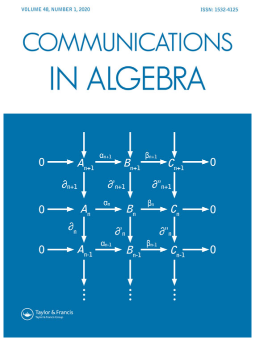

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Twisted conjugacy in free products

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ABSTRACT

Let $\phi : G \to G$ be an automorphism of a group which is a free product of finitely many groups each of which is freely indecomposable and two of the factors contain proper finite index characteristic subgroups. We show that G has infinitely many ϕ-twisted conjugacy classes. As an application, we show that if G is the fundamental group of a three-manifold that is not irreducible, then G has property R_∞, that is, there are infinitely many ϕ-twisted conjugacy classes in G for every automorphism ϕ of G.

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1. Introduction

Let G be an infinite group. Given an automorphism $\phi : G \to G$, one has an action of G on itself, known as the ϕ-twisted conjugation, defined as $g.x = gx\phi(g^{-1})$. The orbits of this action are the ϕ-twisted conjugacy classes. Let $\mathcal{R}(\phi)$ denote the orbit space. We denote by $R(\phi)$ the cardinality of $\mathcal{R}(\phi)$ if it is finite, and, when $\mathcal{R}(\phi)$ is infinite we set $R(\phi) := \infty$ and $R(\phi)$ is called the Reidemeister number of ϕ. One says that G has the R_∞-property, or that G is an R_∞-group, if $R(\phi) = \infty$ for every automorphism ϕ of G. The notion of Reidemeister number first arose in the Nielsen-Reidemeister fixed point theory. Classifying (finitely generated) groups according to whether or not they have the R_∞-property is an interesting problem and has emerged as an active research area that has enriched our understanding of finitely generated groups.

The fundamental group of a closed connected three-dimensional manifold is an important invariant of the manifold as it carries a lot of information concerning its topology. The main motivation for this work is to understand which manifolds have the property that their fundamental groups have the R_∞-property. We have not been able to completely answer this question. However, we obtain a very general result showing that a wide class of groups have the R_∞-property. This yields a partial answer, to the above question covering a large class of compact three-manifolds.

Recall that a closed connected three-dimensional manifold M is said to be *prime* if $M = M_1 M_2$ implies that at least one of the M_i is a 3-sphere. One says that M is *irreducible* if every embedded 2-sphere is the boundary of a 3-disk in M.[1] Every irreducible manifold is prime, but the converse is not true: $\mathbb{S}^2 \times \mathbb{S}^1$ is an example of a prime manifold which is not irreducible. If M is

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[1]We work in the PL or smooth category. Note that every three-manifold admits (unique) PL and smooth structures.

irreducible and has infinite fundamental group, then the sphere theorem (due to C. D. Papakyriakopoulos) implies that M is a $K(\pi, 1)$-space. A fundamental result in three-manifold theory is that every closed connected orientable 3-manifold M, can be expressed as a connected sum: $M \cong M_1 \# \cdots \# M_k$ where each M_j is prime (and not the 3-sphere). Moreover, the decomposition is unique (up to reordering of the factors). When M is non-orientable, one still has a prime decomposition. However, the uniqueness part fails. If P is nontrivial $\mathbb{S}^2$-bundle over $\mathbb{S}^1$, then $P\#N = (\mathbb{S}^2 \times \mathbb{S}^1)\#N$ when N is non-orientable. In view of this, in the case when M is non-orientable, one may assume that none of its prime factors M_i is $\mathbb{S}^2 \times \mathbb{S}^1$. With this restriction the uniqueness part is valid. See [5, Chapter 3]. As for any finitely generated group, $\pi_1(M)$ may be decomposed as a free product of groups $\pi_1(M) = G_1 * \cdots * G_r$ where each G_i is freely indecomposable. It turns out that $r = k$ and after reordering of indices $G_i = \pi_1(M_i), 1 \leq i \leq k$.

Our main result is the following.

Theorem 1. *Let $k \geq 2$. Suppose that $G = G_1 * \cdots * G_k$ where (i) each G_i is freely indecomposable, and, (ii) G_i has a proper characteristic subgroups of finite index for $i = 1, 2$. Then G has the R_∞-property.*

As an application of the above theorem, we shall establish the following.

