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F.A. Pino, P.A. Morettin

and

R.P. Mentz

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MODELLING AND FORECASTING LINEAR COMBINATIONS OF TIME SERIES

Francisco A. Pino[1], Pedro A. Morettin[2] and Raúl P. Mentz[3]

[1] Institute of Agricultural Economics, C.P 8114, São Paulo, Brazil

[2] University of São Paulo, C.P 20570, São Paulo, Brazil

[3] University of Tucumán and CONICET, C.C 209, Tucumán, Argentina

Summary

This paper reviews and extends several aspects of the analysis of linear combinations of time series. Special cases are temporal and contemporaneous aggregations and systematic sampling. We present some simple examples, a unified notation, references to the literature, and some general results for linear combinations of scalar and vector time series. For basic time series following ARIMA models in scalar cases we derive the ARIMA models of the linear combinations as functions of those of the basic series in the non-seasonal and seasonal cases. For vector time series we compare the forecast efficiences of two alternative approaches: first model and forecast and then form the linear combination, and first form the linear combination and then model and forecast; for this analysis we use the moving average representation of a stationary time series. A final section contains an application to milk production and milk productivity series, monthly data, for the State of São Paulo, Brazil.

Key words: ARIMA models, modelling, efficiency of forecast, linear combination, aggregation, systematic sampling.

1. INTRODUCTION

Linear combinations of the observations of one or more time series have shown to be of considerable interest in statistics, econometrics and elsewhere. The special cases of temporal and contemporaneous aggregation and systematic sampling are often discussed. In this paper we consider a general linear combination of k time series over a period of H time intervals. Two basic problems will be entertained, namely those of modelling and forecasting such linear combinations, under specific assumptions.

1.1. Motivation

We motivate the purpose of this paper with some simple examples. Let us consider a stationary stochastic process satisfying a first-order autoregressive model, denoted AR(1),

$$z_t = \phi z_{t-1} + a_t, \quad t = 0, \pm 1, \pm 2, \ldots, \qquad (1.1)$$

where the $\{a_t\}$ form a white noise sequence (that is, they are independent, identically distributed random variables with zero mean and variance σ_a^2), and $|\phi| < 1$. For the sake of definiteness let us consider that t ranges over semesters. We then consider the situations that arise when certain linear combinations of the z_t are taken.

Case 1: Moving Average. This smoothing technique corresponds to forming the "overlapping" linear combination

$$x_t = \sum_{h=0}^{H-1} w_h z_{t-h}, \quad t=0,\pm1,\pm2\ldots \qquad (1.2)$$

By overlapping we mean that for two different values of t, x_t may include some of the same z_t. A very simple case of (1.2) is when $H = 2$, that is, when the moving average covers one year. Then,

$$x_t = w_0 z_t + w_1 z_{t-1} = w_0(\phi z_{t-1}+a_t) + w_1(\phi z_{t-2}+a_{t-1})$$

$$= \phi x_{t-1} + w_0 a_t + w_1 a_{t-1}$$

and then,

$$x_t = \phi x_{t-1} + a_t^* + (w_1/w_0)a_{t-1}^*, \qquad (1.3)$$

which shows that x_t follows an ARMA (1.1) model with the same autoregressive coefficient as in (1.1) and a white noise sequence a_t^* with variance $w_0^2\sigma_a^2$.

Case 2: Temporal Aggregation, Flow Variable. If the variable in (1.1) is a flow variable (for example, production of cars per semester) we can form the yearly series by aggregation as follows:

$$Y_T = z_t + z_{t-1}, \quad T = 0,\pm1,\pm2,\ldots, \qquad (1.4)$$

where $t = 2T$ and 0 is (arbitrarily) taken as the origin. Note that (1.4) looks like a special case of (1.2), but here the linear combination is "non-overlapping", in that for two different values of T, Y_T includes different z_t. Then

$$z_t+z_{t-1} = \phi(z_{t-1}+z_{t-2}) + a_t + a_{t-1}$$

$$= \phi^2(z_{t-2}+z_{t-3}) + \phi(a_{t-1}+a_{t-2})+a_t+a_{t-1}$$

and it follows that

$$Y_T = \phi^2 Y_{T-1}+a_t+(1+\phi)a_{t-1}+\phi a_{t-2}. \qquad (1.5)$$

Now

$$\text{cov}\,\{a_{t+2u}+(1+\phi)a_{t+2u-1}+\phi a_{t+2u-2}, a_t+(1+\phi)a_{t-1}+\phi a_{t-2}\} =$$

$$\begin{aligned} &= 2\sigma_a^2(1+\phi+\phi^2), & u = 0 \\ &= \sigma_a^2 \cdot \phi, & u = 1 \qquad (1.6) \\ &= 0, & u > 1 \end{aligned}$$

hence (1.5) may be written as

$$Y_T = \phi^2 Y_{T-1} + b_T + \theta^* b_{T-1}, \qquad (1.7)$$

and we conclude that Y_T is ARMA (1,1). A further question is what b_T and θ^* will have (1.6) for covariances. This will be discussed later in sections 2 and 4.

Case 3: Systematic Sample, Stock Variable. If the variable in (1.1) is a stock variable (for example, end-of-semester cash balances), we can form a yearly series by recording the values corresponding to one of the semesters only. Hence,

$$Y_T = z_{2T},\ T = 0,\pm 1,\pm 2,\dots, \qquad (1.8)$$

and we have that

$$z_{2T} = \phi z_{2T-1}+a_{2T} = \phi(\phi z_{2T-2}+a_{2T-1})+a_{2T},$$

or

$$Y_T = \phi^2 Y_{T-1} + b_T, \qquad (1.9)$$

where $b_T = a_{2T} + \phi a_{2T-1}$ form and iid sequence, with mean zero and variance $\sigma^2 (1+\phi^2)$. Hence, we showed that Y_T is also AR(1), with change in parameter from ϕ to ϕ^2 and in the variance of the white noise sequence from σ_a^2 to $\sigma_a^2(1+\phi^2)$.

Case 4: Contemporaneous aggregation. Let z_{1t} and z_{2t} be two series measured in semesters, both AR(1),

$$z_{1t} = \phi_1 z_{1,t-1}+a_{1t},\ z_{2t}= \phi_2 z_{2,t-1}+a_{2t},\ t=0,\pm 1,\pm 2,\dots, \qquad (1.10)$$

where the a_{1t} are iid$(0,\sigma_1^2)$ and the a_{2t} are iid$(0,\sigma_2^2)$. Consider the aggregate

$$z_t^* = z_{1t} + z_{2t}. \qquad (1.11)$$

If $\phi_1 = \phi_2 = \phi$, and further the a_{1t} and a_{2t} are independent over time, then

$$z_t^* = \phi z_{t-1}^* + a_t^*, \qquad (1.12)$$

where the a_t^* are i.i.d$(0,\sigma_1^2+\sigma_2^2)$, that is, z_t^* is AR(1) with the same parameter ϕ and a change in the innovation variance.

Several observations can be deduced from these examples:

(a) Questions of practical interest can be interpreted as arising from linear transformations of one or more observable time series; (b) If we assume that the basic series satisfy models belonging to the ARIMA family, it is reasonable to expect that linear transforma tions will remain in the family; even if this is the case there are questions about changes in the orders and in the nature of the parameters, including the innovation variances; (c) One important objective of time series analysis is forecasting: the question then arises as to how should the forecasting of linear combinations of time series be studied; (d) The models considered so far are non-seasonal: similar questions as those discussed above can be raised for seasonal models.

