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The notion of postprojective partition of the module category mod$A$ for an artin algebra A (or, more generally, of a subcategory of mod$A$) was introduced by Auslander and Smalø in [4] (under the name of preprojective partition) in order to generalise the covering property of the projective modules. The problems involving these partitions are mostly of two kinds: either to find algorithms for their calculation or to establish connections with other research areas of the representation theory of algebras. We deal here with some aspects of both types of problems.

The situation we consider is the following. Let A be an artin algebra, T_A be a tilting module and B =End (T_A). Our objective is to describe the postprojective partitions for each of the classes $\mathcal{X}$ and $\mathcal{Y}$ of the torsion pair $(\mathcal{X}, \mathcal{Y})$ induced by T_A in mod$B$, in particular in the situation where A is hereditary, and thus B is tilted. Our starting point is the notion of relative postprojective partitions. Indeed, it was observed by Betzler in [2] that the notion of postprojective partition in a subcategory $\mathcal{C}$ of mod$A$ depends upon certain properties of epimorphisms. Abstracting these properties led to the definition of an epiclass in $\mathcal{C}$ (see section 1 below) and, attached to each epiclass, to a relative postprojective partition in $\mathcal{C}$. As expected, the epiclass consisting of all the epimorphisms yields the usual postprojective partition of $\mathcal{C}$. An example of epiclass is given by covariant functors, namely, if $F: \mathcal{C} \longrightarrow \mathcal{A}$ is a covariant functor where $\mathcal{C}$ is an abelian category and $\mathcal{A}$ is the category of the abelian groups, then the class of all the morphisms f in $\mathcal{C}$ such that $F(f)$ is an epimorphism in $\mathcal{A}$, forms an epiclass. It has been shown in [2] that, if F is a finitely generated covariant functor, then the above epiclass induces a postprojective partition $\{\mathcal{P}_i^F(\mathcal{C})\}$.

Our first aim is to study the relative postprojective partitions induced by the tilting functors. One key observation is that, while the usual postprojective partition are (obviously) not preserved by the tilting functors, the relative postprojective partitions are (see 2.1).

In order to state our main result, we need the following notation. Let (A, T, B) be a tilting triple and denote by $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{X}, \mathcal{Y})$ the torsion pairs induced by T_A in mod$A$ and mod$B$, respectively. For simplicity, we use sometimes the notation $(X, -)$ and ${}^1(X, -)$ for the functors $\mathrm{Hom}_A(X, -)$ and $\mathrm{Ext}^1_A(X, -)$, respectively. Letting D denote the standard duality between mod$A$ and modA^{op}, we denote by $\mathcal{S}$ the additive subcategory of mod$B$ generated by $\tau_B^{-1}(\mathrm{Hom}_A(T_A, \mathrm{D}A))$, and, for an indecomposable module $X \in \mathcal{X}$, we define $n_{\mathcal{S}}(X)$ to be the length of a shortest path of irreducible morphisms from a module in $\mathcal{S}$ to X if such a path exists and infinity otherwise. With this notation, we have the following theorem.

Theorem. Let (A, T, B) be a tilting triple. Then

(a) The functors $\mathrm{Hom}_A(T,-)$ and $-\otimes_B T$ induce quasi-inverse category equivalences between $\mathcal{P}_n^{(T,-)}(\mathcal{T})$ and $\mathcal{P}_n(\mathcal{Y})$ for each n, and the functors $\mathrm{Ext}_A^1(T,-)$ and $\mathrm{Tor}_1^B(-,T)$ induce quasi-inverse category equivalences between $\mathcal{P}_n^{1(T,-)}(\mathcal{F})$ and $\mathcal{P}_n(\mathcal{X})$ for each n.

(b) If T_A is a splitting tilting module, then $\mathcal{P}_n(\mathcal{Y}) = \mathcal{P}_n(\mathrm{mod}B) \cap \mathcal{Y}$ for each $n \geq 0$.

(c) If A is hereditary, then $\mathcal{P}_n(\mathcal{X}) = \{X \in \mathcal{X} : n_S(X) = n\}$ for each $n \geq 0$.

This paper is organised as follows. After recalling in section 1 the basic results of [2] and proving some preparatory lemmata, we proceed in section 2 to show part (a) of our theorem above, while parts (b) and (c) are proven in section 3. For the basic notions of representation theory, we refer the reader to [3] and, for tilting theory, to [1, 6].

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1 Relative postprojective partitions

Throughout this work, A will denote a basic and connected artin algebra, and $\mathrm{mod}A$ the category of the finitely generated right A-modules . Let $\mathcal{C}$ be an additive subcategory of $\mathrm{mod}A$. The notion of relative postprojective partition in $\mathcal{C}$ relies upon the existence of an epiclass ξ in $\mathcal{C}$, as defined by Betzler in [2]. A class ξ of morphisms in $\mathcal{C}$ is an *epiclass* provided:

(I) 1_M is in ξ for each $M \in \mathcal{C}$.

