

# Congruence filter pairs, equational filter pairs and adjoints

PETER ARNDT, *WE Informatik, University of Düsseldorf, Universitätsstr. 1, Düsseldorf, 40225, Germany.*

HUGO LUIZ MARIANO, *Department of Mathematics, University of São Paulo, Rua do Matão, 1010, São Paulo, 05508-090, São Paulo, Brazil.*

E-mail: hugomar@ime.usp.br

DARLLAN CONCEIÇÃO PINTO, *Department of Mathematics, Federal University of Bahia, Av. Milton Santos, s/n, Salvador, 40.170-110, Bahia, Brazil .*

E-mail: darllan@ufba.br

## Abstract

Filter pairs are a tool for creating and analyzing logics. A filter pair can be seen as a presentation of a logic, given by presenting its lattice of theories as the image of a lattice homomorphism, with certain properties ensuring that the resulting logic is substitution invariant. Every substitution invariant logic arises from a filter pair. Particular classes of logics can be characterized as arising from special classes of filter pairs. We consider so-called congruence filter pairs, i.e. filter pairs for which the domain of the lattice homomorphism is a lattice of congruences for some quasivariety. We show that the class of logics admitting a presentation by such a filter pair is exactly the class of logics having an algebraic semantics. We study the properties of a certain Galois connection coming with such filter pairs. We give criteria for a congruence filter pair to present a logic in some classes of the Leibniz hierarchy by means of this Galois connection, and its interplay with the Leibniz operator.

*Keywords:* filter pairs, congruences, Leibniz operator, quasivarieties

## Introduction

The first main aim of this article is to introduce the notion of filter pair as a tool for creating and analysing logics. A filter pair can be seen as a presentation of a logic, different in style from the usual syntactic presentations in terms of axioms and rules, or semantic presentations in terms of matrices. Roughly, a filter pair is a highly structured lattice homomorphism from some algebraic lattice into the power set of the formula algebra, whose image can be taken as the lattice of theories of a logic. The notion of filter pair can be developed into several different directions, some of which are sketched in Section 5.

In this article we concentrate on one direction, which constitutes its second main aim: to advance the general study of logics having an algebraic semantics. Recall that an algebraic semantics for a logic  $l$  is given by a class of algebras  $\mathbf{K}$  and a set of equations  $\delta(x) = \epsilon(x)$  such that

$$\Gamma \vdash_l \varphi \quad \Leftrightarrow \quad \{\delta(\gamma) = \epsilon(\gamma) : \gamma \in \Gamma\} \vDash_{\mathbf{K}} \delta(\varphi) = \epsilon(\varphi).$$

The notion was considered by Blok-Pigozzi in [12], but in loc. cit. was mainly used as a stepping stone towards the notion of algebraizable logic. As Font writes [21, p.114]: “The bare concept of

## 2 Congruence filter pairs, equational filter pairs and adjoints

algebraic semantics has not been extensively studied. [...] At the time of writing, [14] and [32] are probably the only papers that contain significant results about it.” It turns out that logics having an algebraic semantics are exactly those that can be presented by a so-called equational filter pair - in the above picture of a filter pair as a highly structured lattice homomorphism, these are those homomorphisms whose domain is the lattice of congruences relative to some quasivariety  $\mathbf{K}$ . We find that such a presentation, and the additional structure that comes with it, can be employed to analyze logics with an algebraic semantics. We illustrate this by giving criteria for being algebraizable, weakly algebraizable or truth-equational.

We now sketch the basic idea of the notion of filter pair:

Throughout the article the word *logic* will mean a pair  $(\Sigma, \vdash)$  where  $\Sigma$  is a signature, i.e. a collection of connectives with finite arities, and  $\vdash$  is a Tarskian consequence relation, i.e. an idempotent, increasing, monotonic, finitary and structural relation between subsets and elements of the set of formulas  $Fm_{\Sigma}(X)$  built from  $\Sigma$  and a set  $X$  of variables.

It is well-known that every Tarskian logic gives rise to an algebraic lattice contained in the powerset  $\mathcal{P}(Fm_{\Sigma}(X))$ , namely the lattice of theories. This lattice is closed under arbitrary intersections and directed unions.

Conversely an algebraic lattice  $L \subseteq \mathcal{P}(Fm_{\Sigma}(X))$  that is closed under arbitrary intersections and unions of increasing chains gives rise to a finitary closure operator (assigning to a subset  $A \subseteq Fm_{\Sigma}(X)$  the intersection of all members of  $L$  containing  $A$ ). This closure operator need not be structural – this is an extra requirement.

We observe that the structurality of the logic just defined is equivalent to the *naturality* (in the sense of category theory) of the inclusion of the algebraic lattice into the power set of formulas with respect to endomorphisms of the formula algebra: Structurality means that the preimage under a substitution of a theory is a theory again or, equivalently, that the following diagram commutes:

$$\begin{array}{ccc} Fm_{\Sigma}(X) & & L \xhookrightarrow{i} \mathcal{P}(Fm_{\Sigma}(X)) \\ \sigma \downarrow & \sigma^{-1} \upharpoonright_L & \uparrow \sigma^{-1} \\ Fm_{\Sigma}(X) & & L \xhookrightarrow{i} \mathcal{P}(Fm_{\Sigma}(X)) \end{array}$$

Further, it is equivalent to demand this naturality for all  $\Sigma$ -algebras and homomorphisms instead of just the formula algebra.

We thus arrive at the definition of *filter pair*, Def. 2.1: A filter pair for the signature  $\Sigma$  is a contravariant functor  $G$  from  $\Sigma$ -algebras to algebraic lattices together with a natural transformation  $i: G \rightarrow \mathcal{P}(-)$  from  $G$  to the functor taking an algebra to the power set of its underlying set, which preserves arbitrary infima and directed suprema.

The logic associated to a filter pair  $(G, i)$  is simply the logic associated (in the above fashion) to the algebraic lattice given by the image  $i(G(Fm_{\Sigma}(X))) \subseteq \mathcal{P}(Fm_{\Sigma}(X))$ .

In particular, it is clear that different filter pairs can give rise to the same logic, indeed this will happen precisely if the images of  $i$  for the formula algebra are the same. A filter pair can thus be seen as a *presentation* of a logic, and there can of course be different presentations of the same logic. We could have removed a bit of this ambiguity by requiring that  $i$  be an inclusion, but it is one of the insights of this article that it is beneficial not to do this. Indeed this will give us greater flexibility for the choice of the functor  $G$ , and injectivity of  $i$  can become a meaningful extra feature. Thus, for example, if  $G$  is the functor associating to a  $\Sigma$ -structure the lattice of congruences relative to some

quasivariety  $\mathbf{K}$ , i.e. when we have a so-called *congruence filter pair*, then by Prop. 3.6 the injectivity of  $i$  means that the associated logic is algebraizable.

In this article we show how to recognize classes of logics, like algebraizable or truth-equational logics, through their presentations by filter pairs and how these presentations may permit to use algebraic methods even outside the realm of protoalgebraic logics; see the following overview for a sample of results.

Another theme that we take up is the theme of adjunctions arising in the context of filter pairs. It follows from the definition that the maps  $i$  in a filter pair each have a left adjoint  $\mathcal{E}$  (i.e. are part of a Galois connection). The closure operator obtained as the composition  $i \circ \mathcal{E}$  is exactly the closure operator generating the theories of the logic. For congruence filter pairs we also have another operator back from theories to congruences, namely the Leibniz operator  $\Omega$ . The interplay of the three operators  $i, \mathcal{E}, \Omega$  gives an interesting picture that one can observe for any logic with an algebraic semantics.

### Overview of the article:

In Section 1 we fix terminology and gather some results characterizing different classes of logics, like algebraizable, truth-equational or protoalgebraic logics. No original results are contained in this section.

In Section 2 we introduce the notion of filter pair (Def. 2.1) and show how to get a logic from a filter pair (Prop. 2.3). We show some additional structures that one obtains from a filter pair, for example a left adjoint to each map  $i_A$  ( $A \in \Sigma\text{-Str}$ ) (Def. 2.5) whose composition with  $i$  yields the closure operator of the logic (Prop. 2.6) and a consequence relation on each  $A \in \Sigma\text{-Str}$  (Prop. 2.3). We establish the basic relations between these structures, e.g. that homomorphisms of  $\Sigma$ -structures induce translations between these consequence relations, which are conservative in the case of free algebras (Prop. 2.12), that subsets in the image of  $i$  are filters for the logic, and that in the case of free algebras all filters are in the image of  $i$  (Prop. 2.13).

In Section 3 we consider the special case of filter pairs where the functor  $G$  is given by congruences relative to some class of algebras. In subsection 3.1 we consider the question what the properties of  $G$  demanded in the definition of filter pair enforce on the class  $\mathbf{K}$ . We find that  $\mathbf{K}$  must be closed under subalgebras (Prop. 3.1) and products (Prop. 3.2). This subsection may be skipped and  $\mathbf{K}$  be assumed to be a quasivariety in the remainder of the article. In subsection 3.2 we introduce congruence filter pairs, and give a general construction of them, yielding the so-called equational filter pairs (Thm. 3.8). We give criteria when the associated logic is algebraizable (Prop. 3.6) or truth-equational (Cor. 3.10, Prop. 3.12).

In Section 4 we show and analyze some of the extra structure coming with an equational filter pair, for example give more concrete descriptions of the left adjoint of the  $i_A$  mentioned in the previous paragraph (Prop. 4.2, Prop. 4.3) and investigate its relation with the Leibniz operator (Prop. 4.7 and the following discussion). We conclude from these results that logics having an algebraic semantics are precisely those that admit a presentation by an equational filter pair (Thm. 4.4), and we get another criterion for the algebraizability of a logic (Thm. 4.7).

Finally, in Section 5 we discuss some of the other directions into which one can develop filter pairs.

## 1 Preliminaries

### 1.1 Ordered sets

DEFINITION 1.1 (Galois connections).

Let  $f: P \rightarrow Q, g: Q \rightarrow P$  be order preserving maps between partially ordered sets. Then  $(f, g)$  is a Galois connection, where  $f$  is called left adjoint to  $g$  and  $g$  is called right adjoint to  $f$ , if the following

#### 4 Congruence filter pairs, equational filter pairs and adjoints

equivalence is satisfied:

$$f(p) \leq q \Leftrightarrow p \leq g(q) \quad \forall p \in P, q \in Q.$$

A Galois connection (or adjunction) between posets,  $(f, g)$ , is uniquely determined by its left (respectively right) component, and induces an isomorphism between the subposets  $\hat{P} \subseteq P$  and  $\hat{Q} \subseteq Q$ , where  $\hat{P} := \text{Fix}(g \circ f)$  and  $\hat{Q} := \text{Fix}(f \circ g)$ .

THEOREM 1.2

[33, Thm. 3.6.9] Let  $g: Q \rightarrow P$  be a function between complete (and hence cocomplete) partially ordered sets that preserves arbitrary infima (resp. suprema). Then  $g$  has a left adjoint (resp. right adjoint)  $f: P \rightarrow Q$ , given by  $f(p) = \inf\{q \in Q : p \leq g(q)\}$  (resp.  $f(p) = \sup\{q \in Q : p \geq g(q)\}$ ).

We will be concerned with algebraic lattices.

DEFINITION 1.3 (algebraic lattices).

1. Let  $L$  be a lattice. A element  $a \in L$  is compact if for every (upward) directed subset  $\{d_i\}$  of  $L$  we have  $a \leq \bigvee_i d_i \Leftrightarrow \exists i (a \leq d_i)$ .  $L$  is said to be algebraic if it is a complete lattice such that every element is the join of the compact elements below it.
2. We denote by  $AL$  the category of all algebraic lattices and morphisms that preserve arbitrary meets and directed joins. In particular, these morphisms are monotonic functions that preserve the top elements of the lattices.

REMARK 1.4 (sublattices of algebraic lattices).

Let  $X$  be a set and denote  $(P(X), \subseteq)$  the (algebraic) lattice of all subsets of  $X$ . Note that a subset  $\mathcal{S} \subseteq P(X)$  that is closed under arbitrary intersections and directed unions determines an algebraic sublattice  $(\mathcal{S}, \subseteq) \hookrightarrow (P(X), \subseteq)$ , where the compact elements coincide with the finitely generated elements of  $(\mathcal{S}, \subseteq)$ .

##### 1.2 Structures, logics, filters and matrices

DEFINITION 1.5 (signatures, structures, formulas).

A signature is a sequence of pairwise disjoint sets  $\Sigma = (\Sigma_n)_{n \in \mathbb{N}}$ , where  $\Sigma_n$  is thought of as a set of  $n$ -ary function symbols. A  $\Sigma$ -structure is a set with an interpretation of the function symbols as actual operations. A homomorphism of  $\Sigma$ -structures is a map respecting these operations. We denote by  $\Sigma\text{-Str}$  the category of  $\Sigma$ -structures and homomorphisms.

For a set  $X$  denote by  $Fm_\Sigma(X)$  (or  $Fm(X)$ , if the signature is clear), the absolutely free  $\Sigma$ -structure, i.e. the set of  $\Sigma$ -formulas with variables from  $X$ . A homomorphism  $Fm(X) \rightarrow A$  with domain  $Fm(X)$  is also called a valuation. An endomorphism of  $Fm_\Sigma(X)$  is also called a substitution.

In this work there will be naturally occurring consequence relations on non-free  $\Sigma$ -structures, hence we phrase the definition of logic slightly more generally than customary.

DEFINITION 1.6 (consequence relation, abstract logic, logic).

A Tarskian consequence relation on a  $\Sigma$ -structure  $A$  is a relation  $\vdash \subseteq \mathcal{P}(A) \times A$ , such that, for all sets  $\Gamma, \Delta \subseteq A$  and all elements  $\varphi, \psi \in A$  the following conditions are satisfied:

- **Reflexivity** :If  $\varphi \in \Gamma$ ,  $\Gamma \vdash \varphi$

- **Cut** :If  $\Gamma \vdash \varphi$  and for every  $\psi \in \Gamma$ ,  $\Delta \vdash \psi$ , then  $\Delta \vdash \varphi$
- **Monotonicity** :If  $\Gamma \subseteq \Delta$  and  $\Gamma \vdash \varphi$ , then  $\Delta \vdash \varphi$
- **Finitarity** :If  $\Gamma \vdash \varphi$ , then there is a finite subset  $\Delta$  of  $\Gamma$  such that  $\Delta \vdash \varphi$ .
- **Structurality** :If  $\Gamma \vdash \varphi$  and  $\sigma$  is an endomorphism of  $A$ , then  $\sigma[\Gamma] \vdash \sigma(\varphi)$

An abstract logic (in the sense of [9]) is a pair  $(A, \vdash)$ , where  $A$  is a  $\Sigma$ -structure and  $\vdash$  is a Tarskian consequence relation on  $A$ .

A logic is a pair  $(\Sigma, \vdash)$  where  $\Sigma$  is a signature and  $\vdash$  is a Tarskian consequence relation on  $Fm_\Sigma(X)$ .

A consequence determines, and is determined by, a closure operator on the power set  $\mathcal{P}(A)$ , and we will switch freely between the two.

DEFINITION 1.7 (Filters, matrices).

Let  $l = (\Sigma, \vdash)$  be a logic.

1. Let  $A \in \Sigma\text{-Str}$ . A subset  $F \subseteq A$  is an  $l$ -filter if for every  $\Gamma \cup \{\varphi\} \subseteq Fm_\Sigma(X)$  such that  $\Gamma \vdash \varphi$  and every valuation  $v : Fm(X) \rightarrow A$ , if  $v[\Gamma] \subseteq F$  then  $v(\varphi) \in F$ .
2. In particular, since  $l$  is structural, when  $A = Fm(X)$ , then the  $l$ -filters in  $Fm(X)$  coincide with the theories of  $l$ , i.e. the  $\vdash$ -closed subsets of  $Fm(X)$ .
3. Let  $A \in \Sigma\text{-Str}$  and  $F \subseteq A$  be an  $l$ -filter. Then the pair  $(M, F)$  is then said to be a matrix model of  $l$ , or shortly, an  $l$ -matrix.