1.2. Contents of the Paper

The rest of the paper is organized as follows. Section 1.3. contains a brief survey of the literature on linear combinations of time series. In Section 1.4. we establish the notation that will be used in the sequel. Section 2 discusses the problem of modelling a linear combination of one or more time series, assuming that the basic time series follow ARIMA models; the non-seasonal and seasonal case are treated separately.

Forecasting linear combination is dealt with in Section 3. Two different approaches are used and their efficiences compared. Two applications with real series are shown in Section 4 and some further comments are collected in Section 5.

1.3. References

Temporal aggregation has been well discussed in statistical and econometric literature. It was first investigated in econometrics by Theil (1954), Grunfeld and Griliches (1960),Mundlak (1961), Orcutt, Watts and Edwards (1968), Moriguchi (1970), Zellner and Montmarquette (1971), Aigner and Goldfeld (1973 and 1974),Dunn, Williams and DeChaine (1976), Tiao and Wei (1976), Geweke (1978), Hsiao (1979), Palm and Nijman (1981) and others. Geweke (1979) derived procedures for optimal seasonal adjustment and aggregation.

Derivations of the resulting model for the linear combination series given the model for the original series were presented by Amemiya and Wu (1972) in the flow case for AR model, by Brewer (1973) in the flow and stock cases for ARMA and ARMAX models, by Wei (1979) in the flow case for seasonal and nonseasonal ARIMA models, by Granger and Morris (1976) for the sum of independent ARMA processes and by Rose (1977) for linear combinations of independent ARIMA processes.

The effect of linear combination on parameter estimation was considered by Tiao (1972), Tiao and Wei (1976), Wei (1978 and 1979) and Hsiao (1979). The effect of linear combination on forecasting was studied by Tiao (1972), Amemiya and Wu (1972), Tiao and Wei (1976), Granger and Morris (1976), Rose (1977), Tiao and Guttman (1980), Wei and Abraham (1981), Abraham (1982), Abraham and Ledolter (1982), and Kohn (1982).

Temporal aggregation is related to missing observations problem when time series observations may be divided in two periods: one with data in aggregated form and another with data in disaggregated form (see Harvey and Pierse, 1984).

The case of forecasting contemporaneous time series aggregates was considered by Wei and Abraham (1981) using a Hilbert space approach.

1.4. Notation

In this Section we set down the notation that will be used in the rest of the paper. We shall denote by $\{z_t, t=0,\pm1,\pm2,\ldots\}$ the basic univariate time series in the original time scale. Let B be the backshift operator, such that $B^j z_t = z_{t-j}$.

By an overlapping linear combination of the z_t's in the original time scale we shall mean the series

$$x_t = \sum_{h=0}^{H-1} w_h z_{t-h}, \quad t = 0,\pm1,\pm2,\ldots, \tag{1.13}$$

while

$$Y_T = \sum_{h=0}^{H-1} w_h z_{t-h} = \left(\sum_{h=0}^{H-1} w_h B^h\right) z_t, \quad T = 0,\pm1,\pm2,\ldots, \tag{1.14}$$

where $t = TH$, is a non-overlapping linear combination of the z_t's in the new time scale. In (1.13) and (1.14), $w_0, w_1, \ldots, w_{H-1}$ are real known weights, with $w_0 \neq 0$.

Two interesting special cases of (1.14) are:

(a) If $w_0 = w_1 = \ldots = w_{H-1} = 1$ we have a temporal aggregation (as in the case of a flow variable);

(b) If $w_{h_0} = 1$, for $0 \leq h_0 \leq H-1$ and $w_h = 0$ for $h \neq h_0$, we have a systematic sample (as in the case of a stock variable). To simplify the notation and without loss of generality (we can shift the origin) we let $h_\sigma = 0$. It follows that the process $\{Y_T\}$ in (1.14) is then a systematic sample of $\{x_t\}$ in (1.13), $Y_T = x_{TH}$, $T = 0, \pm 1, \pm 2, \ldots$

Turning to vector time series, we shall denote by $\{\underline{z}_t, t=0,\pm 1,\pm 2,\ldots\}$ a k x 1 basic vector time series in the original time scale. If $\underline{w}_0'$ is a k x 1 vector of known constants, then

$$\underline{w}_0'\underline{z}_t, \quad t = 0, \pm 1, \pm 2, \ldots \tag{1.15}$$

is a contemporaneous linear combination in the original time scale, and

$$Y_T = \sum_{h=0}^{H-1} \underline{w}_h'\underline{z}_{t-h} = \Big(\sum_{h=0}^{H-1} \underline{w}_h' B^h\Big)\underline{z}_t, \quad T = 0, \pm 1, \pm 2, \ldots \tag{1.16}$$

is a non-overlapping linear combination of contemporaneously aggregated series, in the new time scale. Here, as before, $t = TH$ and $\underline{w}_0, \underline{w}_1, \ldots, \underline{w}_{H-1}$ are k x 1 vectors of real known weights, with $\underline{w}_0 \neq \underline{0}$. If they are all equal to $(1,1,\ldots,1)'$ we have an aggregation of contemporaneous sums, and if $\underline{w}_0 = (1,1,\ldots,1)'$ while $\underline{w}_h = \underline{0}$ for $h \neq 0$, we have a systematic sample of contemporaneous sums.

Observe that (1.16) may be also written as

$$Y_T = \sum_{i=1}^{k} \Big(\sum_{h=0}^{H-1} w_{hi} B^h\Big) z_{it}, \tag{1.17}$$

where $\underline{w}_h = (w_{h1}, \ldots, w_{hk})'$, $\underline{z}_t = (z_{1t}, \ldots, z_{kt})'$.

These ideas can be generalized one step further, by considering simultaneously sets of m $(1 \leq m \leq k)$ linear combinations, for example by defining

$$\underset{\sim}{Y}_T = \sum_{h=0}^{H-1} \underset{\sim}{W}_k \underset{\sim}{z}_{t-h} \qquad (1.18)$$

where $\underset{\sim}{W}_h$ are $m \times k$ matrices. Then $\underset{\sim}{W}_0 \underset{\sim}{z}_t$ is a set of m possibly different, contemporaneous aggregates; if $k = m$, the case of $\underset{\sim}{W}_0 = \underset{\sim}{I}$, $\underset{\sim}{W}_h = \underset{\sim}{0}$ for $h \neq 0$, corresponds to a systematic sample of the vector time series $\underset{\sim}{z}_t$, $\underset{\sim}{W}_0 = \underset{\sim}{W}_1 = \ldots = \underset{\sim}{W}_{H-1} = \underset{\sim}{I}$ correspond to the sum $\underset{\sim}{z}_t + \ldots + \underset{\sim}{z}_{t-H+1}$, etc. To study this situation we need to develop models and forecasts for vector aggregates, but we do not consider this extension in the present paper.

2 - MODELLING LINEAR COMBINATIONS

In this Section we derive the ARIMA models for linear combinations of basic time series that also follow ARIMA models, in the non-seasonal and seasonal cases.

