A HIGHER-ORDER NON-AUTONOMOUS SEMILINEAR PARABOLIC EQUATION

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ABSTRACT. In this paper, we study results of well-posedness and regularity of higher order in time abstract non-autonomous semilinear Cauchy problems associated with Newton's binomial theorem and the theory of sectorial operators. Our approach to parabolic problems of arbitrarily order n apparently has never been addressed earlier in the existing literature. Also, we present applications to evolutionary equations involving the fractional Laplacian in bounded smooth domains of $\mathbb{R}^N$.

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CONTENTS

1. INTRODUCTION

In this paper, we consider higher-order non-autonomous semilinear parabolic equations as

$$\left(\frac{d}{dt}+A^{\frac{1}{n}}\right)^n u = f(t,u), \quad \left(\frac{d}{dt}+A^{\frac{1}{n}}\right)^n u = \sum_{k=0}^{n}\binom{n}{k}\frac{d^k}{dt^k}A^{\frac{n-k}{n}}u, \quad \binom{n}{k}=\frac{n!}{k!(n-k)!}, \tag{1.1}$$

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with the initial conditions given by

$$\frac{d^k u}{dt^k}(t_0) = u_k \in X^{\frac{n-k-1}{n}}, \text{ for } k = 0, 1, \dots, n-1, \tag{1.2}$$

where $t_0 \in \mathbb{R}$, X is a separable Hilbert space and $A : D(A) \subset X \to X$ is a linear, closed, densely defined, self-adjoint, and positive definite unbounded operator with A^{-1} being a compact operator on X. From this, we conclude that A is a sectorial operator in the sense of Henry [19, Definition 1.3.1].

This allows us to define the fractional power $A^{-\alpha}$ of order $\alpha \in (0,1)$ according to Amann [1, Formula 4.6.9] and Henry [19, Theorem 1.4.2] by

$$A^{-\alpha} = \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty \lambda^{-\alpha}(\lambda I + A)^{-1} d\lambda. \tag{1.3}$$

Moreover, it follows from Amann [1, Proposition 4.6.3] that for any $0 < \alpha < 1+m$, $m \in \mathbb{N}$,

$$A^{-\alpha} = \frac{\sin(\alpha\pi)}{\pi} \frac{m!}{(1-\alpha)(2-\alpha)\cdots(m-\alpha)} \int_0^\infty \lambda^{-\alpha+m}(\lambda I + A)^{-m-1} d\lambda. \tag{1.4}$$

This operator $A^{-\alpha}$ is bounded and injective, which allow us to define A^α as the inverse of $A^{-\alpha}$, see e.g. Henry [19, Theorem 1.4.2]. Additionally, A^α is a closed, densely defined linear operator and we denote by $X^\alpha = D(A^\alpha)$ for $\alpha \in [0,1)$, taking $A^0 := I$ on $X^0 := X$ when $\alpha = 0$. Recall that X^α is dense in X for all $\alpha \in (0,1]$, for details see Amann [1, Theorem 4.6.5]. The fractional power space X^α endowed with the graph norm

$$\| \cdot \|_{X^\alpha} := \|A^\alpha \cdot \|_X$$

is a Banach space, see [12, (3.0.24)]. It is not difficult to show that $-A^\alpha$ is the generator of a strongly continuous analytic semigroup on X, which we will denote by $\{\exp(-tA^\alpha) : t \geqslant 0\}$, see Kato [21, Theorem 2] for any $\alpha \in [0,1]$. With this notation, we have $X^{-\alpha} = (X^\alpha)'$, the dual space of X^α, for all $\alpha > 0$, see Amann [1] for the characterization of the negative scale.

We require the nonlinearity f to be a map defined in $[t_0, \infty) \times X^{\frac{n-1}{n}}$ taking values on X fulfilling the Hölder condition in the variable t and the Lipschitz continuous condition in the variable u on every bounded subset of $[t_0, \infty) \times X^{\frac{n-1}{n}}$.

Higher-order differential equations have already been extensively studied in the literature in different contexts, see e.g. the works of Balakrishnan [2] on fractional powers of closed operators, where the author mentions the equation

$$\Big(\frac{d^n}{dt^n} \pm A\Big)u = 0, \tag{1.5}$$

with $n \geqslant 2$ as the main motivation to study fractional powers of closed operators, to order of obtaining results of well-posedness for (1.5) in some sense, under suitable spectral conditions on the linear operator A. In [15, 16, 17] the solvability of the equation

$$\Big(\frac{d^n}{dt^n} + A\Big)u = 0,$$

is considered on linear topological spaces under suitable spectral conditions on the linear operator A, in the sense of theory of strongly continuous semigroup of bounded linear operators and cosine family. In [29, 30, 31] the author consider the equation

$$\Big(\frac{d^n}{dt^n}+\frac{d^{n-1}}{dt^{n-1}}A\Big)u+\sum_{k=0}^{n-2}\frac{d^k}{dt^k}B_k u=0, \tag{1.6}$$

and results on solvability of (1.6) under suitable spectral conditions on the linear operators A and B_k for $k=0,1,2,\ldots,n-2$. In these articles, the author consider a matrix approach and this also will use here. We also can quote the references [22, 24, 25, 26, 27] and [37], where the authors consider linear higher-order differential equations as (1.6) and results of well-posedness and regularity are obtained in different contexts. In particular, for third-order differential equation we recommend [6, 7], see also [11, 14, 20, 23, 34, 35, 36].