REMARK 1.8 (Functoriality of filters).

Let  $l = (\Sigma, \vdash)$  be a logic and  $A$  be a  $\Sigma$ -structure. Denote by  $(Fi_l(A), \subseteq) \xrightarrow{i_A} (P(A), \subseteq)$  the (sub)poset of all  $l$ -filters of  $A$ . Clearly,  $Fi_l(A)$  is closed under arbitrary intersections and, since  $l$  is a finitary logic,  $Fi_l(A)$  is closed under directed unions. Thus,  $(Fi_l(A), \subseteq)$  is an algebraic lattice where the compact elements are the finitely generated filters. Further, for a homomorphism of  $\Sigma$ -algebras  $f : A \rightarrow A'$  and a filter  $F' \in Fi_l(A')$ , the preimage  $f^{-1}(F')$  is a filter in  $Fi_l(A)$ . Defining  $Fi_l(f) := f^{-1}$  (inverse image), then  $Fi_l$  becomes a contravariant functor from the category  $\Sigma\text{-Str}$  to the category  $AL$ .

### 1.3 Congruences and equational consequence

A class  $\mathbf{K} \subseteq \Sigma\text{-Str}$  is a quasivariety if it is axiomatizable by quasi-identities in the signature  $\Sigma$ , i.e. formulas of the form

$$(p_1 \approx q_1 \wedge \dots \wedge p_n \approx q_n) \rightarrow p \approx q.$$

Quasivarieties are precisely the classes of  $\Sigma$ -structures that are closed under isomorphisms, subalgebras, products and directed colimits.

A  $\mathbf{K}$ -congruence on a  $\Sigma$ -structure is a  $\Sigma$ -congruence  $\theta \subseteq A \times A$  such that  $A/\theta \in \mathbf{K}$ .

REMARK 1.9 (functoriality of  $\mathbf{K}$ -congruences).

Let  $\mathbf{K}$  be a quasivariety. For a  $\Sigma$ -structure  $A$  denote by  $(Co_{\mathbf{K}}(A), \subseteq) \xrightarrow{i_A} (P(A \times A), \subseteq)$  the (sub)poset of all  $\mathbf{K}$ -congruences. As  $\mathbf{K}$  is closed under isomorphisms, subalgebras and products,  $Co_{\mathbf{K}}(A) \subseteq P(A \times A)$  is closed under arbitrary intersections and, since  $\mathbf{K}$  is closed under directed colimits,  $Co_{\mathbf{K}}(A) \subseteq P(A \times A)$  is closed under directed unions. Thus,  $(Co_{\mathbf{K}}(A), \subseteq)$  is an algebraic lattice where the compact elements are its finitely generated members. If  $f : A \rightarrow A'$  is a homomorphism and  $\theta' \in Co_{\mathbf{K}}(A')$  then, since  $\mathbf{K}$  is closed under isomorphisms and subalgebras,  $(f \times f)^{-1}(\theta') \in Co_{\mathbf{K}}(A)$ .

## 6 Congruence filter pairs, equational filter pairs and adjoints

Defining  $Co_{\mathbf{K}}(f) := (f \times f)^{-1}$ , then  $Co_{\mathbf{K}}$  becomes a contravariant functor from the category  $\Sigma\text{-Str}$  to the category  $AL$ .

Let  $\Sigma$  be a signature and  $\mathbf{K} \subseteq \Sigma\text{-Str}$  a class of algebras. Let  $Eq := Fm(X)_{\{x\}} \times Fm(X)_{\{x\}}$  be the set of pairs of formulas in at most the variable  $x$ . We think of such a pair of formulas  $(\delta, \epsilon)$  as an equation between the terms of both sides and write, following Blok-Pigozzi,  $\delta(x) \approx \epsilon(x)$ .

DEFINITION 1.10 (equational consequence).

Given a class of algebras  $\mathbf{K}$  over the signature  $\Sigma$ , and a set of variables  $X$ , the associated equational consequence is the relation  $\models_{\mathbf{K}}^X$  between a set of equations  $\Gamma$  and a single equation  $\varphi \approx \psi$  over  $\Sigma$  defined by:

$$\Gamma \models_{\mathbf{K}}^X \varphi \approx \psi \text{ iff for every } A \in \mathbf{K} \text{ and every } \Sigma\text{-homomorphism } h : Fm(X) \rightarrow A, \\ \text{if } h(\eta) = h(v) \text{ for all } \eta \approx v \in \Gamma, \text{ then } h(\varphi) = h(\psi).$$

In the context of amalgamation we will consider equational consequence for several sets of variables simultaneously. The following is the basic relation that we will need.

LEMMA 1.11

Let  $\mathbf{K}$  be a class of algebras over the signature  $\Sigma$ ,  $X \subseteq Y$  sets of variables, and  $\Gamma \cup \{\varphi \approx \psi\}$  sets of equations with variables in  $X$ . Then,

$$\Gamma \models_{\mathbf{K}}^Y \varphi \approx \psi \Leftrightarrow \Gamma \models_{\mathbf{K}}^X \varphi \approx \psi.$$

PROOF.  $\Leftarrow$ : Suppose  $\Gamma \models_{\mathbf{K}}^X \varphi \approx \psi$ . Let  $A \in \mathbf{K}$  and  $h : Fm(Y) \rightarrow A$  be a  $\Sigma$ -homomorphism such that  $h(\eta) = h(v)$  for all  $\eta \approx v \in \Gamma$ . We need to show that  $h(\varphi) = h(\psi)$ . For this, just precompose with the inclusion  $i : Fm(X) \hookrightarrow Fm(Y)$ . As  $\Gamma \cup \{\varphi \approx \psi\} \subseteq Fm(X)$ , we have  $(h \circ i)(\eta) = (h \circ i)(v)$  for all  $\eta \approx v \in \Gamma$ . From the assumption  $\Gamma \models_{\mathbf{K}}^X \varphi \approx \psi$  it follows that  $h(\varphi) = (h \circ i)(\varphi) = (h \circ i)(\psi) = h(\psi)$ .

$\Rightarrow$ : Suppose  $\Gamma \models_{\mathbf{K}}^Y \varphi \approx \psi$ . Let  $A \in \mathbf{K}$  and  $h : Fm(X) \rightarrow A$  be a  $\Sigma$ -homomorphism such that  $h(\eta) = h(v)$  for all  $\eta \approx v \in \Gamma$ . We need to show that  $h(\varphi) = h(\psi)$ .

Choose a left inverse  $Y \rightarrow X$  to the inclusion  $X \subseteq Y$ . These two maps induce homomorphisms  $i : Fm(X) \hookrightarrow Fm(Y)$  and  $r : Fm(Y) \rightarrow Fm(X)$  such that  $r \circ i = id_{Fm(X)}$ . The composition  $h \circ r : Fm(Y) \rightarrow Fm(X) \rightarrow A$  is a homomorphism such that  $(h \circ r)(\eta) = (h \circ r)(v)$  for all  $\eta \approx v \in \Gamma$ . As  $\Gamma \models_{\mathbf{K}}^Y \varphi \approx \psi$ , we have that  $(h \circ r)(\varphi) = (h \circ r)(\psi)$ . Precomposing with  $i$  yields  $h(\varphi) = (h \circ r \circ i)(\varphi) = (h \circ r \circ i)(\psi) = h(\psi)$ .  $\square$

It is shown in chapter 2 of [12] that  $\models_{\mathbf{K}}^X$  satisfies the axioms of a structural consequence relation, which is finitary if and only if  $\models_{\mathbf{K}}^X = \models_{\mathbf{K}^Q}^X$ , where  $\mathbf{K}^Q$  denotes the quasivariety generated by  $\mathbf{K}$ . As usual, the consequence relation  $\models_{\mathbf{K}}^X$  corresponds to a closure operator  $Cn_{\mathbf{K}}^X$  on  $\mathcal{P}(Eq)$ . A set of equations  $\Gamma \subseteq Eq$  is called a  $\mathbf{K}$ -theory if it is closed under  $\mathbf{K}$ -consequence, i.e. if  $Cn_{\mathbf{K}}^X(\Gamma) = \Gamma$ .

For the following Lemma keep in mind that for a set  $\Gamma \subseteq Eq$  we have two viewpoints: we can see  $\Gamma$  as a set of equations, or as a set of pairs of formulas. From the latter view point it makes sense to ask whether a set is a congruence relation or not. We suppose all the formulas to be in  $Fm(X)$  for some set of variables  $X$ .

LEMMA 1.12

For a set of equations  $\Gamma = \{(\gamma_1^i, \gamma_2^i) : i \in I\} \subseteq Eq$  its closure  $Cn_{\mathbf{K}}^X(\Gamma)$  is the congruence relation generated by  $\Gamma$ . In particular, a set  $\Gamma \subseteq Eq$  is a  $\mathbf{K}$ -theory if and only if it is a congruence relation.

PROOF. Denote the  $\mathbf{K}$ -consequence relation generated by  $\Gamma$  by  $\hat{\Gamma}$ .

$Cn_{\mathbf{K}}^X(\Gamma) \supseteq \hat{\Gamma}$ : Let  $(\varphi(x), \psi(x)) \in \hat{\Gamma}$ . Note that  $\hat{\Gamma}$  is the smallest congruence  $\theta$  on  $Fm(X)$  such that  $\gamma_1^i = \gamma_2^i$  in  $Fm(X)/\theta$ . Thus, every quotient  $Fm(X)/\theta$  of  $Fm(X)$  in which  $\gamma_1^i(x) = \gamma_2^i(x)$  ( $i \in I$ ) holds will be itself a quotient of  $Fm(X)/\hat{\Gamma}$ .

Now suppose that for some algebra  $A \in \mathbf{K}$  and some  $a \in A$  we have  $(\gamma_1^i)^A(a) = (\gamma_2^i)^A(a)$ . Then there is a homomorphism  $h: Fm(X) \rightarrow A$  sending  $x$  to  $a$  and the other variables to arbitrary elements. The image  $Im h$  of this homomorphism is isomorphic to a quotient of  $Fm(X)$  and a subalgebra of  $A$  in which  $(\gamma_1^i)^A(a) = (\gamma_2^i)^A(a)$  holds. Thus,  $Im h$  is also isomorphic to a quotient of  $Fm(X)/\hat{\Gamma}$ , and since  $(\varphi(x), \psi(x)) \in \hat{\Gamma}$ , we have  $\varphi^A(a) = \psi^A(a)$  in  $Im h \subseteq A$ . This shows  $(\varphi(x), \psi(x)) \in Cn_{\mathbf{K}}(\Gamma)$ .

$Cn_{\mathbf{K}}^X(\Gamma) \subseteq \hat{\Gamma}$ : Let  $(\varphi(x), \psi(x)) \in Cn_{\mathbf{K}}^X(\Gamma)$ . Then since in  $Fm(X)/\hat{\Gamma}$  we have  $\gamma_1^i(x) = \gamma_2^i(x)$  ( $i \in I$ ), we also have  $\varphi(x) = \psi(x)$ . Therefore,  $(\varphi(x), \psi(x)) \in \hat{\Gamma}$ .  $\square$

#### 1.4 Algebraicity of logics

We summarize several ways in which a logic can be tied to, and its properties determined by, algebra.

DEFINITION 1.13 (algebraic semantics).

A logic  $l = (\Sigma, \vdash_l)$  has an algebraic semantics in a class of algebras  $\mathbf{K}$  given by a set of equations  $\delta(x) \approx \epsilon(x)$  if the following equivalence holds:

$$\Gamma \vdash_l \varphi \Leftrightarrow \{\delta(\gamma) \approx \epsilon(\gamma) : \gamma \in \Gamma\} \vdash_{\mathbf{K}} \delta(\varphi) \approx \epsilon(\varphi)$$

In the following ‘‘algebraizable logic’’ will mean ‘‘algebraizable’’ in the sense of Blok-Pigozzi [12].

DEFINITION 1.14 (algebraizable).

A logic  $l = (\Sigma, \vdash_l)$  is algebraizable with equivalent algebraic semantics  $\mathbf{K}$  if

- (1)  $l$  has an algebraic semantics in  $\mathbf{K}$  given by a set of equations  $\delta(x) \approx \epsilon(x)$
- (2) there is a finite set  $\Delta_j(p, q), j = 1, \dots, m$  of formulas in two variables such that for every set of equations  $\Gamma$  and for every equation  $\varphi \approx \psi$ , we have that

$$\Gamma \vdash_{\mathbf{K}} \varphi \approx \psi \Leftrightarrow \{\xi \Delta \eta : \xi \approx \eta \in \Gamma\} \vdash_l \varphi \Delta \psi.$$

- (3) For every  $\psi \in Fm(X)$  we have that

$$\psi \dashv\vdash_l \Delta(\tau(\psi)).$$

DEFINITION 1.15 (compatible congruence, Leibniz operator, reduced matrix).

Let  $\Sigma$  be a signature,  $A$  be a  $\Sigma$ -algebra and  $F \subseteq A$ .

(a) Let  $\theta$  be a congruence in  $A$ .  $\theta$  is said to be compatible with  $F$  if, for all  $a, b \in A$ , if  $a \in F$  and  $(a, b) \in \theta$  then  $b \in F$ . Given an algebra  $A$  and a subset  $F$  of its domain there always exists a greatest congruence of  $A$  compatible with  $F$  (Theorem 1.5, [12]), denoted by  $\Omega^A(F)$ . The function  $\Omega^A$  with domain the set of all subsets of  $A$  is called the Leibniz operator on  $A$ .

(b) A matrix  $(A, F)$  for a logic  $l = (\Sigma, \vdash)$  is reduced when  $\Omega^A(F) = \text{Diag}(A) = \{(a, b) \in A \times A : a = b\}$ . We will denote  $\text{Matr}_l^*$  the class of all reduced matrices for  $l$ ;  $\text{Alg}_l^*$  stands for the class of all  $\Sigma$ -algebras underlying to some reduced matrix.

## 8 Congruence filter pairs, equational filter pairs and adjoints

## THEOREM 1.16

(The Isomorphism Theorem, first version [12, Thm 5.1]).

Let  $l$  be a logic and  $\mathbf{K}$  a quasivariety. The following are equivalent.

1.  $l$  is algebraizable with equivalent semantics  $\mathbf{K}$
2. For every algebra  $A$  the Leibniz operator  $\Omega^A : Fi_l(A) \rightarrow Co_{\mathbf{K}}(A)$  is an isomorphism of algebraic lattices.

Moreover, the quasivariety  $\mathbf{K}$  is unique for providing an equivalent semantics by [12, Thm 2.15].

## THEOREM 1.17

(The Isomorphism Theorem, 2nd version [21, Thm. 3.58]).

Let  $l$  be a logic and  $\mathbf{K}$  be a quasivariety. The following conditions are equivalent:

1.  $l$  is algebraizable with equivalent algebraic semantics the class  $\mathbf{K}$ .
2. For every algebra  $A$  there is an isomorphism  $\Phi^A$  between the algebraic lattices  $Fi_l(A)$  and  $Co_{\mathbf{K}}(A)$  that commutes with endomorphisms, i.e. for every  $F \in Fi_l(A)$  and every  $h \in End(A)$ ,  $\Phi^A h^{-1}(F) = h^{-1} \Phi^A F$ .
3. There is an isomorphism of algebraic lattices  $\Phi^{Fm} : Th(l) \rightarrow Co_{\mathbf{K}}(Fm)$  that commutes with substitutions, i.e. for every  $T \in Th(l)$  and every  $\sigma \in End(Fm)$ ,  $\Phi \sigma^{-1} T = \sigma^{-1} \Phi T$ .

If the above conditions are satisfied, the isomorphism of (iii) is unique and given by the Leibniz operator  $\Omega$ . It can be expressed by via set  $\Delta(p, q)$  of formulas in two variables as  $\Omega(F) = \{(\varphi, \psi) : \Delta(\varphi, \psi) \subseteq F\}$  and its inverse can be expressed via a set of equations  $\delta(x) = \epsilon(x)$  as  $\Omega^{-1}(\theta) = \{\varphi \in Fm : (\delta(x), \epsilon(x)) \subseteq \theta\}$ .