A preliminary question relates to the covariance structure of the resulting linear combination. For simplicity we consider the scalar case (1.14), but the vector case (1.16) can be handled similarly.

Lemma 2.1

Let Y_T be defined by (1.14) and let $\underset{\sim}{\Sigma}_Y = (\sigma_Y(|i-j|))$ denote the $n \times n$ covariance matrix of $\underset{\sim}{Y} = (Y_1, Y_2, \ldots, Y_n)'$, $\underset{\sim}{\Sigma}_Z = (\sigma_Z(|i-j|))$ denote the $nH \times nH$ covariance matrix of

$\underset{\sim}{Z} = (z_1, z_2, \ldots, z_{nH})'$ and

$$\underset{\sim}{W} = \begin{bmatrix} \underset{\sim}{w} & \underset{\sim}{0} & \cdots & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{w} & \cdots & \underset{\sim}{0} \\ \vdots & \vdots & & \vdots \\ \underset{\sim}{0} & \underset{\sim}{0} & \cdots & \underset{\sim}{w} \end{bmatrix} \tag{2.1}$$

where $\underset{\sim}{w} = (w_{H-1}, \ldots, w_1, w_0)'$ and $w_0 \neq 0$. Then,

(a) $\underset{\sim}{\Sigma}_Y = \underset{\sim}{W}' \underset{\sim}{\Sigma}_Z \underset{\sim}{W}$;

(b) $\sigma_Y(|U|) = \sum_{h=0}^{H-1} \sum_{i=0}^{H-1} w_h w_i \sigma_Z(|UH+h-i|)$;

(c) $\underset{\sim}{\Sigma}_Z$ positive definite implies $\underset{\sim}{\Sigma}_Y$ positive definite.

<u>Proof</u>.

We have that

$$\underset{\sim}{Y} = \underset{\sim}{W}' \underset{\sim}{Z}, \tag{2.2}$$

so that (a) and (b) follow directly. Next, $\underset{\sim}{W}$ is nH x n and of rank n, since $w_0 \neq 0$ and this proves (c). See Anderson (1984), for example.

In systematic sampling $\underset{\sim}{w} = (0, 0, \ldots, 1)'$, and $\underset{\sim}{\Sigma}_Y$ is obtained from $\underset{\sim}{\Sigma}_Z$ by deletion of rows and columns.

In order to treat formally the cases of aggregation and systematic sampling at the same time, let us define

$$r = 1 + \max\{h: 0 \leq h \leq H-1, w_h \neq 0\}. \tag{2.3}$$

In the case of aggregation, $r = H$, in the case of systematic sampling, $r = 1$. We also use the notation $[x]$ to mean the largest integer contained in x.

2.1. - Non-seasonal Case

The model for Y_T, when z_t follows a non-seasonal ARIMA model is given in the following theorem.

Theorem 2.1

Let Y_T, $T = 0, \pm 1, \pm 2, \ldots$ be as in (1.14) and suppose that z_t follows an ARIMA (p,d,q) model. Then, Y_T follows an ARIMA (p,d,q^*) model, where

$$q^* = [\{\frac{1}{H}(H-1)(p+d)+q+r-1\}]. \qquad (2.4)$$

Proof.

Let us consider the overlapping linear combination (1.13), namely

$$x_t = \sum_{h=0}^{H-1} w_h z_{t-h} = W_{H-1}(B)\, z_t, \qquad (2.5)$$

where $W_{H-1}(B) = \sum_{h=0}^{H-1} w_h B^h$. From (2.5) we obtain Y_T by systematic sampling: $Y_T = x_{TH}$. If the ARIMA (p,d,q) model for z_t is written

$$\phi_p(B)\,(1-B)^d z_t = \theta_q(B)\, a_t, \qquad (2.6)$$

where $\phi_p(B) = 1-\phi_1 B - \ldots - \phi_p B^p$ is a p-order autoregressive operator and $\theta_q(B) = 1+\theta_1 B+\ldots+\theta_q B^q$ is a q-order moving average operator (see Box and Jenkins, 1976), and defining $Z_t = (1-B)^d z_t$ and $X_t = (1-B)^d x_t$, then

$$X_t = (1-B)^d W_{H-1}(B) z_t = W_{H-1}(B) Z_t$$
$$= W_{H-1}(B) \{1-\phi_p(B)\} Z_t + W_{H-1}(B) \theta_q(B) a_t$$

and

$$X_t = \{1 - \phi_p(B)\} X_t + \bar{\theta}_{q+r-1}(B) a_t^* \tag{2.7}$$

where the moving average operator $\bar{\theta}_{q+r-1}(B)$ has coefficients $\bar{\theta}_j = \sum_{i=0}^{j} \theta_i w_{j-1} / w_0$, if we define $w_{r+1} = w_{r+2} = \ldots = w_{q+r-1} = \theta_{q+1} = \theta_{q+2} = \ldots = \theta_{q+r-1} = 0$, and the white noise sequence $a_t^* = w_0$ has mean zero and variance $w_0^2 \sigma_a^2$.

Therefore X_t follows an ARMA (p,q+r-1) model and x_t follows an ARIMA (p,d,q+r-1) model. Now, whereas Y_T is a systematic sample of x_t, it follows (see Brewer, 1973) that Y_T follows an ARIMA (p,d,q*) model where $q^* = [p + d + \frac{1}{H}(q + r - 1 - p - d)]$, and (2.4) holds.

<u>Example</u>.

Let z_t be MA(1), i.e., $z_t = a_t + \theta a_{t-1}$, and consider aggregation as for flow variable, where H = 2. By theorem 2.1, using (2.4), Y_T follows a MA(1) model. In fact,

$$\begin{aligned} \mathrm{cov}(z_{t+2u} + z_{t+2u-1}, z_t + z_{t-1}) &= 2\sigma_a^2(1 + \theta + \theta^2), \quad u = 0, \\ &= \sigma_a^2, \quad |u| = 1 \\ &= 0, \quad |u| > 1, \end{aligned}$$

or

$$\sigma_Y(U) = \text{cov}(Y_{T+U}, Y_T) = 2\sigma_a^2(1 + \theta + \theta^2), \quad U = 0,$$
$$= \sigma_a^2 \quad , |U| = 1, \qquad (2.8)$$
$$= 0 \quad , |U| \quad 1,$$

which agrees with the covariance structure of a MA(1) model, as expected.

By lemma 2.1 we note that (2.8) is a positive definite covariance sequence. It remains to see (cf. section 1.1, case 2) what a_T^* and θ^* will have (2.8) for covariances. For positive definite sequences Anderson (1971, p.224-5) gives the procedure to find θ^*, with $\sigma_a^* = \sigma_a$, that is, $a_T^* = a_T$. For MA(1), the covariance generating function is $\sigma_Y(-1) + \sigma_Y(0)z + \sigma_Y(1)z^2 = 0$, which has two roots: one is $|z_1| \leq 1$ and the other is $1/z_1$. Then (see Anderson, 1971, page 225), $z - z_1 = \alpha_0^* z + \theta^* = z + \theta^*$ and we have that

$$\theta^* = -z_1 = \frac{\sigma_Y(0) \mp \{\sigma_Y^2(0) - 4\sigma_Y^2(1)\}^{\frac{1}{2}}}{2\sigma_Y(1)} \qquad (2.9)$$

whichever is less than or equal to one in absolute value, where $\sigma_Y(U)$ are given in (2.8).