Theorem 5. *Let M be a non-prime compact connected three-manifold. Then $\pi_1(M)$ has the R_∞-property.*

The main tool used in the proof of Theorem 1 is Kuroš subgroup theorem. It is well-known that no group is both a nontrivial free product and a nontrivial direct product. See [8, Observation, p. 177]. Thus, if $H = H_0 \times H_1, H_0, H_1$ are any two nontrivial groups with H_0 is a finite group with the trivial center and if H_1 is torsionless, then H is freely indecomposable and admits finite index characteristic subgroup, namely H_1. To see this we note that (i) any automorphism of H maps H_0 to itself since H_1 is torsion-free, the centralizer of H_1 in H contains H_0, and, (iii) the only element of H_0 in the centralizer of any element $(h_0, h_1) \in H$ is the trivial element. So H_1 is characteristic in H. Therefore we see that the hypotheses on the free factors of G in Theorem 1 hold for a large family of groups.

Theorem 5 follows easily from Theorem 1 using the fact that the fundamental group of a compact three-manifold is residually finite. (See [10, Theorem 3.3], [6]).

We should point out that Fel'shtyn outlined in [2] the main steps of a proof which shows that finitely generated non-elementary relatively hyperbolic groups have property R_∞. This proof relies on group actions on $\mathbb{R}$-trees and other notions from geometric group theory. Thus Fel'shtyn's result will imply that any finite free product of freely indecomposable finitely generated groups has property R_∞ from which Theorem 5 will follow. On the other hand, Theorem 1 does not assume that the free factors are finitely generated and the proof uses elementary techniques from combinatorial group theory. Hence, Theorem 1 does not follow from the result of [2]. For instance, if G is a freely indecomposable torsion-free group containing a proper finite index characteristic subgroup and if $H = \bigoplus_p \mathbb{Z}_p$ where p varies over the set of all primes, then $G * H$ has property R_∞ while H is not finitely generated (see also [9]).

2. The R_∞-property of a free product

Our goal here is to establish the R_∞-property for a free product $G = G_1 * \cdots * G_n, n \geq 2$, for a wide class of groups G_i. The main tool will be the Kuroš theorem that reveals the structure of a subgroup of a free product. The strategy of proof would be to first establish our goal when all the G_i are finite. Here the case $n = 2$ is well-known. We then reduce the general case, under suitable hypotheses on the G_i, to the case of free product of finite groups.

We begin by recalling the Kuroš subgroup theorem. Let G be a free product of groups $G = G_1 * \cdots * G_n$ and let K be a subgroup of G. Then K is itself a free product of groups

$$K = F_0 * H_1 * \cdots * H_n \tag{*}$$

where each H_j is a free product of a family of subgroups $\{\alpha_{i,j} H_{i,j} \alpha_{i,j}^{-1}\}_{i \in J_j}$ of G for suitable elements $\alpha_{i,j} \in G$ and suitable subgroups $H_{i,j} \leq G_j, i \in J_j$ for some indexing set $J_j, 1 \leq j \leq n$.

The following lemma is a standard application of the Kuroš subgroup theorem. We include a proof for the sake of completeness.

Lemma 2. *Let $G = G_1 * \cdots * G_n$ where each $G_i, 1 \leq i \leq n$ is a finite nontrivial group. Then G is virtually free and hence has the R_∞-property if $n \geq 2$.*

Proof. The statement that G is virtually free is trivially valid when $n = 1$. So assume that $n \geq 2$. We consider the kernel of projection $\eta : G \to G_1 \times \cdots \times G_n$, denoted K. Note that η maps any conjugate of G_i isomorphically onto G_i. Therefore, if H_i is a subgroup of G_i and $g \in G$, then $\eta(gH_ig^{-1})$ maps onto a conjugate of H_i. It follows that writing $K = F_0 * K_1 * \cdots * K_n$ as in $(*)$, we see that K_i is trivial for all i. Therefore $K = F_0$ is a free group. Since $G/K = \prod G_i$ is finite, the index of K in G is finite. Since G is finitely generated, the same is true of K.

If $n = 2$ and $G_1 \cong G_2 \cong \mathbb{Z}_2$, then G is infinite dihedral and it is known that G has the R_∞-property (see [3]). In all other cases, with $n \geq 2$, K is a non-abelian free group of finite rank. It follows that G is finitely generated non-elementary word hyperbolic and thus has the R_∞-property by [7]. □

We say that a nontrivial group is *freely indecomposable* if it cannot be expressed as a free product of two nontrivial groups. The only nontrivial free group which is freely indecomposable is the infinite cyclic group.