This paper contributes especially to the study of the case $n\geqslant 3$, since the matter of solvability, regularity and fractional approximations of (1.1) for $n=1$ or $n=2$ are well known, see e.g. [3], [4], [5], [13]. Nevertheless, for higher order cases, the semilinear Cauchy problems associated with (1.1) have not yet been addressed in the literature with a sectorial operator approach, as we propose here.

To better present our results, we introduce some notation and terminology. Consider the phase space

$$Y=X^{\frac{n-1}{n}}\times X^{\frac{n-2}{n}}\times X^{\frac{n-3}{n}}\times\cdots\times X$$

which is a Banach space equipped with the norm given by

$$\|\cdot\|_Y^2=\|\cdot\|^2_{X^{\frac{n-1}{n}}}+\|\cdot\|^2_{X^{\frac{n-2}{n}}}+\|\cdot\|^2_{X^{\frac{n-3}{n}}}+\cdots+\|\cdot\|^2_X.$$

Here, we consider distinct norms on each factor of the product space Y due to the dissipativity theory of linear operators in the theory of strongly continuous semigroups in Banach spaces, see e.g. Amann [1] and Pazy [33]. Moreover, the choice of this particular phase space Y becomes clear once we pose the problem as a system of n equations, discussed in the sequel.

We can restate the initial value problem associated with (1.1) in X as a semilinear Cauchy problem in Y, letting $v_1=u$, $v_2=\frac{du}{dt}$, $v_3=\frac{d^2u}{dt^2}$, $\ldots$, $v_n=\frac{d^{n-1}u}{dt^{n-1}}$,

$$\mathbf{u}=\begin{bmatrix} v_1\\ v_2\\ v_3\\ \vdots\\ v_n\end{bmatrix},$$

and we consider the problem

$$\begin{cases}\dfrac{d\mathbf{u}}{dt}+\Lambda_n\mathbf{u}=F(t,\mathbf{u}), \quad t>t_0,\\ \mathbf{u}(t_0)=\mathbf{u}_0=(u_0,u_1,u_2,\ldots,u_{n-1})\in Y,\end{cases} \tag{1.7}$$

where the unbounded linear operator $\Lambda_n : D(\Lambda_n)\subset Y\to Y$ is defined by

$$D(\Lambda_n)=X^1\times X^{\frac{n-1}{n}}\times X^{\frac{n-2}{n}}\times\cdots\times X^{\frac{1}{n}}, \tag{1.8}$$

and

$$\Lambda_n \mathbf{u} = \begin{bmatrix} 0 & -I & 0 & \cdots & 0 & 0 \\ 0 & 0 & -I & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -I \\ A & \binom{n}{1}A^{\frac{n-1}{n}} & \binom{n}{2}A^{\frac{n-2}{n}} & \cdots & \binom{n}{n-2}A^{\frac{2}{n}} & \binom{n}{n-1}A^{\frac{1}{n}} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{n-1} \\ v_n \end{bmatrix}$$

$$:= \begin{bmatrix} -v_2 \\ -v_3 \\ -v_4 \\ \vdots \\ -v_n \\ Av_1 + \binom{n}{1}A^{\frac{n-1}{n}}v_2 + \binom{n}{2}A^{\frac{n-2}{n}}v_3 + \cdots + \binom{n}{n-2}A^{\frac{2}{n}}v_{n-1} + \binom{n}{n-1}A^{\frac{1}{n}}v_n \end{bmatrix}, \tag{1.9}$$

for

$$\mathbf{u} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} \in D(\Lambda_n).$$

The nonlinearity F in (1.7) is given by

$$F\left(t, \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ f(t,v_1) \end{bmatrix}. \tag{1.10}$$

From now on, we denote

$$Y^1 = D(\Lambda_n) = X^1 \times X^{\frac{n-1}{n}} \times X^{\frac{n-2}{n}} \times \cdots \times X^{\frac{1}{n}}.$$

In order to get the well-posedness of equation (1.1), it will be necessary to study the properties of the linear operator Λ_n, such as the description of its inverse, the localization of its spectrum, sectoriality, and, in consequence, generation of an analytic semigroup. We can then connect that information with the evolutionary equation (1.7) using the semigroup theory applied to PDEs as in [33].

A similar approach to semigroup generation by matrix operators was performed by Nagel in [28]. However, the author focus on 2×2 matrix operators. Our approach to parabolic problems of arbitrarily order n apparently has never been addressed earlier in the existing literature.

This paper is organized as follows. In Section 2 we study the spectral behavior of the unbounded linear operator Λ_n and we prove the main results of this paper which treats the sectoriality of Λ_n for any $n \in \mathbb{N}$ and the well-posedness of the semilinear Cauchy problems (1.7) and consequently (1.1). Finally, in Section 3 we present applications to evolutionary equations involving the fractional Laplacian in bounded smooth domains of $\mathbb{R}^N$.

2. Spectral behavior

In this section, we study the spectral behavior of the unbounded linear operator Λ_n and the semilinear Cauchy problem (1.7) in Y.