For a more general algebra  $A$ , the same set of formulas, resp. set of equations, induce isomorphisms and there is at most one way to express an isomorphism by equations in this way [21, Exercise 3.40].

The following corollary was the main motivation to define the congruence filter pairs in order to establish conditions to get algebraizable logics knowing previously a quasivariety of some similarity type  $\Sigma$ .

## COROLLARY 1.18

(The Isomorphism Theorem, 3rd version)

Let  $l$  be a logic and  $\mathbf{K}$  be a quasivariety. Then following conditions are equivalent:

1.  $l$  is an algebraizable logic with equivalent algebraic semantics the class  $\mathbf{K}$ .
2. There is a natural isomorphism between the functors  $Fi_l$  and  $Co_{\mathbf{K}}$ .

PROOF. "1  $\Rightarrow$  2" Suppose that  $l$  is an algebraizable logic. By theorem 1.16 we have that for every  $A \in \Sigma\text{-Str}$ , the Leibniz operator  $\Omega^A : Fi_l(A) \rightarrow Co_{\mathbf{K}}(A)$  is an isomorphism. Let  $\Omega = (\Omega^A)_{A \in \Sigma\text{-Str}}$ . We prove that  $\Omega$  is a natural transformation. In order to do that, it is enough to prove that given  $f \in Hom_{\Sigma\text{-Str}}(A, B)$ , the following diagram commutes:

$$\begin{array}{ccc} Fi_l(A) & \xrightarrow{\Omega^A} & Co_{\mathbf{K}}(A) \\ f^{-1} \uparrow & & \uparrow (f \times f)^{-1} \\ Fi_l(B) & \xrightarrow{\Omega^B} & Co_{\mathbf{K}}(B) \end{array}$$

Let  $F \in Fi_l(B)$ . Firstly, we prove that  $f^{-1}(\Omega^B(F))$  is compatible with  $f^{-1}(F)$ . Let  $(a, b) \in f^{-1}(\Omega^B(F))$  and suppose that  $a \in f^{-1}(F)$ . Then  $(f(a), f(b)) \in \Omega^B(F)$  and  $f(a) \in F$ . Therefore,  $f(b) \in F$ , thus  $b \in f^{-1}(F)$ . Thus,  $(a, b) \in \Omega^A((f \times f)^{-1}(F))$ .

Now let  $(a, b) \in \Omega^A(f^{-1}(F))$ , then by algebraizability of  $l$ , we have  $\Delta^A(a, b) \in f^{-1}(F)$ . Thus,  $\Delta^B(f(a), f(b)) = f(\Delta^A(a, b)) \subseteq F$ . Therefore,  $(f(a), f(b)) \in \Omega^B(F)$  and finally  $(a, b) \in f^{-1}(\Omega^B(F))$ . So  $\Omega^A(f^{-1}(F)) = f^{-1}(\Omega^B(F))$ , which means that  $\Omega$  is natural.

"2  $\Rightarrow$  1" Suppose that there is a natural isomorphism  $\Phi : Fi_l \rightarrow Co_{\mathbf{K}}$ . In particular, we have that for every  $A \in \Sigma\text{-Str}$ ,  $\Phi^A : Fi_l(A) \rightarrow Co_{\mathbf{K}}(A)$  is an isomorphism and commutes with endomorphisms. By theorem 1.17 we have that  $l$  is an algebraizable logic.  $\square$

We consider now some special kinds of logics that are generalizations for algebraizable logics (see [21]).

DEFINITION 1.19 (protoalgebraic, truth-equational).

Let  $l = (\Sigma, \vdash)$  a logic:

- $l$  is a protoalgebraic logic if for any theory  $T \in Th(l)$ ,

$$\text{if } (\varphi, \psi) \in \Omega(T) \text{ then } T, \varphi \vdash \psi \text{ and } T, \psi \vdash \varphi.$$

- A class of matrices  $M$  has its filters equationally definable by a set of equations  $\tau(p)$  if for every matrix  $(A, F) \in M$ , for every  $a \in A$ ,

$$a \in F \text{ iff } \delta^A(a) = \varepsilon^A(a), \text{ for every } \delta \approx \varepsilon \in \tau(p).$$

$l$  is a truth-equational logic if the class of reduced matrices  $Matr_l^*$  has its filters equationally definable.

- Let  $K$  be a pointed class of algebras of a similarity type  $\Sigma$ .  $l$  is the assertional logic of  $K$  if,

$$\Gamma \vdash \varphi \Leftrightarrow \{\gamma \approx T \mid \gamma \in \Gamma\} \models_K \varphi \approx T$$

Now we recall some characterizations of the classes of logics defined above.

THEOREM 1.20

Let  $l$  be a logic:

- $l$  is protoalgebraic iff  $\Omega$  is monotone on set of theories  $Th(l)$ .
- $l$  is equivalential iff  $(\Omega^A)_{A \in \Sigma\text{-Str}}$  commutes with homomorphism (categorical naturality) and  $\Omega$  is monotone.
- $l$  is truth-equational iff there exists a set of equations  $\tau(p)$  such that for every algebra  $A$  and every  $F \in Fi_l(A)$ ,

$$F = \{a \in A; \tau^A(a) \subseteq \Omega^A(F)\}$$

The following diagram summarizes the relations between the different classes of logics mentioned in this section.

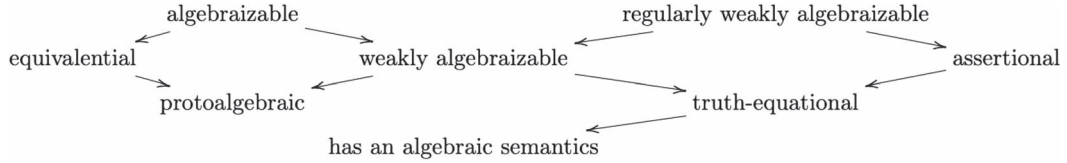
## 2 Finitary filter pairs

To motivate the notion of filter pair, we recall from Remark 1.8 that, given a logic  $l = (\Sigma, \vdash)$ , one can associate to each  $\Sigma$ -structure  $M$  its collection of  $l$ -filters  $Fi_l(M)$  and that, together with taking

## 10 Congruence filter pairs, equational filter pairs and adjoints

inverse image, this constitutes a functor  $Fi_l: \Sigma\text{-Str} \rightarrow AL$ , from  $\Sigma$ -structures to algebraic lattices. Also recall from Remark 1.8 that for  $M \in \Sigma\text{-Str}$  the inclusion  $i_M: (Fi_l(M), \subseteq) \hookrightarrow (\mathcal{P}(M), \subseteq)$  preserves arbitrary infima and directed suprema, i.e. it is a morphism in the category  $AL$  (in particular preserves order).

Moreover, given a morphism  $h: M \rightarrow N$  we have the following commutative diagram:



This collection of data is the motivating example for the notion, and the name, of filter pair.

### DEFINITION 2.1

Let  $\Sigma$  be a signature. A **filter pair** over  $\Sigma$  is a pair  $(G, i)$ , consisting of a contravariant functor  $G: \Sigma\text{-Str} \rightarrow AL$  and a collection of maps  $i = (i_M)_{M \in \Sigma\text{-Str}}$  such that for any  $M \in \Sigma\text{-Str}$  the function  $i_M: G(M) \rightarrow (\mathcal{P}(M), \subseteq)$  satisfies the following properties:

1. For any  $M \in \Sigma\text{-Str}$ ,  $i_M$  preserves arbitrary infima (in particular  $i_M(\top) = M$ ) and directed suprema.

2. Given a morphism  $h: M \rightarrow N$  the following diagram commutes:

$$\begin{array}{ccccc}
 M & & Fi_l(M) & \xrightarrow{i_M} & (\mathcal{P}(M), \subseteq) \\
 h \downarrow & & h^{-1} \uparrow & & \uparrow h^{-1} \\
 N & & Fi_l(N) & \xrightarrow{i_N} & (\mathcal{P}(N), \subseteq)
 \end{array}$$

### REMARK 2.2

Condition 2. says that  $i$  is a natural transformation from  $G$  to the functor  $\mathcal{P}: \Sigma\text{-Str}^{op} \rightarrow AL$  sending a  $\Sigma$ -structure to the power set of its underlying set and a homomorphism of  $\Sigma$ -structures to its associated inverse image function.

From a filter pair we obtain a consequence relation on every  $\Sigma$ -structure, i.e. an abstract logic in the sense of Definition 1.6.

### PROPOSITION 2.3

Let  $(G, i)$  be a filter pair over the signature  $\Sigma$  and  $A$  a  $\Sigma$ -structure. Then there is a generalized logic  $\mathcal{I}_G^A = (A, \vdash_G^A)$ , defined as follows: Given  $\Gamma \cup \{\varphi\} \subseteq A$ , define

$$\Gamma \vdash_G \varphi \text{ iff for any } a \in G(Fm(X)), \text{ if } \Gamma \subseteq i_{Fm(X)}(a) \text{ then } \varphi \in i_{Fm(X)}(a).$$

PROOF. It is easy to see that  $\vdash_G^A$  satisfies reflexivity, cut and monotonicity.

The structurality is a consequence of condition 2 (naturality). Indeed, let  $\sigma \in \text{hom}(Fm(X), Fm(X))$  and  $\Gamma \cup \{\varphi\} \subseteq Fm(X)$  such that  $\Gamma \vdash_G \varphi$ . Consider  $a \in G(Fm(X))$  such that  $\sigma[\Gamma] \subseteq i_{Fm(X)}^G(a)$ . This implies  $\Gamma \subseteq \sigma^{-1}(i_{Fm(X)}^G(a))$ . By naturality we have  $\sigma^{-1}(i_{Fm(X)}^G(a)) = i_{Fm(X)}^G(G(\sigma)(a))$ . Therefore,  $\varphi \in i_{Fm(X)}^G(G(\sigma)(a)) = \sigma^{-1}(i_{Fm(X)}^G(a))$  and finally  $\sigma(\varphi) \in i_{Fm(X)}^G(a)$ .

Now we are going to prove the finitariness. Let  $\Gamma \cup \{\varphi\} \subseteq Fm(X)$ . Consider the set  $S = \{\Gamma' \subseteq Fm(X) : \Gamma' \subseteq_{fin} \Gamma\}$ . Notice that  $S$  is a directed set. Suppose that for any  $\Gamma' \in S$ ,  $\Gamma' \not\vdash_G \varphi$ , hence there is  $a \in G(Fm(X))$  such that  $\Gamma' \subseteq i_{Fm(X)}(a)$  and  $\varphi \notin i_{Fm(X)}(a)$ . Denote by  $a_{\Gamma'} = \bigwedge \{a \in G(Fm(X)) : \Gamma' \subseteq i_{Fm(X)}(a)\}$ .  $i_{Fm(X)}$  preserves inf, thus  $\Gamma' \subseteq i_{Fm(X)}(a_{\Gamma'})$  and  $\varphi \notin i_{Fm(X)}(a_{\Gamma'})$ . We obtain that the set  $s = \{a_{\Gamma'} : \Gamma' \in S\}$  is a directed set.

By the assumption  $i_{Fm(X)}$  preserves directed suprema, hence

$$\Gamma = \bigcup S \subseteq \bigcup_{\Gamma' \in S} i_{Fm(X)}(a_{\Gamma'}) = i_{Fm(X)}(\bigvee s).$$

On the other hand,  $\varphi \notin \bigcup_{\Gamma' \in S} i_{Fm(X)}(a_{\Gamma'}) = i_{Fm(X)}(\bigvee s)$ . Therefore,  $\Gamma \not\vdash_G \varphi$ . □

REMARK 2.4

The main instance of interest of the construction of Prop. 2.3 is the case  $A = Fm_{\Sigma}(X)$ , the formula algebra over some set  $X$  of variables. In this case one obtains a logic in the usual sense of the word, which we will denote by  $(\Sigma, \vdash^X)$ . Thus, one can see a filter pair (together with a set  $X$  of variables) as a *presentation* of a logic, different in style from the usual presentations by axioms and rules or by matrices. It is clear from the definition that the logic defined in this way does not depend on the values of the filter pair at  $\Sigma$ -structures other than  $Fm_{\Sigma}(X)$ , and indeed it only depends on *the image of the map*  $G(Fm_{\Sigma}(X)) \rightarrow \mathcal{P}(Fm_{\Sigma}(X))$ , as this is exactly the collection of theories of the logic, by definition.

Thus, just as an algebraic structure can have many different presentations by generators and relations, a logic can have presentations by different filter pairs, each of which can be useful for different purposes.

EXAMPLE 1

Given a Tarskian logic  $l = (\Sigma, \vdash)$ , by Remark 1.8, defining  $Fi_l(A)$  to be the set of  $l$ -filters on a  $\Sigma$ -structure  $A$  provides a functor  $Fi_l : \Sigma\text{-Str}^{op} \rightarrow AL$ , and hence a filter pair  $(Fi_l, i)$  where  $i$  is the inclusion of filters into all subsets. As filters on the formula algebra are exactly the theories, this shows that every logic admits a presentation by a filter pair.

There is a direct description of the closure operator associated to the consequence relation of Prop. 2.3 through an application of the notion of adjunctions between posets. Since the natural transformation  $i$  of a filter pair  $(G, i)$  preserves infima, by Theorem 1.2 for each algebra  $A$ , the map  $i_A$  has a (unique) left adjoint.

DEFINITION 2.5

Let  $(G, i)$  be a filter pair. The left adjoint of  $i_A$  will be denoted by  $\mathcal{E}(i)_A$ .

Note that in general there is no reason that the collection of left adjoints  $\mathcal{E}(i)_A$  ( $A \in \Sigma\text{-Str}$ ) should form a natural transformation. Nevertheless, a resource that comes for free with a filter pair and can be usefully employed to understand the properties of the associated logics.

PROPOSITION 2.6

Let  $(G, i^G)$  be a filter pair and  $A$  a  $\Sigma$ -structure. Then the closure operator on  $A$  associated to the consequence relation  $\vdash_G^A$  on  $A$  of Prop. 2.3 is the one given by  $i_A \circ \mathcal{E}(i)_A$ .

## 12 Congruence filter pairs, equational filter pairs and adjoints

PROOF. Recall the definition of  $\vdash_G^A$ :

$$D \vdash_G^A a \text{ iff for any } z \in G(A), \text{ if } D \subseteq i_A(z) \text{ then } a \in i_A(z).$$

Consider the set  $Z_D := \{z \in G(A) : D \subseteq i_A(z)\}$ . We can rephrase the definition of  $\vdash_G^A$  by saying

$$D \vdash_G^A a \text{ iff } \forall z \in Z_D, a \in i_A(z)$$

Since  $G(A)$  is complete, there exists an infimum, say  $z'$ , of  $Z_D$ . Since  $i_A$  preserves arbitrary infima, we have  $i_A(z') = \bigcap_{z \in Z_D} i_A(z)$ . Since we have  $D \subseteq i_A(z) \forall z \in Z_D$ , we also have  $D \subseteq \bigcap_{z \in Z_D} i_A(z) = i_A(z')$ . Therefore,  $z' \in Z_D$ , i.e. it is the minimal  $z'$  for which  $i_A(z)$  contains  $D$ . Since  $i_A$  is order preserving, if  $a \in i_A(z')$ , then  $a \in i_A(z)$  for every  $z \in Z_D$  and hence  $D \vdash_G^A a$ . The opposite direction is also true, i.e. if  $D \vdash_G^A a$ , then  $a \in i_A(z')$ , just because  $z' \in Z_D$ .

Thus we can describe the consequence relation by  $D \vdash_G^A a$  iff  $a \in i_A(z')$ .