(S) If $f_i \colon M_i \longrightarrow N_i$, where $1 \leq i \leq n$, belong to ξ, then $\bigoplus_{i=1}^{n} f_i : \bigoplus_{i=1}^{n} M_i \longrightarrow \bigoplus_{i=1}^{n} N_i$ lies also in ξ.

(K) If $f \colon L \longrightarrow M$ and $g \colon M \longrightarrow N$ are in ξ, then $gf \colon L \longrightarrow N$ is also in ξ.

(E) If $f: L \longrightarrow M$ and $g: M \longrightarrow N$ are such that gf is in ξ, then g is in ξ.

A morphism in ξ is called a *ξ-epimorphism.*

It follows from (I) and (E) that any split epimorphism between modules in $\mathcal{C}$ belongs to ξ (see [2](2.1)). In particular, if $M = \bigoplus_i M_i$ lies in $\mathcal{C}$, then the natural projections $p_j: M \longrightarrow M_j$ belong to ξ.

Given an epiclass ξ in $\mathcal{C}$, one can define *ξ-covers* of $\mathcal{C}$ and *splitting ξ-projectives* in $\mathcal{C}$ in the usual fashion (see [4]) by restricting to ξ-epimorphisms. Denote by $\mathcal{O}^\xi$ the class of the indecomposable modules N in ind$\mathcal{C}$ for which the zero morphism $0 \longrightarrow N$ belongs to ξ. Using (E), we infer that, if $N \in \mathcal{O}^\xi$, then any morphism $f: M \longrightarrow N$ also belongs to ξ. Moreover, if $g: N \longrightarrow M$ is in ξ, with M, N indecomposable modules and $N \in \mathcal{O}^\xi$, then M belongs to $\mathcal{O}^\xi$ (by (K)). Now, a partition $\mathcal{P}_0^\xi, \mathcal{P}_1^\xi, \cdots, \mathcal{P}_n^\xi, \cdots, \mathcal{P}_\infty^\xi$ of the class ind$(\mathcal{C} \setminus \mathcal{O}^\xi)$ of all isomorphism classes of indecomposable modules in $\mathcal{C}$ not in $\mathcal{O}^\xi$ is called a *ξ-postprojective partition of* $\mathcal{C}$ provided

(a) $\mathcal{P}_i^\xi \cap \mathcal{P}_j^\xi = \emptyset$ if $i \neq j$, $\cup_{i \leq \infty} \mathcal{P}_i^\xi = \text{ind } (\mathcal{C} \setminus \mathcal{O}^\xi)$;

(b) For each $j < \infty$, $\mathcal{P}_j^\xi$ is a finite minimal ξ-cover of ind$(\mathcal{C} \setminus (\mathcal{P}_0^\xi \cup \cdots \mathcal{P}_{j-1}^\xi))$.

A module $X \in \mathcal{C}$ is called *ξ-postprojective* provided each indecomposable summand of X belongs to $\mathcal{P}^\xi = \cup_{i<\infty} \mathcal{P}_i^\xi$. Also, for $X \in \mathcal{P}_n^\xi$, we write $\pi^\xi(X) = n$, and call it the *ξ-level* of X.

The next result is essentially due to Betzler. We recall that a subcategory $\mathcal{C}$ of mod$A$ is *covariantly finite* if for each X in mod$A$, there exists a morphism $f_X: C_X \longrightarrow X$, with $C_X \in \mathcal{C}$ such that $\text{Hom}_A(f_X, D)$ is an epimorphism for each $D \in \mathcal{C}$.

Theorem 1.1 Let $\mathcal{C}$ be a covariantly finite subcategory of mod$A$, and let ξ be an arbitrary epiclass in $\mathcal{C}$. Then $\mathcal{C}$ has a unique ξ-postprojective partition.

Proof: If $\mathcal{C}$ is covariantly finite, then by [4], it has left almost split morphisms. The result now follows from [2](1.5). □

For more details and examples of relative postprojective partitions we refer the reader to [2]. For the rest of this section, we assume that $\mathcal{C}$ is an additive subcategory of mod$A$ with epiclass ξ admitting a ξ-postprojective partition $\{\mathcal{P}_i^\xi\}$. For our first results, we need the following definition. Let $f: N \longrightarrow \bigoplus_{i=1}^{n} M_i$ be a ξ-epimorphism in $\mathcal{C}$, with all M_i indecomposable. We say that f is *componentwise non-split* provided, for each j, the composite $p_j f$ is not split, where $p_j: \bigoplus_{i=1}^{n} M_i \longrightarrow M_j$ denotes the natural projection.

Lemma 1.2 Let $f: \bigoplus_{i=1}^{n} N_i \longrightarrow \bigoplus_{j=1}^{m} M_j$ be a componentwise non-split ξ-epimorphism, where the N_i and the M_j are indecomposable modules. Then there exists an l such that $\pi^\xi(N_l) < \min\{\pi^\xi(M_j): 1 \leq j \leq m\}$.

Proof: Follows directly from the definitions. □

We denote, for an A-module M, by add$M$ the additive subcategory of mod$A$ consisting of the direct sums of indecomposable summands of M.