Proposition 2.1. *Let Λ_n be the unbounded linear operator defined in* (1.8)-(1.9). *Then the following hold.*

i) Λ_n is closed and densely defined;

ii) $0 \in \rho(\Lambda_n)$, where $\rho(\Lambda_n)$ denotes the resolvent set of the operator Λ_n, and

$$\Lambda_n^{-1}\mathbf{u} = \begin{bmatrix} \binom{n}{1}A^{-\frac{1}{n}} & \binom{n}{2}A^{-\frac{2}{n}} & \cdots & \binom{n}{n-2}A^{\frac{2-n}{n}} & \binom{n}{n-1}A^{\frac{1-n}{n}} & A^{-1} \\ -I & 0 & \cdots & 0 & 0 & 0 \\ 0 & -I & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -I & 0 & 0 \\ 0 & 0 & \cdots & 0 & -I & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{n-1} \\ v_n \end{bmatrix}$$

$$:= \begin{bmatrix} \binom{n}{1}A^{-\frac{1}{n}}v_1+\binom{n}{2}A^{-\frac{2}{n}}v_2+\cdots+\binom{n}{n-2}A^{\frac{2-n}{n}}v_{n-2}+\binom{n}{n-1}A^{\frac{1-n}{n}}v_{n-1}+A^{-1}v_n \\ -v_1 \\ -v_2 \\ -v_3 \\ \vdots \\ -v_{n-1} \end{bmatrix},$$

for any

$$\mathbf{u} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} \in Y.$$

Moreover, Λ_n^{-1} is a compact operator on Y and Λ_n has compact resolvent.

Proof: *i)* Firstly, note that the inclusion $Y^1 \subset Y$ is dense (the inclusions $X^\alpha \subset X^\beta$ are dense for $\alpha \geq \beta \geq 0$). Secondly, the operator Λ_n is closed. Indeed, if $\mathbf{u}_j = \begin{bmatrix} v_{1,j} \\ v_{2,j} \\ v_{3,j} \\ \vdots \\ v_{n,j} \end{bmatrix} \in D(\Lambda_n)$ with $\mathbf{u}_j \to \mathbf{u} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix}$ in Y as $j \to \infty$, and $\Lambda_n \mathbf{u}_j \to \varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \vdots \\ \varphi_n \end{bmatrix}$ in Y as $j \to \infty$, then for each $k \in \{2, \ldots, n\}$

$$v_{k,j} \to v_k \text{ in } X^{\frac{n-k}{n}}, \quad v_{k,j} \to -\varphi_{k-1} \text{ in } X^{\frac{n-k+1}{n}}, \text{ as } j \to \infty$$

and consequently, for each $k \in \{2, \ldots, n\}$ we have $v_k = -\varphi_{k-1} \in X^{\frac{n-k+1}{n}}$.

Therefore, $\mathbf{u} = [v_1\ v_2\ \cdots\ v_{n-1}\ v_n]^T$ is such that $v_2 \in X^{\frac{n-1}{n}} = X^{\frac{n-k+1}{n}}$, $v_3 \in X^{\frac{n-2}{n}}$, $\cdots$, $v_{n-1} \in X^{\frac{n-(n-1)+1}{n}} = X^{\frac{2}{n}}$ and $v_n \in X^{\frac{1}{n}}$. It remains to check that $v_1 \in X^1$ in order to conclude that $\mathbf{u} \in Y^1$.

Next, we have

$$Av_{1,j} + \binom{n}{1}A^{\frac{n-1}{n}}v_{2,j} + \binom{n}{2}A^{\frac{n-2}{n}}v_{3,j} + \cdots + \binom{n}{n-2}A^{\frac{2}{n}}v_{n-1,j} + \binom{n}{n-1}A^{\frac{1}{n}}v_{n,j}$$

converges to φ_n in X as $j \to \infty$; that is,

$$A\left(v_{1,j} + \binom{n}{1}A^{-\frac{1}{n}}v_{2,j} + \binom{n}{2}A^{-\frac{2}{n}}v_{3,j} + \cdots + \binom{n}{n-2}A^{\frac{2-n}{n}}v_{n-1,j} + \binom{n}{n-1}A^{\frac{1-n}{n}}v_{n,j}\right)$$

converges to φ_n in X as $j \to \infty$.

Moreover

$$v_{1,j}+\binom{n}{1}A^{-\frac{1}{n}}v_{2,j}+\binom{n}{2}A^{-\frac{2}{n}}v_{3,j}+\cdots+\binom{n}{n-2}A^{\frac{2-n}{n}}v_{n-1,j}+\binom{n}{n-1}A^{\frac{1-n}{n}}v_{n,j}$$

converges to

$$v_{1}+\binom{n}{1}A^{-\frac{1}{n}}v_{2}+\binom{n}{2}A^{-\frac{2}{n}}v_{3}+\cdots+\binom{n}{n-2}A^{\frac{2-n}{n}}v_{n-1}+\binom{n}{n-1}A^{\frac{1-n}{n}}v_{n}$$

in X as $j\to\infty$, and consequently, since A is a closed operator, we conclude that

$$v_{1}+\binom{n}{1}A^{-\frac{1}{n}}v_{2}+\binom{n}{2}A^{-\frac{2}{n}}v_{3}+\cdots+\binom{n}{n-2}A^{\frac{2-n}{n}}v_{n-1}+\binom{n}{n-1}A^{\frac{1-n}{n}}v_{n}$$

belongs to X^1, which allows us to conclude that $v_1\in X^1$ and

$$A\left(v_{1}+\binom{n}{1}A^{-\frac{1}{n}}v_{2}+\binom{n}{2}A^{-\frac{2}{n}}v_{3}+\cdots+\binom{n}{n-2}A^{\frac{2-n}{n}}v_{n-1}+\binom{n}{n-1}A^{\frac{1-n}{n}}v_{n}\right)=\varphi_n.$$

Therefore, $\mathbf{u}\in D(\Lambda_n)$ and $\Lambda_n\mathbf{u}=\varphi$.

Item *ii*) follows from the definition of Λ_n^{-1} which takes bounded subsets of Y into bounded subsets of Y^1, the latter space being compactly embedded in Y. □

Proposition 2.2. *Let Λ_n be the unbounded linear operator defined in* (1.8)-(1.9). *Then $-\Lambda_n$ is not a dissipative operator in Y, according to Pazy* [33, Definition 4.1, Chapter 1].