2.2 - Seasonal Case

The following theorem extends the result to the seasonal case.

Theorem 2.2

Let Y_T, $T = 0, \pm 1, \pm 2, \ldots$, be as in (1.14), and suppose that z_t follows an ARIMA $(p,d,q)\times(P,D,Q)_s$ model, where S is an integer such that SH = s. Then, Y_T follows an ARIMA (p, d, q*) x (P, D, Q)$_S$ model, with q* given by (2.4).

Proof.

By hypothesis,

$$\Phi_P(B^s)\phi_p(B)(1-B^s)^D (1-B)^d z_t = \Theta_Q(B^s)\theta_q(B)\, a_t, \tag{2.10}$$

where $\Phi_P(B^s)$ and $\phi_p(B)$ are autoregressive operators, $\Theta_Q(B^s)$ and $\theta_q(B)$ are moving average operators and $\{a_t,\ t=0,\pm1,\pm2,\ldots\}$ is a white noise sequence with variance σ_a^2. Then,

$$\Phi_P(B^s)\ (1-B^s)^D\ z_t = \Theta_Q(B^s)\ c_t, \tag{2.11}$$

where $\phi_p(B)\ (1-B)^d\ c_t = \theta_q(B)\ a_t$, that is, c_t follows an ARIMA (p,d,q) model. Defining

$$V_T = \left(\sum_{h=0}^{H-1} w_h\, B^h\right) c_{TH}, \tag{2.12}$$

by theorem 2.1, V_T follows an ARIMA(p,d,q*) model, with q* given by (2.4), that is,

$$\bar{\phi}_p(\mathcal{B})\ (1-\mathcal{B})^d\ V_T = \bar{\theta}_{q^*}(\mathcal{B})\ b_T, \tag{2.13}$$

where $\beta = B^H$, $\beta^s = B^{SH} = B^s$, $\bar{\phi}_q(\beta)$ is an autoregressive operator, $\bar{\theta}_{q*}(\beta)$ is a moving average operator and b_T is a white noise sequence. Now,

$$\theta_Q(B^s)\, V_T = (\sum_{h=0}^{H-1} w_h\, B^h)\, \theta_Q(B^s)\, a_{TH}$$

$$= (\sum_{h=0}^{H-1} w_h\, B^h)\, \phi_p(B^s)(1-B^s)^D\, z_{TH}, \qquad (2.14)$$

using (2.13). From (1.14) and (2.14),

$$\phi_p(\beta^s)\, (1-\beta^s)^D\, Y_T = \theta_Q(\beta^s)\, V_T. \qquad (2.15)$$

Multiplying both sides of (2.15) by $\bar{\phi}(\beta)\ (1-\beta)^d$ and using (2.13) we obtain

$$\phi_p(\beta^s)\bar{\phi}_p(\beta)(1-\beta^s)^D(1-\beta)^d\, Y_T = \theta_Q(\beta^s)\bar{\theta}_{q*}(\beta) b_T \qquad (2.16)$$

and the theorem is proved.

Under the conditions of theorems 2.1 or 2.2 we see that

$$\lim_{H\to\infty} q^* = p + d + [\lim_{H\to\infty} r/H], \qquad (2.17)$$

with $0 \le \lim_{H\to\infty} r/H \le 1$, that is, q^* approaches $p + d$ if r does not increase as much as H (as in systematic sampling), otherwise, it approaches $p+d+1$ (as in temporal aggregation).

Consider the special cases of aggregation $(r = H)$ and systematic sampling $(r = 1)$. The models for Y_T, given several models for z_t, are presented in Table 1.

Table 1 - Models for z_t and Y_T in the cases of aggregation and systematic sampling

MODEL FOR z_t	MODEL FOR Y_T: AGGREGATION	MODEL FOR Y_T: SYSTEMATIC SAMPLING
AR(p)	ARMA(p,q*) (1) $q^* = [\frac{(H-1)(p+1)}{H}]$	ARMA(p,q*) $q^* = [\frac{(H-1)p}{H}]$
MA(q)	MA(q*) $q^* = [1 + \frac{q-1}{H}]$	MA(q*) $q^* = [\frac{q}{H}]$
ARMA(p,q)	ARMA(p,q*) (2) $q^* = [\frac{(H-1)(p+1)+q}{H}]$	ARMA(p,q*) (2) $q^* = [\frac{(H-1)p+q}{H}]$
ARIMA(p,d,q)	ARIMA(p,d,q*) $q^* = [\frac{(H-1)(p+d+1)+q}{H}]$	ARIMA(p,d,q*) (3) $q^* = [\frac{(H-1)(p+d)+q}{H}]$
ARIMA(p,d,q)x$(P,D,Q)_s$ $s = SH$	ARIMA(p,d,q*)x$(P,D,Q)_S$ (4) $q^* = [\frac{(H-1)(p+d+1)+q}{H}]$	ARIMA(p,d,q*)x$(P,D,Q)_S$ $q^* = [\frac{(H-1)(p+d)+q}{H}]$

(1) This result was obtained by Brewer (1973). Amemiya and Wu (1972) obtained $q^* = [\{(H-1)(p+1)+1\}/H]$, if $H<p+1$, and $q^* = p$, if $H \geq p+1$.

(2) These results were obtained by Brewer (1973).

(3) This result was obtained by Abraham and Ledolter (1982).

(4) This result was obtained by Wei (1979).

3. FORECASTING LINEAR COMBINATIONS

We now consider forecasting the linear combinations introduced in Section 1.4. We assume that the basic time series are stationary and have a moving average representation, that in the univariate case we write as

$$z_t = \sum_{j=0}^{\infty} \Psi_j a_{t-j}, \qquad t = 0,\pm1,\pm2,\ldots, \tag{3.1}$$

where $\Psi_0 = 1$, $\Sigma_{j=0}^{\infty} \Psi_j^2 < \infty$, and $\{a_t\}$ is a white noise stochastic process with variance $\sigma_a^2 > 0$. In the vector case we use the moving average representation

$$\underline{z}_t = \sum_{j=0}^{\infty} \underline{\Psi}_j \underline{a}_{t-j}, \qquad t = 0,\pm1,\pm2,\ldots, \tag{3.2}$$

where now the $\underline{\Psi}_j$ are kxk matrices, $\underline{\Psi}_0 = \underline{I}$, $\{\underline{a}_t\}$ is a sequence of independent random vectors with mean $\underline{0}$ and positive definite covariance matrix $\underline{\Sigma}_a$, and $\Sigma_{j=0}^{\infty} \underline{\Psi}_j \underline{\Sigma}_a \underline{\Psi}_j'$ is convergent.

The class of time series having this kind of moving average representation is very large. Essentially the representation follows from Wold's theorem, and is valid for non-deterministic stationary time series; see, for example, Anderson (1971,Chapter 7) or Hannan (1970,Chapter III).

We could add constant means to the right hand sides of (3.1) and (3.2), since they may be important in applications. However, this does not affect what follows and for simplicity we omit them.