Proposition 1.3 Let $0 < n < \infty$, then the following conditions are equivalent for $X \in$ ind$\mathcal{C}$:

(a) X belongs to $\mathcal{P}_n^\xi$;

(b) There exists a chain of ξ-epimorphisms

$$(*) \qquad X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \longrightarrow \cdots \longrightarrow X_{t-1} \xrightarrow{f_t} X_t = X$$

with add$X_{i-1} \cap$ add $X_i = \emptyset$ for $i = 1, \cdots, t$, and length $t = n$, and any chain as $(*)$ has length $t \leq n$;

(c) There exists a chain of componentwise non-split ξ-epimorphisms

$$(**) \qquad Y_0 \xrightarrow{g_1} Y_1 \xrightarrow{g_2} Y_2 \longrightarrow \cdots \longrightarrow Y_{t-1} \xrightarrow{g_t} Y_t = X$$

with length $t = n$, and any chain as $(**)$ has length $t \leq n$.

Proof: (a) implies (b). We construct easily a chain as $(*)$ by descending induction, choosing, for each $j = 1, \cdots, n$ the morphism f_j to be a $\mathcal{P}^{\xi}_{j-1}$-cover of X_j. Suppose now that we have a chain of ξ-epimorphisms

$$Z_0 \xrightarrow{h_1} Z_1 \xrightarrow{h_2} Z_2 \cdots Z_{t-1} \xrightarrow{h_t} Z_t = X$$

with add$Z_{i-1} \cap$ add $Z_i = \emptyset$ for $i = 1, \cdots, t$. Fix an i and write $Z_i = \oplus_{j=1}^{n_i} Z_{ij}$ with Z_{ij} indecomposable. Denoting by p_j the canonical projection $Z_i \longrightarrow Z_{ij}$, the ξ-epimorphisms $h_{ij} = p_j h_i : Z_{i-1} \longrightarrow Z_{ij}$ do not split because add$Z_{i-1} \cap$ add $Z_i = \emptyset$. By 1.2, there exists an indecomposable summand Z'_{i-1} of Z_{i-1} such that $\pi^{\xi}(Z'_{i-1}) < \min\{\pi^{\xi}(Z_{ij}) : j = 1, \cdots, n_i\}$. Proceeding inductively, we infer that, for each i, there exists an indecomposable summand Z'_i of Z_i with $\pi^{\xi}(Z'_i) \leq n - t + i$, leading to a contradiction if $t > n$.

(b) implies (c). Observe that a chain as $(*)$ satisfies the required condition for $(**)$. Indeed, any ξ-epimorphism $M \longrightarrow N$ with add$M \cap$ add$N = \emptyset$ is componentwise non-split. Let now

$$Z_0 \xrightarrow{h_1} Z_1 \xrightarrow{h_2} Z_2 \longrightarrow \cdots \longrightarrow Z_{t-1} \xrightarrow{h_t} Z_t = X$$

be a chain of componentwise non-split ξ-epimorphisms. If $t > n$, then the same argument used in the implication (a) implies (b) above yields that $\pi^{\xi}(X) \geq t$. Therefore, there exists a chain as $(*)$ of length $t > n$ which contradicts our hypothesis.

(c) implies (a). The hypothesis that a chain as $(**)$ exists yields that $\pi^{\xi}(X) \geq n$. On the other hand, if $\pi^{\xi}(X) = m > n$, then, by the implications (a) implies (b) implies (c) we get a longer chain, a contradiction to our hypothesis. □

2 Relative postprojective partition given by functors

In this section, we investigate the case where the epiclass is defined by a covariant functor. As observed in [2], given an additive category $\mathcal{C}$ and a covariant functor $F: \mathcal{C} \longrightarrow \mathcal{A}$, the class of morphisms $\xi = \{f: F(f) \text{ is an epimorphism }\}$ is an epiclass in $\mathcal{C}$. Also, if F is finitely generated, then $\mathcal{C}$ has a ξ-postprojective partition (see [2](2.1)). In this situation where the epiclass is defined by the functor F, we write F-epimorphisms, F-postprojective partition and so on.

Let $\mathcal{C}$ and $\mathcal{D}$ be two additive categories and $G: \mathcal{C} \longrightarrow \mathcal{D}$ be an equivalence with quasi-inverse $G': \mathcal{D} \longrightarrow \mathcal{C}$. Let also $F: \mathcal{C} \longrightarrow \mathcal{A}$ be a finitely generated covariant functor and $F' = FG': \mathcal{D} \longrightarrow \mathcal{A}$. As observed above, $\mathcal{C}$ and $\mathcal{D}$ have relative postprojective partitions $\{\mathcal{P}_i^F(\mathcal{C})\}$ and $\{\mathcal{P}_i^{F'}(\mathcal{D})\}$, respectively. With this notation, we have the following result.

Proposition 2.1 Let f be a morphism in $\mathcal{C}$. Then

(a) f is an F-epimorphism if and only if $G(f)$ is an F'-epimorphism.

(b) f is a split F-epimorphism if and only if $G(f)$ is a split F'-epimorphism.

(c) For each n, $G(\mathcal{P}_n^F(\mathcal{C})) = \mathcal{P}_n^{F'}(\mathcal{D})$.