Proof: Let $\mathbf{u}=\begin{bmatrix}v_1\\v_2\\v_3\\\vdots\\v_n\end{bmatrix}\in Y^1$. Then

$$\begin{aligned}&\Big\langle -\Lambda_n\begin{bmatrix}v_1\\v_2\\v_3\\\vdots\\v_n\end{bmatrix},\begin{bmatrix}v_1\\v_2\\v_3\\\vdots\\v_n\end{bmatrix}\Big\rangle_Y\\&=\langle v_2,v_1\rangle_{X^{\frac{n-1}{n}}}+\langle v_3,v_2\rangle_{X^{\frac{n-2}{n}}}+\langle v_4,v_3\rangle_{X^{\frac{n-3}{n}}}+\cdots+\langle v_n,v_{n-1}\rangle_{X^{\frac{n-1}{n}}}\\&-\langle Av_1+\binom{n}{1}A^{\frac{n-1}{n}}v_2+\binom{n}{2}A^{\frac{n-2}{n}}v_3+\cdots+\binom{n}{n-2}A^{\frac{2}{n}}v_{n-1}+\binom{n}{n-1}A^{\frac{1}{n}}v_n,v_n\rangle_X.\end{aligned}$$

In particular, if $v_1=v_2\in X^1$ and $v_1\neq 0$ and $v_j=0$ for $j\in\{3,\dots,n\}$, then

$$\Big\langle -\Lambda_n\begin{bmatrix}v_1\\v_2\\v_3\\\vdots\\v_n\end{bmatrix},\begin{bmatrix}v_1\\v_2\\v_3\\\vdots\\v_n\end{bmatrix}\Big\rangle_Y=\|v_1\|^2_{X^{\frac{n-1}{n}}}>0.$$

□

Explicitly, this means that $-\Lambda_n$ is not an infinitesimal generator of a strongly continuous semigroup of contractions in Y. As a matter of fact, as we shall see in the sequel, we will be able to prove that $-\Lambda_n$ generates an analytic semigroup in Y and we will obtain the estimate

$$\|(\lambda-\Lambda_n)^{-1}\|_Y\leq\frac{M}{|\lambda|}$$

for the resolvent of Λ_n in a given sector for some $M > 0$. The fact that $-\Lambda_n$ does not generate a semigroup of contraction in Y means that this constant M is strictly greater than 1.

Proposition 2.3. *Let Λ_n be the unbounded linear operator defined in* (1.8)-(1.9). *Then*

$$\sigma(-\Lambda_n) = \Big\{\lambda \in \mathbb{C} : \lambda = -\mu_j^{\frac{1}{n}},\ \mu_j \in \sigma(A),\ j \in \mathbb{N}\Big\},$$

where $\sigma(-\Lambda_n)$ and $\sigma(A)$ denote the spectrum set of $-\Lambda_n$ and A, respectively.

Proof: Note that

$$(\lambda I + \Lambda_n)\mathbf{u} = 0 \Leftrightarrow \begin{bmatrix} \lambda v_1 - v_2 \\ \lambda v_2 - v_3 \\ \lambda v_3 - v_4 \\ \vdots \\ \lambda v_{n-1} - v_n \\ \lambda v_n + A v_1 + \binom{n}{1} A^{\frac{n-1}{n}} v_2 + \binom{n}{2} A^{\frac{n-2}{n}} v_3 + \cdots + \binom{n}{n-2} A^{\frac{2}{n}} v_{n-1} + \binom{n}{n-1} A^{\frac{1}{n}} v_n \end{bmatrix} = 0.$$

From the $n-1$ first lines of the above equality, we obtain $v_j = \lambda v_{j-1}$, where $j \in \{2, ..., n\}$. Therefore, $v_j = \lambda^{j-1} v_1$. It follows from the last line of the matrix equality above that

$$\lambda(\lambda^{n-1} v_1) + A v_1 + \binom{n}{1} A^{\frac{n-1}{n}} \lambda v_1 + \binom{n}{2} A^{\frac{n-2}{n}} \lambda^2 v_1 + \cdots + \binom{n}{n-2} A^{\frac{2}{n}} \lambda^{n-2} v_1 + \binom{n}{n-1} A^{\frac{1}{n}} \lambda^{n-1} v_1 = 0,$$

which is equivalent to $(\lambda I + A^{\frac{1}{n}})^n v_1 = 0$.

Therefore, $(\lambda I + \Lambda_n)$ is injective if and only if $(\lambda I + A^{\frac{1}{n}})^n$ is injective. Since Λ_n has compact resolvent (Proposition 2.1), its spectrum consists entirely of isolated eigenvalues and

$$\begin{aligned} \sigma(-\Lambda_n) &= \{\lambda \in \mathbb{C} :\ (\lambda I + A^{\frac{1}{n}})^n \text{ is not injective}\} \\ &= \{\lambda \in \mathbb{C} :\ (\lambda I + A^{\frac{1}{n}}) \text{ is not injective}\} \\ &= \{\lambda \in \mathbb{C} :\ \lambda = -\mu_j^{\frac{1}{n}},\ \mu_j \in \sigma(A)\}. \end{aligned}$$

□

Recall that A is positive definite, self-adjoint and has compact resolvent. Its spectrum is given by $\sigma(A) = \{\mu_j : 0 < \mu_1 < \mu_2 < ... < \mu_j < ...\}$. It follows from the above proposition that Λ_n has a spectrum given by $\sigma(\Lambda_n) = \{\mu_j^{\frac{1}{n}} : 0 < \mu_1 < \mu_2 < ... < \mu_j < ...\}$.

