is absolutely continuous and such that $u''(t)$ verifies the equality (1.5) for a.e. $t \in \mathbb{R}$. We will prove an existence result, theorem 3.2, for positive and T-periodic solutions of (1.5) and, as a consequence, we obtain the existence of positive periodic solutions of the following superlinear delay differential equation

$$-u''(t) = a(t)g(u(t), u(t-\tau)),$$

in which

(i) $\tau > 0$ is given,
(ii) $a : \mathbb{R} \to \mathbb{R}$ is T-periodic, for a given $T > 0$, L^1 if restricted to $[0,T]$ and essentially bounded,
(iii) $g : [0,+\infty) \times [0,+\infty) \to \mathbb{R}$ is continuous.

To obtain our existence result, we shall also assume some other special conditions on a and g, namely conditions (a_1), (a_2) e (g_1)–(g_4) in §4, that generalize analogous conditions in [2, theorem 3.1], in the sense that they include the dependence on a delay.

Theorems 3.2 are proven by a topological approach, based on Mawhin's coincidence degree and proceeding in the general spirit of Feltrin-Zanolin's paper [2].

2. Preliminaries

(a) Estimates of the solutions

This subsection is devoted to recalling some technical results concerning the behaviour of the periodic solutions of a delay differential equation

$$-u''(t) = h(t, u(t), u(t-\tau), u'(t)) \tag{2.1}$$

which is slightly different from (1.5) in the sense that h is defined in $\mathbb{R}^4$. On the other hand, h is, as f, L^1-Carathéodory and T-periodic with respect to the first variable. The importance of the results of this subsection lies in the fact that they help to establish various *a priori* bounds for the solutions of problem (1.5), as will be clear in §3. The next two results are special versions of the maximum principle and are essentially proven in [2, Lemma 6.1]. We give the easy proofs for the sake of completeness.

Theorem 2.1 (Weak Maximum principle). *Let $h:\mathbb{R}^4 \to \mathbb{R}$ be an L^1-Carathéodory map, which is T-periodic with respect to the first variable for a given positive T. Assume also that $h(t,x,y,z) > 0$ if $x < 0$, for a.e. $t \in \mathbb{R}$ and every $y, z \in \mathbb{R}$. Then, any T-periodic solution of (2.1) is non-negative.*

Proof. Assume the contrary and suppose that u is a T-periodic solution which is negative for some $\bar{t} \in \mathbb{R}$. We have two cases. In the first one, we suppose the existence of t_0 such that $u(t_0) = 0$. Define

$$a = \inf\{s : u \text{ is negative in } (s, \bar{t}]\}, \quad b = \sup\{s : u \text{ is negative in } [\bar{t}, s)\}.$$

It is clear that a, b are finite because u is T-periodic. One has that $u(a) = u(b) = 0$, $u(t) < 0$ in (a, b), and, due to the sign assumption on h, u is concave in $[a, b]$. These facts lead to a contradiction. In the second case, we suppose that u is negative on $\mathbb{R}$. As it is not constant since $h(t, u(t), u(t-\tau), u'(t)) > 0$ for almost every t, we deduce that u is strictly concave contradicting its periodicity. ■

Theorem 2.2 (Strong maximum principle). *Let $h:\mathbb{R}^4 \to \mathbb{R}$ be an L^1-Carathéodory map, which is T-periodic with respect to the first variable for a fixed $T > 0$. Assume that, for any compact subset J of $[0, +\infty)$, there exist a positive δ and an L^1 map q such that*

$$|h(t, x, y, z)| \le q(t)(x + |z|) \tag{2.2}$$

for a.e. $t \in [0, T]$, any $x \in \mathbb{R}$, $|z| \in [0, \delta]$ and any y in J. Then, any non-trivial and non-negative T-periodic solution u is such that $u(t) > 0$ for all $t \in \mathbb{R}$.

Proof. Consider a non-negative T-periodic solution u of (2.1). Suppose that $u(\bar{t}) = 0$ for a given $\bar{t} \in \mathbb{R}$, which can be chosen in $[0, T]$. Then, clearly, $u'(\bar{t}) = 0$. We want to prove that, in this case, u is identically zero. Depending on the compact set $J := \{u(t), t \in [0, T]\}$, consider a positive δ and an L^1 map q such that inequality (2.2) is satisfied for a.e. $t \in [0, T]$, any x, $|z| \in [0, \delta]$ and any $y \in J$.

Let $\eta > 0$ be such that $0 \le u(t) \le \delta$ and $0 \le |u'(t)| \le \delta$ for every $t \in [\bar{t} - \eta, \bar{t} + \eta]$. Given $t \in (\bar{t}, \bar{t} + \eta]$, we have

$$\begin{aligned}
0 \le u(t) + |u'(t)| &= u(\bar{t}) + \int_{\bar{t}}^{t} u'(s)\,\mathrm{d}s + \left| u'(\bar{t}) - \int_{\bar{t}}^{t} h(s, u(s), u(s-\tau), u'(s))\,\mathrm{d}s \right| \\
&\le \int_{\bar{t}}^{t} u'(s)\,\mathrm{d}s + \int_{\bar{t}}^{t} \left| h(s, u(s), u(s-\tau), u'(s)) \right| \mathrm{d}s \le \int_{\bar{t}}^{t} \Big(|u'(s)| + q(s)\left(u(s) + |u'(s)|\right) \Big)\,\mathrm{d}s \\
&\le \int_{\bar{t}}^{t} (q(s) + 1)\left(|u'(s)| + u(s)\right)\mathrm{d}s.
\end{aligned}$$

Applying the Gronwall Lemma to $t \mapsto u(t) + |u'(t)|$, we deduce that this map and, consequently, u vanish for all $t \in (\bar{t}, \bar{t} + \eta]$. An analogous computation shows the same result in the interval $[\bar{t} - \eta, \bar{t})$. By this argument, u is a continuous map having the following property: if u is zero at a point t, then it is zero in an open interval containing t. This clearly implies that u is identically zero. Therefore, if u is a non-negative and non-trivial T-periodic solution of (2.1), it cannot vanish at any point. This completes the proof. ■

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Remark 2.3. If $\bar{t}=T$ in the above proof, the integration of q over $[\bar{t},\bar{t}+\eta]$ is defined by taking its T-periodic extension. The case $\bar{t}=0$ is analogous. In addition, the reader can observe that, in the statement in the theorem, h can be defined in $\mathbb{R}\times[0,+\infty)\times[0,+\infty)\times\mathbb{R}$ and not necessarily in $\mathbb{R}^4$.

The next result is an *a priori* bounds theorem for a family of delay differential equations and is the analogue of [2, lemma 6.2]) which deals with ordinary equations.