(d) $\mathcal{O}^F(\mathcal{C}) = \mathcal{O}^{F'}(\mathcal{D})$.

Proof: (a) By definition, f is an F-epimorphism if and only if $F(f)$ is an epimorphism, and $G(f)$ is an F'-epimorphism if and only if $F'(G(f))$ is an epimorphism. The statement now follows from $F'G(f) \cong FG'G(f) \cong F(f)$.

(b) Observe that $G(f)$ is a split F'-epimorphism if and only if there exists a morphism g such that $1 = F'G(f) \cdot F'G(g) = F(f) \cdot F(g)$ if and only if f is a split F-epimorphism.

(c) We first show the result for $n = 0$. Let M be the direct sum of the modules in $\mathcal{P}_0^F(\mathcal{C})$. We claim that $Y = G(M)$ is a relative generator for $\mathcal{D}$. However, for X in $\mathcal{D}$, we have $X \cong G(N)$, with $N \in \mathcal{C}$. By definition, there exists an F-epimorphism $f: M' \longrightarrow N$, with M' in add$M$. By part (a), we have an F'-epimorphism $G(f): G(M') \longrightarrow G(N)$, as required. Conversely, suppose $Y = G(M)$ in $\mathcal{D}$ is a relative generator for $\mathcal{D}$. So, for each N in $\mathcal{D}$, there exists an F'-epimorphism $g': Y' \longrightarrow G(N)$, with Y' in add$Y$. Since $Y' = G(M')$, for some M' in add$M$, and $g' = G(g)$ for some morphism $g: M' \longrightarrow N$, we infer that g is an F-epimorphism. Therefore, M is a relative generator for $\mathcal{C}$.

Suppose now $n > 0$. It follows from (a) and (b) that the following conditions are equivalent:

(1) $M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \longrightarrow \cdots \longrightarrow M_{t-1} \xrightarrow{f_t} M_t$ is a chain of F-epimorphisms in $\mathcal{C}$ with add$M_{i-1} \cap$ add $M_i = \emptyset$ for $i = 1, \cdots, t$; and

(2) $G(M_0) \xrightarrow{G(f_1)} G(M_1) \xrightarrow{G(f_2)} G(M_2) \longrightarrow \cdots \longrightarrow G(M_{t-1}) \xrightarrow{G(f_t)} G(M_t)$ is a chain of F'-epimorphisms in $\mathcal{D}$ with add$G(M_{i-1}) \cap$ add $G(M_i) = \emptyset$ for $i = 1, \cdots, t$.

The conclusion now follows from 1.2.

(d) Clearly, $\mathcal{O}^F(\mathcal{C}) = \{M : F(M) = 0\}$. The statement now follows at once. □

Corollary 2.2 Assume that, above, $\mathcal{C}$ and $\mathcal{D}$ are full additive subcategories of two module categories mod$A$ and mod$B$ respectively, and let $F': \mathcal{D} \longrightarrow \mathcal{A}$ denote the forgetful functor, then $\mathcal{P}_n(\mathcal{D}) = G(\mathcal{P}_n^{F'G}(\mathcal{C}))$ for each $n \geq 0$.

From now on, we specialise our result to tilting functors. The following result, which is just part (a) of our main theorem, is now a direct consequence of the above discussion.

Corollary 2.3 Let (A, T, B) be a tilting triple, and denote by $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{X}, \mathcal{Y})$ the torsion pairs induced by T in mod$A$ and mod$B$, respectively. Then the functors $\mathrm{Hom}_A(T, -)$ and $- \otimes_B T$ induce quasi-inverse category equivalences between $\mathcal{P}_n^{(T,-)}(\mathcal{T})$ and $\mathcal{P}_n(\mathcal{Y})$, and the functors $\mathrm{Ext}^1_A(T, -)$ and $\mathrm{Tor}_1^B(-, T)$ induce quasi-inverse category equivalences between $\mathcal{P}_n^{1(T,-)}(\mathcal{F})$ and $\mathcal{P}_n(\mathcal{X})$.

It is useful to characterise the epiclasses given by the tilting functors.

Proposition 2.4 Let $f\colon M \longrightarrow N$ be a morphism in mod$A$.

(a) Suppose $M, N \in \mathcal{T}$. Then f is a $(T,-)$-epimorphism if and only if f is an epimorphism and Ker$f \in \mathcal{T}$.

(b) Suppose $M, N \in \mathcal{F}$. Then f is an ${}^1(T,-)$-epimorphism if and only if Coker$f \in \mathcal{T}$.