Proposition 2.4. *Let Λ_n be the unbounded linear operator defined in* (1.8)-(1.9)*. Then for each $\lambda \in \rho(-\Lambda_n)$ we have*

$$(\lambda I + \Lambda_n)^{-1} =$$

$$\begin{bmatrix} \sum_{k=1}^{n}\binom{n}{k}\lambda^{k-1}A^{\frac{n-k}{n}} & \sum_{k=2}^{n}\binom{n}{k}\lambda^{k-2}A^{\frac{n-k}{n}} & \sum_{k=3}^{n}\binom{n}{k}\lambda^{k-3}A^{\frac{n-k}{n}} & \cdots & \lambda I+nA^{\frac{1}{n}} & 1 \\ -A & \sum_{k=2}^{n}\binom{n}{k}\lambda^{k-1}A^{\frac{n-k}{n}} & \sum_{k=3}^{n}\binom{n}{k}\lambda^{k-2}A^{\frac{n-k}{n}} & \cdots & \lambda^2 I-n\lambda A^{\frac{1}{n-1}} & \lambda I \\ -\lambda A & -\sum_{k=1}^{2}\binom{n}{k-1}\lambda^{k-1}A^{\frac{n-k+1}{n}} & \sum_{k=3}^{n}\binom{n}{k}\lambda^{k-1}A^{\frac{n-k}{n}} & \cdots & \lambda^3 I-n\lambda^2 A^{\frac{1}{n-1}} & \lambda^2 I \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\lambda^{n-3}A & -\sum_{k=1}^{2}\binom{n}{k-1}\lambda^{k+n-5}A^{\frac{n-k+1}{n}} & -\sum_{k=1}^{3}\binom{n}{k-1}\lambda^{k+n-6}A^{\frac{n-k+1}{n}} & \cdots & \lambda^{n-1}I+n\lambda^{n-2}A^{\frac{1}{n-1}} & \lambda^{n-2}I \\ -\lambda^{n-2}A & -\sum_{k=1}^{2}\binom{n}{k-1}\lambda^{k+n-4}A^{\frac{n-k+1}{n}} & -\sum_{k=1}^{3}\binom{n}{k-1}\lambda^{k+n-5}A^{\frac{n-k+1}{n}} & \cdots & -\sum_{k=1}^{n-1}\binom{n}{k-1}\lambda^{k-1}A^{\frac{n-k+1}{n}} & \lambda^{n-1}I \end{bmatrix}$$

$$\cdot (\lambda I + A^{\frac{1}{n}})^{-n}\cdot$$

$$= [e_{i,j}] \cdot (\lambda I + A^{\frac{1}{n}})^{-n},$$

where

$$e_{ij} = \begin{cases} \displaystyle\sum_{k=j}^{n}\binom{n}{k}\lambda^{k-j+i-1}A^{\frac{n-k}{n}}, & \text{if } i \leqslant j, \\ \displaystyle -\sum_{k=1}^{j}\binom{n}{k-1}\lambda^{k-j+i-2}A^{\frac{n-k+1}{n}}, & \text{if } i > j, \end{cases}$$

Proof: This result is an immediate consequence of (1.8)-(1.9). $\square$

As the main result of this subsection, we show that (1.1) is a parabolic equation, thanks to the same arguments used in the proof of [36, Theorem 2.5.2]. Namely, the same circular sector of the sectorality of $A^{\frac{1}{n}}$ allows for ensuring the sectorality of the operator Λ_n.

Theorem 2.5. *Let Λ_n be the unbounded linear operator defined in* (1.8)-(1.9)*. Then Λ_n is a sectorial operator.*

Proof: In this proof, K will denote a positive constant, not necessarily the same one. First, we note that the operator $A^{\frac{1}{n}} : D(A^{\frac{1}{n}}) \subset X \to X$ is a positive sectorial operator; that is, there exist $\phi \in (0, \frac{\pi}{2})$ and $M \geqslant 1$ such that the resolvent set $\rho(A^{\frac{1}{n}})$ contains the sector

$$\Sigma_\phi = \{\lambda \in \mathbb{C}; \phi \leqslant |\arg(\lambda)| \leqslant \pi\}$$

and for any $\lambda \in \Sigma_\phi$

$$\|(\lambda I - A^{\frac{1}{n}})^{-1}\|_{\mathcal{L}(X)} \leqslant \frac{M}{|\lambda|}.$$

It follows that, for each $k = 1, 2, 3, \dots$, $(\lambda I - A^{\frac{1}{n}})^{-k}$ is a bounded linear operator on X and

$$\|(\lambda I - A^{\frac{1}{n}})^{-k}\|_{\mathcal{L}(X)} \leqslant \frac{M^k}{|\lambda|^k}, \tag{2.1}$$

for any $\lambda \in \Sigma_\phi$. Moreover, for each $\lambda \in \Sigma_\phi$, we have the following identity

$$A^{\frac{1}{n}}(\lambda I - A^{\frac{1}{n}})^{-n} = -(\lambda I - A^{\frac{1}{n}})^{-(n-1)} + \lambda(\lambda I - A^{\frac{1}{n}})^{-n}, \tag{2.2}$$

and from this we deduce

$$A^{\frac{2}{n}}(\lambda I - A^{\frac{1}{n}})^{-n} = (\lambda I - A^{\frac{1}{n}})^{-(n-2)} - 2\lambda(\lambda I - A^{\frac{1}{n}})^{-(n-1)} + \lambda^2(\lambda I - A^{\frac{1}{n}})^{-n}. \tag{2.3}$$