By the formula for left adjoints from Theorem 1.2, for a subset  $D \subseteq A$  we have  $\mathcal{E}(i)_A(D) = \bigwedge \{z \in G(A) : D \subseteq i_A(z)\} = z'$ . Altogether we obtain

$$D \vdash_G^A a \text{ iff } a \in i_A(\mathcal{E}(i)_A(D))$$

This means exactly that the closure operator  $i_A \circ \mathcal{E}(i)_A$  is the closure operator associated to the consequence relation  $\vdash_G^A$ .  $\square$

In the remainder of the section we will explore the relationship between filters of the logic  $(\Sigma, \vdash^X)$  associated to a filter pair  $(G, i)$ , and sets in the image of  $i_A$  for a  $\Sigma$ -structure  $A$ . We start by giving a name to the latter.

### DEFINITION 2.7

Let  $(G, i)$  be a filter pair and  $A$  a  $\Sigma$ -structure. An  $i$ -filter in  $A$  is a subset  $F \subseteq A$  such that for all  $D \subseteq A$  the following implication holds:

$$(D \subseteq F \text{ and } D \vdash_G^A a) \Rightarrow a \in F.$$

In other words,  $i$ -filters in  $A$  are the theories for the abstract logic  $(A, \vdash)$ .

### LEMMA 2.8

Let  $(G, i^G)$  be a filter pair and  $A$  a  $\Sigma$ -structure. The image of  $i_A$  in  $\mathcal{P}(A)$  consists exactly of the  $i$ -filters in  $A$ .

PROOF. The set  $i - Fi(A)$  of Filters in a  $\Sigma$ -algebra  $A$  is defined by the following condition:

$$F \in i - Fi(A) \text{ iff } [(D \subseteq F \text{ and } D \vdash_G^A a) \Rightarrow a \in F].$$

In view of the previous proposition this means

$$F \in i - Fi(A) \text{ iff } [(D \subseteq F \text{ and } a \in (i_A \circ \mathcal{E}(i)_A)(D) \Rightarrow a \in F].$$

It is enough to check the condition on the right hand side on the maximal  $D \subseteq F$ , i.e. on  $F$  itself, which leads to

$$F \in i - Fi(A) \text{ iff } [a \in (i_A \circ \mathcal{E}(i)_A)(F) \Rightarrow a \in F].$$

Now since  $i_A \circ \mathcal{E}(i)_A$  is a closure operator, this means exactly that  $F$  is closed for this operator, i.e.  $F = (i_A \circ \mathcal{E}(i)_A)(F)$ , hence  $F$  is in the image of  $i_A$ . Vice versa, it is clear from the properties of Galois adjunctions that sets in the image of  $i_A$  are closed.  $\square$

The notions of  $i$ -filter for a filter pair and filter for a logic associated to that filter pair are, in general, different. The content of the next proposition is that one notion subsumes the other. It can be shown in relevant examples that this inclusion is strict.

**PROPOSITION 2.9**

Let  $(G, i)$  be a filter pair,  $X$  a set and  $l = (\Sigma, \vdash^X)$  the associated logic with set of variables  $X$ . Then for any algebra  $A$  the subsets in the image of  $i_A$  are  $l$ -filters.

**PROOF.** Let  $A$  be an algebra,  $F = i_A(x) \subseteq A$  a subset in the image of  $i$ ,  $\Gamma \cup \{\varphi\} \subseteq Fm(X)$  formulas with  $\Gamma \vdash^X \varphi$  and  $v: Fm(X) \rightarrow A$  a homomorphism with  $v(\Gamma) \subseteq F$ . We need to show that  $\varphi \in F$ . For this consider the naturality square

$$\begin{array}{ccccc}
 M & & G(M) & \xrightarrow{i_M^G} & (\mathcal{P}(M); \subseteq) \\
 \downarrow h & & \uparrow G(h) & & \uparrow h^{-1} \\
 N & & G(N) & \xrightarrow{i_N^G} & (\mathcal{P}(N); \subseteq)
 \end{array}$$

We have  $\Gamma \subseteq v^{-1}(F) = v^{-1}(i_A(x)) = i_{Fm(X)}(G(v)(x))$ ,

So  $v^{-1}(F)$ , lying in the image of  $i_{Fm(X)}$ , is a theory containing  $\Gamma$ . The assumption  $\Gamma \vdash^X \varphi$  then implies  $\varphi \in v^{-1}(F)$ , i.e.  $v(\varphi) \in F$ .  $\square$

**PROPOSITION 2.10**

Given a filter pair  $(G, i)$  and a set  $X$ , the consequence relation  $\vdash$  on  $Fm(X)$  defined by

$$\begin{aligned}
 \Gamma \vdash \varphi & \text{ iff for any } \Sigma\text{-structure } M, \text{ for any } a \in G(M) \text{ and any valuation } v: Fm(X) \rightarrow M, \\
 & \text{ if } v(\Gamma) \subseteq i_M(a) \text{ then } v(\varphi) \in i_M(a)
 \end{aligned}$$

coincides with the consequence relation  $\vdash^X$ .

**PROOF.** Suppose  $\Gamma \vdash \varphi$ . Then, taking the identity as valuation, one has that if  $\Gamma \subseteq i_{Fm(X)}(a)$  then  $\varphi \in i_{Fm(X)}(a)$  for all  $a \in G(Fm(X))$ . Since the sets in the image of  $i_{Fm(X)}$  are precisely the theories of the logic  $l := (\Sigma, \vdash^X)$ , this shows  $\Gamma \vdash^X \varphi$ .

On the other hand suppose  $\Gamma \vdash^X \varphi$ . Then for any  $l$ -matrix  $(M, F)$  and valuation  $v: Fm(X) \rightarrow M$  one has that if  $v(\Gamma) \subseteq F$  then  $v(\varphi) \in F$ . Since by Proposition 2.9 all subsets in the image of  $i_M$  are filters, this implies that the defining condition for  $\Gamma \vdash \varphi$  is satisfied.  $\square$

**REMARK 2.11**

1. Call a matrix  $(A, F)$  an  $i$ -matrix if the filter  $F$  is in the image of  $i_A$ . Denoting the collection of all  $i$ -matrices by  $i\text{-Matr}$ , and the collection of all matrices for the logic  $l = (\Sigma, \vdash^X)$  by  $l\text{-Matr}$ , Proposition 2.9 can be subsumed as stating an inclusion  $i\text{-Matr} \subseteq l\text{-Matr}$ . Proposition 2.10 then says that, although this inclusion can be strict, the two classes of matrices always define the same logic.
2. The inclusions of  $i$ -filters into all filters, established by Proposition 2.9 are also easily seen to form a natural transformation. Thus we can consider the natural transformation  $i^G: G \Rightarrow Fi_l$ . This exhibits  $(Fi_l, i)$  as a weakly terminal filter pair among all filter pairs presenting the same

## 14 Congruence filter pairs, equational filter pairs and adjoints

logic  $l$ . The collection of filter pairs presenting a given fixed logic will be studied in a follow-up work.

The following two propositions establish that, although the notions of  $i$ -filter and  $l$ -filter (where  $l = (\Sigma, \vdash^X)$  is the logic with variables  $X$ ) may differ for general algebras, this is not the case for an absolutely free algebra  $Fm(Z)$  (with a set of generators  $Z$  possibly different from  $X$ ).

### PROPOSITION 2.12

Let  $(G, i)$  be a filter pair over the signature  $\Sigma$ .

- (i) For any homomorphism of  $\Sigma$ -structures  $f: A \rightarrow B$  and  $\Gamma \cup \{\varphi\} \subseteq A$ :

$$\Gamma \vdash^A \varphi \Rightarrow f[\Gamma] \vdash^B f(\varphi).$$

- (ii) For any injective map of sets  $f: X \hookrightarrow Y$  and  $\Gamma \cup \{\varphi\} \subseteq Fm(X)$ :

$$\Gamma \vdash^X \varphi \Leftrightarrow f[\Gamma] \vdash^Y f(\varphi).$$

PROOF. (i) Suppose  $\Gamma \vdash^A \varphi$ . Let  $z \in G(B)$  such that  $f[\Gamma] \subseteq i_B(z)$ . Then  $\Gamma \subseteq f^{-1}(i_B(z)) = i_A(G(f)(z))$ . Since  $\Gamma \vdash^A \varphi$ , we have that  $\varphi \in i_A(G(f)(z))$ . Therefore,  $f(\varphi) \in i_B(z)$ . As  $z$  was arbitrary we have  $f[\Gamma] \vdash^B f(\varphi)$ .

(ii) Let  $f: X \rightarrow Y$  be injective. By  $l$  we have that  $\Gamma \vdash^X \varphi \Rightarrow f[\Gamma] \vdash^Y f(\varphi)$ . It remains to prove the converse. Let  $z \in G(Fm(X))$  such that  $\Gamma \subseteq i_X^G(z)$ . Since  $f$  is injective there is a  $g: Y \rightarrow X$  such that  $g \circ f = Id_X$ . Hence,  $g \circ f[\Gamma] = \Gamma$ . Therefore,  $f[\Gamma] \subseteq g^{-1}(i_X(z)) = i_Y(G(g)(z))$ . Since  $f[\Gamma] \vdash^Y f(\varphi)$ , we have  $f(\varphi) \in i_Y(G(g)(z)) = g^{-1}(i_X^G(z))$ . Therefore,  $\varphi = g(f(\varphi)) \in i_X^G(z)$ .  $\square$

### PROPOSITION 2.13

Let  $(G, i)$  be a filter pair and  $l := (Fm(X), \vdash^X)$  the associated logic with an infinite set of variables  $X$ . Let  $Z$  be another set and  $F \subseteq Fm(Z)$  an  $l$ -filter. Then  $F = i_Z(\mathcal{E}(F))$ . In particular, all filters in absolutely free algebras lie in the image of  $i$ .

PROOF. From the general properties of adjunctions we know  $F \subseteq i_Z(\mathcal{E}(F))$ . It remains to show  $F \supseteq i_Z(\mathcal{E}(F))$ . So let  $\varphi \in i_Z(\mathcal{E}(F))$ . By Prop. 2.6 this means  $F \vdash^Z \varphi$ .

By Prop. 2.3 the logic  $(Fm(Z), \vdash^Z)$  is finitary, so there is a finite subset  $F' \subseteq F$  such that  $F' \vdash^Z \varphi$ . Then there is a finite set  $V \subseteq Z$  of variables occurring in the formulas of  $F' \cup \{\varphi\}$ . Choose a map  $h: Z \rightarrow X$ , which is injective on  $V$  (this is possible since  $X$  is infinite) and a map  $v: X \rightarrow Z$  such that  $v \circ h \upharpoonright_V = id_V$ . We continue denoting the induced maps on formula algebras by  $h$ , resp.  $v$ . By construction we then have  $v(h(F)) = F$  and  $v(h(\varphi)) = \varphi$ .

From  $F' \vdash^Z \varphi$  and Prop. 2.12(i) it follows that  $h(F') \vdash^X h(\varphi)$ , i.e.  $h(F')$  implies  $h(\varphi)$  in the logic  $l$ .

Now  $v$  is a valuation such that  $v(h(F')) = F' \subseteq F$  and  $F$  was supposed to be an  $l$ -filter, so  $\varphi = v(h(\varphi)) \in F$ .  $\square$

## 3 Congruence filter pairs

Motivated by the example of algebraizable logics, we introduce in this section a special class of filter pairs, namely the filter pairs whose functor part is given by  $Co_{\mathbf{K}}: \Sigma\text{-Str} \rightarrow AL$ , where  $\mathbf{K} \subseteq \Sigma\text{-Str}$  is a class of  $\Sigma$ -structures, and for  $M \in \Sigma\text{-Str}$   $Co_{\mathbf{K}}(M)$  denotes the ordered set of congruences whose associated quotient lies in  $\mathbf{K}$ . We will call such filter pairs *congruence filter pairs*.

We begin by studying the following question: “which classes  $\mathbf{K} \subseteq \Sigma\text{-Str}$  are eligible for the formation of such a filter pair, i.e. when  $Co_{\mathbf{K}}$  becomes a functor with values in algebraic lattices?”

We will provide some examples of congruence filter pairs and present results relating these filter pairs with classes of logics in the Leibniz hierarchy.

### 3.1 Functors of congruences

To begin, we record some conditions on a class  $\mathbf{K} \subseteq \Sigma\text{-Str}$  for obtaining a functor  $Co_{\mathbf{K}}: \Sigma\text{-Str} \rightarrow \text{Ord}$ , where  $\text{Ord}$  denotes the category of partially ordered sets and order preserving maps.

**PROPOSITION 3.1**

Let  $\Sigma$  be a signature and  $\mathbf{K} \subseteq \Sigma\text{-Str}$  a class of algebras that is closed under subalgebras and isomorphisms. Then there is a functor  $Co_{\mathbf{K}}: \Sigma\text{-Str}^{op} \rightarrow \text{Ord}$  associating to a  $\Sigma$ -structure  $A$  the set of congruences  $\theta$  such that  $A/\theta \in \mathbf{K}$  and to a homomorphism  $f: A \rightarrow B$  the function

$$Co_{\mathbf{K}}(f): Co_{\mathbf{K}}(B) \ni \theta \mapsto (f \times f)^{-1}(\theta) = \{(a, a') : (f(a), f(a')) \in \theta\} \in Co_{\mathbf{K}}(A)$$

Conversely, if  $\mathbf{K} \subseteq \Sigma\text{-Str}$  is a class of algebras such that  $Co_{\mathbf{K}}: \Sigma\text{-Str}^{op} \rightarrow \text{Ord}$  is a functor, which on morphisms is given by  $Co_{\mathbf{K}}(f) = (f \times f)^{-1}$ , then  $\mathbf{K}$  is closed under substructures.

**PROOF.** The set-theoretic preimage of a congruence is a congruence again: Indeed, congruences are exactly the set-theoretic preimages of the equality relation (Denoted  $\Delta$ ) along homomorphisms. So, if  $\theta \in Co_{\mathbf{K}}(B)$  is of the form  $(g \times g)^{-1}(\Delta_C)$  for some  $g: B \rightarrow C$ , then  $(f \times f)^{-1}(\theta) = ((g \times g) \circ (f \times f))^{-1}(\Delta_C)$ , and hence is a congruence again.

To see that the quotient by this congruence lies in  $\mathbf{K}$ , suppose that  $\theta \in Co_{\mathbf{K}}(B)$ . From the isomorphism theorems of universal algebra we obtain a commutative square

$$\begin{array}{ccc} Fm(X) & G(Fm(X)) \xrightarrow{i_{Fm(X)}^G} & (\mathcal{P}(Fm(X)); \subseteq) \\ v \downarrow & G(v) \uparrow & \uparrow v^{-1} \\ A & G(A) \xrightarrow{i_A^G} & (\mathcal{P}(A); \subseteq) \end{array}$$

where the lower horizontal map is injective. So  $A/f^{-1}(\theta)$  is isomorphic to a subalgebra of  $B/\theta$ , hence lies in  $\mathbf{K}$  again.

Conversely, suppose that  $Co_{\mathbf{K}}$  is a functor as described and let  $B \in \mathbf{K}$  and  $h: A \hookrightarrow B$  a substructure. Then  $\Delta_B \subseteq B \times B$  is in  $Co_{\mathbf{K}}(B)$ . Hence, so is  $(h \times h)^{-1}(\Delta_B) = \Delta_A$ , hence  $A = A/\Delta_A \in \mathbf{K}$ . □

The following result is essentially a result attributed to Malcev (see [6] Theorem 1.37, pp 869):

**PROPOSITION 3.2**

Let  $\Sigma$  be a signature and  $\mathbf{K} \subseteq \Sigma\text{-Str}$  a class of algebras that is closed under subalgebras and isomorphisms and  $A \in \Sigma\text{-Str}$ . Are equivalent:

- (i)  $Co_{\mathbf{K}}(A)$  has a minimal element;
- (ii)  $\mathbf{K}$  is closed under products;
- (iii)  $\mathbf{K}$  is a reflective subcategory of  $\Sigma\text{-Str}$ .

**COROLLARY 3.3**

Assuming Vopěnka’s principle, under the conditions of Prop. 3.2 the class  $\mathbf{K}$  is a quasivariety.