Proof: (a) Suppose first that f is a $(T,-)$-epimorphism, that is, $\mathrm{Hom}_A(T,f)$ is an epimorphism. Applying the right exact functor $-\otimes_B T$, yields that f is an epimorphism. Consider now the short exact sequence $0 \longrightarrow K \longrightarrow M \xrightarrow{f} N \longrightarrow 0$. Applying $\mathrm{Hom}_A(T,-)$ yields an exact sequence

$$\mathrm{Hom}_A(T,M) \xrightarrow{(T,f)} \mathrm{Hom}_A(T,N) \longrightarrow \mathrm{Ext}^1_A(T,K) \longrightarrow \mathrm{Ext}^1_A(T,M)$$

Since $M \in \mathcal{T}$, we have $\mathrm{Ext}^1_A(T,M) = 0$, and since $\mathrm{Hom}_A(T,f)$ is an epimorphism, we infer that $\mathrm{Ext}^1_A(T,K) = 0$. So $K \in \mathcal{T}$ as required.
Conversely, consider the short exact sequence $0 \longrightarrow K \longrightarrow M \xrightarrow{f} N \longrightarrow 0$, which is entirely in $\mathcal{T}$ by hypothesis. Applying $\mathrm{Hom}_A(T,-)$ to it, we get that $\mathrm{Hom}_A(T,f)$ is an epimorphism, that is, f is a $(T,-)$-epimorphism.
(b) Let $M \xrightarrow{f} N \xrightarrow{p} C \longrightarrow 0$ be exact, and consider the short exact sequence $0 \longrightarrow K \xrightarrow{g} N \xrightarrow{p} C \longrightarrow 0$ and the epimorphism $h\colon M \longrightarrow K$ such that $f = gh$. Applying $\mathrm{Hom}_A(T,-)$, yields an exact sequence

$$\mathrm{Hom}_A(T,C) \longrightarrow \mathrm{Ext}^1_A(T,K) \xrightarrow{{}^1(T,g)} \mathrm{Ext}^1_A(T,N) \longrightarrow \mathrm{Ext}^1_A(T,C) \longrightarrow 0$$

Clearly, if $\mathrm{Ext}^1_A(T,f)$ is an epimorphism, then so is $\mathrm{Ext}^1_A(T,g)$. Therefore $\mathrm{Ext}^1_A(T,C) = 0$, or equivalently, $C \in \mathcal{T}$. Conversely, if $C \in \mathcal{T}$, then $\mathrm{Ext}^1_A(T,g)$ is an epimorphism. Since $\mathrm{Ext}^1_A(T,h)$ is an epimorphism (because h is an epimorphism and pd $T \leq 1$), $\mathrm{Ext}^1_A(T,f) = \mathrm{Ext}^1_A(T,g)\cdot \mathrm{Ext}^1_A(T,h)$ is also an epimorphism. □

Remark 2.5 (a) It is well-known (see [1](1.8)) that, if $M \in \mathcal{T}$, then there exists a short exact sequence $0 \longrightarrow M' \longrightarrow T_0 \xrightarrow{f} M \longrightarrow 0$, with $T_0 \in \mathrm{add}T$, and $M' \in \mathcal{T}$. It follows from the above proposition that f is a $(T,-)$-epimorphism. Thus, every module in $\mathcal{T}$ is the target of a $(T,-)$-epimorphism. Similarly, every module in $\mathcal{F}$ is the source of an $^1(T,-)$-epimorphism: namely, if $M \in \mathcal{F}$, we claim that there exists a short exact sequence $0 \longrightarrow M \xrightarrow{f} (\tau_A T)^d \longrightarrow C \longrightarrow 0$, with $C \in \mathcal{T}$. Indeed, since M is cogenerated by $\tau_A T$, we consider a generating set $\{f_1, \cdots, f_d\}$ of $\mathrm{Hom}_A(M, \tau T)$, write $f = [f_1, \cdots, f_d]$, and consider the short exact sequence $0 \longrightarrow M \xrightarrow{f} (\tau_A T)^d \longrightarrow C \longrightarrow 0$. Since, by construction, $\mathrm{Hom}_A(f, \tau_A T) = 0$, applying the functor $\mathrm{Hom}_A(-, \tau_A T)$ yields $\mathrm{Ext}^1_A(T, C) \cong \mathrm{DHom}_A(C, \tau_A T) = 0$, and so $C \in \mathcal{T}$.

(b) It is interesting to describe the classes $\mathcal{P}_0^{(T,-)}(\mathcal{T})$ and $\mathcal{P}_0^{^1(T,-)}(\mathcal{F})$. First, clearly, $\mathcal{P}_0^{(T,-)}(\mathcal{T}) = \mathrm{ind}T$. Next, we observe that $\mathcal{P}_0^{^1(T,-)}(\mathcal{F}) = \{^1(T,-)$-splitting projectives in $\mathcal{F}\} = \{$ splitting projectives in $\mathcal{F}\} = \{$ Ext-projectives in $\mathcal{F}\}$. Denoting by tM the torsion part of an A-module M, we deduce from [1](1.4) that $\mathcal{P}_0^{^1(T,-)}(\mathcal{F}) = \{P/tP : P$ projective $\}$. Since $\mathrm{Ext}^1_A(T, P/tP) \cong \mathrm{Ext}^1_A(T, P)$, this implies that $\mathrm{Ext}^1_A(T, \mathcal{P}_0^{^1(T,-)}(\mathcal{F})) = \mathrm{add}\ \mathrm{Ext}^1_A(T, A)$.