Using (2.2) and (2.3) we derive

$$\begin{aligned} A^{\frac{3}{n}}(\lambda I - A^{\frac{1}{n}})^{-n} = -(\lambda I - A^{\frac{1}{n}})^{-(n-3)} + 3\lambda(\lambda I - A^{\frac{1}{n}})^{-(n-2)} - 3\lambda^2(\lambda I - A^{\frac{1}{n}})^{-(n-1)} \\ + \lambda^3(\lambda I - A^{\frac{1}{n}})^{-n}, \end{aligned} \tag{2.4}$$

and following with this argument we obtain

$$A^{\frac{k}{n}}(\lambda I - A^{\frac{1}{n}})^{-n} = \sum_{i=0}^{k}(-1)^i\binom{k}{i}\lambda^{k-i}(\lambda I - A^{\frac{1}{n}})^{-(n-i)}, \quad \binom{k}{i} = \frac{k!}{i!(k-i)!}, \tag{2.5}$$

for any $k = 1, 2, \dots, n-1$.

Thus, for each $k = 1, 2, \dots, n-1$, the linear operators $A^{\frac{k}{n}}(\lambda I - A^{\frac{1}{n}})^{-n}$ are bounded on X, and

$$\|A^{\frac{k}{n}}(\lambda I - A^{\frac{1}{n}})^{-n}\|_{\mathcal{L}(X)} \leqslant \frac{K}{|\lambda|^{n-k}}. \tag{2.6}$$

Using the expression for the resolvent of Λ_n obtained in Proposition 2.4, we have, for $\lambda \in \Sigma_\phi$ and $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} \in Y$ with $\|\mathbf{u}\|_Y \leqslant 1$,

$$(\lambda I + \Lambda_3)^{-1}\mathbf{u} = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \vdots \\ \varphi_{n-1} \\ \varphi_n \end{bmatrix}.$$

The estimates (2.6) are applied in each φ_j above, and we obtain

$$\begin{aligned} \|\varphi_1\|_{X^{\frac{n-1}{n}}} &\leqslant \frac{K}{|\lambda|}, \\ \|\varphi_2\|_{X^{\frac{n-2}{n}}} &\leqslant \frac{K}{|\lambda|}, \\ \|\varphi_3\|_{X^{\frac{n-3}{n}}} &\leqslant \frac{K}{|\lambda|}, \\ &\vdots \\ \|\varphi_{n-1}\|_{X^{\frac{1}{n}}} &\leqslant \frac{K}{|\lambda|}, \\ \|\varphi_n\|_X &\leqslant \frac{K}{|\lambda|}. \end{aligned}$$

Therefore

$$\left\|\begin{bmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\\ \vdots\\ \varphi_{n-1}\\ \varphi_n\end{bmatrix}\right\|_Y \leqslant \frac{K}{|\lambda|}.$$

□

As a consequence of this last theorem, we have the following result.

Corollary 2.6. *Let Λ_n be the unbounded linear operator defined in* (1.8)-(1.9)*. Then $-\Lambda_n$ is the infinitesimal generator of an analytic semigroup in Y.*

Proposition 2.7. *Let F be the function defined in* (1.10)*. Then, F fulfills the Hölder condition in the variable t and the Lipschitz continuous condition in the variable $\mathbf{u}$ on every bounded subsets of $[t_0,\infty)\times Y$.*

Proof: The results follows from the fact that the nonlinearity f to fulfill the Hölder condition in the variable t and the Lipschitz continuous condition in the variable u on every bounded subsets of $[t_0,\infty)\times X^{\frac{n-1}{n}}$. □

It follows from Proposition 2.3 and Theorem (2.5) that Λ_n is a sectorial operator with $\mathrm{Re}\sigma(\Lambda_n)>0$. Therefore, existence results presented in [12, Theorem 4.3.3], or [19], can be applied to the problem being studied here and we obtain the following results.

Theorem 2.8. *Let Λ_n be the unbounded linear operator defined in* (1.8)-(1.9)*, and let F be the function defined in* (1.10)*. Since F fulfills the Hölder condition in the variable t and the Lipschitz continuous condition in the variable $\mathbf{u}$ on every bounded subsets of $[t_0,\infty)\times Y$, for each $\mathbf{u}_0\in Y$, there exists a unique $Y-$solution $\mathbf{u}=\mathbf{u}(t,\mathbf{u}_0)$ of* (1.7) *defined on its maximal interval of existence $[t_0,\tau_{\mathbf{u}_0})$ such that*

$$\mathbf{u}\in C([t_0,\tau),Y)\cap C^1((t_0,\tau),Y^\beta)\cap C((t_0,\tau),Y^1),\ \ \beta\in[0,1),\ \ \tau=\tau_{\mathbf{u}_0},$$

where $Y^\alpha=D(\Lambda_n^\alpha)$ for $\alpha\in[0,1)$ is the domain of the fractional power operator Λ_n^α defined as inverse of the bounded, injective and linear operator $\Lambda_n^{-\alpha}$.