If $\alpha : G \to H$ is an isomorphism and if $C \subset G$ is a characteristic subgroup of G, then $\alpha(C)$ is a characteristic subgroup of H which is independent of the choice of α. Indeed, if $\beta : G \to H$ is another isomorphism then $\beta \circ \alpha^{-1} : H \to H$ is an automorphism. Since $\alpha(C)$ is characteristic, we have $\alpha(C) = \beta \circ \alpha^{-1}(\alpha(C)) = \beta(C)$.

Lemma 3. *Let $G = G_1 * \cdots * G_n$ where each $G_j, 1 \leq j \leq n,$ is freely indecomposable and not infinite cyclic. Let $C_j \subset G_j$ be a characteristic subgroup of G_j, $1 \leq j \leq n$. Fix an isomorphism $\alpha_{ij} : G_i \to G_j$ whenever G_i, G_j are isomorphic. Then the subgroup K of G generated by the family $\mathcal{C}$ of subgroups $gC_jg^{-1}, g\alpha_{ij}(C_i)g^{-1} \subset G, g \in G, 1 \leq i,j \leq n$, is characteristic in G.*

Proof. Evidently, K is normal in G since the family $\mathcal{C}$ is closed under conjugation. We need only show that the $\mathcal{C}$ is closed under any automorphism of G.

Let $\phi : G \to G$. Consider the subgroup $\phi(G_j)$. Since G_j is freely indecomposable and is not infinite cyclic, the same is true of $\phi(G_j)$. By the Kuroš subgroup theorem, $\phi(G_j)$ is contained in $g_jG_{k_j}g_j^{-1}$ for some $k_j \leq n$ and $g_j \in G$. Therefore $\phi(G) = \phi(G_1) * \cdots * \phi(G_n) \subset g_1G_{k_1}g_1^{-1} * \cdots * g_nG_{k_n}g_n^{-1} \subset G$. Since $\phi(G) = G$ we must have equality $\phi(G_j) = g_jG_{k_j}g_j^{-1}$ for all j. In particular $\iota_{g_j^{-1}}|_{G_{k_j}} \circ \phi|_{G_j} : G_j \to G_{k_j}$ is an isomorphism, which we shall denote by κ_j. Here ι_g denotes the inner automorphism $x \mapsto gxg^{-1}$ of G.

Let $A_j \subset G_j$ be any characteristic subgroup of G_j. Then $\kappa_j(A_j) = \alpha_{jk_j}(A_j)$. Therefore $\phi(A_j) = g_j(\alpha_{jk_j}(A_j))g_j^{-1}$.

Taking A_j to be C_j or $\alpha_{ij}(C_i)$, it follows that the family $\mathcal{C}$ is closed under any automorphism of G. Hence K is characteristic in G. □

Remark 4. (i) In our application, we shall choose the characteristic subgroups C_j so that whenever G_i and G_j are isomorphic, C_i corresponds to C_j under an isomorphism $G_i \to G_j$. In this case, $K \subset G$ is generated as a normal subgroup by the finite collection of subgroups $C_j, 1 \le j \le n$.

(ii) We remark that a finite index subgroup of a freely indecomposable group is not necessarily freely indecomposable. For example, $SL(2, \mathbb{Z})$ is virtually free with a finite index non-abelian free subgroup but is freely indecomposable.

Proof of Theorem 1. By relabeling if necessary, we assume that (i) $G_1, \ldots, G_n$ are the free factors of G such that either G_i is infinite cyclic or is isomorphic to one of the groups G_1, G_2, and, (ii) the group G_j is not isomorphic to any of the groups $G_1, G_2, \mathbb{Z}$, for $n < j \le k$. Note that $n \ge 2$.

Let K be the kernel of the natural projection $G \to G_1 * \cdots * G_n$ that maps each G_i identically onto $G_i, 1 \le i \le n$ and maps G_j to the trivial group for $j > n$. Then, by Lemma 3, K is characteristic. To show that G has the R_∞-property, we need only prove that $G_0 := G_1 * \cdots * G_n$ has the R_∞ property.

The proof will be divided into three cases depending on the number of groups $G_i, 1 \le i \le n$, that are isomorphic to $\mathbb{Z}$ being zero, or one or at least two. We shall denote this number by r. Relabeling if necessary, we assume that $G_j \cong \mathbb{Z}$ if $1 \le j \le r$ in case $r > 0$.

Let $C_i \subset G_i$ be a proper finite index characteristic subgroup of G_i. We assume, as we may, that whenever $G_i \cong G_j$, then C_j corresponds to C_i (under an isomorphism $G_i \to G_j$).