We shall present all the results in this section in terms of moving average representations. For Y_T defined by (1.14) or (1.16) we shall use

$$Y_T = \sum_{j=0}^{\infty} \gamma_j b_{T-j}, \quad T = 0,\pm1,\pm2,..., \tag{3.3}$$

where $\gamma_0 = 1$, $\sum_{j=0}^{\infty} \gamma_j^2 < \infty$, and $\{b_T\}$ is a white noise stochastic sequence, with variance $\sigma_b^2 > 0$.

3.1 Forecasts

From standard results in the literature (see, for example, Box and Jenkins, 1976) we know that for z_t as in (3.1) the unbiased minimum mean square error (MMSE) forecast of z_{t+m}, at origin t, can be written as

$$\hat{z}_t(m) = \sum_{j=0}^{\infty} \psi_{m+j} a_{t-j}, \tag{3.4}$$

its forecast error is

$$e_t(m) = z_{t+m} - \hat{z}_t(m) = \sum_{j=0}^{m-1} \psi_j a_{t+m-j}, \tag{3.5}$$

and the variance of the forecast error is

$$V[e_t(m)] = \sigma_a^2 \sum_{j=0}^{m-1} \psi_j^2 . \tag{3.6}$$

The class of optimum linear forecasts is closed under linear operations, from which the following results are obtained.

Theorem 3.1

Suppose that z_t satisfies (3.1) and Y_T is given by (1.14).

Then:

(i) the unbiased MMSE forecast of Y_{T+M}, at origin T, is given by

$$\bar{Y}_T(M) = \sum_{h=0}^{H-1} w_h \hat{z}_{TH}(MH-h); \qquad (3.7)$$

(ii) the forecast error is (where 1 corresponds to k = 1)

$$e_T(H,M,1) = Y_{T+M} - \bar{Y}_T(M) = \sum_{h=0}^{H-1} \sum_{j=0}^{MH-h-1} \psi_j a_{(T+M)H-h-j}; \qquad (3.8)$$

(iii) the variance of the forecast error is

$$V\,[e_T(H,M,1)] = \sigma_a^2 \sum_{j=0}^{H-1} \sum_{j=0}^{MH-h-1} \sum_{i=0}^{G} w_h w_i \psi_j \psi_{h+j-i}, \qquad (3.9)$$

where $G = \min\{H-1, j+h\}$.

Box and Jenkins (1976, page 128) consider (3.7). The other results follow from the definitions, but we omit those details here.

Special Cases

a) Aggregation. If $w_h = 1$ for all h, then we have the case presented by Abraham (1982).

b) Systematic sampling. If $w_0 = 1$ and $w_h = 0$ for $h > 0$, then we have the case presented by Abraham and Ledolter (1982).

The previous results extend to the vector case. In effect, we have that the MMSE forecast vector of $\underline{z}_{t+m}$, at origin t, can be written as

$$\hat{\underline{z}}_t(m) = \sum_{j=0}^{\infty} \underline{\psi}_{m+j} \, \underline{a}_{t-j}, \qquad (3.10)$$

its forecast error is the vector

$$\underset{\sim}{e}_t(m) = \underset{\sim}{z}_{t+m} - \hat{\underset{\sim}{z}}_t(m) = \sum_{j=0}^{m-1} \underset{\sim}{\Psi}_j \, \underset{\sim}{a}_{t+m-j}, \tag{3.11}$$

and the variance of the forecast error is the kxk matrix

$$V[\underset{\sim}{e}_t(m)] = \sum_{j=0}^{m-1} \underset{\sim}{\Psi}_j \, \underset{\sim}{\Sigma}_a \, \underset{\sim}{\Psi}_j'. \tag{3.12}$$

See, for example, Tiao and Box (1981). Then we have:

__Theorem 3.2__

Suppose that z_t satisfies (3.2) and $\underset{\sim}{Y}_T$ is given by (1.16). Then:

(i) the unbiased MMSE forecast of Y_{T+M}, at origin T, is given by

$$\hat{Y}_T(M) = \sum_{j=0}^{H-1} \underset{\sim}{w}_h' \, \hat{\underset{\sim}{z}}_{TH}(MH-h); \tag{3.13}$$

(ii) the forecast error is

$$e_T(H,M,k) = \sum_{j=0}^{H-1} \sum_{j=0}^{MH-H-1} \underset{\sim}{w}_h' \underset{\sim}{\Psi}_j \underset{\sim}{a}_{(T+M)H-h-j}; \tag{3.14}$$

(iii) the variance of the forecast error is

$$V[e_T(H,M,k)] = \sum_{h=0}^{H-1} \sum_{j=0}^{MH-h-1} \sum_{i=0}^{G} \underset{\sim}{w}_h' \underset{\sim}{\Psi}_j \underset{\sim}{\Sigma}_a \underset{\sim}{\Psi}_{j+h-i}' \underset{\sim}{w}_i \tag{3.15}$$

__Special Cases__

a) Contemporaneous aggregation. If H = 1 we have that $\underset{\sim}{w}_0'\underset{\sim}{z}_t = \Sigma_{i=1}^k w_{0i}' z_{it}$ is formed. If further $\underset{\sim}{w}_0 = (1,1,\ldots,1)'$ we obtain $\sum_{i=1}^k z_{it}$ a contemporaneous sum of k time series, as considered by Wei and Abraham (1981).

b) Aggregation. If $\underset{\sim}{w}_h = (1,1,\ldots,1)'$ for all h, Y_T becomes an aggregate (through time) of contemporaneous aggregates (sums).

c) Systematic sampling. If $\underset{\sim}{w}_0 \neq \underset{\sim}{0}$ but $\underset{\sim}{w}_h = \underset{\sim}{0}$ for $h > 0$, we obtain in Y_T a systematic sample of contemporaneous aggregates, while if $\underset{\sim}{w}_0 = (1,1,\ldots,1)'$ the latter are (unweighted) sums.

3.2 Efficiency in forecasting linear combinations

We now consider forecasting the general linear combination $Y_T = \Sigma_{h=0}^{H-1} \underset{\sim}{w}_h' \underset{\sim}{z}_{t-h}$ defined in (1.16). We notice that besides the approach given by (3.13), there are other possibilities. Following Wei and Abraham (1981) we consider three alternative procedures:

Method 1. Forecast Y_T from an univariate model for the linear combinations of basic time series. Let the MMSE forecast of Y_{T+M} obtained by using this method be denoted by $\bar{Y}_T(M)$.

Method 2. Obtain MMSE forecasts of each component of $\underset{\sim}{z}_t$ from individual univariate models, then form the vector of those forecasts, and then the forecast of the linear combination, denoted by $\hat{Y}_T(M)$.

Method 3. Forecast $\underset{\sim}{z}_t$ by means of a k-dimensional vector model, and then form $\tilde{Y}_T(M)$ as in Theorem 3.3.

In the case of $\underset{\sim}{z}_t$ scalar ($k=1$), Methods 2 and 3 coincide. For the case of contemporaneous (or contemporal) sums, that is (1.15) with $\underset{\sim}{w}_0 = (1,1,\ldots,1)'$, Wei and Abraham (1981) were able to prove that in terms of mean square error of prediction $\tilde{Y}_T(M)$ dominates both $\bar{Y}_T(M)$ and $\hat{Y}_T(M)$, but that no unique solution exists between $\bar{Y}_T(M)$ and $\hat{Y}_T(M)$. They used a Hilbert space approach, which is relevant because all possible predictors are linear combinations of the available past, and finding a given MMSE predictor corresponds to choosing from a set formed by them (the Hilbert space spanned by the available past) that element which is closest to the value to be predicted, namely its orthogonal projection onto the space.