3 Torsion and torsion-free postprojective modules

In the previous section, we have investigated the connections between the postprojective partitions given by the epiclasses induced by the tilting functors and the postprojective partitions of $\mathcal{X}$ and $\mathcal{Y}$. We now investigate further the latter. We initially consider a tilting triple (A, T, B) and assume that T is splitting, that is, the induced torsion pair $(\mathcal{X}, \mathcal{Y})$ in $\mathrm{mod}B$ splits. We now prove part (b) of our main result.

Proposition 3.1 Let (A, T, B) be a tilting triple, and assume that T_A is splitting. Then, for each n, we have $\mathcal{P}_n(\mathcal{Y}) = \mathcal{P}_n(\text{mod}B) \cap \mathcal{Y}$.

Proof: We prove it by induction, the case $n = 0$ being trivial because the indecomposable projective B-modules are in $\mathcal{Y}$. Suppose $n > 0$ and let $Y \in \mathcal{P}_n(\mathcal{Y})$. By 1.3, there exists a chain of epimorphisms

$$(*) \qquad Y_0 \xrightarrow{f_1} Y_1 \xrightarrow{f_2} Y_2 \longrightarrow \cdots \longrightarrow Y_{t-1} \xrightarrow{f_t} Y_t = Y$$

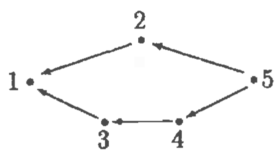
with $t = n$, such that $Y_i \in \mathcal{Y}$ and add$Y_{i-1} \cap$ add $Y_i = \emptyset$ for $i = 1, \cdots, t$. Moreover, any such chain has length at most n. Clearly, $Y \in \mathcal{P}_m(\text{mod}B) \cap \mathcal{Y}$ for some $m \geq n$. If $m > n$, then by 1.3, there exists a chain of epimorphisms

$$Z_0 \xrightarrow{g_1} Z_1 \xrightarrow{g_2} Z_2 \longrightarrow \cdots \longrightarrow Z_{t-1} \xrightarrow{g_t} Z_m = Y$$

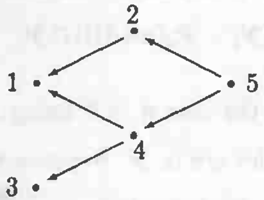
with add$Z_{i-1} \cap$ add $Z_i = \emptyset$ for $i = 1, \cdots, t$ and $Z_i \in \mathcal{Y}$. The maximality of $(*)$ then yields a contradiction. Hence $\mathcal{P}_n(\mathcal{Y}) \subset \mathcal{P}_n(\text{mod}B) \cap \mathcal{Y}$. Similarly, one shows that $\mathcal{P}_n(\text{mod}B) \cap \mathcal{Y} \subset \mathcal{P}_n(\mathcal{Y})$. □

The following example shows that the assumption that T_A is splitting is necessary for the statement to hold.

Example 3.2 Let A be the k-algebra given by the commutative quiver



Then the tilting module $T_A = P_4 \oplus P_5 \oplus \tau^{-1} P_2 \oplus \tau^{-2} P_1 \oplus \tau^{-2} P_4$ (where P_i represents the indecomposable projective A-module associated with the vertex i) is not splitting. The endomorphism algebra B is given by the commutative quiver



and it is easily verified that the indecomposable B-module M of dimension-vector $\underline{\dim} M = (1,1,0,1,1)$ lies in $\mathcal{Y}$, in $\mathcal{P}_1(\mathcal{Y})$, but also in $\mathcal{P}_2(\mathrm{mod}B) \cap \mathcal{Y}$. □

Since T_A is splitting, the class $\mathcal{P}(\mathrm{mod}B) = \cup_{n<\infty} \mathcal{P}_n(\mathrm{mod}B)$ of all postprojective B-modules equals the disjoint union $(\mathcal{P}(\mathrm{mod}B) \cap \mathcal{X}) \cup (\mathcal{P}(\mathrm{mod}B) \cap \mathcal{Y})$. Clearly, $\mathcal{P}_0(\mathrm{mod}B) \cap (\mathcal{P}(\mathrm{mod}B) \cap \mathcal{X}) = \emptyset$. Also, there is no path of irreducible morphisms from a module in $\mathcal{P}(\mathrm{mod}B) \cap \mathcal{X}$ to a module in $\mathcal{P}(\mathrm{mod}B) \cap \mathcal{Y}$. Now, there exists a path of irreducible morphisms from a module in $\mathcal{P}(\mathrm{mod}B) \cap \mathcal{Y}$ to a module in $\mathcal{P}(\mathrm{mod}B) \cap \mathcal{X}$ if and only if $\mathcal{P}(\mathrm{mod}B) \cap \mathcal{X} \neq \emptyset$. Indeed, if $Z \in \mathcal{P}(\mathrm{mod}B) \cap \mathcal{X}$, then there exists a path of irreducible morphisms from an indecomposable projective P, which lies in $\mathcal{P}(\mathrm{mod}B) \cap \mathcal{Y}$, to Z (see [4]). In this case, there exists an irreducible morphism $Y \longrightarrow X$, with $Y \in \mathcal{P}(\mathrm{mod}B) \cap \mathcal{Y}$ and $X \in \mathcal{P}(\mathrm{mod}B) \cap \mathcal{X}$. The next result describes this situation.