As a consequence of well-known results associated with fractional powers of linear operators, see [21, Theorem 2], with $\mathrm{Re}\sigma(\Lambda_n^\alpha)>0$ for $\alpha\in(0,1)$. Moreover, we also have the following result for the fractional counterpart of (1.7) given by

$$\begin{cases}\dfrac{d\mathbf{u}^\alpha}{dt}+\Lambda_n^\alpha\mathbf{u}^\alpha=F(t,\mathbf{u}^\alpha),\ t>t_0,\ 0<\alpha<1,\\ \mathbf{u}^\alpha(t_0)=\mathbf{u}_0^\alpha,\end{cases}\tag{2.7}$$

namely

Theorem 2.9. *Let Λ_n be the unbounded linear operator defined in* (1.8)-(1.9)*, and let F be the function defined in* (1.10)*. Since F fulfills the Hölder condition in the variable t and the Lipschitz continuous condition in the variable $\mathbf{u}$ on every bounded subset of $[t_0,\infty)\times Y$, for* *each for each* $\mathbf{u}_0^\alpha \in Y$*, there exists a unique* $Y-$*solution* $\mathbf{u} = \mathbf{u}(t, \mathbf{u}_0^\alpha)$ *of* (2.7) *defined on its maximal interval of existence* $[t_0, \tau_{\mathbf{u}_0^\alpha})$ *and such that*

$$\mathbf{u} \in C([t_0, \tau), Y) \cap C^1((t_0, \tau), Y^\beta) \cap C((t_0, \tau), Y^\alpha),\ \beta \in [0, \alpha),\ \alpha \in (0, 1),\ \tau = \tau_{\mathbf{u}_0}.$$

In the autonomous case, with f independent of t, thanks to results in [5] we can obtain a result of convergence of the solution for (2.7) as $\alpha \to 1^-$ on bounded subsets of Y.

3. Applications

Let $n \geqslant 3$ be a natural number. We can consider a bounded domain $\Omega \subset \mathbb{R}^N$ with smooth (at least $C^{2,\alpha}$) boundary with $N \in \mathbb{N}$, $N \geqslant 1$ and the unbounded linear operator $A = -\Delta_D$, where Δ_D denotes the Laplacian operator with homogeneous Dirichlet boundary condition. Its $L^2(\Omega)$-normalized eigenfunctions are denoted by w_j, and its eigenvalues counted with their multiplicities are denoted by μ_j; that is,

$$-\Delta_D w_j = \mu_j w_j. \tag{3.1}$$

It is well know that $0 < \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_j \leqslant \cdots$, $\mu_j \to \infty$ as $j \to \infty$, and that $-\Delta_D$ is a positive self-adjoint operator in $L^2(\Omega)$ with domain $D(-\Delta_D) = H^2(\Omega) \cap H_0^1(\Omega)$, and that Δ_D generates a compact analytic C^0-semigroup in $L^2(\Omega)$, see Henry [19].

This allows us to consider the following application of our previous analysis. We consider the semilinear evolution equation of third-order in time

$$\left(\partial_t + (-\Delta_D)^{\frac{1}{n}}\right)^n u = f(u), \quad \left(\partial_t + (-\Delta_D)^{\frac{1}{n}}\right)^n u = \sum_{k=0}^{n} \binom{n}{k} \partial_t^k (-\Delta_D)^{\frac{n-k}{n}} u, \tag{3.2}$$

with initial conditions given by

$$u(0) = u_{n,0} \in X^{\frac{n-1}{n}},\ \partial_t u(0) = u_{n,1} \in X^{\frac{n-2}{n}}, \ldots, \partial_t^{n-1} u(0) = u_{n,n-1} \in X, \tag{3.3}$$

where $X^\alpha = D((-\Delta_D)^\alpha)$ for $\alpha \in [0, 1)$ ($X = L^2(\Omega)$) is endowed with the graph norm

$$\|\cdot\|_{X^\alpha} := \|(-\Delta_D)^\alpha \cdot\|_X,$$

and the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ in (3.2) is a continuously differentiable function satisfying for some $1 < \rho \leqslant \frac{nN}{nN-4(n-1)}$ the growth condition

$$|f'(s)| \leqslant C(1 + |s|^{\rho-1}). \tag{3.4}$$

The case $n = 2$ in (3.2) refers to the semilinear strongly damped wave equation

$$\partial_t^2 u + 2(-\Delta_D)^{\frac{1}{2}} \partial_t u - \Delta_D u = f(u), \tag{3.5}$$

with initial conditions given by

$$u(0) = u_{2,0} \in H_0^1(\Omega),\ \partial_t u(0) = u_{2,1} \in L^2(\Omega). \tag{3.6}$$

Results on regularity of solutions and smoothing estimates for strongly damped wave equations have been a subject of long studies, see e.g. [8, 9, 10, 13] and our analysis includes some of these results for (3.5)-(3.6).

For $n = 3$ in (3.2) we have the equation

$$\partial_t^3 u + 3(-\Delta_D)^{\frac{1}{3}}\partial_t^2 u + 3(-\Delta_D)^{\frac{2}{3}}\partial_t u - \Delta_D u = f(u) \tag{3.7}$$

with initial conditions given by

$$u(0) = u_{3,0} \in X^{\frac{3}{2}},\ u(0) = u_{3,1} \in X^{\frac{1}{3}},\ \partial_t u(0) = u_{3,2} \in X. \tag{3.8}$$

Results on existence and regularity of solutions, as well as smoothing estimates for them, have been a subject of studies in the last years and our analysis include some of these results for (3.7)-(3.8).

The following result is a direct consequence of (3.4) via Mean Value's Theorem.

Lemma 3.1. *Let f be a real function of one real variable such that condition* (3.4) *holds. Then*

$$|f(s_1) - f(s_2)| \leqslant 2^{\rho-1}c|s_1 - s_2|\big(1 + |s_1|^{\rho-1} + |s_2|^{\rho-1}\big),$$

for any $s_1, s_2 \in \mathbb{R}$.