Following similar arguments we can prove our next theorem. In view of our purposes (cf. Section 4) we restrict our attention to Methods 1 and 3, that we identify as Approach I, first form the linear combination, then model and forecast, and Approach II, first model and forecast, and then form the linear combination, respectively.

Theorem 3.3

Let z_t, Y_T and $\tilde{Y}_T(M)$ be as in Theorem 3.2. Let $\{Y_T\}$ have the moving average representation (3.3), so that

$$\overline{Y}_T(M) = \sum_{j=0}^{\infty} \gamma_{M+j}\, b_{T-j} \tag{3.16}$$

Then,

$$E[Y_{T+M} - \tilde{Y}_T(M)]^2 \leq E\,[Y_{T+M} - \overline{Y}_T(M)]^2 \tag{3.17}$$

This result then means that forecasts obtained by Approach II are equally or more precise than those obtained by Approach I, when precision is measured by the mean square error of prediction. Theil (1954) also discussed some advantages of Approach II.

Efficiency of Approach II relative to Approach I can be measured by comparing the corresponding forecast errors, $e_T(H,M,k)$ and $\overline{e}_T(H,M,k)$ respectively, or, alternatively, by relating their variances in the measure

$$E(H,M,k) = \frac{V[e_T(H,M,k)]}{V[\bar{e}_T(H,M,k)]} . \tag{3.18}$$

To use in the following analysis we have that under Approach II

$$\bar{Y}_T(M) = \sum_{j=0}^{H-1} \underline{w}'_h \hat{\underline{z}}_{TH}(MH-h) = \sum_{h=0}^{H-1} \sum_{j=0}^{\infty} \underline{w}'_h \underline{\Psi}_{MH-h-j} \underline{a}_{TH-j} ; \tag{3.19}$$

in (3.15) we split the range $0 \leq j \leq MH-h-1$ into consecutive parts, $0 \leq j \leq H-h-1$, $H-h \leq j \leq 2H-h-1, \ldots, (M-1)H-h \leq j \leq MH-h-1$, so that

$$V[e_T(H,M,k)] = \sum_{\ell=0}^{M-1} A_\ell , \tag{3.20}$$

where

$$A_\ell = \sum_{h=0}^{H-1} \sum_{j=J}^{(\ell+1)H-h-1} \sum_{i=0}^{G} \underline{w}'_h \underline{\Psi}_j \underline{\Sigma}_a \underline{\Psi}'_{j+h-i} \underline{w}_i , \tag{3.21}$$

with $J = \max\{0, \ell H-h\}$, and we already had $G = \min\{H-1, j+h\}$.

Under Approach I, we have for $\bar{Y}_T(M)$ the representation (3.16) and using (3.6)

$$V[\bar{e}_T(H,M,k)] = \sigma_b^2 \sum_{j=0}^{M-1} \gamma_j^2 . \tag{3.22}$$

<u>Theorem 3.4</u>

The efficiency measure (3.18) is given by

$$E(H,M,k) = \frac{\sum_{m=0}^{\infty} \gamma_m^2}{\sum_{m=0}^{M-1} \gamma^2} \quad \frac{\sum_{\ell=0}^{M-1} A_\ell}{\sum_{\ell=0}^{\infty} A_\ell} . \tag{3.23}$$

<u>Proof</u>.

$E(H,M,k)$ is the ratio of (3.20) and (3.22). To evaluate σ_b^2 we use (1.16) and (3.2) to write

$$Y_T = \sum_{j=0}^{H-1} \underline{w}_h' \underline{z}_{t-h} = \sum_{h=0}^{H-1} \sum_{j=0}^{\infty} \underline{w}_h' \underline{\Psi}_j \underline{a}_{t-j-h} \tag{3.24}$$

Hence, using (3.3) we have that

$$\sigma_b^2 \sum_{j=0}^{\infty} \gamma_j^2 = E(Y_T^2) = \sum_{h=0}^{H-1} \sum_{j=0}^{\infty} \sum_{i=0}^{G} \underline{w}_h' \underline{\Psi}_j \underline{\Sigma}_a \underline{\Psi}_{j+h-i}' \underline{W}_i = \sum_{\ell=0}^{\infty} A_\ell \tag{3.25}$$

and (3.23) follows.

We next study conditions under which $V[e_T(H,M,k)] = V[\bar{e}_T(H,M,k)]$, and conditions under which $e_T(H,M,k) = \bar{e}_T(H,M,k)$. In both cases $E(H,M,k) = 1$, that is, predicitions under Approaches I and II have the same forecast variance.

<u>Corollary 3.1</u> (Conditions for $V[e_T(H,M,k)] = V[\bar{e}_T(H,M,k)]$).

a) $V[e_T(H,M,k)] = V[\bar{e}_T(H,M,k)]$ for all $T = 0,\pm1,\ldots$ if and only if

$$\sigma_b^2 = \frac{\sum_{\ell=0}^{M-1} A_\ell}{\sum_{m=0}^{M-1} \gamma_m^2} \quad ; \tag{3.26}$$

b) Equality holds for all $T = 0,\pm1,\ldots$ and for all $M \geq 1$, if and only if

$$A_\ell = \gamma_\ell^2 A_0$$

for all $\ell \geq 1$.

<u>Proof</u>.

Part (a) follows directly by inspection of (3.23).

To prove part (b) we first consider (3.26) for M=1 and M=2.

$$\sigma_b^2 = \frac{A_0}{\gamma_0^2} = A_0 = \frac{A_0 + A_1}{1 + \gamma_1^2}$$

and hence $A_1 = \gamma^2 A_0$. Suppose that (3.27) holds for $\ell = 0,1,\ldots, M-1$; then

$$\sigma_b^2 = A_0 = \frac{\sum_{j=0}^{M} A_j}{\sum_{j=0}^{M} \gamma_j^2} = \frac{A_0 \sum_{j=0}^{M-1} \gamma_j^2 + A_M}{\sum_{j=0}^{M-1} \gamma_j^2 + \gamma_M^2}$$

from which $A_0 \gamma^2{}_M = A_M$ follows, and the proof is complete by induction.

<u>Corollary 3.2</u> (Conditions for $e_T(H,M,k) = \bar{e}_T(H,M,k)$).

a) $e_T(H,M,k) = \bar{e}_T(H,M,k)$ for all $T = 0,\pm1,\pm2,\ldots$, if and only if

$$\sum_{h=0}^{H-1} \sum_{j=0}^{MH-j-1} \underset{\sim}{w}_h' \underset{\sim}{\psi}_j \underset{\sim}{a}_{TH+MH-h-j} = \sum_{j=0}^{M-1} \gamma_j b_{T+M-j} \tag{3.28}$$

for all $T = 0,\pm1,\pm2,\ldots$;

b) $e_T(H,M,k) = \bar{e}_T(H,M,k)$ for all $T = 0,\pm1,\pm2,\ldots$ and for all $M \geq 1$, if and only if

$$\sum_{h=0}^{H-1} \sum_{j=0}^{MH-j-1} \underset{\sim}{w}_h' \underset{\sim}{\psi}_j \underset{\sim}{a}_{TH+MH-h-j} = \sum_{m=0}^{M-1} \sum_{h=0}^{H-1} \sum_{j=0}^{H-h-1} \gamma_m \underset{\sim}{w}_h' \underset{\sim}{\psi}_{(T+M-m)H-h-j} \tag{3.29}$$

for all $M \geq 1$.