Theorem 2.4. *Let $J=[t_1,t_2]$ be a closed interval contained in $[0,T]$ and $q_\infty: J\to\mathbb{R}$ a given L^1 function, which we assume non-trivial and non-negative. Call $\lambda_1>0$ the first eigenvalue of the linear Dirichlet problem in J*

$$-u''=\lambda q_\infty(t)u, \quad u(t_1)=u(t_2)=0,$$

and suppose $\lambda_1<1$. Let $h:\mathbb{R}^4\to\mathbb{R}$ be an L^1-Carathéodory map, which is T-periodic with respect to the first variable, and suppose that it satisfies the following properties:

(i) *$h(t,x,y,z)\geq 0$, for a.e. $t\in J$, all $x\geq 0$ and all $y,z\in\mathbb{R}$;*

(ii)

$$\liminf_{x\to+\infty}\frac{h(t,x,y,z)}{x}\geq q_\infty(t) \tag{2.3}$$

uniformly for a.e. $t\in J$ and all $y,z\in\mathbb{R}$.

Let $\mathcal{K}$ be the set of the L^1-Carathéodory maps $k:\mathbb{R}\times[0,+\infty)\times[0,+\infty)\times\mathbb{R}\to\mathbb{R}$ which are T-periodic with respect to the first variable and such that $k(t,x,y,z)\geq h(t,x,y,z)$, for a.e. $t\in J$, all $x,y\geq 0$ and all $z\in\mathbb{R}$. Then, there exists $R>0$ such that any non-negative, non-trivial, T-periodic solution u of

$$-u''(t)=k(t,u(t),u(t-\tau),u'(t)), \tag{2.4}$$

for some $k\in\mathcal{K}$, verifies $\max_{t\in J}u(t)<R$.

Proof. We proceed by contradiction and we assume the existence of two sequences $(k_n)_n\subseteq\mathcal{K}$ and $(u_n)_n$ such that each u_n is a non-negative and T-periodic solution of (2.4), with respect to k_n, and satisfies

$$R_n:=\max_{t\in J}u_n(t)>n, \quad \forall n\in\mathbb{N}.$$

Due to formula (2.3), define

$$q_n(t)=\min\left\{q_\infty(t), \inf_{x\geq n,\, y,z\in\mathbb{R}} h(t,x,y,z)/x\right\},$$

and observe that $(q_n)_n$ is an increasing sequence of non-negative, non-trivial functions in $L^1(J,\mathbb{R})$, converging to q_∞ uniformly a.e. in J, and such that

$$h(t,x,y,z)\geq q_n(t)x \tag{2.5}$$

for a.e. $t\in J$, all $x\geq n$ and all $y,z\in\mathbb{R}$. Fix

$$0<\varepsilon<\frac{1-\lambda_1}{2}.$$

As $q_n\to q_\infty$, there exists $N\in\mathbb{N}$ such that, for any $n\geq N$, the first positive eigenvalue μ_n of

$$-u''=\lambda q_n(t)u, \quad u(t_1)=u(t_2)=0$$

verifies $\mu_n<\lambda_1+\varepsilon<1-\varepsilon$. Let ϕ_N be a non-negative eigenfunction of

$$-u''=\lambda q_N(t)u, \quad u(t_1)=u(t_2)=0.$$

The map ϕ_N, being non-trivial and concave in J, satisfies $\phi_N(t) > 0$ in (t_1, t_2). In addition $\phi'_N(t_1) > 0$ and $\phi'_N(t_2) < 0$ (the details are left to the reader). For each $n \geq N$ define

$$J'_n = \{t \in J : u_n(t) \geq N\}.$$

Notice that J'_n is a (closed) subinterval of J because u_n is concave in J. In addition, the measure of $J \setminus J'_n$ tends to zero, as $n \to \infty$. To see this, let us first observe that, by the concavity of u_n in J, one has that

$$u_n(t) \geq \frac{R_n}{t_2 - t_1} \min\{t - t_1, t_2 - t\}, \quad \forall t \in J, \tag{2.6}$$

see also [1, p. 420] for an analogous formula. Then, for any $n \geq 2N$ the interval

$$A_n = \left[t_1 + \frac{N}{R_n}(t_2 - t_1), t_2 - \frac{N}{R_n}(t_2 - t_1)\right]$$

is well defined, recalling that $R_n > n$ for every $n \in \mathbb{N}$. By formula (2.6), one has

$$u_n(t) \geq N, \quad \forall t \in A_n, \ \forall n \geq 2N,$$

and hence $A_n \subseteq J'_n$, for any $n \geq 2N$. It is immediate to see that the measure of A_n tends to $t_2 - t_1$, as $n \to \infty$. Consequently, the measure of $J \setminus J'_n$ tends to zero, as $n \to \infty$.

Now, take $n \geq N$. We have

$$\begin{aligned}
0 \geq u_n(t_2)\phi'_N(t_2) - u_n(t_1)\phi'_N(t_1) &= \int_J \frac{\mathrm{d}}{\mathrm{d}t}\Big(u_n(t)\phi'_N(t) - u'_n(t)\phi_N(t)\Big)\mathrm{d}t \\
&= \int_J \Big(u_n(t)\phi''_N(t) - u''_n(t)\phi_N(t)\Big)\mathrm{d}t \\
&= \int_J \Big(-u_n(t)\mu_N q_N(t)\phi_N(t) + k_n\left(t, u_n(t), u_n(t-\tau), u'_n(t)\right)\phi_N(t)\Big)\mathrm{d}t \\
&\geq \int_J \Big(-u_n(t)\mu_N q_N(t)\phi_N(t) + h\left(t, u_n(t), u_n(t-\tau), u'_n(t)\right)\phi_N(t)\Big)\mathrm{d}t.
\end{aligned}$$

By inequality (2.5), we have that the latter integral is greater than or equal to $\alpha_n + \beta_n$, where

$$\alpha_n = \int_{J \setminus J'_n} \Big(-u_n(t)\mu_N q_N(t)\phi_N(t) + h\left(t, u_n(t), u_n(t-\tau), u'(t)\right)\phi_N(t)\Big)\mathrm{d}t$$

and

$$\beta_n = (1 - \mu_N)\int_{J'_n} q_N(t)u_n(t)\phi_N(t)\,\mathrm{d}t.$$

Since $h(t, u_n(t), u_n(t-\tau), u'_n(t)) \geq 0$ for a.e. $t \in J$ and recalling the definition of J'_n, we have

$$\alpha_n \geq -N\mu_N \int_{J \setminus J'_n} q_N(t)\phi_N(t)\,\mathrm{d}t.$$

We have in addition

$$\beta_n \geq \varepsilon N \int_{J'_n} q_N(t)\phi_N(t)\,\mathrm{d}t$$

since $1 - \mu_N > \varepsilon$ and $u_n(t) \geq N$ if $t \in J'_n$. Thus,

$$0 \geq \varepsilon N \int_{J'_n} q_N(t)\phi_N(t)\,\mathrm{d}t - N\mu_N \int_{J \setminus J'_n} q_N(t)\phi_N(t)\,\mathrm{d}t.$$

Since the measure of $J \setminus J'_n$ tends to zero, as $n \to \infty$, then, by the monotone convergence theorem for the Lebesgue integral, we have

$$\lim_{n\to\infty} \int_{J'_n} q_N(t)\phi_N(t)\,\mathrm{d}t = \lim_{n\to\infty} \int_J q_N(t)\phi_N(t)\chi_{J'_n}\,\mathrm{d}t = \int_J q_N(t)\phi_N(t)\,\mathrm{d}t.$$

By the properties of $q_N(t)$ and $\phi_N(t)$, it follows that the latter integral is positive. Therefore, the sequence of the integrals $\int_{J'_n} q_N(t)\phi_N(t)\,dt$ is eventually positive and bounded away from zero, because it is increasing. Consequently, as

$$\lim_{n\to\infty}\int_{J\setminus J'_n} q_N(t)\phi_N(t)\,\mathrm{d}t = 0,$$

it follows, for n sufficiently large, that

$$0 \geq \varepsilon N \int_{J'_n} q_N(t)\phi_N(t)\,\mathrm{d}t,$$

which is a contradiction. This concludes the proof. ■

(b) The topological approach: an equivalent functional version of equation (1.5)

For the reader's convenience, we shall briefly summarize here the main facts about Mawhin's coincidence degree. For details see [3–5].