## 16 Congruence filter pairs, equational filter pairs and adjoints

PROOF. Assuming Vopěnka's principle, by the Corollary after Theorem 6.14 in [4], a class of  $\Sigma$ -structures is a quasivariety if and only if it is closed under subalgebras and products.  $\square$

In view of Cor. 3.3 it seems possible that the question whether under the conditions of Prop. 3.2  $\mathbf{K}$  must be a quasivariety is undecidable. However, for our application to filter pairs we want  $Co_{\mathbf{K}}$  to be taking values in algebraic lattices and can ask whether this condition forces  $\mathbf{K}$  to be a quasivariety. We have not been able to settle this question and leave it as a question.

QUESTION 1.

Let  $\mathbf{K}$  be a class of  $\Sigma$ -structures that is closed under subalgebras and isomorphisms and suppose that the functor  $Co_{\mathbf{K}}$  of Prop. 3.1 takes values in algebraic lattices. Is  $\mathbf{K}$  then necessarily a quasivariety?

REMARK 3.4

One way of approaching the above question is to note that the proof of Proposition 3.2 shows that, in the situation in question  $\mathbf{K}$  is a regular epi-reflective subcategory of  $\Sigma\text{-Str}$ , i.e. a reflexive subcategory for which the unit morphisms of the adjunction are regular epimorphisms. It is a consequence of this, as shown in the proof, that  $\mathbf{K}$  is closed under subalgebras and products. Quasivarieties are precisely the regular epi-reflective subcategories of  $\Sigma\text{-Str}$ , which are also closed under directed colimits (see, e.g. [5], Cor. 10.24 together with the remarks on the next page). A quick proof of the relevant direction proceeds by noting that all we need for having a quasivariety is closure under ultraproducts, and ultraproducts are filtered colimits of diagrams of products. The question can thus be reformulated as asking whether the additional assumption that  $Co_{\mathbf{K}}$  takes values in algebraic lattices forces  $\mathbf{K}$  to be closed under directed colimits, and hence to be a quasivariety.

To see the possible connection, note that the reflection functor is given by taking certain quotients. From this we obtain a reflection of ordered sets congruences, i.e. a left inverse  $r: Co_{\Sigma\text{-Str}}(A) \rightarrow Co_{\mathbf{K}}(A)$  to the inclusion  $Co_{\mathbf{K}}(A) \hookrightarrow Co_{\Sigma\text{-Str}}(A)$ : given a congruence  $\theta \in Co_{\Sigma\text{-Str}}(A)$ , we define  $r(\theta) := (p \times p)^{-1}(\min_{A/\theta}) \in Co_{\mathbf{K}}(A)$  where  $p$  denotes the projection  $p: A \rightarrow A/\theta$ .

By [22, Prop 1.4.12], a class  $\mathbf{K}$  is a quasivariety iff  $Co_{\mathbf{K}}(A) \subseteq Co(A)$  is an algebraic sublattice for every structure  $A$ . Thus the question is, whether the inclusion  $Co_{\mathbf{K}}(A) \subseteq Co(A)$  preserves directed suprema (which could be used to show closure of  $\mathbf{K}$  under directed colimits). A related question is whether the compact elements of  $Co_{\mathbf{K}}(A)$  are exactly the finitely generated  $\mathbf{K}$ -congruences, i.e. the images under the lattice reflection of the compact elements of  $Co_{\Sigma\text{-Str}}(A)$ .

### 3.2 Congruence filter pairs

DEFINITION 3.5

Let  $\Sigma$  be a signature and  $\mathbf{K}$  a class of  $\Sigma$ -structures such that the association  $A \mapsto Co_{\mathbf{K}}(A)$  is (the object part of) a functor from  $\Sigma$ -structures to algebraic lattices. A filter pair of the form  $(Co_{\mathbf{K}}(A), i)$  is called congruence filter pair.

In all examples in this article  $\mathbf{K}$  will actually be a quasivariety, and in this case the set  $Co_{\mathbf{K}}(A)$  of congruences relative to  $\mathbf{K}$ , ordered by inclusion, is an algebraic lattice. The next proposition shows how in this case a presentation by a congruence filter pair is closely linked to the algebraizability of a logic.

PROPOSITION 3.6

Let  $\Sigma$  be a signature and  $\mathbf{K} \subseteq \Sigma\text{-Str}$  a quasivariety. If  $(Co_{\mathbf{K}}, i)$  is a filter pair, such that  $i_{Fm(X)}: Co_{\mathbf{K}}(Fm(X)) \rightarrow \mathcal{P}(Fm(X))$  is injective, then the associated logic is algebraizable.

PROOF. We know from Lemma 2.8 that  $i_{Fm}^{\mathbf{K}}[Co_{\mathbf{K}}(Fm)] = Th(l_{\mathbf{K}})$ . As  $i_{Fm}^{\mathbf{K}}$  is injective, we have that  $i_{Fm}^{\mathbf{K}}$  is bijective. Then  $i_{Fm}^{\mathbf{K}}$  is an isomorphism. Now let  $\sigma \in hom(Fm, Fm)$ . As  $i^{\mathbf{K}}$  is a natural transformation we have the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ A/f^{-1}(\theta) & \xrightarrow{\bar{f}} & B/\theta \end{array}$$

Note that  $\sigma^{-1}(T) \in Th(l_{\mathbf{K}})$  for any  $T \in Th(l_k)$ . Therefore,  $i_{Fm}^{\mathbf{K}}$  is a isomorphism such that commutes with substitutions. By the isomorphism theorem 1.17,  $l_{\mathbf{K}}$  is an algebraizable logic.  $\square$

### REMARK 3.7

The above proposition gives us an alternative proof of theorem 5.2 [13]. The condition of injectivity assumed here is exactly the condition assumed in loc. cit. to get algebraizability.

The next theorem will provide us with an ample supply of examples of congruence filter pairs.

### THEOREM 3.8

Let  $\Sigma$  be a signature,  $\mathbf{K} \subseteq \Sigma\text{-Str}$  a quasivariety and  $\tau$  a finite set of equations in at most one variable. The map  $i^{\mathbf{K}} = (i_M^{\mathbf{K}})_{M \in \Sigma\text{-Str}}$  where:

$$\begin{array}{ccc} i_M^{\tau} : Co_{\mathbf{K}}(M) & \rightarrow & (\mathcal{P}(M), \subseteq) \\ \theta & \mapsto & \{m \in M : \tau^M(m) \in \theta\} \end{array}$$

is a natural transformation and for any  $M \in \Sigma\text{-Str}$ ,  $i_M^{\tau}$  preserves arbitrary infima and directed suprema, i.e.  $(Co_{\mathbf{K}}, i^{\tau})$  is a filter pair.

PROOF. Let  $f \in hom(M, N)$ . Denote here  $f(\tau^M(m)) = \{(f(\varepsilon^M(m)), f(\delta^M(m))) : (\varepsilon, \delta) \in \tau\}$ .

$$\begin{array}{ccccc} Fm & & Co_{\mathbf{K}}(Fm) & \xrightarrow{i_{Fm}^{\mathbf{K}}} & (\mathcal{P}(Fm); \subseteq) \\ \sigma \downarrow & & \uparrow Co_{\mathbf{K}}(\sigma) & & \uparrow \sigma^{-1} \\ Fm & & Co_{\mathbf{K}}(Fm) & \xrightarrow{i_{Fm}^{\mathbf{K}}} & (\mathcal{P}(Fm); \subseteq) \end{array}$$

For  $\theta \in Co_{\mathbf{K}}(N)$  we have

$$\begin{aligned} f^{-1}(i_N^{\mathbf{K}}(\theta)) &= f^{-1}(\{n \in N : \tau^N(n) \subseteq \theta\}) \\ &= \{m \in M : \tau^N(f(m)) \subseteq \theta\} \\ &= \{m \in M : f(\tau^M(m)) \subseteq \theta\} \\ &= \{m \in M : \tau^M(m) \subseteq Co_{\mathbf{K}}(f)(\theta)\} \\ &= i_M^{\mathbf{K}}(f^{-1}(\theta)) \end{aligned}$$

Thus  $i$  is a natural transformation.

18 *Congruence filter pairs, equational filter pairs and adjoints*

Consider a family of congruences  $\theta_i \in Co_{\mathbf{K}}(M)$  ( $i \in I$ ).

$$\begin{aligned} i_M^{\mathbf{K}}(\bigcap_{i \in I} \theta_i) &= \{m \in M : \tau^M(m) \subseteq \bigcap_{i \in I} \theta_i\} \\ &= \{m \in M : \forall i \in I, \tau^M(m) \subseteq \theta_i\} \\ &= \bigcap_{i \in I} \{m \in M : \tau^M(m) \subseteq \theta_i\} \end{aligned}$$

Thus  $i_M$  preserves arbitrary infima.

Now let  $U = \{\theta_i : i \in I\}$  be an upwards directed set.

$$\begin{aligned} i_M^{\mathbf{K}}(\bigvee U) &= \{m \in M : \tau^M(m) \subseteq \bigvee U\} \\ &= \{m \in M : \tau^M(m) \subseteq \bigcup_{i \in I} \theta_i\} \\ &= \bigcup_{i \in I} \{m \in M : \tau^M(m) \subseteq \theta_i\} \end{aligned}$$

Here the last equality holds because  $U$  is a directed set and the set of equations  $\tau$  is finite, hence if each equation in  $\tau(m)$  is contained in some  $\theta_i$  there is a  $\theta$  containing all those  $\theta_i$  and hence all equations. Thus  $i_M$  preserves directed suprema.  $\square$

## DEFINITION 3.9

A filter pair  $(G, i)$  arising as in Theorem 3.8 will be called an equational filter pair. The transformation  $i$  will be said to be given by the equations  $\tau$ . We will sometimes let the defining equations be part of the notation for an equational filter pair, as in  $(Co_{\mathbf{K}}, i^{\tau})$  or  $(Co_{\mathbf{K}}, i^{\delta \approx \epsilon})$

In Thm. 4.4 below we will show that the logics admitting a presentation by a congruence filter pair, are exactly those admitting an algebraic semantics. For example, every algebraizable logic admits such a presentation, with the equations given by the algebraizing pair.

For concreteness, here is a typical example of a logic with a presentation by a congruence filter pair.

## EXAMPLE 2

Consider the signature  $\Sigma = \{\wedge, \vee, \neg, \top, \perp\}$  and let  $\mathbf{K}$  be the variety of pseudocomplemented distributive lattices. Using the single equation  $(x, \top)$

By [12, Thm. 2.6], the logic associated to the filter pair  $(Co_{\mathbf{K}}, i)$  is the implicationless fragment of intuitionistic propositional logic  $IPL^*$ . In [12, Thm. 5.13] it is shown that this logic is not protoalgebraic.

One further, admittedly artificial, example of a logic that is neither protoalgebraic nor truth-equational but has an algebraic semantics:

## EXAMPLE 3

Consider the signature  $\Sigma = (\Sigma)_{n \in \omega}$  that  $\Sigma_1 = \{s\}$  and  $\Sigma_n = \emptyset$  for all  $n \neq 1$ . A  $\Sigma$ -algebra is simply a set with an endomorphism. The free  $\Sigma$ -algebra  $F_{\Sigma}(X)$  on countably many generators  $X = \{x_0, x_1, \dots\}$  is isomorphic to  $\mathbb{N}^X$ , the disjoint union of countably many copies of the natural numbers, where  $x_i \in X$  corresponds to 0 in the  $i$ th copy, and the endomorphism  $s$  acts as the successor function on each copy.

Consider the functor  $Co := Co_{\Sigma-Alg}$  and the map  $i : Co \Rightarrow (\mathcal{P}(\ ), \subseteq)$  given as above by  $\tau = \{(x, s(x))\}$ . We denote the logic associated to this filter pair by  $I_s$ .

For a congruence  $\theta$  on a  $\Sigma$ -algebra  $A$  one has  $i(\theta) = \{a \in A : \bar{a} = s(\bar{a}) \text{ in } A/\theta\}$ . Thus  $i$  assigns to a congruence  $\theta$  the set of elements of  $A$ , which become  $s$ -fixed points in  $A/\theta$ .

The logic  $l_s$  is not protoalgebraic because it is not almost inconsistent, i.e. it is not trivial and has no theorems: the theorems of  $l_s$  are exactly the elements of  $i(\theta_{min})$ , where  $\theta_{min}$  is the minimal congruence relation on  $F_\Sigma(X)$ . As  $F_\Sigma(X)/\theta_{min} = F_\Sigma(X)$ , this is exactly the set of  $s$ -fixed points in  $F_\Sigma(X)$ , which is empty.

After having seen, in Prop. 3.6, a criterion for the algebraizability of a logic in terms of a presentation by an equational filter pair, in the next few items we discuss such a criterion for truth-equationality.

PROPOSITION 3.10

Let  $(Co_{\mathbf{K}}, i)$  be an equational filter pair as in Theorem 3.8. If  $i_A$  is surjective onto the collection of filters for every algebra  $A$ , then the associated logic is truth-equational.

PROOF. By Definition 1.19 a logic is truth-equational if and only if there is a set of equations defining the filters of all its reduced matrices. But if  $i$  is surjective, then *all* filters of all matrices are given by the equations defining  $i$ .  $\square$

The following example shows that the condition of  $i$  being surjective onto filters is not always satisfied, and moreover, that logics associated to equational filter pairs need not be truth-equational.

EXAMPLE 4

Consider the logic  $l_s$  of example 3. Take the  $\Sigma$ -algebra  $A = \mathbb{N} \cup \{z\}$ , where  $z$  is an  $s$ -fixed point and  $\mathbb{N}$  is endowed with the (fixed point-free) successor operation. Then  $\{z\}$  is a filter on  $A$  (belonging to the minimal congruence relation, whose quotient is just  $A$  itself). Now  $\Omega^A(\{z\})$  is the coarsest congruence relation, which does not relate  $z$  with any other element. This is the congruence relation identifying all the elements of  $\mathbb{N}$  and leaving  $z$  alone. Therefore,  $A/\Omega^A(\{z\})$  has two elements (both of which are  $s$ -fixed points). Note that  $i_A(\Omega^A(\{z\}))$  has two elements and is not  $\{z\}$ , so that the natural candidate  $\tau$  for a set of truth-equations does not work.

There is also no other set of equations defining this filter in the way required for truth-equationality: any equation satisfied by the element  $\bar{z}$  in  $A/\Omega^A(\{z\})$  is also satisfied by the other element, because the permutation of the two elements is an automorphism of the algebra  $A/\Omega^A(\{z\})$ .

There is a natural and large class of examples where the condition of Prop. 3.10 is satisfied:

PROPOSITION 3.11

Let  $\mathbf{K}$  be a pointed quasivariety, i.e. a quasivariety over a signature with a constant symbol 0. Consider the set of equations  $\tau = \{(x, 0)\}$ . Then  $\Omega^A$  is a right inverse of  $i_A^\tau$  for any  $A \in \Sigma\text{-Str}$ . In particular  $i_A$  is surjective onto the collection of filters for every  $\Sigma$ -algebra  $A$ .

PROOF. Note that in the associated logic  $l$  we have that  $\vdash_l 0, x, \varphi(x, \bar{z}) \vdash_l \varphi(0, \bar{z})$  and  $x, \varphi(0, \bar{z}) \vdash_l \varphi(x, \bar{z})$  for any  $\varphi(x, \bar{z}) \in Fm$ . Indeed, let  $\theta \in Co_{\mathbf{K}}(Fm)$ , then  $(0, 0) \in \theta$ , thus  $0 \in i_{Fm}^\tau(\theta)$  and thus  $\vdash_l 0$ . Now let  $\theta \in Co_{\mathbf{K}}(Fm)$  and suppose that  $x, \varphi(x, \bar{z}) \in i_{Fm}^\tau(\theta)$ , then  $(x, 0) \in \theta$  and  $(\varphi(x, \bar{z}), 0) \in \theta$ . Since  $\theta$  is a congruence, we have that  $(\varphi(x, \bar{z}), \varphi(0, \bar{z})) \in \theta$ . Therefore,  $(\varphi(0, \bar{z}), 0) \in \theta$ , so  $\varphi(0, \bar{z}) \in i_{Fm}^\tau(\theta)$ . Hence,  $x, \varphi(x, \bar{z}) \vdash_l \varphi(0, \bar{z})$ . The same proof can be used to prove that  $x, \varphi(0, \bar{z}) \vdash_l \varphi(x, \bar{z})$ .