<u>Proof</u>.

$$e_T(H,M,k) = Y_{T+M} - \bar{Y}_T(M) = \sum_{h=0}^{H-1} \underset{\sim}{w}_h' \underset{\sim}{z}_{TH+MH-h} - \bar{Y}_T(M)$$

$$= \sum_{h=0}^{H-1} \sum_{j=0}^{\infty} \underset{\sim}{w}_h' \underset{\sim}{\psi}_j \underset{\sim}{a}_{(T+M)H-h-j} - \sum_{h=0}^{H-1} \sum_{j=0}^{\infty} \underset{\sim}{w}_h' \underset{\sim}{\psi}_{MH-h-j} \underset{\sim}{a}_{TH-j},$$

where we used (3.2) and (3.19); this difference is left-hand-side of (3.28). Next,

$$\bar{e}_T(H,M,k) = Y_{T+M} - \bar{Y}_T(M) = \sum_{j=0}^{\infty} \gamma_j b_{T+M-j} - \sum_{j=0}^{\infty} \gamma_{M+j} b_{T-j},$$

where we used (3.3) and (3.16); this difference is the right-hand-side of (3.28), and the proof of part a) is completed.

To prove part b) we put M=1 in (3.28) to obtain

$$b_{T+1} = \sum_{h=0}^{H-1} \sum_{j=0}^{H-h-1} \underset{\sim}{w}_h' \underset{\sim}{\Psi}_j \underset{\sim}{a}_{(T+1)H-h-j},$$

from which (3.29) follows.

In the special case of contemporaneous aggregation (H=1), it can be seen that $e_t(1,M,K) = \bar{e}_t(1,M,K)$ holds for all t = 0,±1,... and all M ≥ 1 if and only if

$$\underset{\sim}{w}'\underset{\sim}{\Psi}_j = \gamma_j \underset{\sim}{w}' \quad , \qquad \text{for all } j \geq 0, \tag{3.30}$$

that is, $\underset{\sim}{w}$ is an eigenvector for each $\underset{\sim}{\Psi}_j$ and γ_j is the corresponding eigenvalue. Kohn (1982) showed this for M=1 and showed that (3.30) is a necessary condition for $e_t(1,M,k) = \bar{e}_t(1,M,k)$.

In the univariate case (k=1) it can be shown that $V[e_T(H,M,1)] = V[\bar{e}_T(H,M,1)]$ holds for all T = 0,±1,... and all M ≥ 1 if and only if

$$A_\ell^* = \gamma_\ell^2 A_0^* \quad , \qquad \text{for all } \ell \geq 1. \tag{3.31}$$

Abraham and Ledolter (1982) calculated what in our notation is E(H,M.1) in the case of systematic sampling, for the ARIMA(1,0,0) and ARIMA(0,1,1) models.

Finally, note that $\lim_{M\to\infty} E(H,M,k)=1$, that is, Approach I draws near Approach II for long run forecasts.

4 - APPLICATIONS

Now we present two empirical applications to agricultural problems. The models used here are slight modifications of those in Pino and Morettin (1981).

In these applications we have univariate time series that follow seasonal ARIMA$(1,0,0)\times(0,1,0)_{12}$ models, denoted by

$$(1-\phi B)(1-B^{12})z_t = a_t \quad , \tag{4.1}$$

where $\text{var}(a_t) = \sigma_a^2$. We then form the aggregate Y_T as in (1.14), that according to Table 1 follows a seasonal ARIMA$(1,0,1)\times(0,1,0)_1$ model denoted by

$$(1-\phi^* B)(1-B)Y_T = (1-\theta B)b_T \quad , \tag{4.2}$$

where $\text{var}(b_T) = \sigma_b^2$.

In order to use the results of Section 3 we relate σ_b^2 to σ_a^2 as follows. Let $(1-B)z_{TH} \equiv x_{TH}$ and $(1-B)Y_T \equiv y_T$. Then x_{TH} follows an ARIMA(1,0,0) model and y_T follows an ARIMA(1,0,1) model. By repeated substitutions (compare with case 2 in Section 1),

$$x_{TH} = \phi x_{TH-1} + a_{TH} = \phi^2 x_{TH-2} + \phi a_{TH-1} + a_{TH} =$$

$$= \phi^H x_{(T-1)H} + \sum_{j=0}^{H-1} \phi^j a_{TH-j} \quad ,$$

and hence

$$y_T = Y_T - Y_{T-1} = \sum_{j=0}^{H-1} w_h \, x_{TH-h} =$$

$$= \phi^H y_{T-1} + \sum_{h=0}^{H-1} \sum_{j=0}^{H-1} w_h \, \phi^j \, a_{TH-h-j} \qquad (4.3)$$

Comparing (4.2) as an ARIMA(1,0,1) model for y_T with (4.3) we observe that $\phi^* = \phi^H$ and

$$(1 - \theta B) b_T = b_T - \theta b_{T-1} = \sum_{h=0}^{H-1} \sum_{j=0}^{H-1} w_h \, \phi^j \, a_{TH-h-j} \qquad (4.4)$$

Multiplying (4.4) by $b_T - \theta b_{T-1}$ and taking expectations we see that

$$\sigma_b^2 (1 + \theta^2) = \sigma_a^2 F,$$

where σ_a^2 and σ_b^2 are the variances of the processes a_t and b_T, respectively, and where

$$F = \sum_{h=0}^{H-1} \sum_{i=0}^{H-1} \sum_{j=0}^{H-1} \sum_{k=0}^{H-1} w_h \, w_i \, \phi^{j+k},$$

with the sums in F taken only for $h+j = i+k$.

Similarly, multiplying (4.4) by $b_{T-1} - \theta b_{T-2}$ and taking expectations, we obtain

$$-\theta \sigma_b^2 = G \sigma_a^2 \, , \qquad (4.6)$$

where

$$G = \sum_{h=0}^{H-1} \sum_{i=0}^{H-1} \sum_{j=0}^{H-1} \sum_{k=0}^{H-1} w_h \, w_i \, \phi^{j+k} \, ,$$

with the sums in G taken only for $h+j = i+j+H$.

Solving (4.5) and (4.6) we obtain

$$\sigma_b^2 = \sigma_a^2 \, [F \pm (F^2 - 4\,G^2)^{1/2}] \, / \, 2 \tag{4.7}$$

and applying theorem 3.4 we have

$$B(H,1,1) = 2\,A_0^* \, / \, [F \pm (F^2 - 4\,G^2)^{1/2}] \, , \tag{4.8}$$

which is easy to compute in terms of the w's and ϕ. This approach enables us to study the loss in forecast efficiency without estimating the model by Approach I, which would be impossible in the next two examples, anyway.

4.11 - Example 1 : milk production

A simple univariate model for monthly data of milk production (in millions of litres) in the State of São Paulo, Brazil, is $(1 - 0.84\ B)\ (1 - B^{12})\ z_t = a_t$, $\sigma_a^2 = 20.9224$. See Pino and Morettin (1981), for the data used and further details.