Let E and F be real Banach spaces and let Ω be an open bounded subset of E. Let $L:\mathrm{Dom}(L)\subseteq E\to F$ be a linear Fredholm mapping of index zero, that is, $\dim(\ker(L))=\mathrm{codim}(\mathrm{Im}(L))<\infty$ and $\mathrm{Im}(L)$ is closed in F. Then there exist continuous projectors $P:E\to E$ and $Q:F\to F$ such that $\mathrm{Im}(P)=\ker(L)$ and $\ker(Q)=\mathrm{Im}(L)$. Moreover, there exists an isomorphism $J:\mathrm{Im}(Q)\to\ker(L)$. It is readily seen that $L|_{\mathrm{Dom}(L)\cap\ker(P)}:\mathrm{Dom}(L)\cap\ker(P)\to\mathrm{Im}(L)$ is bijective, so it has an algebraic inverse K_P. A continuous map $N:E\to F$ is called L–compact on $\overline{\Omega}$ if the set $QN(\Omega)$ is bounded and the map $K_P(I-Q)N:\overline{\Omega}\to E$ is compact. Since $JQN:\overline{\Omega}\to\ker(L)$ is continuous, we may identify $\ker(L)$ with $\mathbb{R}^n$ and thus define $\deg(JQN,\Omega\cap\ker(L),0)$ as the Brouwer degree of JQN restricted to $\ker L$, provided that QN does not vanish on $\partial\Omega\cap\ker(L)$. A standard homotopy argument connects this degree with the so-called *coincidence degree*, denoted by $D_L(L-N,\Omega)$, which, roughly speaking, may be regarded as an algebraic count of the number of solutions of the problem $Lu=Nu$ in Ω. This is the key idea in the following continuation theorem (e.g. [5]):

Theorem 2.5. *Let L be a Fredholm mapping of index zero, and let N be L-compact on $\overline{\Omega}$. Assume:*

(*a*) *$Lu\neq\lambda Nu$ for $\lambda\in(0,1]$ and $u\in\partial\Omega$.*
(*b*) *$QNu\neq 0$ for $u\in\partial\Omega\cap\ker(L)$.*
(*c*) $\deg(JQN,\Omega\cap\ker(L),0)\neq 0$.

Then

$$|D_L(L-N,\Omega)| = |\deg(JQN,\Omega\cap\ker(L),0)|. \tag{2.7}$$

In particular, the problem $Lu=Nu$ has at least one solution in $\mathrm{Dom}(L)\cap\Omega$.

Remark 2.6. Concerning equality (2.7), which is given in terms of absolute values, we point out the definition of coincidence degree is based on a choice of the orientations of $\ker L$ and $\mathrm{coker}L=F/\mathrm{Im}(L)$ (see [4, section 2.4]), while $\deg(JQN,\Omega\cap\ker(L),0)$ is canonically defined no matter what orientation is chosen on domain and target space of JQN since they coincide (with $\ker L$). Here, we did not enter into details of the construction of the coincidence degree, which goes beyond the purposes of this paper. We limit to observe that actually one has $D_L(L-N,\Omega)=\pm\deg(JQN,\Omega\cap\ker(L),0)$, depending on the chosen orientations in the definition of D_L. This justifies the absolute values in the (2.7) in the present version of the continuation theorem.

We observe as in [2] that a standard strategy for the search of non-trivial solutions consists in using the excision property of the degree. Specifically, if $\overline{\Omega}_1\subseteq\Omega_2$ and $L\neq N$ over $\partial\Omega_1\cup\partial\Omega_2$, then

$$D_L(L-N,\Omega_2)=D_L(L-N,\Omega_1)+D_L(L-N,\Omega_2\setminus\Omega_1).$$

In particular, if $D_L(L-N,\Omega_1)\neq D_L(L-N,\Omega_2)$, then the problem $Lu=Nu$ has at least one solution in $\Omega_2\setminus\Omega_1$.

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In our case, let us take E as the real Banach space of the T-periodic and C^1 maps $u:\mathbb{R}\to\mathbb{R}$, with the usual norm

$$|u|_1 = |u|_\infty + |u'|_\infty.$$

Denote by D the subspace of E of the maps having absolutely continuous derivative. It is known that D is dense and not closed in E (consequently it is not complete). Let F be the space of L^1 and T-periodic maps from $\mathbb{R}$ to $\mathbb{R}$. F is clearly a Banach space with the norm

$$|z|_{L^1} = \int_0^T |z(s)|\,\mathrm{d}s.$$

It turns out to be well defined by the linear operator

$$L:D\to F, \quad Lu = -u''$$

and the reader can easily check that

(i) $\ker L$ is one-dimensional, being composed by the constant maps;
(ii) the image of L, $\mathrm{Im}L$, is the closed subspace of F of those z such that $\int_0^T z(s)\,ds = 0$;
(iii) L is unbounded.

Now consider the following bounded linear projectors:

$$P:E\to E, \quad (Pu)(t) = \frac{1}{T}\int_0^T u(s)\,\mathrm{d}s$$

and

$$Q:F\to F, \quad (Qz)(t) = \frac{1}{T}\int_0^T z(s)\,\mathrm{d}s.$$

Thus, identifying the respective subsets of constant functions of E and F with $\mathbb{R}$, we may assume that $J:\mathbb{R}\to\mathbb{R}$ is the identity function.

In this setting, the operator K_P associates to any L^1 and T-periodic map z, with $\int_0^T z(s)\,\mathrm{d}s = 0$, the unique $v\in D$ satisfying $-v'' = z$ and $\int_0^T v(s)\,\mathrm{d}s = 0$.

The following properties of K_P are well known. We give here just a sketch proof for the sake of completeness.