Now we are able to prove that for any  $A \in \Sigma\text{-Str}$  and  $F \in Fi_{\mathbf{K}}(A)$ ,  $F = i_A^\tau(\Omega^A(F))$ . Let  $a \in F$  and  $\varphi(x, \bar{z}) \in Fm$ . Let  $\bar{c} \in A$  and suppose that  $\varphi^A(a, \bar{c}) \in F$ . Since  $x, \varphi(x, \bar{z}) \vdash_l \varphi(0, \bar{z})$ , we have that  $\varphi^A(0, \bar{c}) \in F$ . Analogously we have that if  $\varphi^A(0, \bar{c}) \in F$ ,  $\varphi^A(a, \bar{c}) \in F$ . Hence,  $(a, 0) \in \Omega^A(F)$ . By definition of  $i_A^\tau$  we have  $a \in i_A^\tau(\Omega^A(F))$ . Thus,  $F \subseteq i_A^\tau(\Omega^A(F))$ . Let  $a \in i_A^\tau(\Omega^A(F))$ , then  $(a, 0) \in \Omega^A(F)$ . Since  $\vdash_l 0$ , we have  $0 \in F$ , therefore  $a \in F$ . Hence,  $i_A^\tau(\Omega^A(F)) \subseteq F$ .  $\square$

With this we have arrived at the well-known fact that assertional logics are truth-equational:

## COROLLARY 3.12

Let  $\mathbf{K} \subseteq \Sigma\text{-Str}$  be a pointed quasivariety. Then the logic associated to the equation  $\tau = \{(x, 0)\}$  (as in Theorem 3.8) is truth-equational.

PROOF. Follows from Prop. 3.11 and Prop. 3.10.  $\square$

#### 4 Filters and adjunctions for equational filter pairs

This section provides a closer analysis of the left adjoint  $\mathcal{E}_A$  of  $i_A$  for equational filter pairs, its relation to the Leibniz operator and some consequences for the filters of the associated logics.

The main statements are Theorems 4.7 and 4.11.

## PROPOSITION 4.1

For a logic presented by an equational filter pair  $(Co_{\mathbf{K}}, i^{\tau})$ , a  $\Sigma$ -structure  $A$  and a congruence  $\theta \in Co_{\mathbf{K}}(A)$ , the filter  $i_A^{\tau}(\theta)$  is compatible with the congruence relation  $\theta$  (in the sense of Def. 1.15).

PROOF. Let the filter pair be given by the equations  $\tau = (\epsilon(x), \delta(x))$ . Let  $\theta \in Co_{\mathbf{K}}(A)$  and  $(\varphi, \psi) \in \theta$ . Then we have

$$\begin{aligned} \varphi \in i(\theta) &\Leftrightarrow (\epsilon(\varphi), \delta(\varphi)) \in \theta \text{ (by definition of } i(\theta)\text{)} \\ &\Leftrightarrow \epsilon(\bar{\varphi}) = \delta(\bar{\varphi}) \in A/\theta \\ &\Leftrightarrow \epsilon(\bar{\psi}) = \delta(\bar{\psi}) \in A/\theta \text{ (because } \bar{\varphi} = \bar{\psi} \in A/\theta\text{)} \\ &\Leftrightarrow \psi \in i(\theta) \text{ (by definition of } i(\theta)\text{)} \end{aligned} \quad \square$$

Next we give an explicit description of the operator  $\mathcal{E}_A$  of Def. 2.5 in the case of an equational filter pair.

## PROPOSITION 4.2

Let  $(Co_{\mathbf{K}}, i)$  be an equational filter pair, with  $i$  given by the equations  $\delta(x) = \epsilon(x)$ . Then for a  $\Sigma$ -algebra  $A$  the left adjoint  $\mathcal{E}_A$  to  $i_A$  maps  $S \subseteq A$  to the congruence relation generated by  $\{(\delta(s), \epsilon(s)) : s \in S\}$ .

PROOF. This follows immediately from the general description of Theorem 1.2 of left adjoints between complete partially ordered sets:

$$\begin{aligned} \mathcal{E}_A(S) &= \bigwedge \{\theta \in Co_{\mathbf{K}}(A) : S \subseteq i_A(\theta)\} \\ &= \bigwedge \{\theta \in Co_{\mathbf{K}}(A) : S \subseteq \{a \in A : \delta(\bar{a}) = \epsilon(\bar{a}) \text{ in } A/\theta\}\} \\ &= \bigwedge \{\theta \in Co_{\mathbf{K}}(A) : \{(\delta(s), \epsilon(s)) : s \in S\} \subseteq \theta\} \end{aligned} \quad \square$$

Recall from Def. 1.10 the notation  $Cn_{\mathbf{K}}(R)$  for the set of equations that is implied in the quasivariety  $\mathbf{K}$  by a set of equations  $R$ . Also recall from and Lemma 1.12 that, if one identifies equations with pairs of elements,  $Cn_{\mathbf{K}}(R)$  is exactly the congruence relative  $\mathbf{K}$  generated by the set of pairs  $R$ . Blok-Pigozzi, in [12, p.28], on the formula algebra  $Fm(X)$  define the operator

$$\Omega_{\mathbf{K}}: \mathcal{P}(Fm(X)) \rightarrow Co_{\mathbf{K}}(Fm(X)), \quad T \mapsto Cn_{\mathbf{K}}(\{\delta(a) \approx \epsilon(a) : a \in T\}).$$

This makes sense on any algebra and gives us an alternative description of  $\mathcal{E}_A$ :

## COROLLARY 4.3

For an algebra  $A$  and a subset  $T \subseteq A$  one has  $\mathcal{E}_A(T) = Cn_{\mathbf{K}}(\{\delta(a) \approx \epsilon(a) : a \in T\})$ .

PROOF. Follows immediately from Lemma 1.12 and Proposition 4.2.  $\square$

With Corollary 4.3 we can now pin down the class of logics admitting a presentation by an equational filter pair.

**THEOREM 4.4**

Let  $\Sigma$  be a signature,  $l$  a logic over  $\Sigma$ ,  $\mathbf{K}$  a class of  $\Sigma$ -algebras and  $\mathbf{K}^Q$  the quasivariety generated by  $\mathbf{K}$ . Then  $l$  has an algebraic semantics in  $\mathbf{K}$ , if and only if it has a presentation by an equational filter pair  $(Co_{\mathbf{K}^Q}, i)$ .

**PROOF.** Assume that  $l$  has an algebraic semantics in  $\mathbf{K}$  given by a set  $\{\delta(x) = \epsilon(x)\}$  of equations over  $\Sigma$ .

Using Theorem 3.8 we can define the filter pair  $(Co_{\mathbf{K}^Q}, i^{\delta=\epsilon})$  and compare its associated logic with  $l$ . Let  $\mathcal{E}_{Fm(X)}$  be the left adjoint of  $i_{Fm(X)}$ .

The result follows now from noting the following equivalences (the single steps are explained below):

$$\begin{aligned}
 \Gamma \vdash_l \varphi &\Leftrightarrow \{\delta(\gamma) = \epsilon(\gamma) : \gamma \in \Gamma\} \models_{\mathbf{K}} \delta(\varphi) = \epsilon(\varphi) \\
 &\Leftrightarrow \{\delta(\gamma) = \epsilon(\gamma) : \gamma \in \Gamma\} \models_{\mathbf{K}^Q} \delta(\varphi) = \epsilon(\varphi) \\
 &\Leftrightarrow \delta(\varphi) = \epsilon(\varphi) \in Cn_{\mathbf{K}^Q}(\{\delta(\gamma) = \epsilon(\gamma) : \gamma \in \Gamma\}) \\
 &\Leftrightarrow \delta(\varphi) = \epsilon(\varphi) \in \mathcal{E}_{Fm(X)}(\Gamma) \\
 &\Leftrightarrow \varphi \in i^{\delta=\epsilon}(\mathcal{E}_{Fm(X)}(\Gamma)) \\
 &\Leftrightarrow \Gamma \vdash_{(Co_{\mathbf{K}^Q}, i^{\delta=\epsilon})} \varphi
 \end{aligned}$$

For the second equivalence, note that by [12, Cor 2.3] one can replace  $\mathbf{K}$  by  $\mathbf{K}^Q$ .

The third equivalence holds by the definition of  $Cn_{\mathbf{K}^Q}$ .

The fourth equivalence holds by Cor. 4.3.

The fifth equivalence holds by the definition of  $i^{\delta=\epsilon}$ .

The sixth equivalence holds by Prop. 2.6

Vice versa it follows from the last four equivalences that a logic presented by an equational filter pair has an algebraic semantics.  $\square$

We dedicate the rest of the section to a study of the properties of the family of adjunctions  $i^\tau, \mathcal{E}$ . To begin, many of the results of Chapter 3 of [12] on the operator  $\Omega_{\mathbf{K}}$ , there proven directly from the definition and only for the formula algebra  $Fm(X)$ , follow now immediately from the fact that it is left adjoint to  $i^\tau$ .

**LEMMA 4.5**

Let  $(Co_{\mathbf{K}}, i^\tau)$  be an equational filter pair and  $A$  a  $\Sigma$ -structure.

1. [12, Lemma 3.3(i)] Write  $C := i_A^\tau \circ \mathcal{E}_A$  for the closure operator of the abstract logic on  $A$ . Then for  $\Gamma \subseteq A$  we have  $\mathcal{E}_A(C(\Gamma)) = \mathcal{E}_A(\Gamma)$ .
2. [12, Lemma 3.3(ii)] The map  $\mathcal{E}_A$  preserves arbitrary suprema.
3. [12, Lemma 3.3(iii)] The map  $\mathcal{E}_A$  preserves unions of directed sets of theories of the abstract logic on  $A$ .
4. [12, Lemma 3.4(i)] For a theory  $T$  of the abstract logic on  $A$  one has  $(i_A^\tau \circ \mathcal{E}_A)(T) = T$
5. [12, Lemma 3.4(ii)] For every  $\theta \in Co_{\mathbf{K}}(A)$  one has  $(\mathcal{E}_A \circ i^\tau)(\theta) \subseteq \theta$ . Equality holds if and only if  $\theta = \mathcal{E}_A(T)$  for some  $A$ -theory  $T$ .
6. [12, Lemma 3.5(i)]  $\mathcal{E}_A$  maps the lattice of  $A$ -theories isomorphically to a compact and join-complete sublattice of  $Co_{\mathbf{K}}(A)$ .

**PROOF.**

1. This is just the triangle equality  $\mathcal{E}_A \circ i^\tau \circ \mathcal{E}_A = \mathcal{E}_A$  coming from the adjointness (Blok-Pigozzi's original statement follows together with Prop. 2.6).
2. The map  $\mathcal{E}_A$  is a left adjoint, hence preserves arbitrary colimits.

## 22 Congruence filter pairs, equational filter pairs and adjoints

3. Since both  $\mathcal{E}_A$  and  $i_A^r$  preserve directed suprema, the composite  $i_A^r \circ \mathcal{E}_A$  preserves directed suprema in  $\mathcal{P}(A)$ , i.e. directed unions, and takes values in the sublattice of theories. Thus, directed suprema in the lattice of theories coincide with set-theoretic unions, hence this follows from part (ii).
4. By Lemma 2.8 theories of the abstract logic on  $A$  (there called  $i$ -filters, see Def. 2.7) are exactly the sets in the image of  $i_A^r$ . Hence, the claim follows from the adjointness equality  $i_A^r \circ \mathcal{E}_A \circ i^r = i^r$ .
5. The inclusion is the counit of the adjunction, the condition for equality is clear from the adjunction properties.
6. Adjunctions induce isomorphisms between the sublattices occurring as the images. The map  $\mathcal{E}_A \circ i^r$  is a coreflection onto the respective sublattice of  $Co_{\mathbf{K}}(A)$  and coreflective sublattices are closed under suprema.  $\square$

As mentioned earlier, the collection of adjoint maps  $(\mathcal{E}_A)_{A \in \Sigma\text{-Str}}$  need not be a natural transformation with respect to preimage maps in general, and its behaviour with respect to homomorphisms of  $\Sigma$ -structures does not follow directly from adjointness. The following lemma records the behaviour with respect to substitutions on the formula algebras.

### LEMMA 4.6

Let  $\sigma : Fm(X) \rightarrow Fm(Y)$  be a homomorphism of free  $\Sigma$ -algebras.

1. For  $\Gamma \subseteq Fm(X) \times Fm(X)$  we have  $(\sigma \times \sigma)(Cn_{\mathbf{K}}^{Fm(X)}(\Gamma)) \subseteq Cn_{\mathbf{K}}^{Fm(Y)}(\sigma \times \sigma(\Gamma))$ .
2. For  $\Gamma \subseteq Fm(X) \times Fm(X)$  we have  $Cn_{\mathbf{K}}^{Fm(Y)}((\sigma \times \sigma)(Cn_{\mathbf{K}}^{Fm(X)}(\Gamma))) = Cn_{\mathbf{K}}^{Fm(Y)}(\sigma(\Gamma))$ .
3. For any homomorphism of free  $\Sigma$ -algebras  $\sigma : Fm(X) \rightarrow Fm(Y)$  and  $T \subseteq Fm(X)$  we have  $\mathcal{E}_{Fm(Y)}(\sigma(T)) = Cn_{\mathbf{K}}^{Fm(Y)}((\sigma \times \sigma)(\mathcal{E}_{Fm(X)}(T)))$
4. If  $\sigma : Fm(X) \rightarrow Fm(Y)$  is the homomorphism induced by an injective map of variables  $X \hookrightarrow Y$  and  $T \subseteq Fm(Y)$ , we have  $\mathcal{E}_{Fm(X)}(\sigma^{-1}(T)) \subseteq (\sigma \times \sigma)^{-1}(\mathcal{E}_{Fm(Y)}(T))$

### PROOF.

1. Equational consequence satisfies, for any substitution  $\sigma : Fm(X) \rightarrow Fm(Y)$ , and set  $\Gamma \cup \{\tau\} \subseteq Fm(X)$  of equations,  $\Gamma \vDash_{\mathbf{K}} \tau \Rightarrow \sigma \Gamma \vDash_{\mathbf{K}} \sigma \tau$ . Hence, we have  $(\sigma \times \sigma)(Cn_{\mathbf{K}}^{Fm(X)}(\Gamma)) = \{(\sigma \times \sigma)(\tau) : \Gamma \vDash_{\mathbf{K}} \tau\} \subseteq \{\tau : (\sigma \times \sigma)(\Gamma) \vDash_{\mathbf{K}} \tau\} = Cn_{\mathbf{K}}^{Fm(Y)}(\sigma(\Gamma))$ .
2. Applying  $Cn_{\mathbf{K}}^{Fm(Y)}$  to the inclusion from (1) yields  $Cn_{\mathbf{K}}^{Fm(Y)}((\sigma \times \sigma)(Cn_{\mathbf{K}}^{Fm(X)}(\Gamma))) \subseteq Cn_{\mathbf{K}}^{Fm(Y)} Cn_{\mathbf{K}}^{Fm(Y)}(\sigma \times \sigma(\Gamma)) = Cn_{\mathbf{K}}^{Fm(Y)}(\sigma \times \sigma(\Gamma))$ .
3. On the other hand  $Cn_{\mathbf{K}}^{Fm(X)}(\Gamma) \supseteq \Gamma$  implies  $(\sigma \times \sigma)(Cn_{\mathbf{K}}^{Fm(X)}(\Gamma)) \supseteq (\sigma \times \sigma)(\Gamma)$ , which implies  $Cn_{\mathbf{K}}^{Fm(Y)}((\sigma \times \sigma)(Cn_{\mathbf{K}}^{Fm(X)}(\Gamma))) \supseteq Cn_{\mathbf{K}}^{Fm(Y)}((\sigma \times \sigma)(\Gamma))$ .
4. See [12, Lemma 3.6] for the case  $X = Y$ . The general proof is no different.
5. In the special case we are considering now,  $\sigma^{-1}$  (resp.  $(\sigma \times \sigma)^{-1}$ ) is just restriction to formulas (resp. pairs of formulas) in  $Fm(X)$ , i.e.  $\sigma^{-1}(T) = T \cap Fm(X)$ .
6. Recall, from the text after Lemma 1.11, the notation  $Cn_{\mathbf{K}}^X$  for the closure operator associated to the equational consequence relation  $\vDash_{\mathbf{K}}^X$  of Def. 1.10. Expressed in terms of these closure operators, Lemma 1.11 says that  $Cn_{\mathbf{K}}^Y(T \cap Fm(X)^2) \cap Fm(X)^2 = Cn_{\mathbf{K}}^X(T \cap Fm(X)^2)$ . Hence,

$$Cn_{\mathbf{K}}^Y(T) \cap Fm(X)^2 \supseteq Cn_{\mathbf{K}}^Y(T \cap Fm(X)^2) \cap Fm(X)^2 = Cn_{\mathbf{K}}^X(T \cap Fm(X)^2),$$

which implies the claim.  $\square$

While in the usual treatments of abstract algebraic logic the Leibniz operator plays a prominent role, we haven't made any use of it so far. Even algebraizable logics can be characterized purely in terms of the natural transformations  $i$  and  $\mathcal{E}$  (namely if  $\mathcal{E}(i(\theta))$  always holds, Thm. 4.7 (a) $\Leftrightarrow$  (e) below).