Proposition 3.3 Let $f: Y \longrightarrow X$ be an irreducible morphism with $X \in \mathcal{P}(\mathrm{mod}B) \cap \mathcal{X}$ and $Y \in \mathcal{P}(\mathrm{mod}B) \cap \mathcal{Y}$. Then

(a) there exists a B-module Z and a morphism $g: Z \longrightarrow X$ such that $0 \longrightarrow \tau_B X \longrightarrow Y \oplus Z \xrightarrow{[f,g]} X \longrightarrow 0$ is a connecting sequence in $\mathrm{mod}B$; and

(b) there exists a simple module S such that, if I and P denote respectively the injective envelope and the projective cover of S, then $X \cong \mathrm{Ext}^1_A(T,P)$ and $Y \cong \mathrm{Hom}_A(T,J)$ for some indecomposable summand J of I/S.

Proof: Since X is not projective, $\tau_B X \neq 0$, and, since $Y \in \mathcal{Y}$, we infer that $\tau_B Y \in \mathcal{Y}$. Then the almost split sequence ending with X is a connecting sequence. By [1](3.5), there exists a simple A-module S such that this sequence is of the form

$$0 \longrightarrow \mathrm{Hom}_A(T,I) \longrightarrow \mathrm{Hom}_A(T,I/S) \oplus \mathrm{Ext}^1_A(T,\mathrm{rad}P) \longrightarrow \mathrm{Ext}^1_A(T,P) \longrightarrow 0$$

The result now follows easily. □

Corollary 3.4 There exist at most finitely many isomorphism classes of indecomposable modules in $\mathcal{P}(\mathrm{mod}B) \cap \mathcal{X}$ which are the target of an irreducible morphism starting in $\mathcal{P}(\mathrm{mod}B) \cap \mathcal{Y}$.

Corollary 3.5 Assume $\mathcal{P}(\mathrm{mod}B) \cap \mathcal{X} \neq \emptyset$. Then there exists an m such that any path of irreducible morphisms from a projective module to $M \in \mathcal{P}_i(\mathrm{mod}B) \cap \mathcal{X}$ with $i \geq m$ passes through a module in $\mathcal{P}_m(\mathrm{mod}B) \cap \mathcal{X}$.

From now on, we restrict to the situation oin which, in the tilting triple (A,T,B), the algebra A is hereditary, and then B is tilted. The next result gives an algorithm for the the calculation of the postprojective partition $\mathcal{P}_j(\mathcal{X})$ similar to the one Todorov has obtained for the postprojective partition for an hereditary algebra (see [7]).

Proposition 3.6 Let (A,T,B) be a tilting triple with A hereditary , $X \in \mathcal{X}$ and $n \geq 0$. Then $X \in \mathcal{P}_n(\mathcal{X})$ if and only if

(a) for each irreducible morphism $Y \longrightarrow X$ with Y indecomposable, then $Y \in \mathcal{P}_{n-1}(\mathcal{X}) \cup \mathcal{P}_n(\mathcal{X})$; and

(b) there exists $Y \in \mathcal{P}_{n-1}(\mathcal{X})$ and an irreducible morphism $Y \longrightarrow X$ if $n \geq 1$.

Proof: Assume $X \in \mathcal{P}_n(\mathcal{X})$. We prove simultaneously (a) and (b) by induction on $n \geq 0$. Clearly, $\mathcal{P}_0(\mathcal{X}) = \tau_B^{-1}(\ \mathrm{Hom}_A(T,\mathrm{D}A))$, and so the result holds for

$n = 0$. We only sketch the inductive step since it follows very closely the ideas in [5, 7]. Assume $n > 0$ and that the result holds for each $j < n$. Let $0 \longrightarrow \tau X \xrightarrow{f} E \xrightarrow{g} X \longrightarrow 0$ be the almost split sequence ending with X. Observe that it lies in $\mathcal{X}$. Let Y be an indecomposable summand of E and assume that $\pi_{\mathcal{X}}(Y) \leq \pi_{\mathcal{X}}(X) - 2$. Then there exists an epimorphism $h: Z \longrightarrow X$ for some $Z \in \text{add }(\mathcal{P}_{n-1}(\mathcal{X}))$. Since h does not split, there exists $h': Z \longrightarrow E$ such that $gh' = h$. The morphism $(f, h'): \tau_B X \oplus Z \longrightarrow E$ is an epimorphism, implying that $\pi_{\mathcal{X}}(\tau_B X) < \pi_{\mathcal{X}}(Y)$ (I). Decompose E as $E' \oplus E''$, where E' is the sum of all indecomposable summands of E which lie in $\mathcal{P}_0(\mathcal{X}) \cup \cdots \cup \mathcal{P}_{n-2}(\mathcal{X})$ and E'' is a direct complement. Since the Auslander-Reiten component of mod$B$ containing $\tau_B^{-1}(\text{ Hom}_A(T,\text{D}A))$ is a connecting component, it is standard. Using the induction hypothesis, we infer that $\text{Hom}_B(Z, E') = 0$ and then $\text{Im}h' \subset E''$ and $g|_{E''}$ is an epimorphism. Hence E'' has an indecomposable summand M in $\mathcal{P}_{n-1}(\mathcal{X})$ (II). Using (I) and (II), we get an irreducible morphism $\tau_B X \longrightarrow M$ and $\pi_{\mathcal{X}}(\tau_B X) < \pi_{\mathcal{X}}(Y) \leq n - 2 = \pi_{\mathcal{X}}(M) - 1$, a contradiction to the induction hypothesis and so $\pi_{\mathcal{X}}(Y) \geq \pi_{\mathcal{X}}(X) - 1$. It should be clear that there exists an indecomposable summand Y' of E with $\pi_{\mathcal{X}}(Y') = n - 1$. Suppose now $\pi_{\mathcal{X}}(Y) = l > n$. By the induction step, we know that $\pi_{\mathcal{X}}(\tau_B X) \leq n - 1$. Using now inductively the arguments above, we infer that there exists an irreducible morphism $M \longrightarrow N$ with $\pi_{\mathcal{X}}(N) - \pi_{\mathcal{X}}(M) \geq 2$, a contradiction to the inductive hypothesis. □