Lemma 2.7. *The operator K_P is not compact. However, it sends equi-integrable subsets of* $\mathrm{Im}L$ *into totally bounded subsets of D' (that is, subsets of D' having compact closure in E).*

Sketch of the proof. The non-compactness of K_P can be proven by considering the following sequence $(z_n)_n$ in $L^1([0,T),\mathbb{R})$:

$$\text{for any } t\in[0,T),\ n>\frac{2}{T}, \quad z_n(t) = \begin{cases} 0 & \text{if } 0\le t\le \dfrac{T}{2}-\dfrac{1}{n} \text{ or } \dfrac{T}{2}+\dfrac{1}{n}\le t\le T \\ n & \text{if } \dfrac{T}{2}-\dfrac{1}{n} < t < \dfrac{T}{2} \\ -n & \text{if } \dfrac{T}{2}\le t < \dfrac{T}{2}+\dfrac{1}{n}. \end{cases}$$

The sequence $(z_n)_n$ is contained in $\mathrm{Im}L$ and it is bounded in L^1, with

$$|z_n|_{L^1} = 2.$$

Abusing of the notation, call z_n the T-periodic extensions to $\mathbb{R}$ of any above function. Denote

$$u_n := K_P z_n, \quad \text{for any } n\in\mathbb{N},\ n>\frac{2}{T}.$$

One can prove $(u_n)_n$ is not totally bounded in the norm of E by showing that the sequence $(v_n)_n$ of the derivatives of $-u_n$ does not admit a convergent subsequence in the sup norm. By the

T-periodicity of the u_n, we have

$$v_n(t) = \int_0^t z_n(s)\,\mathrm{d}s - \frac{1}{T}\int_0^T \left(\int_0^s z_n(\eta)\,\mathrm{d}\eta\right)\mathrm{d}s = \int_0^t z_n(s)\,\mathrm{d}s - \frac{1}{nT}. \tag{2.8}$$

A simple computation shows that $(v_n)_n$ converges pointwise to the discontinuous function

$$v(t) = \begin{cases} 0 & \text{if } t \neq \dfrac{T}{2} \\ 1 & \text{if } t = \dfrac{T}{2}. \end{cases}$$

Therefore, $(v_n)_n$ cannot admit a uniformly convergent subsequence in the sup norm and this shows that K_P is not compact.

Concerning the second part of the lemma, recall that a subset $\mathcal{C}$ of F is called *equi-integrable* if it satisfies the following condition: for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for any measurable subset J of $[0, T]$ with measure less that δ and any $f \in \mathcal{C}$, one has

$$\int_0^T |f(t)|\chi_J(t)\,\mathrm{d}t < \varepsilon.$$

Then, consider an equi-integrable sequence $(z_n)_n$ in ImL. It is easy to see that it is bounded in the L^1-norm. Call $(u_n)_n$ the sequence in D'

$$u_n := K_P z_n, \quad \text{for any } n \in \mathbb{N},$$

and denote $v_n := -u_n'$, for any n. One can show that $(v_n)_n$ is bounded in the sup norm and equi-continuous. The boundedness of $(v_n)_n$, in particular, follows from formula (2.8), which holds also in this case. One can also observe that $(u_n)_n$ is a bounded and equi-uniformly continuous sequence. Therefore, it is possible to apply Ascoli's Theorem, and $(u_n)_n$ admits a convergent subsequence in the C^1-norm. Thus K_P sends equi-integrable subsets of Im L into totally bounded subsets of D' and this is the idea of the proof. ■

Now, let $h : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ be T-periodic with respect to the first variable for a fixed positive T, and L^1-Carathéodory (see the Introduction). Define the Nemitski operator associated with h,

$$N : E \to F, \quad N(u)(t) = h(t, u(t), u(t-\tau), u'(t)).$$

By the assumptions on h, the operator N is well defined, and it is possible to see (we leave the easy proof to the reader) that

(A) N sends bounded sets of E into equi-integrable subsets of F.

It is worth mentioning that the definition of the coincidence degree is based on the Leray–Schauder degree of the map $I - \Phi$ where I is the identity of E and Φ is the following nonlinear operator:

$$\Phi : E \to E, \quad \Phi(u) = Pu + QN(u) + K_P(I - Q)N(u).$$

In the above formula, we can simply consider the composition QN instead of JQN since Im(Q) and ker(L) can be identified (even if it is not technically correct to say that they coincide, being subspaces of different spaces) and we can choose J as the identity. It is possible to check that $u \in D$ is a T-periodic solution of (1.5), if and only if it is a fixed point of Φ. Hence, the problem of finding T-periodic solutions to equation (1.5) is brought back to the search of fixed points of the operator Φ, or, equivalently, to the search of solutions to the nonlinear functional equation

$$u - \Phi(u) = 0. \tag{2.9}$$

Property (A) yields the well known fact that, although K_P is not compact, the Nemitski operator is L-compact and the coincidence degree theory is therefore applicable in order to study equation (2.9). This will be done in the following two sections.

3. The general result

In this section, we prove the existence of T-periodic positive solutions of problem (1.5). As said in the Introduction, we suppose that $f:\mathbb{R}\times[0,+\infty)\times[0,+\infty)\times\mathbb{R}\to\mathbb{R}$ is T-periodic with respect to the first variable and L^1-Carathéodory. In addition, we will assume three further conditions:

(f_1) $f(t,x,y,z)=0$ for a.e. $t\in\mathbb{R}$, every $x,y\geq 0$ such that $xy=0$ and every $z\in\mathbb{R}$;

(f_2) for any compact subset J of $[0,+\infty)$ there exist a positive δ and an L^1 map q such that

$$|f(t,x,y,z)|\leq q(t)(x+|z|)$$

for a.e. $t\in[0,T]$, any x and $|z|$ in $[0,\delta]$ and any y in J;

(f_3) (Nagumo-type condition) for each $\eta>0$ there exists a continuous function $\phi:[0,+\infty)\to[a,+\infty)$, for a suitable $a>0$, satisfying

$$\int_0^{+\infty}\frac{s}{\phi(s)}\,\mathrm{d}s=+\infty,$$

and such that

$$|f(t,x,y,z)|\leq\phi(|z|),\quad \text{a.e. } t\in\mathbb{R},\ 0\leq x,y\leq\eta,\ z\in\mathbb{R}.$$

Define now the following extension of f:

$$\tilde{f}:\mathbb{R}^4\to\mathbb{R},\quad \tilde{f}(t,x,y,z)=\begin{cases} f(t,x,y,z) & \text{if } x,y\geq 0\\ -x & \text{if } x<0,\ y\geq 0\\ -y & \text{if } x\geq 0,\ y<0\\ -x-y & \text{if } x,y<0.\end{cases}\tag{3.1}$$

Remark 3.1. By conditions (f_1) and (f_2), it is immediate to see that $\tilde{f}$ satisfies the assumptions of theorems 2.1 and 2.2. Actually, the (f_2) coincides with the assumption of theorem 2.2 (see also remark 2.3). Observe in addition that the definition of $\tilde{f}$ when $x,y<0$ (and for any z) ensures the continuity of $\tilde{f}$ with respect to these variables, playing no other role.