We will now start considering the interplay of our operators with the Leibniz operator  $\Omega$ .

**THEOREM 4.7**

Let  $(Co_{\mathbf{K}}, i)$  be an equational filter pair, with  $i$  given by equations  $\delta(x) = \epsilon(x)$ . Let  $A$  be a  $\Sigma$ -algebra and  $\theta \in Co_{\mathbf{K}}(A)$  a congruence on  $A$ . Then  $\mathcal{E}_A(i_A(\theta)) \subseteq \theta \subseteq \Omega_A(i_A(\theta))$ . Moreover the following are equivalent:

- (a) one of the inclusions is an equality for every  $\theta \in Co_{\mathbf{K}}(Fm(X))$  (where  $X$  is any enumerable set)
- (b) both inclusions are equalities for all  $\Sigma$ -structures  $A$  and every  $\theta \in Co_{\mathbf{K}}(A)$
- (c)  $\mathcal{E}_A = \Omega_A$  for all  $\Sigma$ -structures  $A$ , as maps from the lattice of filters to  $Co_{\mathbf{K}}(A)$
- (d)  $\mathcal{E}_{Fm(X)} = \Omega_{Fm(X)}$ , as maps from the lattice of theories to  $Co_{\mathbf{K}}(Fm(X))$
- (e) the logic presented by  $(Co_{\mathbf{K}}, i)$  is algebraizable with  $i_{Fm(X)}$  being a substitution preserving lattice isomorphism.

**PROOF.** The first inclusion has been noted in Lemma 4.5(v).

For the second inclusion recall that by Prop. 4.1  $\theta$  is compatible with  $i(\theta)$ . But  $\Omega_A(i_A(\theta))$  is the coarsest congruence relation that is compatible with  $i(\theta)$ , so  $\theta \subseteq \Omega_A(i_A(\theta))$ .

For the equivalences we prove the cycle  $(a) \Rightarrow (e) \Rightarrow (c) \Rightarrow (b) \Rightarrow (d) \Rightarrow (a)$ .

$(a) \Rightarrow (e)$  : If one of the inclusions  $\theta \subseteq \Omega(i(\theta))$ ,  $\theta \subseteq \mathcal{E}(i(\theta))$  is an equality, then  $i$  has a left inverse and hence is injective. Since by definition of the associated logic,  $i$  is also surjective onto the theories, we have a substitution preserving isomorphism between the lattice of congruences and the lattice of theories. By Thm. 1.17 the associated logic is algebraizable.

$(e) \Rightarrow (c)$  : By Thm. 1.17 for any  $\Sigma$ -structure  $A$  the equations defining  $i_A$  induce isomorphisms, and the inverse is given by the Leibniz operator  $\Omega_A$ . Being an inverse of  $i_A$ ,  $\Omega_A$  is in particular a left adjoint. By uniqueness of adjoints, it follows that  $\mathcal{E}_A = \Omega_A$ .

$(c) \Rightarrow (b)$  : Clear, using the two inclusions and the equality  $\mathcal{E}_A = \Omega_A$ .

$(b) \Rightarrow (d)$  : By hypothesis (b) the maps  $\mathcal{E}_{Fm(X)}$  and  $\Omega_{Fm(X)}$  coincide on the filters in the image of  $i_{Fm(X)}$ . But by Prop. 2.13 all theories are in the image of  $i_{Fm(X)}$ , hence  $\mathcal{E}_{Fm(X)} = \Omega_{Fm(X)}$ .

$(d) \Rightarrow (a)$  : Clear, using the two inclusions and the equality  $\mathcal{E}_{Fm(X)} = \Omega_{Fm(X)}$ . □

**LEMMA 4.8**

Let  $(Co_{\mathbf{K}}, i)$  be an equational filter pair, with  $i$  given by equations  $\delta(x) = \epsilon(x)$ . Let  $(A, F)$  be a reduced matrix for the associated logic with  $F = i_A(\theta)$  in the image of  $i_A$ , for a congruence  $\theta \in Co_{\mathbf{K}}(A)$ . Then  $F = \{a \in A : \delta(a) = \epsilon(a)\}$ .

**PROOF.** The assumption that  $(A, F)$  is a reduced matrix means  $\Omega_A(F) = \Delta$  (where  $\Delta$  denotes the diagonal, i.e. the minimal congruence relation). We therefore have  $\Omega_A(F) = \Delta \subseteq \mathcal{E}_A(F)$ , and thus by the inclusions of Theorem 4.7  $\Delta = \mathcal{E}_A(F) = \theta$ . We obtain

$$\begin{aligned} F = i_A(\theta) &= \{a \in A : \delta(\bar{a}) = \epsilon(\bar{a}) \text{ in } A/\theta\} \\ &= \{a \in A : \delta(a) = \epsilon(a) \text{ in } A\} \end{aligned}$$

□

Our considerations about adjoints and filter pairs let us arrive at the following well-known fact about assertional logics.

## 24 Congruence filter pairs, equational filter pairs and adjoints

### COROLLARY 4.9

Let  $(Co_K, i)$  be an equational filter pair, with  $i$  given by equations  $x = \top$ . Let  $(A, F)$  be a reduced matrix for the associated logic. Then  $F = \{\top\}$ .

PROOF. For a filter pair given by the equation  $x = \top$  all filters are in the image of  $i_A$  by Lemma 3.11. By Lemma 4.8 we have  $F = \{a \in A : a = \top\} = \{\top\}$ .  $\square$

We know from Theorem 4.7 that if one of the inclusions  $\mathcal{E}(i(\theta)) \subseteq \theta \subseteq \Omega(i(\theta))$  is actually an equality (and hence  $i$  is injective), then the filter pair in question presents an algebraizable logic. Applying  $i$  to the first inclusion, by general properties of adjunctions, we obtain an equality  $i(\mathcal{E}(i(\theta))) = i(\theta)$ . In contrast, we do not necessarily have the corresponding equality  $i(\theta) = i(\Omega(i(\theta)))$  for the Leibniz operator.

### EXAMPLE 5

Consider again the logic  $I_s$  of Examples 3 and 4. In Example 4 we considered the algebra  $A := \mathbb{N} \cup \{z\}$  with the successor operation on  $\mathbb{N}$  and  $z$  a fixed point for the unary operation  $s$ . As  $\{z\}$  is the  $s$ -fixed point set of  $A$ , we have  $\{z\} = i(\Delta_A)$  (where  $\Delta_A$  denotes the minimal congruence relation on  $A$ ). As noted in Example 4,  $\Omega_A(\{z\})$ , being the coarsest congruence not relating elements of  $\{z\}$  with elements outside  $\{z\}$ , is the congruence collapsing all elements of  $\mathbb{N}$  to a single element and leaving  $z$  alone. The quotient  $A/\Omega_A(\{z\})$  consists of two elements, both of which are  $s$ -fixed points. Hence,  $i(\Omega_A(\{z\}))$ , the set of elements of  $A$ , which become  $s$ -fixed points in  $A/\Omega_A(\{z\})$ , is all of  $A$ . Altogether we have  $i(\Omega_A(i(\Delta_A))) = A \supsetneq \{z\} = i(\Delta_A)$ .

Continuing to consider the previous example we may apply  $\Omega$  to the inequality  $i(\Omega_A(i(\Delta_A))) = A \supsetneq \{z\} = i(\Delta_A)$  and obtain  $\Omega(i(\Omega(i(\Delta_A)))) = \Omega(A) = A \times A \supsetneq \Omega(\{z\}) = \Omega(i(\Delta_A))$  where the strict inequality comes from the fact that the quotient of the left hand congruence yields a one element set and the quotient of the right hand congruence yields a two element set. This shows that one also in general does not have equality in the inclusion  $\Omega \circ i \circ \Omega \circ i \supseteq \Omega \circ i$ . However, in all examples we considered, the iterated application of  $\Omega \circ i$  stabilizes at some point (in the above example this happens in the next step).

### QUESTION 2.

Does iterated application of  $\Omega \circ i$  always stabilize, i.e. is there for any congruence  $\theta$  an  $n \in \mathbb{N}$  such that  $(\Omega \circ i)^{on}(\theta) = (\Omega \circ i)^{o(n+1)}(\theta)$ ? If yes, is there a uniform bound on  $n$  for all subsets  $F$  of a fixed algebra  $A$ ?

While the condition  $\theta = \Omega(i(\theta))$  is equivalent to algebraizability, the weaker condition  $i(\theta) = i(\Omega(i(\theta)))$  can also hold for non-algebraizable logics: In Theorem 4.11 below we will see that for a regularly weakly algebraizable logic  $\Omega$  is a left adjoint of  $i$ . Then the weaker condition holds by the triangle identity. Thus, any regularly weakly algebraizable logic that is not algebraizable shows that the second condition is weaker.

### QUESTION 3.

Can one give a meaningful characterization of the class of equational filter pairs for which  $i = i \circ \Omega \circ i$  holds? Is it precisely the class of filter pairs given by equations of the form  $x = \top$ ?

The following discussion will shed some more light on this question.

We have seen that the operator  $\mathcal{E} \circ i$  associates to a congruence  $\theta$  the *smallest* congruence in whose quotient the same elements get mapped to solutions of the equations  $\tau$  set as in the quotient by  $\theta$  – in particular this smallest congruence exists. If  $i$  has a right adjoint  $R$ , then the operator  $R \circ i$  associates

to a congruence  $\theta$  the *biggest* congruence, which has the same  $\tau$ -solution set as  $\theta$ . This biggest congruence exists if and only if there is a right adjoint, but this need not be the case in general.

One may suggest the mental image that the Leibniz operator is trying to be a right adjoint of  $i$ , but may fail to be so. Indeed, the failure may lie in the fact that it is not even an order preserving map for non-protoalgebraic logics. For protoalgebraic logics, the Leibniz operator is not only order preserving, but also preserves arbitrary infima, and hence by Theorem 1.2 has a left adjoint. The description of the left adjoint offered by that same theorem resembles the description of  $i$  in the case of an assertional logic, as we will see below.

PROPOSITION 4.10

Let  $l$  be a protoalgebraic logic. Then the Leibniz operator from the *filters* to the congruences has a left adjoint given by  $L(\theta) = \bigcap \{F : \theta \text{ does not relate elements of } F \text{ with elements outside } F\}$ .

PROOF. Since it preserves arbitrary infima,  $\Omega$  has a left adjoint  $L$ . By the adjunction formula Thm. 1.2 it is given by:

$$\begin{aligned} L(\theta) &= \bigwedge \{F : \theta \subseteq \Omega(F)\} \\ &= \bigwedge \{F : \theta \subseteq \bigvee \{\rho : F \text{ is a union of } \rho - \text{equivalence classes}\}\} \\ &= \bigcap \{F : \theta \text{ does not relate elements of } F \text{ with elements outside } F\} \end{aligned}$$

Arbitrary intersections of sets that are compatible with a given equivalence relation  $\theta$  are compatible with  $\theta$  again. Arbitrary intersections of filters are filters again, so replacing the infimum on the last step with an intersection is ok.  $\square$

We would like to relate this to the right adjoint  $i$  occurring in a filter pair. Consider the case where  $i$  is given by the equation  $(x, \top)$ , then we have the formula  $i(\theta) = \{a \ a\theta \top\} = [\top]_\theta$ . We have  $i(\theta) \subseteq F$  for each filter  $F$  such that  $\theta \subseteq \Omega(F)$ , thus  $i(\theta) \subseteq L(\theta)$ . The opposite inclusion,  $L(\theta) \subseteq i(\theta)$ , holds too, since  $[\top]_\theta$  occurs among the filters over which we take the intersection in the formula for  $L(\theta)$  (this inclusion is also a consequence of the adjunction, since  $\theta \subseteq \Omega(i(\theta))$ , because  $\theta$  is compatible with  $i(\theta)$ ). **Thus,  $i = L$  !**

The hypothesis that our logic admits a presentation by a congruence filter pair with  $i$  given by the equation  $(x, \top)$  is equivalent to asking that it is assertional. By [21, 6.125], a logic is assertional and protoalgebraic if and only if it is regularly weakly algebraizable.

Thus, we arrive at the following result:

THEOREM 4.11

Let  $l$  be the logic associated to a congruence filter pair  $(Co_K, i)$ , with  $i$  given by the equation  $(x, \top)$ . Then  $l$  is regularly weakly algebraizable if and only if  $\Omega$  is a left adjoint of  $i$ .

PROOF. One direction has been shown in the above discussion: If a logic is regularly weakly algebraizable, then it is protoalgebraic and we have seen above that then the left adjoint of  $\Omega$  coincides with  $i$ .

In the other direction note that if  $i$  is left adjoint to  $\Omega$ , then  $\Omega$ , being a right adjoint, preserves arbitrary infima and hence the logic is protoalgebraic. Assertionality follows directly from the filter pair being given by the equation  $(x, \top)$ .  $\square$

Note that the equation  $i = i \circ \Omega \circ i$ , shown independently before, now follows immediately from the adjointness of  $i$  and  $\Omega$ .

LEMMA 4.12

$\Omega : Fi \rightarrow Co_K$  has a left adjoint if and only if the logic is protoalgebraic.

## 26 Congruence filter pairs, equational filter pairs and adjoints

PROOF.  $\Omega$  has a left adjoint if and only if it preserves arbitrary infima (by Thm. 1.2) if and only if the logic is protoalgebraic, by Thm. 6.4 in [21].  $\square$

### PROPOSITION 4.13

Let  $l$  be the logic associated to a filter pair  $(Co_K, i)$ , with  $i$  given by the equation  $x = \top$ . Suppose that  $l$  is protoalgebraic and that the class of reduced algebras coincides with  $K$ . Then  $l$  is algebraizable.

PROOF. We know already that  $l$  is weakly regularly algebraizable (by a combination of the previous Lemma and Proposition). It remains to show that  $l$  is also equivalential.

For a weakly algebraizable logic, by [21, Thm. 6.117] the Leibniz operator is a bijection  $\Omega^A: Fil_A \cong Con_{Alg^*} A$  from filters to congruences relative the class of reduced algebras for  $l$ . If this class coincides with  $\mathbf{K}$ , then we have that  $\Omega$  is a bijection  $\Omega^A: Fil_A \cong Con_{\mathbf{K}} A$ . Its adjoint  $i$  is then also a bijection and hence the logic is algebraizable. Alternatively, since  $i$  is the adjoint, and hence inverse, of  $\Omega$ , and  $i$  is a natural transformation,  $\Omega$  must be natural as well, which means that  $l$  is equivalential by [21, Cor. 6.69].  $\square$

Note that, in this case,  $\mathbf{K}$  is a quasivariety that is the equivalent algebraic semantics for  $l$ .