We now prove part (c) of our theorem. Denote by $\mathcal{S}$ the class of modules $\tau_B^{-1}(\text{ Hom}_A(T, DA))$. Also, for a module $X \in \mathcal{X}$, define $n_{\mathcal{S}}(X)$ to be the length of a shortest path of irreducible morphisms from a module in $\mathcal{S}$ to X if such a path exists or ∞ otherwise.

Corollary 3.7 For each $j \leq \infty$, we have $X \in \mathcal{P}_j(\mathcal{X})$ if and only if $n_{\mathcal{S}}(X) = j$.

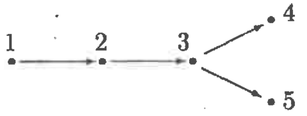
Proof: We first observe that $\mathcal{P}_0(\mathcal{X}) = \mathcal{S}$. Also, it follows from [4] that the modules of $\mathcal{P}(\mathcal{X}) = \cup_{n<\infty}\mathcal{P}_n(\mathcal{X})$ belong to the Auslander-Reiten component Γ of mod$B$ containing $\mathcal{S}$ (which is a connecting component).

Claim. If $X \in \mathcal{P}_j(\mathcal{X})$, then $n_{\mathcal{S}}(X) \leq j$.

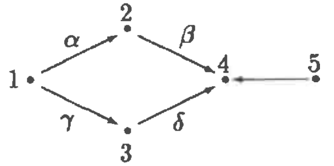
Indeed, using 3.6, it is not difficult to construct a chain of irreducible morphisms from some module of $\mathcal{S}$ to X with length j, which implies the claim. We prove the statement now by induction on $n_{\mathcal{S}}(X) = n$, the case $n = 0$ trivially following from the fact that $\mathcal{P}_0(\mathcal{X}) = \mathcal{S}$. Let $n > 0$, and let $Z \xrightarrow{f} X$ be a minimal right almost split morphism ending with X. Observe that Z has an indecomposable summand Z' with $n_{\mathcal{S}}(Z') = n-1$. By induction, $Z' \in \mathcal{P}_{n-1}(\mathcal{X})$. By 3.6, we infer that $X \in \mathcal{P}_{n-1}(\mathcal{X}) \cap \mathcal{P}_n(\mathcal{X})$. Using the claim we conclude that $X \in \mathcal{P}_n(\mathcal{X})$, as required. □

We finish the paper with an example showing the relative postprojective partitions for $\mathcal{X}$ and $\mathcal{Y}$ for a tilted algebra.

Example 3.8 We start with the hereditary k-algebra A given by the quiver

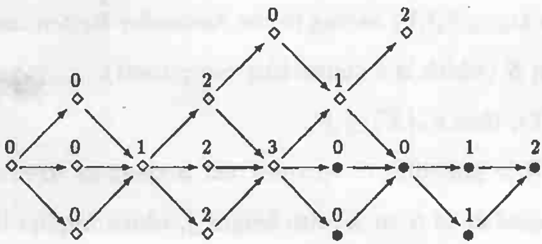


Consider now the tilting module $T_A = P_1 \oplus P_2 \oplus \tau_A^{-2}P_1 \oplus \tau_A^{-2}P_4 \oplus \tau_A^{-2}P_5$ (where P_i indicates the indecomposable projective corresponding to the vertex i). The algebra $B = \text{End}T$ is given by the quiver



with $\beta\alpha = \delta\gamma$

Its Auslander-Reiten quiver has the following shape



where the modules marked with $\diamond$ are the torsion-free modules (those in $\mathcal{Y}$) and the ones marked with $\bullet$ are the torsion modules (those in $\mathcal{X}$). The number next to a module indicate the level of the module in the corresponding postprojective partition $\mathcal{P}_i(\mathcal{Y})$ or $\mathcal{P}_i(\mathcal{X})$, respectivelly. □

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