Case 1: Suppose that none of the G_i is infinite cyclic. Set ${}_{\bar{G}}i = G_i/C_i, 1 \le i \le n$, and let $\bar{G} = {}_{\bar{G}}1 * \cdots * {}_{\bar{G}}n$. Let K_0 be the kernel of the natural projection $G_0 \to {}_{\bar{G}}0$. Then K_0 is normally generated by the finite collection of subgroups $\{C_j | 1 \le j \le n\}$. Hence by Lemma 3, K_0 is characteristic in G_0. By Lemma 2, ${}_{\bar{G}}0$ has the R_∞-property. It follows that G_0 has the R_∞-property.

Case 2: Suppose that $r > 1$ so that $G_j \cong \mathbb{Z}$ for $1 \le j \le r$ and $G_j \ncong \mathbb{Z}$ for $j > n$. If $r = n$, in view of the fact that $n \ge 2$, G_0 is a non-abelian free group of finite rank and so has the R_∞ property.

So suppose that $2 \le r < n$. Set $A := G_1 * \cdots * G_r, B := G_{r+1} * \cdots * G_n$ so that $G_0 = A * B$ where A is a *non-abelian* free group of rank r. Let K_1 be the kernel of the projection $G_0 \to A$. Then K_1 is the free product of the family of groups $\mathcal{C} = \{gG_jg^{-1} | g \in G_0, r < j \le n\}$. Since each $G_j, j > r$, is indecomposable and not infinite cyclic, under any automorphism of G_0, G_j is mapped to a conjugate of a G_i isomorphic to G_j where $r < i \le n$. It follows that $\mathcal{C}$ is stable by any automorphism of G_0. Therefore K_1 is characteristic in G_0. Since A is a free non-abelian group of finite rank, it has the R_∞-property. It follows that G_0 also has the R_∞-property.

Case 3: Suppose that $r = 1$, that is, $G_1 \cong \mathbb{Z}, G_j \ncong \mathbb{Z}$ for $2 \le j \le n$. We consider the canonical projection $\eta : G_0 \to G_1 * \bar{B}$ where $\bar{B}$ is the free product of $G_i/C_i, 2 \le i \le n$. The kernel $K_3 :=$ $\ker(\eta)$ is as a subgroup generated by the collection $\{gC_jg^{-1} | g \in G_0, j \ge 2\}$. Proceeding as in the proof of Lemma 3, we see that K_3 is characteristic in G_0 in view of our hypothesis on the C_j. (See Remark 4 (i).)

Now $\bar{B}$ is *nontrivial* and is virtually free by Lemma 2, possibly finite. Indeed the kernel F of the natural projection $\pi : \bar{B} \to \prod_{2 \le j \le n} G_j/C_j$ is free by Kuroš' theorem. Since $\bar{B}$ is finitely generated, and F has a finite index, F has finite rank. *We claim that $G_1 * \bar{B} \cong \mathbb{Z} * \bar{B}$ has a non-abelian free group of finite rank as a finite index subgroup.* To see this, let S be a transversal to the projection π. We note that the kernel L of natural projection $q : G_1 * \bar{B} \to \prod_{2 \le j \le n} G_j/C_j$ equals the subgroup generated by the family of subgroups F, gG_1g^{-1} as g varies over S. The subgroups $F, gG_1g^{-1}, g \in S,$ generate their free product in $G_1 * \bar{B}$. Since F is free of finite rank, $G_1 \cong \mathbb{Z}$ and S is finite, we see that L is a free group of finite rank. Since $\prod G_j/S_j$ is a finite group, L has finite index in $G_1 * \bar{B}$. The group L is non-abelian since S has at least two elements. This proves our claim. So $G_1 * \bar{B}$ is a finitely generated non-elementary word hyperbolic group and so has the R_∞-property by [7]. Since K_3 is characteristic in G_0, it follows that G_0 also has the R_∞-property.

Thus in all cases, G_0 has the R_∞-property as was to be shown.