### EXAMPLE 6

Consider the filter pair  $(Co_K, i^\tau)$  where  $\mathbf{K}$  is the variety of all groups, axiomatized with a constant symbol  $e$  for the neutral element, and  $i^\tau$  given by the equation  $\tau = (x, e)$ .

The associated logic is protoalgebraic with implication connective  $y^{-1}x$ : This follows from the fact that if  $y^{-1}x = e$  and  $x^{-1}z = e$  then also  $y^{-1}z = y^{-1}xx^{-1}z = ee = e$ . Furthermore, every group has  $\{e\}$  as a reduced filter, so every group is the underlying algebra of a reduced matrix. By Prop. 4.13 the associated logic is algebraizable.

A similar reasoning applies to the filter pair  $(Co_K, i^\tau)$  with  $\mathbf{K}$  the variety of all rings, and  $i^\tau$  given by the equation  $\tau = (x, 0)$ .

On the other hand, classical logic  $l_c$  has as algebraic semantics the quasivariety generated by the orthomodular lattice  $O_6$  (see [21, Ex. 4.79]), which is not the class of boolean algebras. Thus, the criterion of Prop. 4.13 is sufficient but not necessary for a logic to be algebraizable. This comes from the possibility of giving an algebraic semantics to an algebraizable logic that is not an equivalent algebraic semantics.

## 5 Final remarks and future works

We sketch a few directions into which the work on filter pairs may be taken. We begin with three sets of questions upon which we have touched in this work and which may be developed further.

### REMARK 5.1

#### Adjoint operators and the Leibniz hierarchy

With the sample of results of section 4, the topic of adjunctions and filter pairs is far from exhausted. While we have considered the inclusions  $\theta \subseteq \Omega(i(\theta))$ ,  $i(\theta) \subseteq i(\Omega(i(\theta)))$  etc. and the meaning of equality holding in such an inclusion, one can, for example, also consider when equality holds for the opposite compositions  $i(\Omega(F)) = F$ ,  $\Omega(i(\Omega(F))) = \Omega(F)$  and so on. Concrete descriptions of the operator  $\Omega \circ i$  and its companions can be given for equational filter pairs, with assertional logics being the easiest case. Each of these conditions will define a class of logics, possibly extending the Leibniz hierarchy.

A natural further step will be to give corresponding treatments of the Suszko operator and work out the relation to the strong version of a logic.

Furthermore, in [25], [26] there is an interesting approach to the Leibniz hierarchy: Roughly, Leibniz classes are taken to be classes defined by the condition of receiving a translation from a fixed class of logics. It is an interesting question whether the classes of logics defined via properties of the operators  $i, \Omega, \mathcal{E}$  and their adjoints, are Leibniz classes in the sense of [25], [26].

REMARK 5.2

**Bridge theorems through equational filter-pairs.**

In this work we showed how to express standard notions of abstract algebraic logic in the language of filter pairs. But one can also prove theorems within this formalism. For example, it is well-known that for algebraizable logics, the amalgamation property of the associated quasivariety implies the Craig interpolation property of the logic. In [3][Section 5] we use the language of congruence filter pairs to prove that the amalgamation property implies Craig interpolation for a class of algebraic semantics encompassing both equivalent algebraic semantics and regular semantics in regular varieties – the latter case includes also logics that are not even protoalgebraic.

The class of algebraic semantics is characterized by the condition that the operator  $\mathcal{E}$  be a natural transformation. The proof suggests that similar results may be obtained for other operators between  $\mathcal{E}$  and  $\Omega$ .

More broadly, it is a natural task to take further classical bridge theorems, and prove and generalize them in the language of filter pairs, like the ones involving Beth definability or admissible closedness.

REMARK 5.3

**Studying logics by variation of their defining equation.**

Even if one is only interested in a single logic with an algebraic semantics, one may get insight by varying the equation. Suppose that the logic in question has the usual propositional connectives  $\neg, \wedge, \vee$ . Then, maintaining the same variety  $\mathbf{K}$ , one for example has also the logics associated to the equations  $(x, \neg x), (x \wedge \neg x, \perp), (x \vee \neg x, \top)$  or  $(x, \neg\neg x)$ . The theorems of these logics can be seen to be the “contingent formulas” (which are “as true as they are false”, which, e.g. makes sense in many-valued logic), the consistent formulas (which is a non-trivial property for paraconsistent logics), the exhaustive formulas (which, together with their negation exhaust all possible cases) or the classical formulas (e.g. considered in intuitionistic logic as the image of the double negation translation).

The consequence relation of these new logics can, roughly, be understood as  $\Gamma \vdash_{cons} \varphi$  saying that if all  $\gamma \in \Gamma$  are, say, consistent, then so is  $\varphi$ . The relationship to the initially given logic is ensured by taking the same quasivariety  $\mathbf{K}$ .

One can start exploring the new associated logics with the help of the congruence filter pair presentations. For example, suppose that one knows that the initial logic is algebraizable and enjoys the Craig interpolation property. Then one knows that  $\mathbf{K}$  has the amalgamation property and can try to infer Craig interpolation for the new logics by the techniques of [3][Section 5].

While the previous directions of further research are close to the contents of this article, there are also quite different directions into which the idea of filter pairs may be usefully developed.

REMARK 5.4

**Further examples of filter pairs.** In this work we have emphasized the class of congruence filter pairs. But there are other interesting classes. For example one can consider, instead of the lattice of

(relative) congruences on algebras, the lattice of (relative) congruences for some subsignature (or more generally a morphism of signatures with target the signature of the given logic).

Let us consider the extreme example of the empty subsignature: Then the functor part of the filter pair will associate to an algebra the lattice of equivalence relations on its underlying set. In general a quotient by such an equivalence relation does not inherit an algebra structure. It *does*, however, inherit a structure of *multialgebra*, i.e. a gadget with many-valued operations corresponding to the connectives of the initially given signature. This could connect to Avron's non-deterministic semantics [7] [8], and maybe make it possible to relate metalogical properties to non-deterministic matrices in the style of algebraic logic.

Going further, one may take a subsignature and consider consequence relations only with respect to that subsignature. Repeating the considerations of the previous section, one could hope to obtain a partial Craig interpolation theorem, talking only about the fragment given by the subsignature.

Finally, allowing more general signature morphisms, and considering congruences for the pullbacks of algebras along these, one may obtain a framework for remote algebraization from the left – the dual to the possible translation semantics of [10] [11].

Also any functorial reflective sublattice of the lattice of all congruences can be considered, as long as the inclusion preserves directed suprema. The passage to a reflective sublattice will result in a stronger logic (to see this just consider the composition of the two adjunctions in question).

Of course, lattices do not need to consist of congruences at all. One possible direction, still staying on the side of algebra, is to let  $\mathbf{K}$  be any reflexive subcategory of  $\Sigma$ -Str, not necessarily a quasivariety, and for a  $\Sigma$ -algebra  $A$  consider a lattice of maps into certain members of  $\mathbf{K}$  ordered by factorization. In a different direction, both the quotient and the subobject lattices in a locally finitely presentable category are algebraic lattices [30, Thm. 5, Thm. 12]. One might try to take the subobject lattices in a category of coalgebras and obtain a new point of view on coalgebraic semantics.

#### REMARK 5.5

**Algebraic semantics in other categories than Sets.** One may adjust the notion of filter pair by changing the domain category from  $\Sigma$ -structures in the category of sets to  $\Sigma$ -structures in different categories. Not only will this extend the range of semantics that can be given to propositional logics. It should also render the results in [3][Section 5] far more useful: While regular varieties defined over the category of sets are scarce, it is precisely regularity that allows to interpret such varieties in arbitrary symmetric monoidal categories. For example, the variety of rings is not regular, but the variety of monoids is. Hence, one can say what a monoid in a symmetric monoidal category is. Taking the category of abelian groups with tensor product, one recovers the category of rings as a regular variety relative to the category of abelian groups. Associated to this is the equational logic of monoids internal to the category of abelian groups. A relative algebraic semantics for a logic can now be defined by interpreting it into this equational logic. All of this is readily expressed in the language of filter pairs.

#### REMARK 5.6

**On categories of infinitary logics and infinitary filter pairs.** Motivated by a question on the existence of natural extensions of a logic, posed by Cintula and Noguera in [19] and in the meantime solved by Přenosil [31], we have extended in [2] the notion of finitary filter pair to an infinitary version called  $\kappa$ -filter pair, where  $\kappa$  is any regular cardinal. We show that a  $\kappa$ -filter pair gives rise to a logic of cardinality  $\kappa$  and every logic of cardinality  $\kappa$  comes from a  $\kappa$ -filter pair, extending the result for  $\kappa = \omega$  contained in this work. Together with an analogue of Prop. 2.12 this yields a

solution to the construction of natural extensions, which turns out to be identical to one of Přenosil's solutions, but obtained on a different route.

Due to the possible non-uniqueness of natural extensions, the ambiguity of different presentations of a single logic by different filter pairs is bigger than in the finitary case. This suggests a point of view of a filter pair as a logic together with a family of natural extensions to sets of variables of all arities.

The natural task arises to extend the results of this work to the infinitary case. This is carried out in [2] for the results of Section 2.

#### REMARK 5.7

**On a representation theory of logics.** Taking adequate notions of morphisms, we show that the category of logics of cardinality  $\kappa$  and translation morphisms is (isomorphic to) a full reflective subcategory of the category of  $\kappa$ -filter pairs: Thus, this can be seen as a natural sequel of the work initiated in [29], where there is established a full correspondence of certain functors between categories of  $\Sigma$ -structures and translations between algebraizable logics.

The presentation of functorial encodings of morphisms of logics, will play a role in the long term project of establishing a representation theory of logics, i.e. studying arbitrary logics through their translations into a class of "well-behaved logical objects", such as the class of algebraizable logics ([28], [29]) or the class of filter pair matrices ([1]).

A possibly interesting use of this machinery is to establish local-global principles in the realm of Propositional Logic, in particular in the study of meta-logical properties of logical systems. A concrete goal is to establish sufficient conditions for a meta-logical property to be preserved under constructions such as products, filtered colimits, among others.

## Acknowledgements

The authors express their gratitude to the referees for their contribution in improving the presentation.

## References

- [1] P. Arndt, H. L. Mariano and D. C. Pinto. Finitary filter pairs and propositional logics. *South American Journal of Logic*, **4**, 257–280, 2018.
- [2] P. Arndt, H. L. Mariano and D. C. Pinto. Filter pairs and natural extensions of logics. *Archive for Mathematical Logic*, **62**, 113–145, 2023.
- [3] P. Arndt, H. L. Mariano and D. C. Pinto. Congruence filter pairs. *Adjoints and Leibniz Hierarchy*, arXiv, 2021. <https://arxiv.org/abs/2109.01065>.
- [4] J. Adámek and J. Rosický. *Locally Presentable and Accessible Categories*. Lecture Notes Series of the LMS, vol. 189. Cambridge University Press, Cambridge, Great Britain, 1994.
- [5] J. Adámek, J. Rosický and E. M. Vitale. Algebraic theories. *Cambridge Tracts in Mathematics*, **184**, 2011.
- [6] V. A. Artamonov. *Quasivarieties*. Handbook of Algebra, vol. 3, pp. 861–883, 2003.
- [7] A. Avron. Non-deterministic matrices and modular semantics of rules. In *Logica Universalis*, J.-Y. Béziau., ed. Birkhäuser, Basel, 2005.
- [8] A. Avron and A. Zamansky., Non-deterministic Semantics for Logical Systems, to appear in *Handbook of Philosophical Logic*, <http://www.cs.tau.ac.il/aa/articles/nmatrices-survey.pdf>

- [9] S. Bloom, D. Brown and R. Suszko. Some theorems on abstract logics. *Algebra Logika*, **9**, 274–280, 1970.
- [10] J. Bueno, M. E. Coniglio and W. A. Carnielli. Finite algebraizability via possible-translations semantics. In *Proceedings of CombLog 04 - Workshop on Combination of Logics: Theory and Applications*, W. A. Carnielli and F. M. D. e. P. Mateus., eds, pp. 79–86, 2004.
- [11] J. Bueno-Soler, M. E. Coniglio and W. A. Carnielli. Possible-translations Algebraizability. In *Paraconsistency with No Frontiers*, J.-Y. Beziau and W. Carnielli., eds. North-Holland, 2006.
- [12] W. J. Blok and D. Pigozzi. Algebraizable logics. In *Memoirs of the AMS*, vol. 396. American Mathematical Society, Providence, USA, 1989.
- [13] W. J. Blok and J. G. Raftery. Ideals in quasivarieties of algebras, in *Models, Algebras and Proofs* ( X. Caicedo and C.H. Montenegro., eds.). *Lecture Notes in Pure and Applied Mathematics*, **203**, 167–186, 1999.
- [14] W. J. Blok and J. Rebagliato. Algebraic semantics for deductive systems. *Abstract Algebraic Logic, Part II (Barcelona, 1997)*. *Studia Logica*, **74**, 153–180, 2003.
- [15] J. Czelakowski. *Protoalgebraic Logic, Trends in Logic*. Studia Logica Library. Kluwer Academic Publishers, 2001.
- [16] J. J. Czelakowski. Logical matrices and the amalgamation property. *Studia Logica*, **41**, 329–342, 1982.
- [17] J. Czelakowski. *The Equationally Defined Commutator*. Birkhäuser, 2015.
- [18] J. Czelakowski and D. Pigozzi. Amalgamation and interpolation in abstract algebraic logic. In *Models, Algebras and Proofs*, X. Caicedo and C. H. Montenegro., eds, pp. 187–265. Marcel Dekker, ISBN 0-8247-1970-0, New York - Basel, 1999.
- [19] P. Cintula and C. Noguera. A note on natural extensions in abstract algebraic logic. *Studia Logica*, **103**, 815–823, 2015.
- [20] J. M. Font, R. Jansana and D. Pigozzi. A survey of abstract algebraic logic. *Studia Logica*, **74**, 2003.
- [21] J. M. Font. *Abstract Algebraic Logic: An Introductory Textbook*. College Publications, London, 2016.
- [22] V. A. Gorbunov. *Algebraic Theory of Quasivarieties*. Siberian School of Algebra and Logic, Consultants Bureau, New York, 1998.
- [23] A. Horn and N. Kimura. The category of semilattices. *Algebra Universalis*, **1**, 26–38, 1971/72.
- [24] E. Hoogland. Algebraic characterizations of various Beth definability properties. *Studia Logica*, **65**, 91–112, 2000.
- [25] R. Jansana and T. Moraschini. The poset of all logics I: Interpretations and lattice structure. *The Journal of Symbolic Logic*, **86**, 935–964, 2021. <https://doi.org/10.1017/jsl.2021.48>.
- [26] R. Jansana and T. Moraschini. The poset of all logics II: Leibniz classes and hierarchy. *The Journal of Symbolic Logic*, **88**, 324–362, 2021. <https://doi.org/10.1017/jsl.2021.49>.
- [27] J. Łós and R. Suszko. Remarks on sentential logics. *Proceedings Koninklijke Nederlandse Akademie van Wetenschappen, Series A*, **61**, 177–183, 1958.
- [28] H. L. Mariano and D. C. Pinto., Representation Theory of Logics: A Categorical Approach. arXiv preprint, <http://arxiv.org/abs/1405.2429>, 2014.
- [29] H. L. Mariano and D. C. Pinto. Algebraizable logics and a functorial encoding of its morphisms. *Logic Journal of the IGPL*, **25**, 524–561, 2017.
- [30] H.-E. Porst., Algebraic lattices and locally finitely presentable categories. *Algebra Universalis*, **65** (2011), Nr. **3**, p. 285–298
- [31] A. Přenosil. Constructing natural extensions of propositional logics. *Studia Logica*, **194**, 1179–1190, 2016.

- [32] J. G. Raftery. The equational definability of truth predicates. *Reports on Mathematical Logic*, **41**, 95–149, 2006.
- [33] P. Taylor., *Practical Foundations of Mathematics*. Cambridge Studies in Advanced Mathematics **59**, Cambridge University Press, 1999.

Received 19 July 2023