Available data (60 observations, from January 1975 to December 1979) did not allow to estimate a model to produce directly yearly forecasts. Therefore, Approach II ad to be used, with $w_0 = \dots = w_{11} = 1$. Using theorem 2.1, the one year ahead (1980) forecast turned out to be 1,622 ± 127, for the observed production of 1,695 (see Table 2).

From Theorem 2.2, we see that yearly production follows an ARIMA(1,1,1) model. The estimated forecast efficiency, using (4.8), without incorporating the estimation error of ϕ (see Abraham and Ledolter, 1982, for a discussion of this point), is

$$E(12,1,1) = 0.82.$$

This means that there is a loss of 18% in efficiency when forecasting with Approach I, instead of with Approach II.

5.2 - Example 2 : milk productivity

The estimated univariate model for monthly data of milk productivity (in daily litres per cow) in the State of São Paulo, Brazil, resulted to be $(1 - 0.91\,B)(1 - B^{12})z_t = a_t$, $\sigma_a^2 = 0.0095$.

As before, Approach II had to be used to obtain the yearly forecast for 1980, with $w_0 = w_2 = w_4 = w_6 = w_7 = w_9 = w_{11} = 31/366$, $w_3 = w_5 = w_8 = w_{10} = 30/366$ and $w_1 = 29/366$.

Using Theorem 2.1, the one-year ahead forecaste turned out to be 4.11 ± 0.29, for the observed value of 4.17 (see Table 2). The resulting aggregate process is an ARIMA(1,1,1), according to the theorem 2.2. The estimated forecast efficiency, without incorporating the estimation error of ϕ, resulted to be

$$E(12,1,1) = 0.74,$$

meaning that there is a loss of 26% in efficiency when using Approach I instead of Approach II.

Table 2 - Milk data

	Milk production			Milk productivity		
Month	Observed 1979	Observed 1980	Forecast 1980	Observed 1979	Observed 1979	Forecast 1980
Jan.	158.84	149.96	150.19	4.11	4.19	4.12
Feb.	146.36	145.27	139.20	4.41	4.50	4.42
Mar.	143.06	142.80	137.08	3.99	4.16	4.00
Apr.	136.59	132.88	131.56	4.11	4.19	4.12
May	131.66	129.91	127.39	3.89	4.04	3.90
June	128.03	127.50	124.48	3.93	4.09	3.94
July	121.26	134.06	118.36	3.78	3.85	3.79
Aug.	123.18	135.97	120.74	3.77	3.85	3.77
Sept.	137.34	138.43	135.24	4.17	4.18	4.17
Oct.	141.57	144.82	139.88	4.13	4.13	4.13
Nov.	151.22	151.56	149.76	4.42	4.34	4.42
Dec.	149.28	162.45	148.06	4.17	4.50	4.57
Year	-	1,695.41	1,622.00	-	4.17	4.11

5. Concluding Remarks

In this paper we showed that temporal and contemporaneous aggregation, (temporal) systematic sampling, and other operations of interest, can be profitably studied as linear combinations of univariate or vector basic time series with a known system of weights. We studied in general k-dimensional vectors and aggregation over H time periods, so that a linear combination is $Y_T = \sum_{h=0}^{H-1} \underset{\sim}{w}'_h \underset{\sim}{z}_{t-h}$, where $t = TH$.

One advantage of the linear-combination approach is that it makes clear the following: if the basic time series is second order stationary the linear combination Y_T also has this property(*); if the basic time series has a positive definite covariance sequence so has the linear combination, and the covariances are related by a simple formula (cf. Lemma 2.1).

We considered basic univariate time series that follow ARIMA(p,d,q) or ARIMA(p,d,q)x(P,D,Q)$_s$ models, in the non-seasonal and seasonal cases, respectively, and asked what model does the linear combination follow. We found that the needed model is also in the ARIMA family, with a possible change from q to q* (cf. (2.4)), and from s to S in the new time scale, where s = SH. The coefficients of the resulting ARIMA model and the variance of the innovations may differ from those in the original ARIMA model, and the cases of Section 1 showed various possibilities; the coefficients of the MA operator of Y_T may be related to those of the MA operator of z_t in

(*) It can also be shown that if z_t is strictly stationary the same is true for Y_T.

non-trivial ways, as one example showed, and the same is true for the AR operator.

Another point refers to the invertibility of AR and MA operators in the ARIMA model for Y_T. We may assume that z_t is second-order stationary and "causal" (dependence is only on present and past values), so that its AR operator is invertible (this is a necessary and sufficient condition). Under this condition the AR operator in the model for Y_T corresponds to second-order stationarity, and is invertible. The MA operators always define stationary stochastic processes, so that invertibility has to be studied separately, but we did not consider this point here; Teräsvirta (1977) studied the invertibility of MA operators under contemporaneous sums, for univariate and vector time series.

We also studied the forecasting of the linear combination Y_T, for M periods ahead, at origin T, assuming that all parameters are known. We considered two approaches: Approach I, first form the linear combination, then model Y_T and forecast, and Approach II, first model $\underline{z}_t$ and forecast, then form the linear combination. For these studies we enlarged the class of time series under consideration, since we now only assumed that they possess convergent, one-sided, infinite MA representations.

We showed that, in terms of mean square error of forecasting, Approach II should be preferred to Approach I.

Besides being more precise, there are at least two cases in which Approach II is intuitively more appealing: (a) when the number of available observations is small (cf. Section 4); in fact, when parameters have to be estimated, as is usually the case, a reasonable number of observations is needed to produce good parameter estimates; (b) when both leves (e.g., monthly and annual) are of interest to the user.

In spite of its relatively larger mean square forecast error, we may consider Approach I at least in two situations: (a) when disaggregate date are scarce, or have larger observation error than aggregate data, as pointed out by Aigner and Goldfeld (1974); (b) when it is difficult, or costly, to develop a vector model for $\underline{z}_t$.

There is still a third possible forecasting approach when k > 1, namely to model each component of $\underline{z}_t$ separately, forecast them, and then form the aggregate forecast. This is reasonable if the basic univariate time series are independent, since otherwise the development of a vector model that takes into account the correlations among series is superior. Hence, we did not consider this approach in our presentation.

The loss in forecast efficiency of Approach I relative to Approach II was studied theoretically. In two numerical illustrations it was shown that the loss due to the difference in approach can be substantial (18% and 26%). In these calculations the parameters were taken as known; in practice the effect of the estimation procedure can be also taken into account, as for example in Abraham and Ledolter (1982).

Necessary and sufficient conditions for Approaches I and II to have equal forecast efficiency were developped. In the simpler cases of H=1, k=1, the analysis led to simple relations.

A situation not considered in the paper is the resulting model for the combined times series, when the basic time series follows a vector ARMA model. This will be pursued elsewhere.

The case of linear transformations of vector ARMA processes is discussed by Lütkepohl (1984). Related references are Engel (1984) and Weiss (1984).

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RÉSUMÉ

Cet article présent quelques examples assez simples, une notation

unifié, les references sur la litterature et quelques resultates générales pour les combinaisons linéaires du séries chronologiques univariées et multivariées. On considére deux problémes: modelisation et prédiction des combinaisons linéaires sous des hypothéses specifiées.

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