As an immediate corollary, we obtain

Theorem 5. *Let M be a non-prime compact connected three-manifold. Then $\pi_1(M)$ has the R_∞-property.*

Proof. Since M is not prime, it admits a prime decomposition: $M = M_1 \# \cdots \# M_k$ where each M_i is a prime manifold and $k \geq 2$. Thus $\pi_1(M) = \pi_1(M_1) * \cdots * \pi_1(M_k)$. (See [5, Chapter 3].) Note that since M_i is prime, $\pi_1(M_i)$ is freely indecomposable in view of [5, Theorem 7.1]. Also, it is known that $\pi_1(M_i)$ is residually finite as a consequence of the geometrization theorem and the work of Thurston [10, Theorem 3.3]. (See also [6]) So we may take G_i to be $\pi_1(M_i)$ in Theorem 1 and we see that $\pi_1(M)$ has the R_∞-property. □

Remark 6. (i) The same arguments as above also yields the following: if M is a connected sum $M_1 \# \cdots \# M_r, r \geq 2,$ where each M_j is a connected n-manifold ($n \geq 3$) and M_1 and M_2 each admits a nontrivial finite *characteristic cover*, then $\pi_1(M)$ has the R_∞ property. (A cover is characteristic if it corresponds to a characteristic subgroup of the fundamental group.)

(ii) Let $P \subset \mathbb{N}$ be a nonempty proper subset of primes. Suppose that $p \notin P$. Let $\mathbb{Z}(P) \subset \mathbb{Q}$ be subring $\mathbb{Z}[1/q | q \in P]$. Since $p \notin P,$ we have a natural surjective ring homomorphism $\mathbb{Z}(P) \to \mathbb{Z}/p\mathbb{Z}$. The kernel is a proper characteristic subgroup of $\mathbb{Z}(P)$. (It consists precisely of elements which are expressible as $px, x \in \mathbb{Z}(P)$.) Evidently $\mathbb{Z}(P)$ is freely indecomposable. It is readily seen that $\mathbb{Z}(P)$ is not isomorphic to $\mathbb{Z}(P')$ if $P \neq P'$ so the collection of such groups has cardinality the continuum. It is known that $\mathbb{Z}(P)$ is the fundamental group of an open three-manifold $M(P)$, which is, in fact, an aspherical space. This can be derived from constructing a non-compact 3-manifold (in fact, the complement of a solenoid in $\mathbb{S}^3$) whose fundamental group is $\mathbb{Q}$ (see e.g., [1, p. 209]). If $M_j := M(P_j), 1 \leq j \leq k,$ are such manifolds (where we do *not* assume that the P_j are pairwise distinct) then their connected sum $M_1 \# \cdots \# M_k$ is an open three-manifold whose fundamental group has the R_∞-property, provided $k \geq 2$ and at least two sets, say, P_1, P_2 are nonempty proper subsets of the set of all primes.

Remark 7. (i) Let M be a closed connected three-manifold such that $\pi_1(M)$ is infinite cyclic. Then M is prime. We claim that M is a 2-sphere bundle over the circle. If M is irreducible, then by the sphere theorem, $\pi_2(M)$ is trivial. Since $\pi_1(M)$ is infinite, it is a $K(\mathbb{Z}, 1)$-space. Therefore it is homotopic to a circle. This is a contradiction since $H_3(M; \mathbb{Z}_2) \cong \mathbb{Z}_2$. So M is not irreducible. By [5, Lemma 3.13] M is a 2-sphere bundle over the circle.

(ii) When a three-manifold M admits a geometric structure, in some cases it is known whether or not $\pi_1(M)$ has the R_∞-property. For example, when M admits spherical geometry, the fundamental group is finite and it is trivial that $\pi_1(M)$ does not have the R_∞-property. On the other hand, when the manifold admits hyperbolic geometry, then $\pi_1(M)$ has the R_∞-property as an immediate consequence of the work of Levitt and Lustig [7]. In the case of $\mathbb{S}^2 \times \mathbb{R}$-geometry, we see that $\pi_1(\mathbb{S}^2 \times \mathbb{S}^1) \cong \mathbb{Z}$ does not have R_∞-property whereas $\pi_1(\mathbb{R}P^3 \# \mathbb{R}P^3) \cong \mathbb{Z}_2 * \mathbb{Z}_2$ has the R_∞-property; see [3]. It can be shown that the fundamental groups of Seifert fiber spaces have the R_∞-property provided the base surface has genus at least 2. However, the complete classification for geometric three-manifolds will take us too far a field. It will be carried out in a forthcoming article [4].

In general, by Thurston's geometrization conjecture (Perelman's theorem), for any orientable prime three-manifold M that is not geometric, the fundamental group of M is an iterated amalgamated free product of fundamental groups of geometric manifolds where the amalgamating group is $\mathbb{Z}^2$. Our approach to Theorem 1 fails in this setting.

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