Theorem 3.2. *Let $f:\mathbb{R}\times[0,+\infty)\times[0,+\infty)\times\mathbb{R}\to\mathbb{R}$ be an L^1-Carathéodory function, which is T-periodic with respect to the first variable. Suppose f verifies properties (f_1), (f_2) and (f_3). In addition, assume that, for suitable $0<r<R$, the following conditions hold:*

(H_m)

$$\int_0^T f(t,r,r,0)\,\mathrm{d}t<0.$$

(H_r) *Any T-periodic solution u of the parametrized problem*

$$-u''(t)=\theta f(t,u(t),u(t-\tau),u'(t)),\tag{3.2}$$

for some $0<\theta\leq 1$, such that $u(t)>0$ for all real t, verifies $|u|_\infty\neq r$.

(H_R) *There exist a positive α_0 and a non-negative, not identically zero, essentially bounded and T-periodic function v, whose restriction to $[0,T]$ is L^1, such that any non-negative T-periodic solution u of*

$$-u''(t)=f(t,u(t),u(t-\tau),u'(t))+\alpha v(t),\tag{3.3}$$

for any given $0<\alpha<\alpha_0$, satisfies $|u|_\infty\neq R$. In addition, any (possible) non-negative T-periodic solution u of (3.3) for $\alpha=\alpha_0$ is such that $|u|_\infty>R$.

Then, (1.5) admits at least one T-periodic solution u such that $u(t)>0$ for all t and $r<|u|_\infty<R$.

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Proof. Consider the parametrized equation

$$-u''(t) = \theta \widetilde{f}(t, u(t), u(t-\tau), u'(t)), \tag{3.4}$$

in which $\widetilde{f}$ is the map defined in formula (3.1). Let u be a T-periodic solution for a given $\theta > 0$. Since $\theta\widetilde{f}$ satisfies the assumptions of theorem 2.1, u turns out to be non-negative and thus it is a T-periodic solution of (3.2) as well (for the same θ). In addition, property (f_2) implies that $\theta\widetilde{f}$ satisfies the assumptions of theorem 2.2. Hence, $u(t) > 0$ for every $t \in \mathbb{R}$, if u is not identically zero. According to condition (f_3), with $\eta = r$, corresponding to r, let $M_r > 0$ and $\phi : [0, +\infty) \to \mathbb{R}$, continuous and positive with

$$2r < \int_0^{M_r} \frac{s}{\phi(s)} \, ds,$$

be such that

$$|f(t, x, y, z)| \le \phi(|z|), \quad \text{a.e. } t \in \mathbb{R},\ 0 \le x, y \le r,\ z \in \mathbb{R}.$$

We claim that:

($\mathcal{P}$) if u is a T-periodic solution of (3.2) for a given $0 < \theta \le 1$, satisfying $|u|_\infty \le r$, then $|u'|_\infty < M_r$.

In fact, if the above property were false, there would exist $t_0, t_1 \in \mathbb{R}$, with $t_0 < t_1$ and $t_1 - t_0 < T$, such that $u'(t_0) = 0$ and $u'(t_1) = M_r$. This implies, recalling the Nagumo condition (f_3) applied to the map θf,

$$2r < \int_0^{M_r} \frac{s}{\phi(s)} \, ds = -\int_{t_0}^{t_1} \frac{u'(t)}{\phi(u'(t))} \theta f(t, u(t), u(t-\tau), u'(t)) \, dt \le \int_{t_0}^{t_1} u'(t) = u(t_1) - u(t_0) \le 2r,$$

which is a contradiction, and our assertion is proven.

Now, let E, F, P, Q and J be as in §2b, take the open and bounded subset Ω_r of E given by

$$\Omega_r = \{u \in E : |u|_\infty < r,\ |u'|_\infty < M_r\}$$

and set N as the Nemitski operator associated with $\widetilde{f}$.

By condition (H_r) and property ($\mathcal{P}$), one can see that equation (3.4) has no T-periodic solution on $\partial\Omega_r$ for any $0 < \theta \le 1$. Recalling that ImP and ImQ coincide with the set of real constant functions, if u solves the equation $QN(u) = 0$, then u is constant, say $u(t) \equiv c$, that is, $\int_0^T f(t, c, c, 0) \, dt = 0$. By assumption ($H_m$), u cannot belong $\partial\Omega_r$. Therefore, we are in the position to apply the Continuation theorem 2.5, obtaining

$$|D_L(L - N, \Omega_r)| = |\deg(\mathcal{F}, \Omega_r \cap \text{Im}P, 0)|,$$

where

$$\mathcal{F}(c) = -\frac{1}{T} \int_0^T \widetilde{f}(t, c, c, 0) \, dt$$

Thanks to condition (H_m), property (f_1) and the definition of $\widetilde{f}$, it follows that $\mathcal{F}(c) < 0$ if $c < 0$, $\mathcal{F}(0) = 0$, and $\mathcal{F}(r) > 0$. Therefore, as a basic fact in degree theory, one has

$$\deg(\mathcal{F}, \Omega_r \cap \text{Im}P, 0) = 1,$$

and this implies, by theorem 2.5, that problem (1.5) admits at least one T-periodic solution u in Ω_r. Any T-periodic solution is non-negative and, if non-trivial, strictly positive in $\mathbb{R}$.

However, the previous computation does not allow us to understand if (1.5) has only the trivial solution or also some positive ones. Thus, in order to prove the existence of positive T-periodic solutions and, in particular, to conclude the proof of the theorem, we employ the excision property of the degree as follows. Consider the parametrized equation

$$-u''(t)=\widetilde{f}(t,u(t),u(t-\tau),u'(t))+\alpha v(t). \tag{3.5}$$

Since the map

$$\widetilde{f}_\alpha(t,x,y,z):=\widetilde{f}(t,x,y,z)+\alpha v(t)$$

verifies the assumption of theorem 2.1 for every $\alpha \geq 0$, then any T-periodic solution of (3.5) for some $\alpha \geq 0$ is non-negative and thus it is a T-periodic solution of equation (3.3). Now, consider

$$f_\alpha(t,x,y,z):=f(t,x,y,z)+\alpha v(t), \quad 0\leq \alpha \leq \alpha_0.$$

Then, apply again condition (f_3) to f: corresponding to R, let $N_R>0$ and $\phi:[0,+\infty)\to\mathbb{R}$, continuous and positive with

$$2R<\int_0^{N_R}\frac{s}{\phi(s)}\,\mathrm{d}s,$$

be such that

$$|f(t,x,y,z)|\leq \phi(|z|), \quad \text{a.e. } t\in\mathbb{R},\ 0\leq x,y\leq R,\ z\in\mathbb{R}.$$

Define

$$\phi_0(z)=\phi(z)+\alpha_0\widehat{v},$$

where

$$\widehat{v}:=\text{ess sup}\, v(t)$$

is the essential supremum of v in $\mathbb{R}$, which is finite and positive according to assumption (H_R). Recalling that ϕ is non-negative, it is immediate to check that, thanks to (f_3), one has

$$|f_\alpha(t,x,y,z)|\leq \phi_0(|z|), \quad \text{a.e. } t\in\mathbb{R},\ 0\leq x,y\leq R,\ z\in\mathbb{R}, \quad \text{and} \quad \forall\,\alpha\in[0,\alpha_0].$$