Moreover, we have the following result.

Lemma 3.2. *Given $s \in [0, \frac{nN}{4})$, let f be a real function of one real variable such that condition* (3.4) *holds with $1 < \rho \leqslant \frac{nN+4s}{nN-4(n-1)}$. Then the Nemitskiĭ operator $f^e : X^{\frac{n-1}{n}} \to X^{-\frac{s}{n}}$ given by $f^e(u)(x) = f(u(x))$ for any $u \in X^{\frac{n-1}{n}}$ and $x \in \Omega$ is Lipschitz continuous in bounded subsets of $X^{\frac{n-1}{n}}$.*

Proof: Let B be bounded subsets of $X^{\frac{n-1}{n}}$ and $u_1, u_2 \in B$. Since $X^\gamma \hookrightarrow H^{2\gamma}(\Omega)$ for any $\gamma > 0$, we derive for all $s \in [0, \frac{nN}{4})$

$$X^{\frac{s}{n}} \hookrightarrow H^{\frac{2s}{n}}(\Omega) \hookrightarrow L^{\frac{2nN}{nN-4s}}(\Omega).$$

Therefore $L^{\frac{2nN}{nN+4s}}(\Omega) \hookrightarrow X^{-\frac{s}{n}}$. Now by Lemma 3.1 and Hölder's inequality with $\frac{2nN}{nN+4s}$, $\frac{2nN}{nN-4(n-1)}$ and $\frac{nN}{2(s+(n-1))}$ we obtain

$$\begin{aligned}
\|f^e(u_1) - f^e(u_2)\|_{X^{-\frac{s}{n}}} &\leqslant c_0\|f^e(u_1) - f^e(u_2)\|_{L^{\frac{2nN}{nN+4s}}(\Omega)} \\
&\leqslant c_0\left(\int_\Omega [2^{\rho-1}c|u_1 - u_2|(1 + |u_1|^{\rho-1} + |u_2|^{\rho-1})]^{\frac{2nN}{nN+4s}}dx\right)^{\frac{nN+4s}{2nN}} \\
&\leqslant c_1\|u_1 - u_2\|_{L^{\frac{2nN}{nN-4(n-1)}}(\Omega)}\left(\int_\Omega (1 + |u_1|^{\rho-1} + |u_2|^{\rho-1})^{\frac{nN}{2(s+(n-1))}}dx\right)^{\frac{2(s+(n-1))}{nN}} \\
&\leqslant c_2\|u_1 - u_2\|_{L^{\frac{2nN}{nN-4(n-1)}}(\Omega)}\left(1 + \|u_1\|^{\rho-1}_{L^{\frac{nN(\rho-1)}{2(s+(n-1))}}(\Omega)} + \|u_2\|^{\rho-1}_{L^{\frac{nN(\rho-1)}{2(s+(n-1))}}(\Omega)}\right),
\end{aligned}$$

where $c_0 > 0$ is the embedding constant from $L^{\frac{2nN}{nN+4s}}(\Omega)$ to $X^{-\frac{s}{n}}$.

From Sobolev embeddings

$$X^{\frac{n-1}{n}} \hookrightarrow H^{\frac{2(n-1)}{n}}(\Omega) \hookrightarrow L^{\frac{nN(\rho-1)}{2(s+(n-1))}}(\Omega),$$

for all $1 < \rho \leqslant \frac{nN+4s}{nN-4(n-1)}$, it follows that

$$\|f^e(u_1) - f^e(u_2)\|_{X^{-\frac{s}{n}}} \leqslant C\|u_1 - u_2\|_{X^{\frac{n-1}{n}}}(1 + \|u_1\|^{\rho-1}_{X^{\frac{n-1}{n}}} + \|u_2\|^{\rho-1}_{X^{\frac{n-1}{n}}}),$$

for some constant $C > 0$. $\square$

Remark 3.3. *Since $L^{\frac{2nN}{(nN-4(n-1))\rho}}(\Omega) \hookrightarrow L^2(\Omega)$ for all $1 < \rho \leqslant \frac{nN}{nN-4(n-1)}$, it follows from the proof of the Lemma 3.2 that $f^e : X^{\frac{n-1}{n}} \to L^2(\Omega)$ is Lipschitz continuous in bounded subsets; that is,*

$$\|f^e(u) - f^e(v)\|_{L^2(\Omega)} \leqslant \tilde{c}\,\|f^e(u) - f^e(v)\|_{L^{\frac{2nN}{(nN-4(n-1))\rho}}(\Omega)} \leqslant \tilde{\tilde{c}}\|u - v\|_{X^{\frac{n-1}{n}}}.$$

The scheme below describes this situation:

$$X^{\frac{n-1}{n}} \hookrightarrow H^{\frac{2(n-1)}{n}}(\Omega) \hookrightarrow L^{\frac{2nN}{nN-4n+4}}(\Omega) \overset{f(u)\approx u^\rho}{\longmapsto} L^{\frac{2nN}{(nN-4(n-1))\rho}}(\Omega) \hookrightarrow L^2(\Omega).$$

A direct consequence of Lemma 3.2 and Remark 3.3 is the following result.

Corollary 3.4. *If f is as in Lemma 3.2, then the function $F : Y \to Y$ given by* (1.10) *is Lipschitz continuous in bounded subsets of Y.*

Now, Theorem 2.8 and [33, Theorem 1.4] guarantees local well posedness for the semilinear Cauchy problem (1.7) on Y with $A = -\Delta_D$ and f as in Lemma 3.2.

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All authors contributed to the study conception and design.

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