In addition, since $\phi(s)\geq a>0$ for every $s\geq 0$, one has that

$$\int_0^{+\infty}\frac{s}{\phi_0}(s)\,\mathrm{d}s=+\infty, \tag{3.6}$$

and hence one can choose M_R sufficiently large such that

$$\int_0^{M_R}\frac{1}{\phi_0}(s)\,\mathrm{d}s>2R.$$

Let us observe that, if ϕ merely were a positive function, the integral in (3.6) could be finite. An example is shown in ([6], pp. 46–47). Proceeding analogously to the proof of property $(\mathcal{P})$, we obtain that any f_α, with $0\leq\alpha\leq\alpha_0$ satisfies the Nagumo-type condition (f_3), uniformly with respect to α, in the sense that, corresponding to the R given in the statement of the theorem, the constant M_R and the map ϕ_0 do not depend on α. Consequently, we have the following

($\mathcal{P}'$) if u is a T-periodic solution of (3.3) for a given $0<\alpha\leq\alpha_0$, satisfying $|u|_\infty\leq R$, then $|u'|_\infty < M_R$.

This fact and assumption (H_R) imply that, given

$$\Omega_R=\{u\in E:|u|_\infty<R,\ |u'|_\infty<M_R\},$$

we have that

(i) for every $\alpha\in[0,\alpha_0]$ (3.5) has no T-periodic solution on $\partial\Omega_R$,
(ii) (3.7) has no T-periodic solution on $\overline{\Omega_R}$ if $\alpha=\alpha_0$.

Call now $N_\alpha : E \to F$ the Nemitski operator associated to the nonlinear part of equation (3.5), that is, $N_\alpha(u)(t) = \tilde{f}(t, u(t), u(t-\tau), u'(t)) + \alpha v(t)$. By properties (i) and (ii), we can apply the homotopy invariance property of the coincidence degree, obtaining:

$$D_L(L - N, \Omega_R) = D_L(L - N_0, \Omega_R) = D_L(L - N_{\alpha_0}, \Omega_R) = 0.$$

We conclude that

$$D_L(L - N, \Omega_R \backslash \Omega_r) = -D_L(L - N, \Omega_r) = \pm 1 \neq 0$$

and, consequently, problem (1.5) admits a T-periodic solution $\bar{u} \in \Omega_R \setminus \overline{\Omega}_r$ (recall also the strong maximum principle to ensure that any solution is positive). This completes the proof. ∎

We conclude this section by observing that theorem 3.2 can be proven by replacing assumption (f_3) with a different Nagumo-type condition:

(f_{3b}) for each $\eta > 0$ there exist $M_\eta > 0$ and two functions $b : \mathbb{R} \to \mathbb{R}$, L^1, positive and T-periodic, $\phi : [0, +\infty) \to [a, +\infty)$, for a suitable $a > 0$, continuous, with

$$\int_0^{+\infty} \frac{1}{\phi(s)}\, \mathrm{d}s = +\infty,$$

and such that

$$|f(t, x, y, z)| \leq b(t)\phi(|z|), \quad \text{a.e. } t \in \mathbb{R},\ 0 \leq x, y \leq \eta,\ z \in \mathbb{R}.$$

More precisely, we have the following

Theorem 3.3. *Suppose* $(f_1), (f_2), (f_{3b}), (H_m), (H_r)$ *and* (H_R) *hold, just removing the essential boundedness condition concerning the map* v *in the* (H_R)*. Then, the conclusion of theorem* 3.2 *holds as well.*

The proof is basically the same as the one just carried out for theorem 3.2. We limit to observe that, in the proof of theorem 3.2, properties $(\mathcal{P})$ and $(\mathcal{P}')$ have been proven by applying the (f_3), and they can be proven as well by the (f_{3b}). Indeed, apply condition (f_{3b}): corresponding to r, let $M_r > 0$, $b : \mathbb{R} \to \mathbb{R}$, L^1, positive and T-periodic, $\phi : [0, +\infty) \to [a, +\infty)$, for a suitable $a > 0$, continuous with

$$\int_0^T b(t)\, \mathrm{d}t < \int_0^{M_r} \frac{1}{\phi(s)}\, \mathrm{d}s,$$

be such that

$$|f(t, x, y, z)| \leq b(t)\phi(|z|), \quad \text{a.e. } t \in \mathbb{R},\ 0 \leq x, y \leq r,\ z \in \mathbb{R}.$$

This implies property $(\mathcal{P})$, that is, any T-periodic solution u of (3.2) for a given $0 < \theta \leq 1$, with $\|u\|_\infty \leq r$, satisfies $\|u'\|_\infty < M_r$. If this were false, there would exist $t_0, t_1 \in \mathbb{R}$, with $t_0 < t_1$ and $t_1 - t_0 < T$, such that $u'(t_0) = 0$ and $u'(t_1) = M_r$. Therefore,

$$\int_0^{M_r} \frac{1}{\phi(s)}\, \mathrm{d}s = -\int_{t_0}^{t_1} \frac{1}{\phi(u'(t))} \theta f(t, u(t), u(t-\tau), u'(t))\, \mathrm{d}t \leq \int_{t_0}^{t_1} b(t)\, \mathrm{d}t,$$

which is a contradiction, and thus $(\mathcal{P})$ holds.

Let us now show that $(\mathcal{P}')$ holds as well. Consider the parametrized equation

$$-u''(t) = \tilde{f}(t, u(t), u(t-\tau), u'(t)) + \alpha v(t). \tag{3.7}$$

Since the map

$$\tilde{f}_\alpha(t, x, y, z) := \tilde{f}(t, x, y, z) + \alpha v(t)$$

verifies the assumption of theorem 2.1 for every $\alpha \geq 0$, then any T-periodic solution of (3.7) for some $\alpha \geq 0$ is non-negative and thus it actually is a T-periodic solution of equation (3.3). Now,

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consider

$$f_\alpha(t,x,y,z) := f(t,x,y,z) + \alpha v(t), \quad 0 \le \alpha \le \alpha_0.$$

Then, apply condition (f_{3b}) to f: corresponding to R, let $N_R > 0$, $b : \mathbb{R} \to \mathbb{R}$, L^1, positive and T-periodic, $\phi : [0, +\infty) \to [a, +\infty)$, continuous with

$$\int_0^T b(t)\,\mathrm{d}t < \int_0^{N_R} 1/\phi(s)\,\mathrm{d}s,$$

be such that

$$|f(t,x,y,z)| \le b(t)\phi(|z|), \quad \text{a.e. } t \in \mathbb{R},\ 0 \le x, y \le R,\ z \in \mathbb{R}.$$

Define

$$b_0(t) := b(t) + \alpha_0 v(t), \quad \text{and} \quad \phi_0(z) = \phi(z) + 1.$$

Recalling that b, v and ϕ are non-negative, it is immediate to check that, thanks to the (f_{3b}), one has

$$|f_\alpha(t,x,y,z)| \le b_0(t)\phi_0(|z|), \quad \text{a.e. } t \in \mathbb{R},\ 0 \le x, y \le R,\ z \in \mathbb{R}, \quad \text{and} \quad \forall\, \alpha \in [0, \alpha_0].$$

As for the analogous formula in the proof of theorem 3.2, one has

$$\int_0^{+\infty} \frac{1}{\phi_0}(s)\,\mathrm{d}s = +\infty.$$

Therefore, choosing M_R sufficiently large such that

$$\int_0^{M_R} \frac{1}{\phi_0}(s)\,\mathrm{d}s > \int_0^T b_0(t)\,\mathrm{d}t,$$

it follows that any f_α, with $0 \le \alpha \le \alpha_0$ satisfies the Nagumo-type condition (f_{3b}), uniformly with respect to α, in the sense that, corresponding to R, the constant M_R and the maps b_0 and ϕ_0 do not depend on α. Consequently, property $(\mathcal{P}')$ is satisfied, that is, if u is a T-periodic solution of (3.3) for a given $0 < \alpha \le \alpha_0$, satisfying $|u|_\infty \le R$, then $|u'|_\infty < M_R$. In conclusion, the validity of properties $(\mathcal{P})$ and $(\mathcal{P}')$ allows us to prove theorem 3.3 with the same strategy applied in the proof of theorem 3.2.

Remark 3.4. Conditions (f_3) and (f_{3b}) are different in the sense that neither is a generalization of the other. The (f_3) allows, for example, quadratic growth of the map ϕ, but this more general assumption on ϕ has a 'cost' that is paid by the essential boundedness condition on f in the first variable, which is not necessary in the (f_{3b}). Observe, in addition, that (f_{3b}) does not require that the map v in the step (H_R) of both theorems is essentially bounded. We stress, on the other hand and repeating what was just said, that (f_{3b}) requires a stronger condition than (f_3) about ϕ.

4. Applications to superlinear problems

In this section, we apply theorem 3.2 to prove the existence of positive T-periodic solutions to the following delay equation:

$$-u''(t) = a(t)g(u(t), u(t-\tau)), \tag{4.1}$$

in which

(i) $\tau > 0$ is given,
(ii) $a : \mathbb{R} \to \mathbb{R}$ is T-periodic, for a given $T > 0$, L^1 if restricted to $[0, T]$ and essentially bounded,
(iii) $g : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ is continuous.

In addition, we shall assume that the following conditions on a and g hold:

(a_1) there exist a finite number of closed and pairwise disjoint subintervals $I_1, \ldots, I_m$ of $[0, T]$ such that

$$a(t) \geq 0 \text{ for a.e. } t \in \bigcup_{i=1}^{m} I_i, \text{ and } a \not\equiv 0 \text{ on each } I_i, \; i = 1, \ldots, m$$

and

$$a(t) \leq 0 \text{ for a.e. } t \in [0, T] \setminus \bigcup_{i=1}^{m} I_i;$$

(a_2) $\int_0^T a(t)\,\mathrm{d}t < 0;$

(g_1) $\begin{cases} g(x,y) > 0 & \text{if } x \cdot y \neq 0 \\ g(x,y) = 0 & \text{if } x \cdot y = 0. \end{cases}$

(g_2) $\lim_{x \to 0} \frac{g(x,y)}{x} = 0$, uniformly for y in the compact subsets of $[0, +\infty)$;

(g_3) $\lim_{\substack{x \to 0 \\ \omega_1, \omega_2 \to 1}} \frac{g(\omega_1 x, \omega_2 x)}{g(x,x)} = 1;$

(g_4) $g_\infty := \liminf_{x \to +\infty} \frac{g(x,y)}{x} > \max_{i=1,\ldots,m} \lambda_1^i$, uniformly in $y \in [0, +\infty)$, where, for any given $i = 1, \ldots m$, λ_1^i is the first eigenvalue of the Dirichlet problem in the interval I_i

$$-u'' = \lambda a(t) u, \quad u|_{\partial I_i} = 0,$$

and this is positive, since a is non-negative in each I_i.

Remark 4.1. Condition (g_3) above can be regarded as a generalization of the notion of regular oscillation of a function. The reader can see e.g. [2, Section 1] for a discussion and some references on this concept.

Theorem 4.2. *Suppose a and g verify the above conditions. Then, problem* (4.1) *has at least one positive periodic T-periodic solution.*

Proof. We will apply theorem 3.2. Consequently, we need to prove that

$$f(t, x, y, z) := a(t) g(x, y)$$

satisfies the assumptions of that theorem. This will be obtained in a number of steps.

Step 1. The map f is clearly L^1-Carathéodory and T-periodic with respect to the first variable. Condition (f_1) and (f_2) on f follow directly from (g_1) and (g_2), respectively. Property (f_3) is an immediate consequence of conditions (ii) and (iii) on a and g. Property (H_m) is verified, since, by (a_2) and (g_1), one has

$$\int_0^T f(t, x, y, z)\,\mathrm{d}t < 0, \quad \forall x, y > 0.$$

Step 2: verification of (H_r). We show here that f satisfies condition (H_r) of theorem 3.2: more precisely we prove that there exists $\widehat{r} > 0$ such that condition (H_r) is satisfied for any $r \in (0, \widehat{r}]$. To see this, assume the contrary and suppose the existence of two sequences $(\theta_n)_n$ and $(u_n)_n$, with $0 < \theta_n \leq 1$ for any $n \in \mathbb{N}$, such that

— u_n is a positive T-periodic solution of $-u''(t) = \theta_n a(t) g(u(t), u(t - \tau))$,
— $r_n := |u_n|_\infty$ converges to zero, as $n \to \infty$.

Define the sequence of maps

$$v_n(t) := \frac{u_n(t)}{r_n},$$

which clearly verify

$$-v_n''(t) = \theta_n a(t) \frac{g(u_n(t), u_n(t-\tau))}{u_n(t)} v_n(t), \quad \forall t \in \mathbb{R},\ \forall n \in \mathbb{N}.$$

Let $t_n \in [0, T)$ be such that $v_n'(t_n) = 0$ (t_n exists since v_n is periodic). Thus,

$$v_n'(t) = -\theta_n \int_{t_n}^{t} a(s) \frac{g(u_n(s), u_n(s-\tau))}{u_n(s)} v_n(s)\, ds.$$

Therefore,

$$|v_n'|_\infty \le \int_0^T |a(s)| \frac{g(u_n(s), u_n(s-\tau))}{u_n(s)}\, ds.$$

By condition (g_2), it follows that $|v_n'|_\infty \to 0$, as $n \to \infty$. This convergence and the fact that $|v_n|_\infty = 1$ for all n imply that $(v_n)_n$ converges uniformly to 1. Now, recalling that u_n and consequently u_n' are periodic and that $\theta_n \neq 0$, we have

$$0 = \int_0^T a(t) g(u_n(t), u_n(t-\tau))\, dt.$$

Thus, we have

$$0 = \int_0^T \Big(a(t) \big[g(r_n, r_n) + g(r_n v_n(t), r_n v_n(t-\tau)) - g(r_n, r_n) \big] \Big)\, dt,$$

and hence, since $g(r_n, r_n) \neq 0$, for all n, and applying (g_1),

$$-\int_0^T a(t)\, dt = \int_0^T a(t) \frac{g(r_n v_n(t), r_n v_n(t-\tau)) - g(r_n, r_n)}{g(r_n, r_n)}\, dt.$$

The first integral above, by (a_2), is non-zero. Thus,

$$\begin{aligned} 0 &< \left| \int_0^T a(t)\, dt \right| < \int_0^T |a(t)|\, dt \cdot \max_{t\in[0,T]} \left| \frac{g(r_n v_n(t), r_n v_n(t-\tau))}{g(r_n, r_n)} - 1 \right| \\ &= \int_0^T |a(t)|\, dt \cdot \left| \frac{g(r_n v_n(\eta_n), r_n v_n(\eta_n - \tau))}{g(r_n, r_n)} - 1 \right| =: \widetilde{\alpha}_n \end{aligned}$$

for suitable $\eta_n \in [0, T]$.

By the uniform convergence (in the sup norm) of v_n to 1 and condition (g_3), one has that $\widetilde{\alpha}_n \to 0$ as $n \to \infty$, leading to a contradiction. Therefore there exists $\widehat{r} > 0$ such that any positive T-periodic solution u of

$$-u''(t) = \theta a(t) g(t, u(t), u(t-\tau)), \quad 0 < \theta \le 1$$

verifies $|u|_\infty \neq r$ for each $0 < r \le \widehat{r}$, that is, $|u|_\infty > \widehat{r}$.

Step 3: verification of (H_R). Consider the linear Dirichlet problems

$$-u'' = \lambda g_\infty a(t) u, \qquad u|_{\partial I_i} = 0, \tag{4.2}$$

in which g_∞ and the intervals I_i have been defined in the assumptions (g_4) and (a_1), respectively. As $g_\infty > \max_{i=1,\dots,m} \lambda_1^i$, each first eigenvalue of (4.2), $i = 1, \dots, m$, is less than 1. In addition, by condition (g_4), the map

$$(t, x, y, z) \mapsto a(t) g(x, y)$$

satisfies assumptions (i) and (ii) of theorem 2.4 with respect to the map $g_\infty a(t)$ in any interval I_i. Hence, we can apply theorem 2.4 to equation (4.1) in every interval I_i and we obtain that there exists $R > 0$ such that, for any L^1-Carathéodory function $k : \mathbb{R} \times [0, +\infty) \times [0, +\infty) \to \mathbb{R}$, T-periodic with respect to the first variable, with

$$k(t, x, y) \geq a(t)g(x, y), \quad \text{a.e. } t \in I_i \text{ and } \forall x, y \geq 0,$$

every non-negative T-periodic solution u of

$$-u''(t) = k(t, u(t), u(t - \tau)),$$

satisfies

$$\max_{\bigcup_{i=1}^m I_i} u(t) < R. \tag{4.3}$$

Choose $R > \widehat{r}$, where $\widehat{r}$ has been introduced in step 2. Let $w \in L^1([0, T], \mathbb{R})$ be a non-trivial function such that

$$w(t) \geq 0 \text{ for a.e.} t \in \bigcup_{i=1}^m I_i$$

and

$$w(t) = 0 \text{ for a.e. } t \in [0, T] \setminus \bigcup_{i=1}^m I_i.$$

Assume in addition that w is essentially bounded. Call α_0 a positive constant such that

$$\alpha_0 > \frac{\int_0^T |a(t)| \, \mathrm{d}t \cdot \max_{x,y \in [0,R]} g(x, y)}{\int_0^T |w(t)| \, \mathrm{d}t},$$

and consider the parametrized equation, which is the analogue of (3.3),

$$-u''(t) = a(t)g(u(t), u(t - \tau)) + \alpha \widetilde{w}(t), \quad 0 \leq \alpha \leq \alpha_0, \tag{4.4}$$

where $\widetilde{w}$ is the T-periodic extension of w to $\mathbb{R}$. By (a_1), (g_1) and the definition of w, the restriction to $[0, T]$ of any T-periodic solution u of (4.4), for a given $\alpha \in [0, \alpha_0]$, is concave in each I_i and convex in any subinterval of $[0, T] \setminus \bigcup_{i=1}^m I_i$. Therefore,

$$\max_{[0,T]} u(t) = \max_{\bigcup_{i=1}^m I_i} u(t). \tag{4.5}$$

Since u is non-negative, (4.3) and (4.5) imply

$$|u|_\infty < R.$$

Concerning the second part of the (H_R), suppose that u is a T-periodic solution of

$$-u''(t) = a(t)g(u(t), u(t - \tau)) + \alpha_0 \widetilde{w}(t),$$

such that $0 \leq u(t) \leq R$ for every $t \in \mathbb{R}$. Integrating, we have

$$\alpha_0 \int_0^T \widetilde{w}(t) \, \mathrm{d}t \leq \int_0^T |a(t)| g(u(t), u(t - \tau)) \, \mathrm{d}t \leq \int_0^T |a(t)| \, \mathrm{d}t \max_{t \in [0,T]} g(u(t), u(t - \tau)),$$

and this contradicts the definition of α_0. This completes the verification of (H_R).

Summarizing the above arguments, the map $f(t, x, y, z) := a(t)g(x, y)$ verifies the assumptions of theorem 3.2 with respect to suitable positive and different $\widehat{r}$ and R, and hence (4.1) admits at least one T-periodic solution u such that $u(t) > 0$ for all t and verifying $\widehat{r} < \|u\|_\infty < R$. This concludes the proof. ∎

Remark 4.3. In the above section we have seen that an existence result for positive T-periodic solutions of the equation (1.5) depends on Nagumo-type conditions, which can be formulated in different ways, obtaining theorems 3.2 and 3.3, neither being a generalization of the other, as recalled in remark 3.4. In the case of equation (4.1) the situation is different, since we have no explicit dependence on the derivative of the solution. Therefore, it is immediate to observe that (f_{3b}) is satisfied by simply eliminating the condition that the map a (and the map v) is essentially bounded. In this case ϕ is actually a constant map. In other words, theorem 4.2 can be immediately generalized, replacing condition (ii) with the following

(ii_b) $a:\mathbb{R}\to\mathbb{R}$ is T-periodic, for a given $T>0$ and L^1 if restricted to $[0,T]$.

Remark 4.4. In §3, we studied an equation like (4.1) without an explicit dependence on the derivative of the unknown. Our intention has been to focus on the superlinear condition of the map g. However, interesting problems can be tackled with the explicit dependence on the derivative in the equation. In a recent and interesting paper [7], Boscaggin, Feltrin and Zanolin obtain existence results for positive solutions of an equation of the type

$$u'' + cu' + \lambda a(t)g(u) = 0,$$

with various boundary conditions, and discussing the difficulties that the presence of the first-order term entails. From our point of view, it is interesting to study the above family of problems by adding a dependence of a delay. This will be done in a future investigation.

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